

## Article

# The Effect of a Spiral Density Wave on the Galaxy's Rotation Curve, as Applied to the Andromeda Galaxy (M31)

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**Abstract:** The rotational velocity curve, which is the circular velocity profile of the stars and gas in a spiral galaxy as a function of their distance from the galactic center, plays an important role in the kinematic and dynamic investigation of spiral galaxies. There are observations of approximately flat rotation curves (RC) at large distances that have introduced mass discrepancy between the theoretically derived RC and the observed one. In this paper, we derive a rotational velocity expression using a nonlinear spiral density wave solution for the surface mass density (SMD) within the disk. We show that the proposed nonlinear spiral solution is able to support the observed flat rotational velocity curve for large distances with no mass deficiency. The aim of the paper is to confirm the crucial importance of the mass distribution on the rotation curve profile. Although the model is limited by the fluid description of the galactic disk, it provides an improved rotational velocity expression and a rotation curve with no mass discrepancy in the outer part of the disk due to the inclusion of the spiral mass distribution. The disk mass has not been averaged within the exponential disk approximation, but it rather follows the observed spiral pattern given by the analytical solution of the nonlinear equation. The M31 galaxy has been chosen as the closest and well mapped spiral galaxy, similar in many aspects to our host galaxy, in order to apply a rotational velocity expression that accounts for nonlinear effects and derive RC. The obtained result can have a strong influence on large-scale gravity dynamics, as well.



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## 1. Introduction

The missing mass problem, mentioned by Zwicky as a result of applying virial theorem to the velocities of galaxies in the galaxy cluster [1], started the dark matter concept. More observations of the spiral galaxy rotation curves led to the conclusion that the observed lack of mass is a general property of any galactic system, not only on the large scale. The present research is focused on the dynamics of spiral galaxies, which are recognized as the strongest support for the dark matter hypothesis.

In the general model of spiral galaxies, one solves the Poisson's equation in cylindrical coordinates by approximating the surface mass density fitting the observation, and then the obtained gravitational potential is used to derive a rotational velocity expression [2,3]. In either case, the bulge and disk masses were not enough to fulfill the flat shape of rotation curve, so that an additional mass component distributed over the spherical halo surrounding each spiral galaxy has been involved [3].

Recently, more dynamical models of our galaxy have been explored by [4] using complex gravitational potential that is generated by three discs (gas and both thin and thick stellar discs), a bulge and a dark halo.

There are other approaches trying to explain these observations with no additional mass inclusion; modified Newtonian dynamics (MOND) [5–7], modify the acceleration and gravitational potential differently from the classical expressions [8,9]. However, both theories use additional parameters that have no physical explanation, yet.

A recent publication on the MOND theory provides a good review of two important effects on the relation between the rotational velocity and mass distribution [10]: to test the mass versus asymptotic speed relation, it is necessary to obtain a careful measurement of both the mass and the asymptotic speed; another effect is the flatness of the disk.

The importance of the accurate measurements accompanied by the interpretation of the measured data is given in [11], giving better insight into the rise of the galactic rotation problem in favor of no need for dark matter.

In this paper, we show that the mass of the galaxy itself is not enough to derive any reliable conclusion on the rotation curve shape. It is rather an important mass distribution. The importance of the mass distribution is very well demonstrated in the Figure 2.17 of [12], where in the graph, there are three different velocity profiles of the same mass distributed in different ways. It is clear that any averaging process of the mass would lead to different shapes of the rotation curve. The more rough the averaging is, the higher the discrepancy between the theoretical and observed curve is. In favor of this statement, we can add the importance of the mass distribution in the Yukawa potential [13]; in this model, a part of the exponential disk mass distribution, it is necessary to vary the gravitational constant  $G$  with respect to radius in order to recover the RC flat shape.

The importance of the mass distribution is well demonstrated in the following papers [14,15]. In the first one, luminosity data have been used as an independent constraint to identify differences between total density and light-constrained stellar density with the clear conclusion that dark matter is unnecessary to model galactic rotation and to explain the sliding M/L phenomenon, based on the 214 galaxies sample; in the second one, constructed parameter-free inverse models have been constructed that uniquely specify the mass inside any given radius and thus directly constrain density dependence on radius, solely from velocity  $v(r)$  and galactic aspect ratios. Calculated mass distributions, in this way, are consistent with visible luminosity and require no non-baryonic component.

The aim of this paper is to apply the rotational velocity expression previously derived for the Milky Way [16] to show the qualitative and quantitative influence of spiral mass distribution within the disk on the shape of rotational velocity curve. In order to do this, we chose the M31 galaxy as the closest spiral galaxy with quite accurate observational data, necessary to determine certain parameters in order to derive RC. It was discovered by Babcock that the M31 galaxy rotation curve rises as the luminosity decreases [17]. The observed, rather flat curve was confirmed by Rubin and Ford for the same galaxy using a different method [18]. Recent data on the flat rotational velocity curve for the M31 galaxy extended out to 150 arc min away from the galaxy center are given by [19]. It will be shown that the angular velocity plays an essential role in the dynamics, so that in the RC shape, the dynamics are solved self-consistently in a nonlinear regime.

Advantage of the proposed model is in its recovering of the flat shape of the RC without calling any unknown background, just by reinvestigation of the two main assumptions: linearization of governing equations and surface mass distribution. The first one led to a fully nonlinear self-consistently derived solution for the surface mass density, or equivalently gravity potential, while the second one allowed for overcoming the problem of the flatness of the rotation curve.

The paper is organized as follows: in Section 1 a standard model of the spiral galaxy is presented that will be used in order to derive expression for rotational velocity, and nonlinear dynamics are discussed over linear dynamics, including its advantages and consequences. In Section 2, a method for using the surface mass density solution of a nonlinear equation to derive an explicit formula for rotational velocity is discussed; in the same section, details are given for the estimation of the parameters involved in the rotation velocity expression. In Section 3, a derived formula is applied to the specific M31 spiral galaxy, as the closest one with well-mapped surface mass density; parameters are estimated using observed data for that galaxy, and after, the derived rotation velocity curve is compared to one obtained using direct observations, resulting in possible consequences on the rotation period of the galaxy. Finally, in Section 4, the obtained results are discussed

with a proposal of possible new method for the estimation of the mass and angular velocity of the galaxy.

