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$SU(5) \times U(1)'$ Models with a Vector-like Fermion Family

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Abstract: Motivated by experimental measurements indicating deviations from the Standard Model predictions, we discuss F-theory-inspired models, which, in addition to the three chiral generations, contain a vector-like complete fermion family. The analysis takes place in the context of $SU(5) \times U(1)'$ GUT embedded in an E_8 covering group, which is associated with the (highest) geometric singularity of the elliptic fibration. In this context, the $U(1)'$ is a linear combination of four abelian factors subjected to the appropriate anomaly cancellation conditions. Furthermore, we require universal $U(1)'$ charges for the three chiral families and different ones for the corresponding fields of the vector-like representations. Under the aforementioned assumptions, we find 192 models that can be classified into five distinct categories with respect to their specific GUT properties. We exhibit representative examples for each such class and construct the superpotential couplings and the fermion mass matrices. We explore the implications of the vector-like states in low-energy phenomenology, including the predictions regarding the B-meson anomalies. The rôle of R-parity violating terms appearing in some particular models of the above construction is also discussed.

Keywords: F-theory phenomenology; LHCb anomalies; vector-like particles



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1. Introduction

The quest for New Physics (NP) phenomena beyond the Standard Model predictions is a principal and interesting issue. Numerous extensions of the Standard Model (SM), including Grand Unified Theories (GUTs) and String Theory-derived effective models, incorporate novel ingredients in their spectra. The latter could manifest themselves through exotic interactions and their novel predictions. Amongst the most anticipated ones are additional neutral gauge bosons, leptoquark states forming couplings with quarks and leptons, additional neutral states (such as sterile neutrinos) and vector-like families.

Current experimental data of the Large Hadron Collider (LHC) and elsewhere, on the other hand, provide significant evidence of the existence of possible novel interactions mediated by such exotic states, although nothing is conclusive yet. Some well-known persisting LHCb data that are in tension with the SM predictions, for example, are related to various B-meson decay channels. In particular, measurements of the ratio of the branching ratios $Br(B \rightarrow K\mu^+\mu^-)/Br(B \rightarrow Ke^+e^-)$ associated with the semi-leptonic transitions $b \rightarrow s\mu^+\mu^-$, $b \rightarrow se^+e^-$ indicate that lepton flavor universality is violated [1,2]. Possible explanations of the effect involve leptoquark states, Z' neutral bosons coupled differently to the three fermion families and vector-like generations [3–9].

In a previous study [10] (see also [11]), we performed a systematic analysis of a class of semi-local F-theory models with $SU(5) \times U(1)'$ gauge symmetry obtained from a covering E_8 gauge group through the chain

$$E_8 \supset SU(5) \times SU(5)' \supset SU(5) \times U(1)^4 \supset SU(5) \times U(1)', \quad (1)$$

where $U(1)'$ stands for any linear combination of the four abelian factors incorporated in $SU(5)'$. In this framework, we have derived all possible solutions of the anomaly-free $U(1)'$

factors and have shown that many of these cases entail non-universal couplings to the three chiral families. Next, we considered the case where the spontaneous breaking of the $U(1)'$ symmetry occurs at a few TeV scales and examined the implications in low-energy phenomenology, computing observables of several exotic processes in the effective theory.

Despite the rich structure and the variety of the non-universal $U(1)'$ factors, strong lower bounds coming from the $K - \bar{K}$ system [12] on the mass of the associated Z' boson far outweigh any observable effects in B-meson anomalies and the non-universal contributions to $Br(B \rightarrow K\mu^+\mu^-)/Br(B \rightarrow Ke^+e^-)$ are completely depleted. It was shown that in the so-derived effective F-theory models, only the existence of additional vector-like families could interpret the LHCb data [10].

In the present article, we expand on previous work [10,11] on F-theory-inspired $SU(5) \times U(1)'$ models by including vector-like fermion generations in the low-energy spectrum. More precisely, we are interested in models that allow the existence of a complete family of extra fermions in addition to the spectrum of the Minimal Supersymmetric Standard Model (MSSM). To avoid severe constraints for the Kaon system, we look for models where the regular MSSM fermion matter fields acquire universal charges under the additional $U(1)'$ symmetry and are chosen to be different from the corresponding states of the vector-like family. This way the non-universality effects are strictly induced from the considered vector-like states [13–16].

