

Dynamics in Interacting Scalar-Torsion Cosmology

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Abstract: In a spatially flat Friedmann–Lemaître–Robertson–Walker background space, we consider a scalar-torsion gravitational model which has similar properties to the dilaton theory. This teleparallel model is invariant under a discrete transformation similar to the Gasperini–Veneziano duality transformation. Moreover, in the gravitational action integral, we introduce the Lagrangian function of a pressureless fluid source which is coupled to the teleparallel dilaton field. This specific gravitational theory with interaction in the dark sector of the universe was investigated by using methods of the dynamical system analysis. We calculate that the theory provides various areas of special interest for the evolution of the cosmological history. Inflationary scaling solutions and the de Sitter universe are recovered. Furthermore, we calculate that there exist an attractor which provides a stable solution where the two fluid components, the scalar field and the pressureless matter, contribute in the cosmological fluid. This solution is of special interest because it can describe the present epoch. Finally, the qualitative evolution of the cosmographic parameters is discussed.

Keywords: teleparallel; scalar field; dilaton field; scalar-torsion; interaction



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1. Introduction

The detailed analysis of the recent cosmological data indicates that General Relativity may need to be modified in order to describe the observations; for a recent review, we refer the reader to [1]. The late-time cosmic acceleration has been attributed to a fluid, so-called dark energy, which has negative pressure and anti-gravity effects [2]. In order to explain the anti-gravitational effects, cosmologists have proposed the modification of the Einstein–Hilbert action by using geometric invariants [3]. In this direction, new geometrodynamical terms are introduced in the field equations which can drive the dynamics in order to explain, with a geometric approach, the observational phenomena [4,5].

Teleparallelism [6,7] includes a class of modified theories of gravity which have been widely studied in recent years [8–15]. The fundamental invariant of teleparallelism is the torsion scalar of the antisymmetric connection which plays an important role, instead of the Levi–Civita connection in general relativity. In previous studies, it has been discussed that teleparallel gravity may violate Lorentz symmetry [16], and while Lorentz violation has not yet been observed, it is common in various subjects of quantum gravity [17]. However, recent studies have shown that Lorentz symmetry can be preserved in teleparallelism, as can be seen, for instance, in [18,19]. Specifically, in [18], it was found that the introduction of a scalar field in the gravitational action integral of teleparallelism preserves the Lorentz symmetry. For a recent review on teleparallelism, we refer the reader to [20]. In the literature, teleparallel cosmology has been widely studied. For instance, in $f(T)$ teleparallel theory, the cosmological perturbations were investigated in [21,22], while in [22] it was found that $f(T)$ theory can mimic dynamical dark energy models. The mechanism which describes the Higgs inflation era in scalar-torsion theory was studied in [23]. An extension of the scalar-torsion theories with the introduction of the boundary term was introduced in [24]. For other recent studies on teleparallelism, we refer the reader to [25–29] and the references therein.

In this study, we focused on the scalar-torsion or teleparallel dark energy models [30–32] which can be seen as the analogue of the scalar-tensor theories in teleparallelism. In scalar-torsion theory, a scalar field is introduced in the gravitational action integral which is non-minimally coupled to the fundamental scalar of teleparallelism, the torsion scalar. There are various studies in the literature which indicate that the scalar-torsion theories can explain the recent cosmological observations [33,34]. In the following, we consider the existence of a matter source with zero pressure in the gravitation action which interacts with the scalar field [35–38]. In our analysis, the interaction between the scalar field and the pressureless fluid is inspired by the interaction provided in the Weyl integrable theory [39,40]. The plan of the paper is as follows.

In Section 2, we introduce the model of our consideration which is the teleparallel dilaton model coupled to a pressureless fluid source. This model belongs to the family of scalar-torsion theories from which the field equations are derived. Section 3 includes the new material of this study. We performed a detailed study of the asymptotic dynamics for the gravitational field equations for the model of our consideration. We determined the stationary points and we investigated their stability as we discussed the physical properties of the exact solutions described by the stationary points. This analysis provides important information about the cosmological viability of the proposed theory. It is clear that our model can explain the major eras of cosmological evolution and it can be used as a dark energy candidate. Furthermore, in Section 4, we discuss the evolution of the cosmographic parameters as they are provided by our model. Finally, in Section 5, we summarize our results and we draw our conclusions.

2. Teleparallel Dilaton Model

The gravitational model of our consideration is an extension of the teleparallel dilaton theory known as scalar-torsion theory where the gravitational action integral is defined:

$$S = \frac{1}{16\pi G} \int d^4x e \left[e^{-\frac{\phi}{2}} \left(T + \frac{\omega}{2} \phi_{;\mu} \phi^{;\mu} + V(\phi) + L_m \right) \right], \quad (1)$$

in which T is the torsion scalar of the antisymmetric curvatureless connection, $\phi(x^k)$ is a scalar field with potential function $V(\phi)$, ω is a nonzero constant, L_m is the Lagrangian function for the additional matter source and e is the determinant of the vierbein fields. Action integral (1) belongs to the family of gravitational models known as teleparallel dark energy models, or scalar-torsion models [30,31,41,42]. Scalar-torsion models can be seen as the analogue of scalar-tensor models in teleparallelism, in which instead of the Ricci scalar, the torsion scalar T is used for the definition of the action integral. The gravitational field equations are of second-order, however, under conformal transformations, the scalar-torsion theories are now equivalent with the quintessence model [41]. Under a conformal transformation, the scalar-torsion action integral is equivalent with that of a modified higher-order theory known as $f(T, B)$ where B is the boundary term which differs from the torsion T and the Ricci scalar [41]. This makes the scalar-torsion and scalar-tensor theories totally different from one another, which means that there is not any conformal transformation which may connect the solutions of the two theories. In this study, we assume that the matter source and the scalar field are interacting, i.e., from (1), the mixed term $e^{-\frac{\phi}{2}} L_m$ exists.