## 2. Model: Non-Linear Integrable Equation and Bright Soliton Solution

A convectional picture of dynamical models of spiral galaxies uses complex gravitational potential generated by three discs (gas and both thin and thick stellar discs), a bulge and a dark halo in order to derive rotational velocity profile theoretically. We will show that a flat rotational velocity curve of spiral galaxies is possible to recover without dark halo inclusion using a density wave description in a nonlinear regime within the galactic disk. Bulge contribution is achieved by the implementation of the unperturbed variables in the fluid. We show in this work that any additional mass distributed in the surrounding halo is unnecessary, at least for the observed, flat rotation curve restoration. The gas component of the disk has been neglected as less effective on the gravity potential than the stellar one. Since its contribution is crucial at the very outer part of the galactic disk, we will discuss the possible nonlinear behavior of the gas in Section 2.2.

The motivation to apply a nonlinear, stable density perturbation solution, known as a soliton, in order to derive the rotation velocity expression is as follows: if, for the dynamical system, there exists an integrable nonlinear wave equation, then the dynamics of that system are represented by a long-lasting stable wave with constant amplitude and group velocity.

A soliton, or solitary wave, is a self-reinforcing wave packet that maintains its shape while it propagates at a constant velocity. It is a unique kind of wave that is much more stable than ordinary waves and can propagate for long distances; even after collisions, these solitary waves continue propagating nearly unperturbed.

The important property of non-linear propagating solitary waves, solitons, is not limited to the fluid dynamic properties of water or light but is also a property of the macromolecules of the biological system. The non-negligible advantage of such a wave is that it can enormously mathematically simplify exploration of very complex systems.

The soliton creation and nonlinear effects are underestimated in the astrophysical environment, although there is a number of evidence in that favor; the one is definitely a long lasting spiral structure. For the spiral galaxies, such a wave has already been derived; it is a one-dimensional curved wave due to the rotation following the spiral shape as observed. Theoretical research concerning nonlinear effects was given by [20] for the first time. There is also number of non-linear approaches to density waves in numerical studies using simulations [21,22].

The derivation of a nonlinear equation with a soliton solution for the stellar component of a thin disk galaxy has been performed by [23]. It is possible to derive the same type of nonlinear equation with the same solution but with different parameters, namely the group velocity of the wave, for the gaseous component; this will be discussed in detail in Section 2.2.

Physically, this type of solution for the surface mass density, stellar or gaseous, means that density is enhanced along the spiral due to gravitational potential, which traps the stars and gas, although they can belong to separate spirals. Here, the emphasis is on the spiral distribution of the baryonic matter. Mathematically, such a solution has a constant group velocity, which means that material (stars and/or gas) trapped by spiral gravitational potential has an enlarged velocity at the outer part of the disk than is expected by linear theory. Comparing the velocity of the trapped material at different radii of the disk, it will remain constant as long as the necessary conditions for the soliton's existence are satisfied due to the constant group velocity of the soliton. That condition is the marginal stability of the disk [23].

The rotation velocity curve, derived using this model, follows the observed curve due to the constant ratio of differential rotation (epicyclic frequency) to SMD, although these two parameters are radius-dependent. The parameters involved in the rotational velocity expression are consequences of the nonlinear density wave solution, and they have to be derived from direct observational data, not by any fitting procedure. Note that by using

the exponential density profile in the derivation of the rotational velocity, a certain amount of the mass trapped in the spiral arms has been averaged.

We have noticed that there are many misunderstandings on the importance of non-linear effects and soliton solutions, so a part of the procedure in order to derive a non-linear equation for the galactic disk, altogether with the basic properties and advantages of solitons, will be explained in detail in Section 2.1.

The proposed model is consistent with the observed level of non-axisymmetric photometric and kinematic structures in real galaxies (the former using near-infrared photometry [24], and the latter using high-resolution 2D kinematic maps [25]). The latter shows that non-circular motions are typically 4.5 percent, on average, of the mean circular velocity in a sample of about 20 spiral galaxies. The proposed nonlinear spiral solution for the density profile is valid for a radius larger than 1kpc; therefore, velocity deviations from the circular motion estimated using the nonlinear density wave solution is even less than observed by THINGS. The argument is the property of the soliton group velocity and its direction due to galactic disk marginal stability conditions. The motivation for this nonlinear approach and its implication on the spiral dynamics, as well as on the group velocity direction, will be discussed in detail at the end of Section 2.1.

The derived rotation velocity profile expression is useful in the following ways: the validity of the expression can be tested by N body simulations substituting the spiral density distribution instead of the exponential one within the disk [26]; it is possible to compare our result with MOND theory to identify a possible physical explanation for the parameters introduced in MOND. The proposed spiral, nonlinear density wave solution, accompanied by a vortex soliton derived for the inner part of galaxy [27], can be used for larger-scale investigations; it is possible to study dynamics of galactic clusters via soliton interactions, considering each galaxy in a cluster as a single solitary vortex.

Inspecting relevant parameters, it is possible to establish a general method for the estimation of spiral galaxy mass using this approach, as well as the angular velocity profile, independent of the RC. We show that, in any case, this model is sufficient to support the observed flat rotational velocity curve shape.

### 2.1. Stellar Component of the Disk

The stellar component of the galaxy is described by standard fluid equations, together with Poisson’s equation. The model is considered an infinitesimally thin disk following the Lin–Shu model [28]. In order to keep nonlinear terms in the calculations, the reductive perturbation method (RPM) developed by [29] has been used. The scale transformation for coordinates, introduced by Gardner and Morikawa, may be derived from the linearized asymptotic behavior of long waves [29]. Using a combination of this transformation of coordinates with a perturbation expansion of the dependent variables, one can obtain a single non-linear equation of the Korteweg de Vrie (KdV) or the nonlinear Schrödinger equation (NLSE) type.

In the case of a stellar galactic disk, the linearized dispersion relation reads as:

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G\rho_0|k|, \tag{1}$$

where  $\omega - m\Omega$  is the Doppler-shifted frequency,  $\Omega$  is the angular velocity,  $\kappa$  is the epicyclic frequency due to differential rotation,  $G$  is the gravitational constant,  $\rho_0$  is the unperturbed surface mass density, and  $k$  and  $\omega$  are the wave number and frequency, respectively.

The stability parameter is defined by the critical wave number  $k_2 = \frac{\kappa^2}{2\pi G\rho_0}$ , so all waves with  $k < k_2$  are purely stable. This wave number is subject to the relevant stability regime for the galaxy disc: it must be close to the threshold of instability [30]. In this case, the new transformation of variables has to be introduced, different from the pure stable case (the reason is that in marginal stability, the frequency goes to zero, so the group velocity becomes infinite). Stretched coordinates have to be introduced for this case, according to [31], contrary to the stable case, as follows:

$$\zeta = \epsilon(\tau - Cr), \eta = \epsilon^2 r \tag{2}$$

where  $\tau = t + \Omega^{-1}\varphi$ ,  $t$  is time,  $\varphi$  is the polar angle and  $C$  is the inverse of the group velocity of the nonlinear wave.