The paper is organized as follows: in Sections 2 and 3, we describe the origin of the gauge symmetry of the model, the anomaly cancellation and flux constraints and define the content in terms of the flux parameters. In Section 3, in particular, all models with one vector-like family are sorted into five classes distinguished by their $U(1)'$ properties. The phenomenological analysis of the models, their superpotential couplings and mass matrices are presented in Section 4. Section 5 deals with the implications on flavor processes and particularly B-meson anomalies. A short discussion is devoted to the possible implications of R-parity violating terms in Section 6. A summary and conclusions are found in Section 7.

2. Flux Constraints for a Spectrum with a Complete Vector-like Family

In this section, we present a short description of the GUT model, focusing mainly on the basic constraints and characteristics coming from its F-theory embedding. Further technical details can be found in [10].

The (semi-local) F-theory construction in the present work is assumed to originate from an E_8 singularity under the reduction shown in Equation (1). The Cartan generators $Q_k = \text{diag}\{t_1, t_2, t_3, t_4, t_5\}$, $k = 1, 2, 3, 4$ corresponding to the four $U(1)$ factors in Equation (1), subjected to the $SU(5)$ tracelessness condition $\sum_{i=1}^5 t_i = 0$, are taken to be

$$Q_a = \frac{1}{2}\text{diag}(1, -1, 0, 0, 0), \quad Q_b = \frac{1}{2\sqrt{3}}\text{diag}(1, 1, -2, 0, 0), \quad (2)$$

$$Q_\psi = \frac{1}{2\sqrt{6}}\text{diag}(1, 1, 1, -3, 0), \quad Q_\chi = \frac{1}{2\sqrt{10}}\text{diag}(1, 1, 1, 1, -4). \quad (3)$$

To ensure a tree-level top-quark mass, a Z_2 monodromy $t_1 \leftrightarrow t_2$ is imposed, “breaking” $U(1)_a$ while leaving invariant the remaining three abelian factors. In addition, appropriate fluxes [17] can be turned on along the remaining $U(1)$ ’s in such a way that some linear combination $U(1)'$ of the abelian factors remains unbroken at low energies. Thus, the gauge symmetry of the effective model under consideration is

$$G_S = SU(5) \times U(1)'. \quad (4)$$

The $U(1)'$ factor assumed to be left unbroken in the effective model is a linear combination of the symmetries surviving the monodromy action, namely:

$$Q' = c_1 Q_b + c_2 Q_\psi + c_3 Q_\chi, \quad (5)$$

with the coefficients c_1, c_2, c_3 satisfying the normalization condition

$$c_1^2 + c_2^2 + c_3^2 = 1. \tag{6}$$

The latter is also subject to anomaly cancellation conditions, which have been analyzed in detail elsewhere [10,11]. After imposing the Z_2 monodromy, the $10, \bar{10}$ and $5, \bar{5}$ representations accommodating the massless fields reside on four matter curves $\Sigma_{10_j}, j = 1, 2, 3, 4$ and seven $\Sigma_{5_i}, i = 1, 2, \dots, 7$, respectively [18].

The $U(1)$ fluxes mentioned above also determine the chiralities of the $SU(5)$ representations. Their effect on the representations of the various matter curves, Σ_{10_j} and Σ_{5_i} can be parametrized in terms of integers M_j and m_j as follows:

$$n_{10_j} - n_{\bar{10}_j} = m_j \quad n_{5_i} - n_{\bar{5}_i} = M_i, \tag{7}$$

while, in order to accommodate the three fermion families, the chirality condition $\sum_j m_j = -\sum_i M_i = 3$ should be imposed. Furthermore, turning on a hypercharge flux \mathcal{F}_Y , the $SU(5)_{GUT}$ symmetry is broken down to $SU(3) \times SU(2) \times U(1)_Y$. Parametrizing the hypercharge flux with integers N_i and N_j , the various multiplicities of the SM representations are given by