In the case of a spatially flat Friedmann–Lemaître–Robertson–Walker metric (FLRW) background space:

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \quad (2)$$

the gravitational field equations for (1) in the case of the vacuum are invariant under a discrete transformation with the origin $O(d, d)$ symmetry [43]. The resulting discrete transformation is a generalization of the Gasperini–Veneziano scale-factor duality transformation for the dilaton field in scalar tensor theory [44]. The existence of this discrete transformation is important for the study of the pre-Big Bang period of the universe as it

is described by string cosmology. However, when parameter ω is small, the Gasperini–Veneziano transformation is recovered. However, in general, the pre-Big Bang period for the teleparallel dilaton theory differs from that of string cosmology by a term provided by the nonzero constant ω .

In this study, we assumed that the scalar field ϕ interacts with the matter source. For the latter, we assumed it to be a pressureless fluid source, dust fluid, with energy density ρ_m , known as dark matter. Models with interaction in the dark sector of the universe has drawn the attention of cosmologists the last decade [45–49]. Indeed, there are various theoretical studies which show that such models are viable, while from the analysis of the cosmological data, it seems that the interacting models are supported by the observations [50–53]. The action integral (1) can be seen as the teleparallel extension of some scalar tensor models coupled to dark matter [39,40].

For the background space (2), it follows that $T = 6H^2$, $H = \frac{\dot{a}}{a}$, hence, from the action integral (1) and for a dust fluid for the matter source, we derive the modified gravitational field equations:

$$e^{-\frac{\phi}{2}} \left(6H^2 - \left(\frac{\omega}{2} \dot{\phi}^2 + V(\phi) \right) - \rho_m \right) = 0, \tag{3}$$

$$\left(2\dot{H} + 3H^2 \right) + \frac{1}{2} \left(\frac{\omega}{2} \dot{\phi}^2 - V(\phi) \right) - H\dot{\phi} = 0, \tag{4}$$

while for the matter source and the scalar field it follows:

$$\dot{\rho}_m + (3H - \dot{\phi})\rho_m = 0, \tag{5}$$

$$\omega \left(\dot{\phi}^2 - 2\ddot{\phi} - 6H\dot{\phi} \right) + 2(V - V_{,\phi}) = 0. \tag{6}$$

In the case where $\rho_m = 0$ and $V(\phi) = V_0$, the discrete transformation which keeps invariant the field equations is:

$$a \rightarrow a^{p_1} e^{p_2 \phi}, \quad \phi \rightarrow p_3 \ln a + p_4 \phi \tag{7}$$

in which $p_1 = -\frac{1+3\omega}{1-3\omega}$, $p_2 = \frac{\omega}{1-3\omega}$, $p_3 = -\frac{12}{1-3\omega}$ and $p_4 = \frac{1+3\omega}{1-3\omega}$ when $\omega \neq \frac{1}{3}$. However, in the presence of the matter source, the field equations do not remain invariant under the action of the discrete transformation. Moreover, when ω is near to zero, the discrete transformation (7) becomes the Gasperini–Veneziano duality transformation [44]. The discrete transformation (7) follows from the presence of the $O(d, d)$ symmetry for the action integral (1). In addition, the existence of the $O(d, d)$ symmetry indicates the presence of a conservation law for the classical field equations which can be used in order to integrate and write the analytic solution in closed-form expression. Finally, the $O(d, d)$ symmetry is preserved in the quantization process of the theory. Hence, a quantum operator related to the $O(d, d)$ symmetry is determined, which helps us solve the Wheeler–DeWitt Equation [43].

Additionally, from Equation (3), we define the energy density for the scalar field $\rho_\phi = \left(\frac{\omega}{2} \dot{\phi}^2 + V(\phi) \right)$, thus, in order for $\rho_\phi \geq 0$ to not have ghost terms, we assume $\omega > 0$. The pressure component p_ϕ of the scalar field from Equation (4) is defined as $p_\phi = \frac{1}{2} \left(\frac{\omega}{2} \dot{\phi}^2 - V(\phi) \right) - H\dot{\phi}$.

In the following Section, we study the general evolution for the cosmological history as it is provided by the dynamical system (3)–(6).