An expansion of the variables and the details of derivation can be found in [23]. We will just present the final nonlinear equation derived using RPM and the corresponding solution with a physical explanation regarding the direction of the group velocity.

The nonlinear equation of the NLSE type describing the dynamics of the galactic disk reads as:

$$i \frac{\partial}{\partial \eta} \rho^{1,1} + P \frac{\partial^2}{\partial \zeta^2} \rho^{1,1} + Q |\rho^{1,1}|^2 \rho^{1,1} = 0, \tag{3}$$

where  $P = -\frac{k_2}{\kappa} = \frac{1}{2} \frac{\partial k}{\partial \omega} < 0$ , and  $Q = -\frac{3}{2} \kappa k_2 < 0$ , which implies  $PQ > 0$ , and consequently a bright soliton solution in the form:

$$\rho^{1,1}(\zeta, \eta) = \rho_a \frac{e^{i\psi}}{ch(\sqrt{\frac{Q}{2P}} \rho_a (\zeta - 2P\eta))}, \tag{4}$$

$$\psi = P(\frac{Q}{2P} \rho_a^2 - 1)\eta + \zeta, \tag{5}$$

where  $\rho_a$  is the relative amplitude of the soliton,  $V_g = 1/P$  is the group velocity,  $w = \kappa/Q = 1/k_2$  is the width of the soliton, all in dimensionless units, and  $\psi$  is the phase. A bright soliton means that the corresponding variable surface mass density is enhanced. Note that balance between nonlinearity (third term in Equation (3)) and dispersion (second term) does not mean that parameters  $P$  and  $Q$  should be equal. Behind the soliton formation, there are more fine mechanisms balancing these two effects. The KdV type of nonlinear equation has the quadratic type of nonlinearity, while NLSE has the cubic type, which is essential for the spiral description of the galactic disk. Furthermore, NLSE contains fine structures inside the envelope soliton.

The gravity potential gradient is approximated as:

$$\frac{\partial \phi}{\partial r} = r\Omega^2 + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} 2\pi G \epsilon^n \Re(\rho^{(n,m)}(\zeta, \eta) e^{i(kr - \omega\tau)}), \tag{6}$$

where term  $r\Omega^2$  comes from the equilibrium property. We repeat briefly here the main physical background of the model and used approximations. The standard galaxy model is a fluid one defined by the continuity equation, momentum equation and Poisson’s equation in a cylindrical coordinate system. The first two equations can be projected in two spatial dimensions due to the flatness of the disk, while the third one remains in three spatial dimensions. Therefore, the solution of Poisson’s equation for the gravity potential asks for an asymptotic solution evaluated at zero vertical dimension giving an infinitesimally thin disk approximation. Equation (6) is the first approximation of the asymptotic solution of Poisson’s equation (details can be found in [28]).

Since the problem has been solved self-consistently, for the gravity potential, it is possible to obtain the same nonlinear equation with a dark soliton at the same coordinates, which means that enhanced mass is trapped within that potential well.

For typical values ( $\kappa \sim 10^{-15} s^{-1}$ ,  $\rho_0 \sim (4 - 6) \times 10^{-2} gcm^{-2} = (200 - 300) M_{\odot} pc^{-2}$ , and  $\rho_a \sim 0.4$ ), the group velocity is approximately  $V_g = 2\pi G \rho_0 / \kappa \sim 200 kms^{-1}$ . This wave velocity coincides with the rotational velocity of particles as long as the soliton wave exists; the direction of group velocity is tangent on the spiral at given  $r$ , while rotational velocity is tangent on the circle at a given  $r$ . Therefore,  $V_g = V \cos \alpha$ , where  $\alpha$  is the angle between the tangent on the spiral and the tangent on the circle at a given  $r$ . As long as this angle is smaller than  $25^{\circ}$ , we have  $\cos \alpha \simeq 1$  (this angle is known as the pitch angle, and according to 12 [12], p. 472, it is  $\sim 10 - 15^{\circ}$  [12]). Inspection of a large sample of galaxies indicates

that very few of them exceed this value [32–34]. Going further from the center of the galaxy, that angle becomes even smaller.

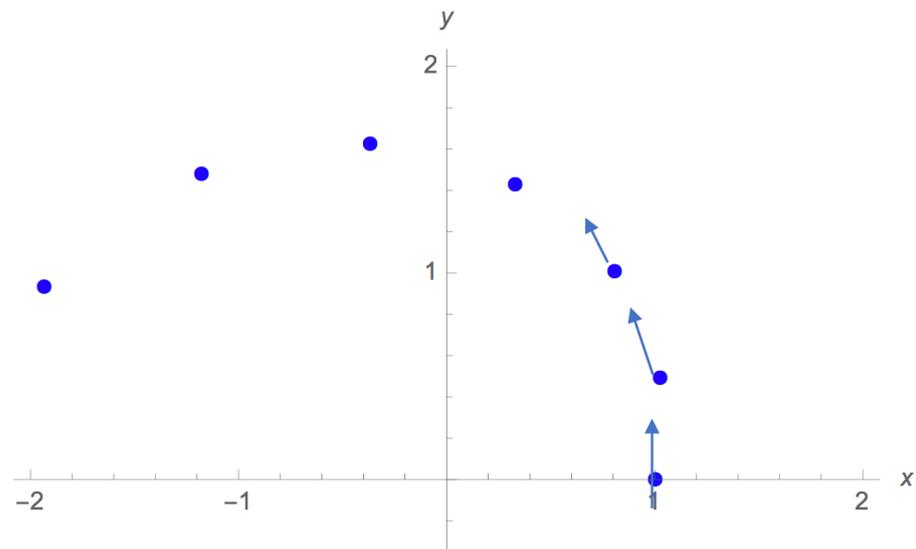
Using values for some typical spiral galaxies, we calculate that the theoretical rotational curve is valid from  $r \sim 1\text{ kpc}$  ( $\kappa \sim 10^{-15}\text{ s}^{-1}$ ,  $\rho_0 \sim (4 - 8) \times 10^{-2}\text{ g cm}^{-2} = (200 - 600)M_{\odot}\text{ pc}^{-2}$ ; a density enhancement of about 5-10 percent implies  $\rho_a \sim (0.2 - 0.8)$ ) up to infinity, theoretically, due to the linear dispersion analysis [23]. The estimation of the amplitude of the wave and enhancement of the density along the spiral can be done using the result of near-infrared photometry [35].