$$10_{t_j} = \begin{cases} n_{(3,2)_{\frac{1}{6}}} - n_{(\bar{3},2)_{-\frac{1}{6}}} = m_j \\ n_{(\bar{3},1)_{-\frac{2}{3}}} - n_{(3,1)_{\frac{2}{3}}} = m_j - N_j \\ n_{(1,1)_{+1}} - n_{(1,1)_{-1}} = m_j + N_j \end{cases}; \quad 5_{t_i} = \begin{cases} n_{(3,1)_{-\frac{1}{3}}} - n_{(\bar{3},1)_{+\frac{1}{3}}} = M_i \\ n_{(1,2)_{+\frac{1}{2}}} - n_{(1,2)_{-\frac{1}{2}}} = M_i + N_i \end{cases}. \tag{8}$$

We start with the flux data and the SM content of each matter curve. For details, we refer to our previous work [10], and here, we only present the properties of the complete spectrum, as shown in Table 1. In order to obtain the desired spectrum, the following constraints were taken into account.

Table 1. Matter curves along with their $U(1)'$ charges, flux data and the corresponding SM content. Note that the flux integers satisfy $N = N_7 + N_8 + N_9$.

Matter Curve	Q'	N_Y	M	SM Content
$\Sigma_{10_{1,\pm t_1}}$	$\frac{10\sqrt{3}c_1 + 5\sqrt{6}c_2 + 3\sqrt{10}c_3}{60}$	$-N$	m_1	$m_1 Q + (m_1 + N)u^c + (m_1 - N)e^c$
$\Sigma_{10_{2,\pm t_2}}$	$\frac{-20\sqrt{3}c_1 + 5\sqrt{6}c_2 + 3\sqrt{10}c_3}{60}$	N_7	m_2	$m_2 Q + (m_2 - N_7)u^c + (m_2 + N_7)e^c$
$\Sigma_{10_{3,\pm t_3}}$	$\frac{\sqrt{10}c_3 - 5\sqrt{6}c_2}{20}$	N_8	m_3	$m_3 Q + (m_3 - N_8)u^c + (m_3 + N_8)e^c$
$\Sigma_{10_{4,\pm t_4}}$	$-\sqrt{\frac{2}{5}}c_3$	N_9	m_4	$m_4 Q + (m_4 - N_9)u^c + (m_4 + N_9)e^c$
$\Sigma_{5_{1,(\pm 2t_1)}}$	$-\frac{c_1}{\sqrt{3}} - \frac{c_2}{\sqrt{6}} - \frac{c_3}{\sqrt{10}}$	N	M_1	$M_1 \bar{d}^c + (M_1 + N)\bar{L}$
$\Sigma_{5_{2,(\pm t_1 + t_2)}}$	$\frac{5\sqrt{3}c_1 - 5\sqrt{6}c_2 - 3\sqrt{10}c_3}{30}$	$-N$	M_2	$M_2 \bar{d}^c + (M_2 - N)\bar{L}$
$\Sigma_{5_{3,(\pm t_1 + t_3)}}$	$-\frac{c_1}{2\sqrt{3}} + \frac{c_2}{\sqrt{6}} - \frac{c_3}{\sqrt{10}}$	$-N$	M_3	$M_3 \bar{d}^c + (M_3 - N)\bar{L}$
$\Sigma_{5_{4,(\pm t_1 + t_4)}}$	$\frac{-10\sqrt{3}c_1 - 5\sqrt{6}c_2 + 9\sqrt{10}c_3}{60}$	$-N$	M_4	$M_4 \bar{d}^c + (M_4 - N)\bar{L}$
$\Sigma_{5_{5,(\pm t_3 + t_4)}}$	$\frac{c_1}{\sqrt{3}} + \frac{c_2}{\sqrt{6}} - \frac{c_3}{\sqrt{10}}$	$N_7 + N_8$	M_5	$M_5 \bar{d}^c + (M_5 + N_7 + N_8)\bar{L}$
$\Sigma_{5_{6,(\pm t_3 + t_5)}}$	$\frac{20\sqrt{3}c_1 - 5\sqrt{6}c_2 + 9\sqrt{10}c_3}{60}$	$N_7 + N_9$	M_6	$M_6 \bar{d}^c + (M_6 + N_7 + N_9)\bar{L}$
$\Sigma_{5_{7,(\pm t_4 + t_5)}}$	$\frac{5\sqrt{6}c_2 + 3\sqrt{10}c_3}{20}$	$N_8 + N_9$	M_7	$M_7 \bar{d}^c + (M_7 + N_8 + N_9)\bar{L}$