3. Asymptotic Dynamics

We define the new dimensionless variables:

$$x = \sqrt{\frac{\omega}{12}} \frac{\dot{\phi}}{H}, \quad y^2 = \frac{V}{6H^2}, \quad \Omega_m = \frac{\rho_m}{6H^2}, \quad \lambda = \frac{V_{,\phi}}{V} \tag{8}$$

where the field equations are written as the following algebraic-differential system:

$$\Omega_m = 1 - x^2 - y^2 \tag{9}$$

$$\frac{dx}{d\tau} = \frac{3}{2}x(x^2 - y^2 - 1) + \sqrt{\frac{3}{\omega}}y^2(1 - \lambda), \tag{10}$$

$$\frac{dy}{d\tau} = y\left(\frac{3}{2}(1 + x^2 - y^2) - \sqrt{\frac{3}{\omega}}x(1 - \lambda)\right), \tag{11}$$

$$\frac{d\lambda}{d\tau} = 2\sqrt{3}x\lambda^2(\Gamma(\lambda) - 1), \Gamma(\lambda) = \frac{V_{,\phi\phi}V}{(V_{,\phi})^2}, \tag{12}$$

in which the new independent variable is $dt = Hd\tau$.

In addition, in the new variables, the equation of state parameter for the effective fluid $w_{eff} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2}$ reads:

$$w_{eff}(x, y, \eta, \lambda) = x^2 - y^2 - \frac{2}{\sqrt{3\omega}}x. \tag{13}$$

As far as the scalar field potential is concerned, we consider the exponential potential:

$$V(\phi) = V_0e^{\lambda_0\phi}. \tag{14}$$

For the exponential potential, we infer that $\lambda = \lambda_0$ is a constant and $\Gamma(\lambda) = 1$ such that the rhs of Equation (12) is always zero. Thus, for the exponential potential, the dimension of the dynamical system is two. Moreover, in order the matter source to be physically accepted, it follows that Ω_m is a constraint as $0 \leq \Omega_m \leq 1$, and from (9), it follows that the variables (x, y) are constrained similarly as $x^2 + y^2 \leq 1$. Furthermore, we observe that the equations are invariant under the change of variables $y \rightarrow -y$, which means that without loss of generality, we can work on the branch $y \geq 0$.

Stationary Points for the Exponential Potential

We proceed our analysis by considering the exponential scalar field potential. For an arbitrary value of the free parameter λ , the dynamical system (10) and (11) admits the following stationary points $P = P(x, y)$:

Point A_1 with coordinates $A_1 = (0, 0)$, in which $\Omega_m(A_1) = 1$, and $w_{eff}(A_1) = 0$. At the stationary point the scalar field is constant, i.e., $\phi = \phi_0$. Hence, the point describes a universe dominated by the dust fluid source, where the scale factor is $a(t) = a_0t^{\frac{2}{3}}$. In order to infer the stability of the stationary points, we linearize Equations (10) and (11) around the stationary point, and we derive the eigenvalues of the linearized matrix, which are calculated as $e_1(A_1) = -\frac{3}{2}, e_2(A_1) = \frac{3}{2}$. Hence, because only one of the eigenvalues is negative, the stationary point is always a saddle point.

The stationary points $A_2^\pm = (\pm 1, 0)$ describe universes where only the scalar field contributes in the cosmological solution, i.e., $\Omega_m(A_2^\pm) = 0$. Because $y(A_2^\pm) = 0$, it means that the scalar field potential does not contribute in the total cosmological evolution, while only the kinematic part contributes, i.e., $V(\phi) \ll \dot{\phi}^2$. The effective equation of the state parameter is derived $w_{eff}(A_2^\pm) = 1 \mp \frac{2}{\sqrt{3\omega}}$. For point A_2^+ , it follows that $w_{eff}(A_2^+) < 1$. Furthermore, the asymptotic solution at A_2^+ describes an accelerated universe when $\omega > \frac{4}{3}$. On the other hand, the asymptotic solution at the stationary point A_2^- gives $w_{eff}(A_2^-) > 1$. As far as the stability properties of the points are concerned, we derive the eigenvalues $e_1(A_2^\pm) = 3 \pm \sqrt{\frac{3}{\omega}}(\lambda - 1), e_2(A_2^\pm) = 3$. Thus, point A_2^+ is a saddle point when $\lambda < 1$ and $\omega < \frac{1}{3}(1 - \lambda)^2$, otherwise A_2^+ is a source. Point A_2^- a saddle point when $\lambda > 1$ and $\omega < \frac{1}{3}(1 - \lambda^2)$.

The stationary point $A_3 = \left(\frac{1-\lambda}{\sqrt{3\omega}}, \sqrt{1 - \frac{(1-\lambda)^2}{3\omega}}\right)$ is physically accepted when $\omega > \frac{1}{3}(1 - \lambda)^2$. Point A_3 describes a scaling solution for an ideal gas with the parameter for the equation of state $w_{eff}(A_3) = -1 + \frac{2\lambda}{3\omega}(\lambda - 1)$, and $\Omega_m(A_3) = 0$. The point describes an accelerated universe when the free parameters are constrained as $\{\lambda \leq 0, \omega > \lambda(\lambda - 1)\}$,

$\{0 < \lambda \leq 1\}$ and $\{\lambda > 1, \omega > \lambda(\lambda - 1)\}$. For $\lambda = 0$ or $\lambda = 1$, the stationary points describe the de Sitter universe, i.e., $w_{eff}(A_3) = -1$. The eigenvalues of the linearized system are derived as $e_1(A_3) = -3 + \frac{2}{\omega}(1 - \lambda)^2$, $e_2(A_3) = -3 + \frac{(1-\lambda)^2}{\omega}$. Therefore, point A_3 is an attractor when $\omega > \frac{2}{3}(1 - \lambda)^2$ and for an arbitrary value for the parameter λ .