A rough estimation of the width of the spiral arm as  $w \sim 1\text{ kpc}$  allows us to more accurately estimate more  $\kappa$  using another property of the solitary wave; even if the amplitude slightly decreases with further radial distances, the width of the wave must increase. This property is a consequence of the stability regime of the system. Increasing the  $w$  with respect to distance from the galactic center suggests decreasing behavior of the  $\kappa$  (and consequently  $\Omega$ ), which is in accordance with the observed function of  $\Omega(r)$ , at least in the case of the Milky Way [36]. In this paper, we point out the importance of the soliton properties in the shape of the RC of typical spiral galaxies, so that any detailed derivation of such parameters is beyond the scope of this research and will be the subject of further investigation. However, this possibility is an argument to apply the results determined for our host galaxy, namely  $\Omega(r) \sim 1/r$ , on the spiral galaxies that have similar structures within a certain range of surface mass density and angular velocity [37].

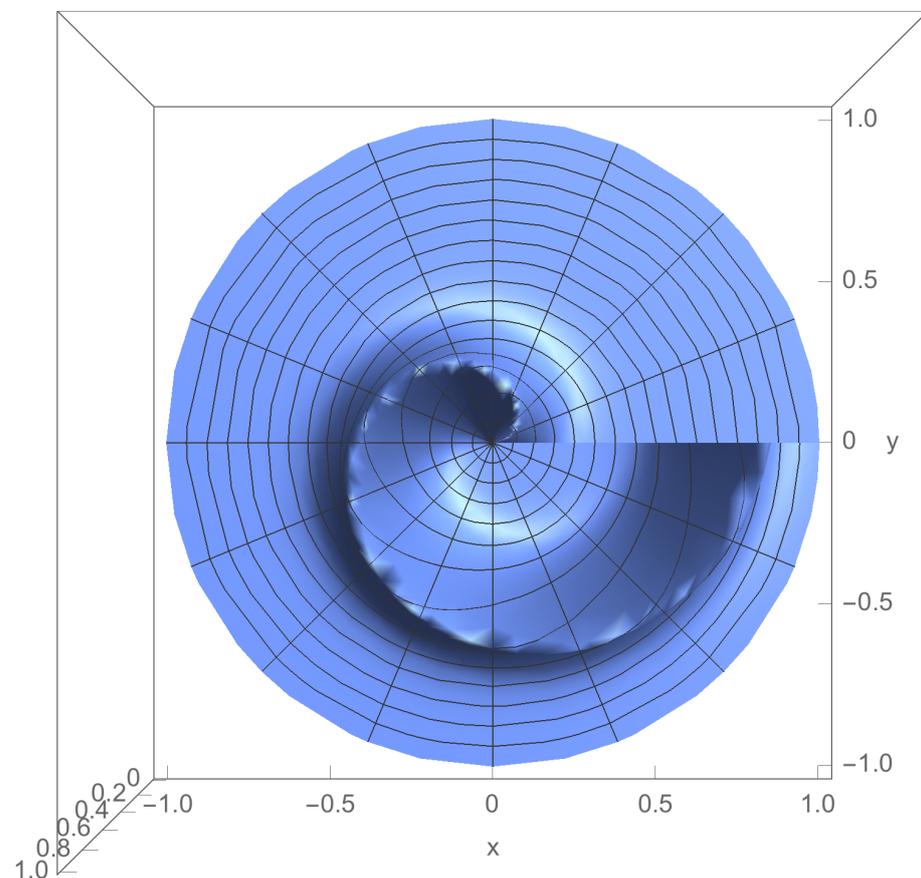
Note:

- We have followed the density wave description conducted by [28];
- We have kept nonlinear terms in the calculations—multiplication of perturbed variables;
- The existence of the NLSE means that these nonlinear terms are balanced by the dispersive properties of the wave;
- The dispersive properties of the wave are due to specific relationships between wave vector and wave frequency—the top of the wave moves faster than the bottom, causing the breakdown of the wave;
- Balance of nonlinear effects and dispersive effects results in the wave of constant amplitude and constant group velocity traveling with unchanged form as long as the conditions for the existence are fulfilled;
- In this particular case, that condition is the marginal stability of the disk (marginal stability of disk galaxies has been discussed in Chapter 15.2 of [30]);
- The azimuthal direction of the group velocity is a direct consequence of the marginal stability condition due to the transformation of specific stretched coordinates;
- The curvature of the one-dimensional wave is due to rotation;
- The gradient of the gravity potential has been approximated using Poisson's equation, given by Equation (11), or (A25) of [28];
- In [28], according to linearized theory, these dispersive effects were responsible for very short wave existence;
- Accompanied by strong rotation, they had no chance to last long enough to explain the observed structure;
- NLSE as a solution has an envelope wave that contains a carrier wave—a fine structure of space with a period much smaller than the width of a soliton, explaining the process of star formation with sharp density gradients;
- Although the fluid model description cannot predict resonances, it is very useful, and the behavior of density waves at resonance can be discussed without involving kinetic theory; the obtained cubic type of nonlinearity provides the main contribution to the Lindblad resonances;
- Nonlinear theory allows that the amplitude of the wave should not be considered as very small; therefore, even facing the resonance, it should not be completely damped, but it rather broadens the resonance function (a detailed discussion on the linearized density wave dynamics at resonance is given in Chapter 17 of [30]).

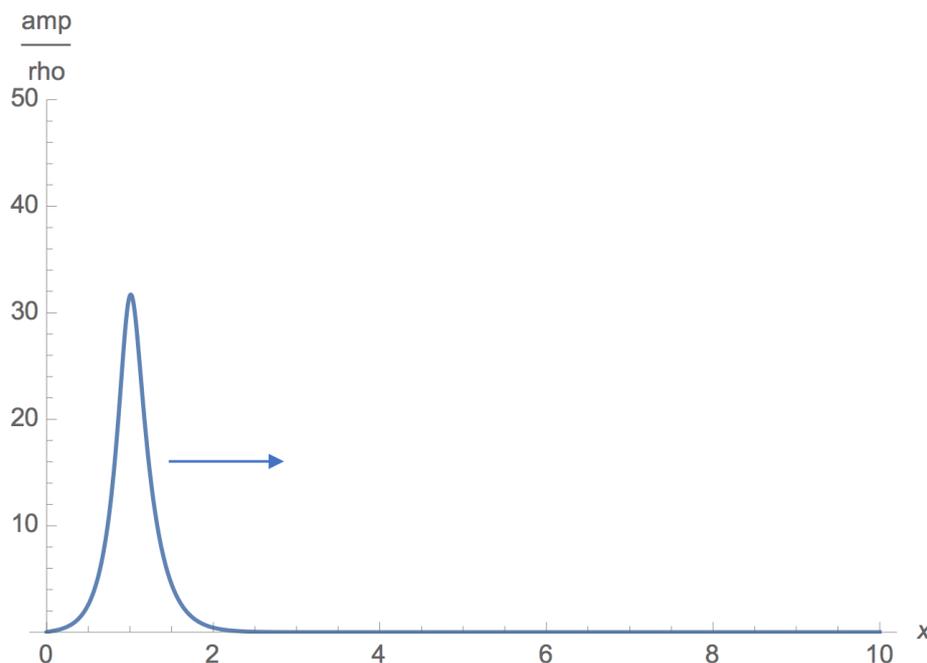
Figure 1 presents the top of the wave moving with the group velocity along the spiral (in the azimuthal direction) with increasing time; it is due to coordinate transformation that it has to be done in just one possible way since the disk is at the threshold of instability. The explicit solution of NLSE given by Equation (4) is presented in Figure 2.