The spectrum of a local F-theory model is determined once a set of the above integers—respecting the aforementioned constraints—is chosen. Thus, to start with, we proceed by accommodating the Higgs doublet H_u on the Σ_{5_1} matter curve. Choosing the associated flux integers to be $M_1 = 0$ and $N = 1$, it can be observed that the H_u remains in the massless spectrum, and at the same time, the down-type color triplet is eliminated. As a consequence of this mechanism [17], proton decay is sufficiently suppressed. Next, focusing on the Σ_{10_1} matter curve, we let m_1 vary in $0 < m_1 < 3$. In addition, thanks to

the Z_2 monodromy [18,19] already discussed, at least one diagonal tree-level up-quark Yukawa coupling, $\lambda_{top} 10_1 10_1 5_1$, is effectuated in the superpotential \mathcal{W} . Furthermore, in order to ensure exactly one extra family of vector-like fermions, in addition to the condition $\sum_j m_j = -\sum_i M_i = 3$, which fixes the number of chiral families to three, we also impose the following conditions into the various flux integers [10,11]:

$$\sum_{j=1}^4 |m_j| = \sum_{i=1}^7 |M_i| = 5, \tag{9}$$

$$|m_1 + 1| + |m_2 - N_7| + |m_3 - N_8| + |m_4 - N_9| = 5, \tag{10}$$

$$|m_1 - 1| + |m_2 + N_7| + |m_3 + N_8| + |m_4 + N_9| = 5, \tag{11}$$

$$1 + |M_2 - 1| + |M_3 - 1| + |M_4 - 1| + |M_5 + N_7 + N_8| + |M_6 + N_7 + N_9| + |M_7 + N_9 + N_8| = 7. \tag{12}$$

Except for m_1, M_1 and $N = N_7 + N_8 + N_9$, which their allowed ranges and values are subjected to the aforementioned conditions, the remaining flux parameters are limited as follows:

We restrict the $m_{2,3,4}$ flux integers characterizing the number of Q, \bar{Q} states in the spectrum in the range $[-1, 2]$. Since the Σ_{10_1} matter curve always hosts at least two u^c 's (due to conditions $M_1 = 0, N = 1, 0 < m_1 < 3$), we bound the other u^c multiplicities ($m_j - N_k$ with $j = 2, 3, 4$ and $k = 7, 8, 9$) to be in the range $[-1, 1]$. Similarly, for the multiplicities of the e^c and \bar{e}^c states, we impose $-1 \leq (m_j + N_k) \leq 3$ for $j = 2, 3, 4$ and $k = 7, 8, 9$.

In the same way, for the d^c 's, we set the values of the corresponding multiplicities of M_i 's ($i = 2, 3, 4, 5, 6, 7$) to vary in the range $[-3, 1]$, while for the multiplicities of \bar{L} 's (see Table 1), the relations are set to vary in the range $[-2, 1]$. We note here that for the latter, in general, we could allow for values in the range $[-3, 1]$, but this leads to mixing the vector-like states with the MSSM ones, something that is against our intention to look for models with vector-like $U(1)'$ charges different than the MSSM ones.

Implementing all the restrictions described above, we receive 1728 flux solutions with one vector-like family in addition to the three standard chiral families of quarks and leptons.

3. Classification of the Models

In order to determine the c_i coefficients and consequently the $U(1)'$ charges for each model described by the above set of fluxes, we impose anomaly cancellation conditions. In particular, we impose only the mixed MSSM- $U(1)'$ anomalies: $\mathcal{A}_{331}, \mathcal{A}_{221}, \mathcal{A}_{YY1}$ and \mathcal{A}_{Y11} . The pure $U(1)'$ cubic anomaly (\mathcal{A}_{111}) and gravitational anomalies (\mathcal{A}_G) can be fixed later by taking into account the dynamics of the singlet fields that typically appear in F-theory models. Furthermore, in the quest for phenomenologically interesting constructions, we shall confine our search in cases where the three MSSM families have universal $U(1)'$ charges, and only the charges of the vector-like fields will differ. This way, from the resulting 1728 models, only 192 of them appear with this property. These 192 models fall into five classes with respect to their $SU(5) \times U(1)'$ properties. Each class contains models that carry the same charges under the extra $U(1)'$, and they only differ in how the SM states are distributed among the various matter curves. We present one model for each class in Tables 2 and 3.