Finally, the stationary points $A_4 = \left(\frac{\sqrt{3}}{2(1-\lambda)}, \sqrt{\frac{3\omega}{4(1-\lambda)^2}}\right)$ exist when $\lambda \neq 1$. The latter stationary points describe scaling solutions with $w_{eff}(A_4) = \frac{1}{\lambda-1}$, for $\lambda \neq 0$, or de Sitter universes $w_{eff}(A_4) = -1$ when $\lambda = 0$. In addition, the contribution of the dust fluid source is nonzero, i.e., $\Omega_m(A_4) = 1 - \frac{3\omega}{2(1-\lambda)^2}$. The point is physically accepted, when $0 < \Omega_m(A_4) < 1$, i.e., $\lambda \neq 1, 0 < \omega < \frac{2}{3}(1 - \lambda)^2$. The eigenvalues of the linearized system are derived $e_1(A_4) = -\frac{3}{4} + \frac{\sqrt{12\omega-7(\lambda-1)^2}}{4(\lambda-1)}$, $e_2(A_4) = -\frac{3}{4} - \frac{\sqrt{12\omega-7(\lambda-1)^2}}{4(\lambda-1)}$. We infer that point A_4 is always an attractor.

The results for the stationary points and their physical properties are summarized in Table 1, while in Table 2, we summarize the stability conditions for the stationary points. In Figure 1, we present the phase-space portrait for the field equations in the plane (x, y) for different values of the free parameters in which point A_3 or A_4 are attractors. In Figure 2, we present the parametric plot with the evolution of the physical variables Ω_m and w_{eff} for numerical solutions of the field equations.

The stationary point A_4 is of special interest, because it can describe an accelerated universe where the dark matter and the dark energy contributes in the cosmological fluid. This point is always an attractor when it exists. Moreover, from the recent cosmological observations, we know that the deceleration parameter $q = \frac{1}{2}(1 + 3w_{eff})$ for the Λ -cosmology is $q_\Lambda \simeq -0.6$ where $\Omega_m \simeq 0.27$ [54]. Thus, the effective equation of state parameter is $w_{eff}^\Lambda \simeq -0.73$. Therefore, for $(\omega, \lambda) \simeq (0.91, -0.37)$, the solution of point A_4 provides physical variables equal to that of the Λ -cosmology.

Table 1. Stationary points of the field equations and their physical properties for the exponential potential.

Point	(x, y)	Existence	Ω_m	w_{eff}	$w_{eff} < -\frac{1}{3}$
A_1	(0,0)	Always	1	0	No, dust solution
A_2^\pm	(1,0)	Always	0	$1 \mp \frac{2}{\sqrt{3\omega}}$	A_2^+ for $\omega > \frac{4}{3}$
A_3	$\left(\pm \frac{1-\lambda}{\sqrt{3\omega}}, \sqrt{1 - \frac{(1-\lambda)^2}{3\omega}}\right)$	$\omega > \frac{1}{3}(1 - \lambda)^2$	0	$-1 + \frac{2\lambda(\lambda-1)}{3\omega}$	Yes under conditions
A_4	$\left(\frac{\sqrt{3\omega}}{2(1-\lambda)}, \sqrt{\frac{3\omega}{4(1-\lambda)^2}}\right)$	$\lambda \neq 1$ $\omega < \frac{2}{3}(1 - \lambda)^2$	$1 - \frac{3\omega}{2(1-\lambda)^2}$	$\frac{1}{\lambda-1}$	Yes under conditions

Table 2. Stationary points of the field equations and their stability properties for the exponential potential.

Point	Eigenvalues	Stability
A_1	$-\frac{3}{2}, \frac{3}{2}$	Saddle
A_2^\pm	$3 \pm \sqrt{\frac{3}{\omega}(\lambda - 1)}, 3$	A_2^+ saddle $\lambda < 1$ and $\omega < \frac{1}{3}(1 - \lambda)^2$ A_2^- saddle $\lambda > 1$ and $\omega < \frac{1}{3}(1 - \lambda)^2$
A_3	$-3 + \frac{2}{\omega}(1 - \lambda)^2, -3 + \frac{(1-\lambda)^2}{\omega}$	Stable for $\omega > \frac{2}{3}(1 - \lambda)^2$
A_4	$-\frac{3}{4} \pm \frac{\sqrt{12\omega-7(\lambda-1)^2}}{4(\lambda-1)}$	Attractor