**Figure 1.** The top of the wave moving with time along the spiral due to the rotation. The group velocity has a tangent direction on the spiral at each point due to the stretched coordinates transformed in the manner caused by the marginal stability of the disk. The width of the wave has not been presented in this figure, just the top the wave; top view. The shape of the wave is shown in Figure 3.



**Figure 2.** The top view of the wave as a solution given by Equation (4) of [23].



**Figure 3.** The shape and travel direction of the soliton in the case of plane geometry for ocean surface gravity waves. The x direction is toward the coast; vertical axes represent normalized wave amplitude.

The width of the soliton is given by the parameter  $Q$  in Equation (3), so by directly observed data, it is possible to estimate it, as well as the amplitude of the soliton (comparing the ambient surface mass density to the surface mass density along the spiral arm). Using the width of the wave and the estimated functional dependence of the surface mass density, the epicyclic frequency function with respect to  $r$  can be achieved with no additional assumptions.

It can be useful to explain the dynamics of the obtained spiral nonlinear wave by a simple ocean gravity wave or narrow-channel waves. In the case of narrow-channel waves, solitons are created if, and only if, the width of the channel is in a certain ratio with the depth of the water. Keeping the width of the channel constant and allowing a slight change in the water depth would destroy the soliton structure. In the case of ocean gravity waves, it has been recently confirmed that under certain conditions, earthquake disturbances of the water surface can create soliton waves that travels with constant velocity (order of magnitude faster than predicted in the linearized case) toward the harbor/seaside with constant amplitude (few times higher than predicted by linearized theory), but upon reaching the coast, the depth is suddenly changed, which causes the destruction of the wave shape, releasing a huge amount of energy. This process is known as a tsunami. The group velocity of the wave is presented in Figure 3 along the x direction, which is toward the coast. Ocean surface gravity waves, as well as the narrow channel waves, are solutions of the KdV type of nonlinear equations.

### 2.2. Gaseous Component of the Disk

The gaseous component of the disk is possible to treat in the same way as the stellar component, but in this case velocity dispersion cannot be neglected, which results in somewhat different dispersion relationships. The linear dispersion relationship for the gas component has the following form:

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G\rho_0|k| + v_s^2k^2, \tag{7}$$

where  $v_s$  is the dimensionless sound velocity, normalized by  $\pi G\rho_0/\kappa$ . It is possible to repeat all steps in RPM and obtain the same type of nonlinear equation with different

coefficients in front of the nonlinear and dispersive terms. Namely,  $Q' = k'_2\kappa$ ,  $P' = k'_2/\kappa$ , while the critical wave number reads as  $k'_2 = \frac{1}{2v_s^2}(1 \pm \sqrt{1 - 4v_s^2})$ . Therefore, the surface mass density of the gas would have a spiral morphology as well, but it would be a spiral of different amplitude and group velocity. The argument in favor of the spiral structure of the gaseous disk component is the observations of molecular gas distribution for the M31 galaxy [38], as seen in Figure 3. Although more flocculent and with strong ring pattern, a faint spiral structure of the gaseous component within the disk is observed. Gas can play an important role when discussing the behavior of the density wave at resonance. The gas component is partially transparent at the outer Lindblad resonance, so short, trailing waves manage to find their way into the outer regions, providing transport of the material and keeping the disk at the threshold of instability. For any detailed comparison of the proposed density wave, a solution with observations in the case of gas asks for the closer inspection of the short-wave existence and propagation; the existence of such waves is restricted by the shortest wavelength being not smaller than the thickness of the disk, while the intensity of sound velocity can prohibit the existence of such a wave.

### 3. Methods

In this paper, a theoretically derived expression for the surface mass density profile in weak non-linear regime is used to obtain the rotational velocity curve. For a stellar, cold disk, circular velocity  $V$  is described by

$$V^2(r) = r \frac{\partial\phi}{\partial r}, \tag{8}$$

where  $r$  is the radius, and  $\phi$  is the gravity potential. Following the widely used exponential disk approximation, one obtains a decreasing rotational velocity curve approaching the asymptotic Keplerian velocity law (details are given in [39], Figure 2). The basic assumption of that approach is an exponential function for SMD given by Equation (3) used to calculate the radial gradient of the gravitational potential in Equation (7) in order to obtain the expression for the rotational velocity in Equation (12) of [39]. As a result, it is necessary to add some additional potential, and consequently mass, in order to recover the observed flat rotational velocity curve for most of the spiral galaxies. This is usually done by the addition of a spherical halo mass distribution represented by the corresponding potential; one of the most convenient is the Miyamoto–Nagai potential [40]. However, this approach ignores certain amounts of mass concentrated within the spiral arms due to averaging over radius.

In this paper, we will use, for the SMD distribution, the exact solution of the nonlinear wave equation obtained for the spiral galaxy instead of the exponential one. The solution has been obtained by treating the galaxy as an ideal fluid self-consistently, keeping nonlinear terms in the calculation. As was already discussed, solitons are stable nonlinear waves that can propagate with unchanged amplitude and group velocity as long as certain conditions are fulfilled. In the case of spiral galaxies, that condition is a constant ratio of angular velocity and equilibrium SMD, representing the group velocity of the wave.

The gravity potential gradient  $\partial\phi/\partial r$  is approximated by Equation (6). Using the derived solution of NLSE for surface density perturbation and taking the real part of Equation (4), we obtain an expression for rotational velocity given by Equation (8) as follows:

$$V(r) = \sqrt{\Omega^2 r^2 + \frac{ar}{\cosh b(T - cr)}}. \tag{9}$$

All parameters and variables are dimensionless.

Returning to the original coordinates, the non-dimensional group velocity is multiplied by  $2\pi G\rho_0/\kappa$ , and  $T = 1 = (t + \varphi/\Omega)$  is multiplied by  $\kappa$ . Evaluating time at  $10^9$  yr (which is  $3 \times 10^{16}$  s), taking  $r$  in [kpc] and  $V$  in [ $kms^{-1}$ ], it is possible to derive the parameters  $a$ ,  $b$ , and  $c$  in Equation (9).