Table 2. Representative flux solutions along with the corresponding c_i s for the five class of models: A, B, C, D and E.

Model	m_1	m_2	m_3	m_4	M_1	M_2	M_3	M_4	M_5	M_6	M_7	N_7	N_8	N_9	c_1	c_2	c_3
A	1	2	1	-1	0	-1	0	0	-1	-2	1	1	0	0	0	$\frac{\sqrt{15}}{4}$	$-\frac{1}{4}$
B	1	2	-1	1	0	0	0	0	-1	-3	1	1	0	0	0	$-\frac{1}{2}\sqrt{\frac{15}{34}}$	$-\frac{11}{2\sqrt{34}}$
C	2	1	1	-1	0	0	0	1	-3	-1	0	0	1	0	$\frac{\sqrt{3}}{2}$	$-\frac{1}{4}\sqrt{\frac{3}{2}}$	$\frac{1}{4}\sqrt{\frac{5}{2}}$
D	2	-1	1	1	0	-1	0	1	-1	-2	0	0	0	1	$-\frac{1}{2}\sqrt{\frac{5}{6}}$	$-\frac{5}{8}\sqrt{\frac{5}{3}}$	$\frac{3}{8}$
E	1	-1	2	1	0	0	1	0	0	-1	-3	0	1	0	$2\sqrt{\frac{10}{93}}$	$-\sqrt{\frac{5}{93}}$	$\frac{4}{\sqrt{31}}$

Table 3. The particle content of models A, B, C, D and E using the data from Table 2.

Model A		Model B		Model C		Model D		Model E	
$\sqrt{10}Q'$	SM	$\sqrt{85}Q'$	SM	Q'	SM	$\sqrt{10}Q'$	SM	$\sqrt{310}Q'$	SM
1/2	$Q + 2u^c$	-2	$Q + 2u^c$	1/4	$2Q + 3u^c + e^c$	-3/4	$2Q + 3u^c + e^c$	9/2	$Q + 2u^c$
1/2	$2Q + u^c + 3e^c$	-2	$2Q + u^c + 3e^c$	-1/2	$Q + u^c + e^c$	-1/2	$\bar{Q} + \bar{u}^c + \bar{e}^c$	11/2	$\bar{Q} + \bar{u}^c + \bar{e}^c$
-2	$\bar{Q} + u^c + e^c$	-1/2	$\bar{Q} + \bar{u}^c + \bar{e}^c$	1/4	$Q + 2e^c$	7/4	$Q + u^c + e^c$	9/2	$2Q + u^c + 3e^c$
-1/2	$\bar{Q} + \bar{u}^c + \bar{e}^c$	11/2	$Q + u^c + e^c$	1/4	$\bar{Q} + \bar{u}^c + \bar{e}^c$	-3/4	$Q + 2e^c$	-8	$Q + u^c + e^c$
-1	H_u	4	H_u	-1/2	H_u	3/2	H_u	-9	H_u
1	$d^c + 2L$	-4	L	-1/4	L	-1/4	$d^c + 2L$	-1	L
-3/2	L	-3/2	L	1/2	L	1	L	-9	\bar{d}^c
1	L	7/2	L	0	\bar{d}^c	3/2	\bar{d}^c	-7/2	L
-3/2	d^c	-3/2	d^c	-1/4	$3d^c + 2L$	9/4	$d^c + L$	1	\bar{L}
1	$2d^c + L$	7/2	$3d^c + 2L$	-3/4	$d^c + L$	-1/4	$2d^c + L$	-27/2	$d^c + L$
3/2	$\bar{d}^c + \bar{L}$	-6	$\bar{d}^c + \bar{L}$	0	\bar{L}	-1	\bar{L}	-7/2	$3d^c + 2L$

Table 2 presents the flux data along with the respective solution¹ for the coefficients c_i of each model. The corresponding $U(1)'$ charges and the spectrum for each model are presented in Table 3. Note that models B and C coincide with the models 5 and 7, respectively, derived in [11]. In addition to the fields presented in Table 3, there are also singlet fields with weights $(t_i - t_j)$ that appear in the present F-theory construction². In the analysis that follows, we will denote these singlet fields as $\theta_{t_i-t_j} = \theta_{ij}$.