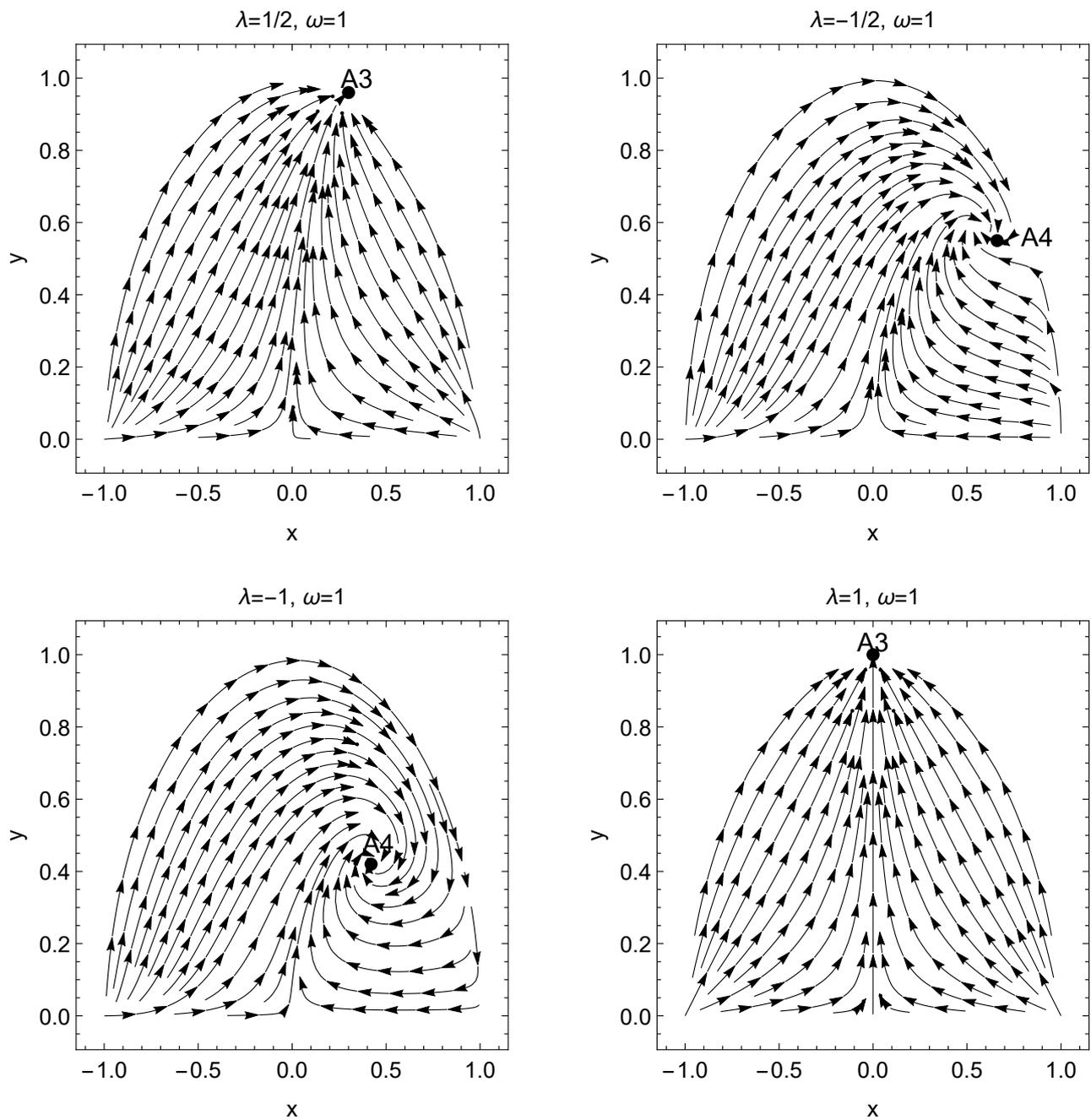


Figure 1. Two-dimensional phase-space diagrams for the dynamical system (10) and (11) in the plane (x, y) . The left figure in the first row is for $(\lambda, \omega) = (\frac{1}{2}, 1)$ with an attractor point A_3 . Right figure in the first row is for $(\lambda, \omega) = (-\frac{1}{2}, 1)$, with attractor point A_4 . For the figures of the second row, the left figure is for $(\lambda, \omega) = (-1, 1)$ with attractor point A_4 , while the right figure is for $(\lambda, \omega) = (1, 1)$ and an attractor for the de Sitter point A_3 .