*Derivation of Parameters*

Parameter  $a$ , related to the amplitude of the wave  $\rho_a$  (density enhancement along the spiral) given as a number that accounts for  $r$  expressed in  $[kpc]$  while rotational velocity is expressed in  $[kms^{-1}]$ , reads as follows:

$$a = 2\pi G\rho_0\rho_a(3 \times 10^{16})[kms^{-2}]. \tag{10}$$

Parameter  $b$  is related to the relative wave amplitude and epicyclic frequency  $\kappa$  (width of the wave), and it reads as follows:

$$b = \frac{1}{2}\kappa\rho_a(3 \times 10^{16})[s^{-1}], \tag{11}$$

while parameter  $c$  is related to a group velocity of the solitary wave

$$c = \frac{1}{V_g}[skm^{-1}]. \tag{12}$$

The epicyclic frequency  $\kappa$  is defined by angular velocity  $\Omega$  [12]: in the central part of galaxy,  $\kappa \simeq 2 \Omega$ , while elsewhere,  $\kappa \simeq \Omega$ , so that  $\Omega < \kappa < 2 \Omega$ . Therefore, we are left with three relevant parameters only: surface mass density, wave amplitude that is normalized by SMD and either  $\Omega$  or  $\kappa$ . Furthermore, there is a relationship between these three parameters, e.g.,  $c = b/a$ , which leads to the fact that estimating only two of them from measurements is enough to derive the rotation curve.

We have not made an a priori assumption on surface density distribution or gravity potential. The system has been treated self-consistently, and relevant parameters to be estimated using observed data are: SMD  $\rho_0$ , amplitude of the wave (density enhancement along the spiral)  $\rho_a$  and angular velocity  $\Omega$ . Since these three parameters could be estimated from observation data for the given galaxy, this means that the derived equation for rotational velocity, Equation (9), is not a parametric one.

Furthermore, since the soliton exists, it has as a consequence a constant group velocity, which means that  $\kappa/\rho_0$  must be a constant value within all galaxy radii. This results in a general expression for the rotational velocity of a spiral galaxy depending only on the ratio of density distribution and epicyclic frequency, meaning that these two functions are correlated within the disk plane and are not independent of each other.

The shape of the rotational velocity solution is sensitive to the mentioned parameters. Since estimated values for angular velocity, surface stellar mass density and relative amplitude of the surface density are method-dependent, some of these values can differ by an order of magnitude for the same galaxy.

In the previous paper, for the Milky Way, as a first step, we plotted the mentioned ratio in order to confirm that it remains almost constant for radii larger than 3 kpc [16]. Data were used for the Milky Way because, for this galaxy, there is an established explicit power law for angular velocity [36], as well as accurate measurements for  $\kappa$  and SMD of the gas for different radii. Later, we used the SMD of stellar component derived by [41] to derive the RC for the Milky Way.

In the case of the M31 galaxy, there are no measured data for the angular velocity as in the case of our host galaxy; it can be derived from the rotation curve measurements. Since we want to prove that RC will remain flat, we will assume some power law for  $\Omega$ , similar to the one for Milky Way, with the argument given in Section 2.1 related to the possibility of accurate estimation of the  $\kappa$  and consequently  $\Omega$ . We can be assured that it will be a decreasing function with respect to  $r$ . As far as the SMD of the stellar component, a function according to [42] will be used; the resulting RC derived using Equation (9) is presented in Figure 1.

Next, we have assumed some averaged constant values for  $\rho_0$  and  $\kappa$ , although they are both  $r$ -dependent; the ratio of these two variables remains constant due to the soliton's existence. The basic property of a soliton wave is the constant group velocity of the wave, as well as the constant wave amplitude along the spiral (more precisely, the ratio of wave amplitude and the width of the wave remains constant). This particular property motivated us to expect possible support of the observed rotational velocity curve with no inclusion of any other but baryonic matter. If one thinks of the orbital velocity of a star, for example, at a certain distance from the center, the star would be trapped by the potential due to a wave that passes by that radius and enlarges the velocity of the star by the velocity of the wave. Since the wave is a soliton, its group velocity will be constant. Therefore, the expression of rotational velocity is theoretically derived by keeping nonlinear effects in the calculation. It is not derived by any kind of fitting process involved, neither in derivation of expression, nor in parameter derivation. Parameters are explicitly calculated using the SMD distribution function employed in the radial gradient of gravitational potential in order to derive the rotational velocity. The SMD function is not assumed but rather derived as a solution of integrable nonlinear equations. All previously mentioned details provide us with a general expression that can be used for any spiral galaxy.

The SMD can be derived from the rotational velocity curve or using the mass-to-light ratio, while the estimation of angular velocity has to be done using the rotational velocity curve. It would be best if it were possible to estimate both values using some method independent of the rotational velocity curve, as was done for the Milky Way [16].