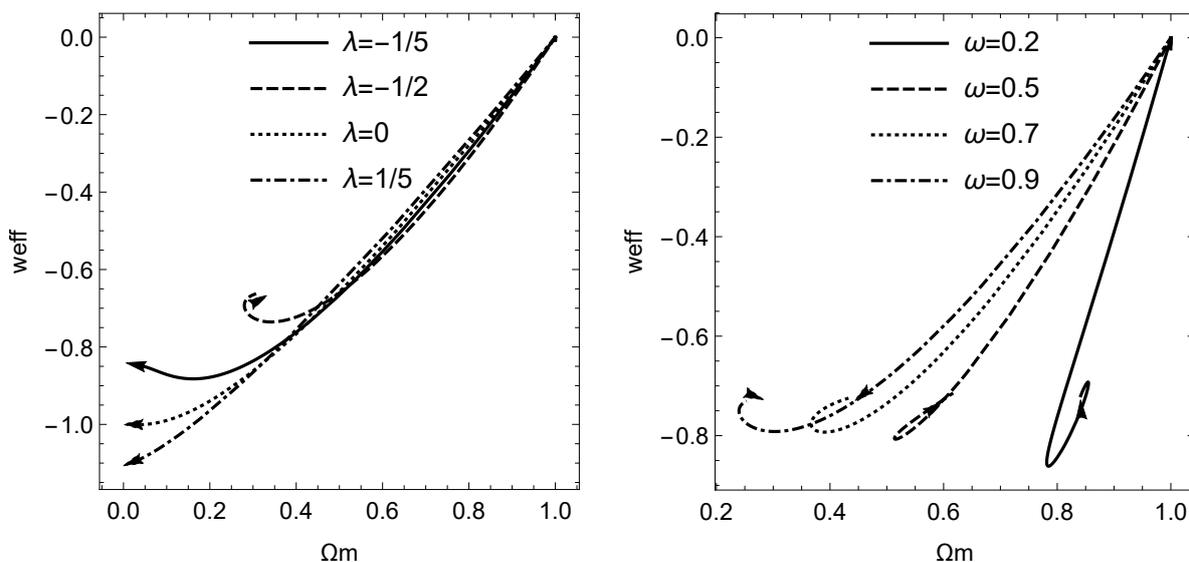


Figure 2. Parametric plot for the qualitative evolution of the Ω_m and of the effective equation of state parameter w_{eff} for a numerical solution of the dynamical system (9)–(11) with initial conditions near to the matter-dominated era $\Omega_m \simeq 1$. In Left Fig., lines are for $\omega = 1$, solid line is for $\lambda = -\frac{1}{5}$, dashed line is for $\lambda = -\frac{1}{2}$, dotted line is for $\lambda = 0$, and the dashed dot line is for $\lambda = \frac{1}{5}$. In the Right Fig., lines are for $\lambda = 0$, the solid line is for $\omega = 0.2$, dashed line is for $\omega = 0.5$, dotted line is for $\omega = 0.7$, and the dashed dot line is for $\omega = 0.9$. The attractor is the de Sitter point A_4 .

4. Cosmographic Parameters

The cosmographic approach is a model independent construction way of the cosmological physical variables [55]. Specifically, the scale factor it is written in the expansion form:

$$\frac{a(t)}{a_0} = 1 + H_0(t - t_0) - \frac{1}{2}q_0^2(t - t_0)^2 + \frac{1}{3!}j_0(t - t_0)^3 + \frac{1}{4!}s_0(t - t_0)^4 + \mathcal{O}[(t - t_0)^5], \quad (15)$$

where H_0 is the value of the Hubble function of today, q_0 is the deceleration parameter of today, j_0 and s_0 are the present value for the jerk and snap parameters. The H, q, j, s are kinematical quantities, which are directly extracted from the spacetime. The kinematic quantities are defined as [56,57]

$$\begin{aligned} H(t) &= \frac{1}{a} \frac{da}{dt}, \\ q(t) &= -\frac{1}{a} \frac{d^2a}{dt^2} \left[\frac{1}{a} \frac{da}{dt} \right]^{-2}, \\ j(t) &= \frac{1}{a} \frac{d^3a}{dt^3} \left[\frac{1}{a} \frac{da}{dt} \right]^{-3}, \\ s(t) &= \frac{1}{a} \frac{d^4a}{dt^4} \left[\frac{1}{a} \frac{da}{dt} \right]^{-4}, \end{aligned}$$

or in terms of the Hubble function, they can be written in the equivalent form:

$$q = -1 - \frac{\dot{H}}{H^2} \quad (16)$$

$$j = \frac{\ddot{H}}{H^3} - 3q - 2 \quad (17)$$

$$s = \frac{\dddot{H}}{H^4} + 4j + 3q(q + 4) + 6, \quad (18)$$

Thus, from the evolution of the cosmographic q, j, s , one can understand the expansion of the universe, the rate of acceleration and its derivative. In Figures 3 and 4 we present the qualitative evolution of the cosmographic parameters for the model of our consideration for initial conditions near the matter dominated era and for the values of the free parameters (λ, ω) which are the same as that of the numerical solutions of Figure 2. From the evolution of the cosmographic parameters presented in Figures 3 and 4, we observe that while the q_0 value for the Λ CDM is recovered, this is not true for the jerk and snap parameters which generally have a different evolution.

From the qualitative evolution, we observed that our model can predict values for the cosmographic parameters as they are given by the cosmological constraints [58].