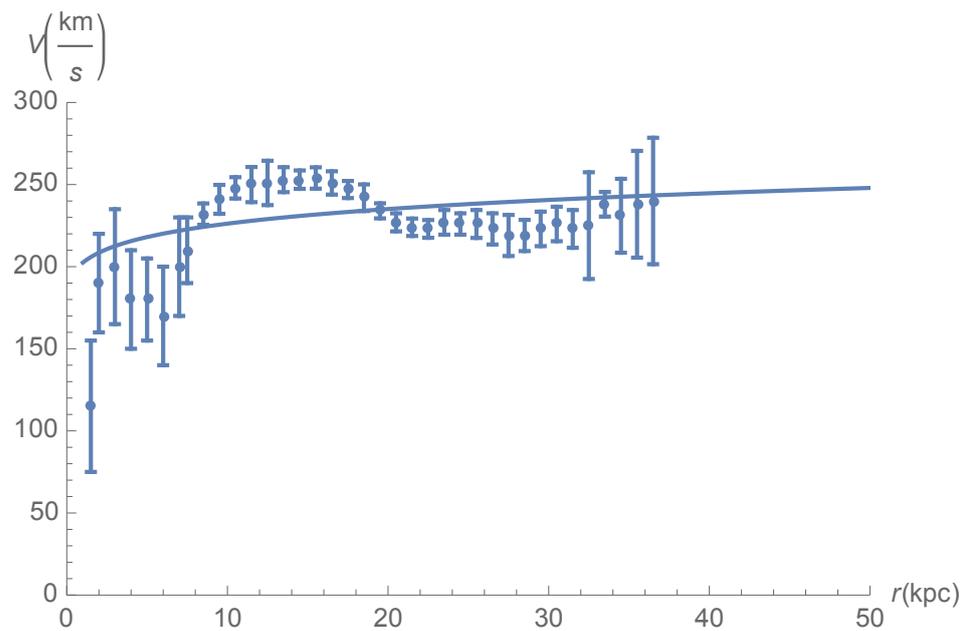
#### 4. Results

Luminosity is one of the most important characteristics of a galaxy: most luminous galaxies show a monotonically declining rotation curve in the outer part, following a broad maximum in the disk. Galaxies with intermediate luminosities have nearly flat rotation; less luminous galaxies have slightly increasing rotation velocities across the optical disk [43]. However, almost 30 percent of galaxies chosen to have similar kinematical properties, out of 30 [44], do not follow the universal velocity curve shape. Therefore, it has to be kept in mind that these mentioned common characteristics should be applied in certain situations; otherwise, it can be misleading. Nevertheless, it has been found that luminosity is not the only relevant parameter in the rotational velocity curve derivation. It has been shown that rotation also plays an essential role in the dynamical properties of a galaxy [16].

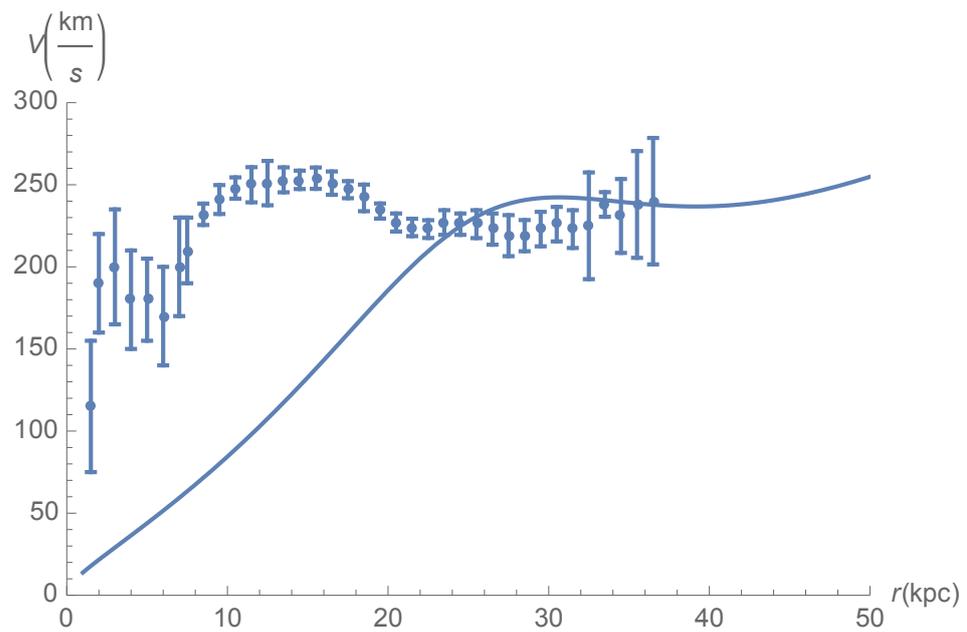
M31 is the nearest spiral galaxy to the Milky Way. Using data taken from [45], we compare the measured curve with RC derived from Equation (9) for the relevant parameters.

First, we estimate parameters  $a$ ,  $b$ , and  $c$  using the profile of surface mass density  $\rho_0$  according to [42], while for  $\Omega$ , we assume a profile similar to the Milky Way with the argument that it must be a decreasing function with respect to radius due to the constant value of group velocity (the ratio of surface mass density function and  $\kappa$  remains constant; see the first paragraph on page 5).

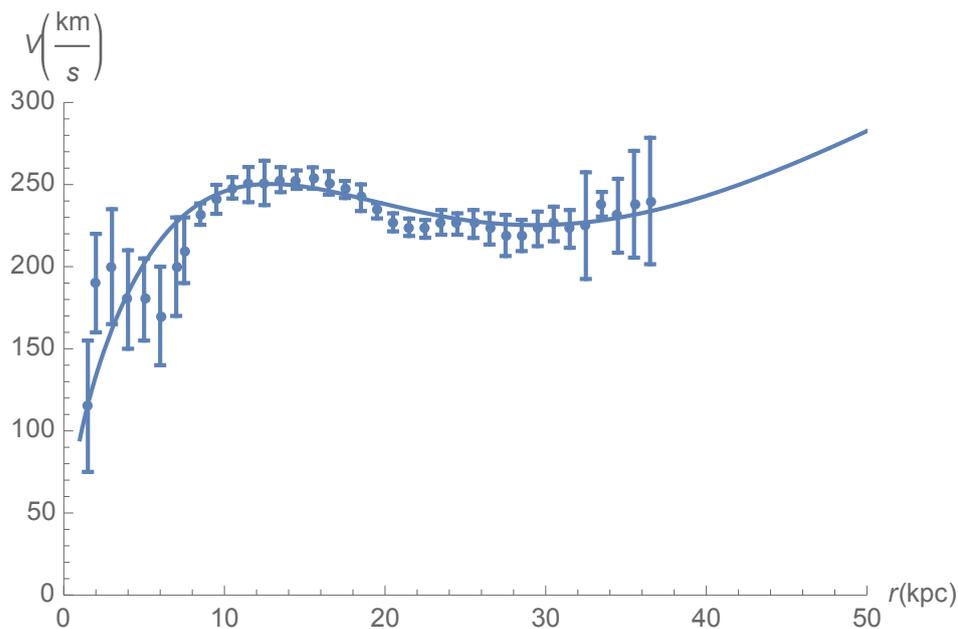
We apply this in Equation (9), and the obtained rotational velocity curve is presented in Figure 4. In the next step, we evaluate each parameter by some constant values in order to make the procedure as simple as possible. However, we discuss the estimation of each parameter since the averaging over radius is not trivial due to the strong dependence of SMD and  $\Omega$  on the radius in the inner part of the disk, while they can be taken as a constant for farther distances. The obtained RC is plotted in Figure 5. Finally, the dependence of the parameter estimation on the time scale is discussed and presented in Figure 6.



**Figure 4.** The rotational velocity curve for the galaxy M31. Dots are observed data [45], and the solid curve is the result of Equation (9) for the following parameters:  $\Omega = 100r^{-0.99} + 100/r$ ,  $a = 1000r^{-0.9}$  according to [42],  $b = 0.01$  and  $c = 0.005$ .



**Figure 5.** The rotational velocity curve for the galaxy M31. Dots are observed data [45], and the solid curve is the result of Equation (9) for the following parameters:  $\Omega = 7$ ,  $a = 1.2 \times 10^3$ ,  $b = 2$ ,  $c = 0.05$ .