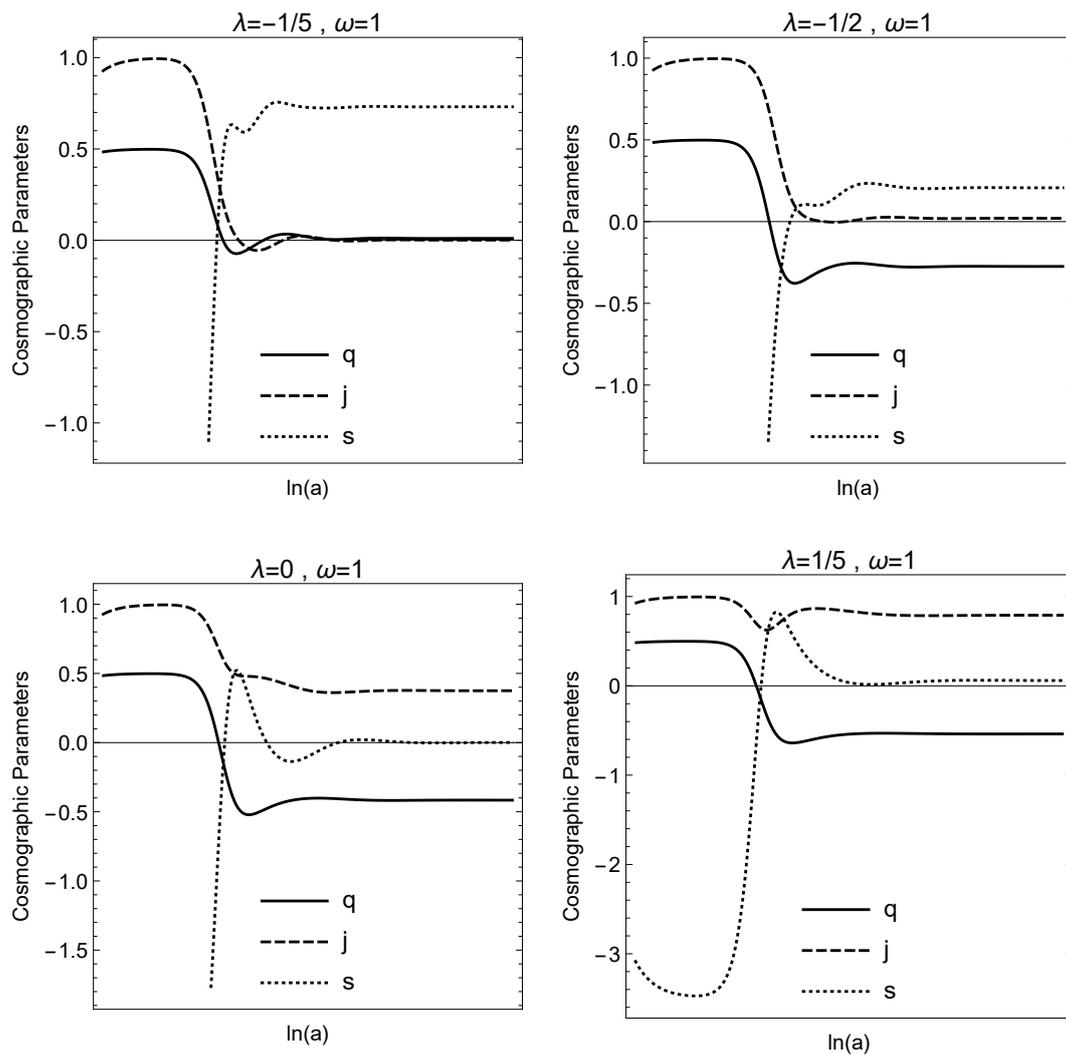


Figure 3. Qualitative evolution for the cosmographic parameters q, j , and s as provided by the field equations (10)–(11) for different values of the free parameters. The initial condition is a point near to the matter dominated solution A_1 . The solid line is for the deceleration parameter q , the dashed line was for the jerk parameter j , and the dotted line was for the snap parameter s .