**Figure 6.** The rotational velocity curve for the galaxy M31. Dots are observed data [45], and the solid curve is the result of Equation (9) for the following parameters:  $a = 8 \times 10^3$ ,  $b = 0.03$ ,  $c = 3.5$ .

#### 4.1. Radial Dependence of Angular Velocity and Surface Mass Density

Using results obtained by [36] and the power-law fit for  $\Omega$  for Milky Way, we assume in the case of M31 following  $\Omega = Ar^{-0.99} + B/r$ , with  $A = B = 100 \text{ kms}^{-1}$  for M31. The angular velocity function of M31 can differ from the Milky Way case in parameters  $A$  and  $B$ , but the radial dependence will remain as  $1/r$  due to the arguments previously discussed at the end of Section 2.1. If possible, parameters  $A$  and  $B$  can be estimated more accurately using adequate observation data and fitting procedures that are beyond the scope of this paper.

The power-law for SMD is approximated according to [42] (Figure 3) as  $\rho_0 = Fr^{-0.9}$ , where  $F = 0.3$  for the M31 galaxy, in the calculation of coefficient  $a$ . In order to obtain coefficients  $b$  and  $c$ , we can use any value of  $\kappa$  and  $\rho_0$  at the same  $r$  in order to calculate  $c = 1/V_g$ , while amplitude is estimated as  $\rho_a \sim 0.3$ , and  $\kappa$  is averaged as  $0.9 \times 10^{-15} \text{ s}^{-1}$  in order to calculate coefficient  $b$ . Using Equations (10)–(12), one obtains  $a = 10^3 r^{-0.9}$ ,  $b = 0.01 \text{ s}^{-1}$  and  $c = 0.005 \text{ km}^{-1}$ .

We substitute the relevant functions and parameter values in the rotation velocity expression in Equation (9), and the resulting RC is presented in Figure 4. The result is compared with data given in [45].

It turns out that the shape of the curve in this case is not sensitive at all to parameters  $b$  and  $c$ . Note that parameters  $a$ ,  $b$  and  $c$  are not derived by any fitting procedure but rather by direct application of the observed functions, although the profile of angular velocity, as well as the profile of SMD could be the subject of the fitting process as observation results.

#### 4.2. Rotation Curve of M31 Galaxy for Constant Parameters

##### 4.2.1. Outer Part of the Disk

Let us estimate parameters  $a$ ,  $b$  and  $c$  from Equations (10)–(12) for radii larger than 20 kpc because we expect slow changes in SMD and  $\Omega$  in that radius range. For such radii, we approximate  $\Omega \sim (5 - 10) \text{ km/skpc}$ , and consequently,  $\kappa \sim 2\Omega \sim 10^{-15} \text{ s}^{-1}$ , while for SMD estimation data given by [42] are used (Figure 3):  $\rho_0 \sim (1 - 2) \times 10^{-2} \text{ g/cm}^2 = (100 - 200) M_\odot / \text{pc}^2$ , and  $\rho_a \sim (0.1 - 0.3)$ .

The estimated values give the following parameters:  $a \sim (1 - 2) \times 10^3$ ,  $b \sim (1 - 3)$  and  $c \sim 10^{-2}$ . The comparison between observed data [45] and RC derived from Equation (9) using calculated parameters for  $r \gtrsim 20 \text{ kpc}$  is presented in Figure 5.

#### 4.2.2. Rotation Period Influence on the Parameters Estimation

It can be noted from Figure 5 that the analytically derived curve seems to be shifted to the observed data. We have inspected parameters  $a$ ,  $b$  and  $c$  for the shorter period of rotation, taking  $T \sim 10^7$  yr. In that case, using the same averaged value for  $\Omega$  and  $\rho_0$  as in the previous case, we calculate the parameters as follows:  $a \sim 8 \times 10^3$ ,  $b \sim 10^{-2}$  and  $c \sim (1 - 10)$ . In this particular case,  $\rho_a$  is taken to be close to 1, which is not in contradiction with the nonlinear theory. In the linearized approach, the wave amplitude must be much smaller than the ambient value. Enhancement of the wave amplitude will be followed by the tightness of the wave width. Going to farther radii, the amplitude of the wave can become smaller, followed by the broadening of the wave. The resulting RC is plotted in Figure 6.

Constant parameter analysis would eventually make it possible to better estimate the SMD and angular velocity of the galaxy, fitting them to the observed rotation curve.

### 5. Discussion

Luminosity is not enough to derive a clear conclusion on the type of rotation curve one could expect in spiral galaxies. It is rather necessary to follow the ratio of two values: surface mass density and angular velocity. An example of the spiral galaxy M31, presented in this paper, shows that it is possible to restore the rotational velocity curve keeping nonlinear terms in the variable expansion in order to derive disk surface mass density distribution with no inclusion of any other but baryonic matter.

It is necessary to test the model on a larger sample of galaxies, which will be the subject of further research. This can be done by optical or near-infrared scanning of the SMD within the galactic disk, corresponding to the SMD as a solution of NLSE (3), deriving  $\rho_0$  and consequently relevant parameters  $P$  and  $Q$  in that equation, and finally identifying the exact profile of epicyclic frequency and consequently angular velocity with no use of RC. Further research would possible prove that the expression of rotational velocity, including nonlinear effects and without averaging surface mass density, is a rather general one.

Derived rotational velocity, including nonlinear effects, has following implications, taking into account the mass trapped within the spiral arms of the disk, it is possible to recover a flat shape of the RC, contrary to the previously used exponential disk approximation; the result of an N-body test applying the proposed model can give better insight into the physical inputs in galactic dynamics simulations; the proposed analytical nonlinear solution could be compared to MOND theory with an aim to identify the physical meaning of parameters introduced in MOND; finally, the perturbation method of the nonlinear soliton solution can be used for studying the dynamics of the merging process of two or more galaxies.

Although the model has some assumptions made within adequate argumentation, it is analytical and self-consistent, and it is able to recover a flat rotation curve shape of spiral galaxies without calling any other but baryonic matter. This result is valid to the small scale only, and it does not resolve the problem of dark matter inclusion for clusters of galaxies, but it can be used to reinvestigate a large-scale phenomena in the nonlinear regime by a method of multiple solitary wave interactions.

Further research would be posed by making an opposite problem: the derivation of parameters  $a$ ,  $b$  and  $c$  by fitting the observed rotation curve, then estimation of angular velocity and SMD of the spiral galaxy. Additionally, some preliminary results conducted by N body simulations suggests that the evolution of the spiral nonlinear density wave leads to ring or flocculant/diffuse morphology of the disk [26].

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