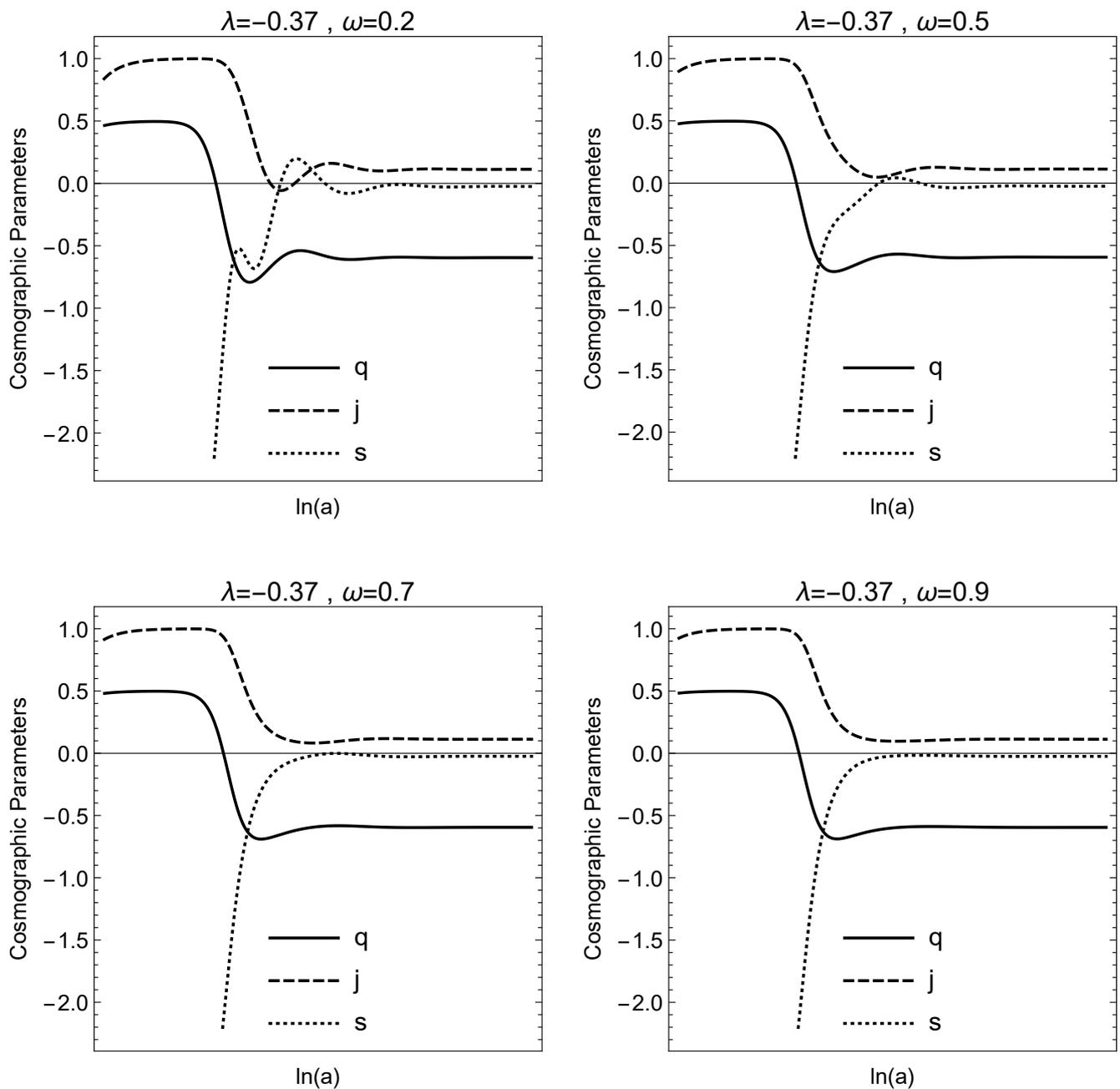


Figure 4. Qualitative evolution for the cosmographic parameters q , j and s as provided by the field Equations (10) and (11) for different values of free parameters. The initial condition is a point near the matter-dominated solution A_1 . The solid line is for deceleration parameter q , dashed line is for the jerk parameter j , and the dotted line is for the snap parameter s .

5. Conclusions

In this study, we considered a scalar-torsion model known as the teleparallel dilaton theory coupled to a pressureless fluid source, which we assumed describes the dark matter. For this cosmological model, the field equations are of second order and we investigated the evolution of the cosmological parameters in a spatially flat FLRW background space by determining the stationary points and studying their stability.

This kind of dynamical analysis is essential for the study of the general evolution of the dynamical system because it provides us with important information in order to infer the cosmological viability of the theory. For the model of our consideration, we wrote the field equations into an equivalent algebraic-differential by using the H -normalization

approach. Every stationary point of the dynamical system describes an exact asymptotic solution for the scale factor which corresponds to a specific epoch of the cosmological history. The stability properties of the stationary points are important to investigate because they tell us about the general evolution of the dynamical system.

For our model, and for the exponential scalar field potential, we found that the field equations admit four stationary points which describe four different cosmological epochs. Point A_1 provides the exact solution of the unstable matter-dominated era; points A_2^\pm describe the universes dominated by the kinetic part of the scalar field, where in contrast to the quintessence of this theory, we calculated $w_{eff}(A_1^\pm) = 1 \mp \frac{2}{\sqrt{3\omega}}$. Hence for the $\omega > \frac{4}{3}$ point, A_2^+ describes the cosmic acceleration. Point A_3 is a scaling solution in general $w_{eff}(A_3) = -1 + \frac{2\lambda(\lambda-1)}{3\omega}$, where for $\lambda(\lambda-1) = 0$, the asymptotic solution at point A_3 is the de Sitter solution. Finally, point A_4 describes an asymptotic solution in which the dark matter and the scalar field contributes to the cosmological fluid. This solution is of special interest because it describes the present cosmological era. Moreover, in order to understand the evolution of the physical variables, we presented the qualitative evolution of the cosmographic parameters from where we found that the values for some of the cosmographic parameters at the present era can be recovered.

From the above results, it is clear that while there are similarities of this model with the dilaton model of scalar tensor theory [59], the two theories are different in the general evolution of the cosmological history. In the same conclusion, we end by comparing the results of this work with those of previous studies on the Weyl integrable theory, where the interaction of the scalar field with the dust fluid is introduced in the gravitational action integral [60]. Consequently, the present model of study has interesting properties which can explain the cosmic history and deserves further study. In addition, in the absence of the dust fluid, we recall that the theory admits a discrete symmetry which can be used to study the cosmic evolution in the pre-Big Bang era, as in string cosmology. However, in contrast to string cosmology and the dilaton field for this teleparallel model, the pre-Big Bang era was not recovered as a reflection of the present epoch.

In a future study, we plan to use the cosmological observations in order to constrain this specific theory as a dark energy candidate.

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