4. Analysis of the Models

In the previous section, we generated five classes of models that all share a common characteristic. The $U(1)'$ charges of the vector-like states differ from the universal $U(1)'$ charges of the SM chiral families. Models with this feature can explain the observed B-meson anomalies, provided there is substantial mixing of the SM fermions with the vector-like exotics. At the same time, lepton universality is preserved among the three chiral families, and severe bounds coming from the Kaon system and other flavor-violating processes are not violated. In the following, we will analyze the models of Table 3 and derive the mass matrices for each model.

4.1. Model A

For this case, we have chosen the following set of fluxes:

$$m_1 = m_3 = -m_4 = 1, \quad m_2 = 2, \quad M_1 = M_3 = M_4 = 0, \quad M_2 = M_6 = -2, \quad M_7 = -M_5 = 1.$$

The corresponding $U(1)'$ charges for the various representations are:

$$10_1 : \frac{1}{2}, \quad 10_2 : \frac{1}{2}, \quad 10_3 : -2, \quad 10_4 : -\frac{1}{2},$$

$$5_1 : -1, \quad 5_2 : -1, \quad 5_3 : \frac{3}{2}, \quad 5_4 : -1, \quad 5_5 : \frac{3}{2}, \quad 5_6 : -1, \quad 5_7 : -\frac{3}{2},$$

while the $\overline{10}, \overline{5}$ representations come with the opposite $U(1)'$ charge. We distribute the fermion generations and Higgs fields into matter curves as follows:

$$10_1 \rightarrow Q_3 + u_{2,3}^c, \quad 10_2 \rightarrow Q_{1,2} + u_1^c + e_{1,2,3}^c, \quad 10_3 \rightarrow Q_4 + u_4^c + e_4^c, \quad \overline{10}_4 \rightarrow \overline{Q}_4 + \overline{u}_4^c + \overline{e}_4^c, \\ 5_1 \rightarrow H_u, \quad \overline{5}_2 \rightarrow d_3^c + L_{2,3}, \quad \overline{5}_3 \rightarrow L_4, \quad \overline{5}_4 \rightarrow H_d, \quad \overline{5}_5 \rightarrow d_4^c, \quad \overline{5}_6 \rightarrow d_{1,2}^c + L_1, \quad 5_7 \rightarrow \overline{d}_4^c + \overline{L}_4.$$

Now we can write down the superpotential and, in particular, the various terms contributing to the fermion mass matrices.

We start with the up-quark sector. The dominant contributions to the up-type quark masses descend from the following superpotential terms:

$$W \supset y_t 10_1 10_1 \overline{5}_1 + \frac{y_1}{\Lambda} 10_1 10_2 \overline{5}_1 \theta_{13} + \frac{y_2}{\Lambda^2} 10_2 10_2 \overline{5}_1 \theta_{13}^2 + \frac{y_3}{\Lambda} 10_3 10_1 \overline{5}_1 \theta_{14} + \frac{y_4}{\Lambda^2} 10_3 10_2 \overline{5}_1 \theta_{13} \theta_{14} \\ + \frac{y_5}{\Lambda^2} 10_3 10_3 \overline{5}_1 \theta_{14}^2 + y_6 10_2 \overline{10}_4 \theta_{53} + y_7 10_1 \overline{10}_4 \theta_{51} + y_8 10_3 \overline{10}_4 \theta_{54} + \frac{y_9}{\Lambda} \overline{10}_4 \overline{10}_4 \overline{5}_4 \theta_{51}, \tag{13}$$

where y_i s are coupling constant coefficients, and Λ is a characteristic high-energy scale of the theory. The operators yield the following mass texture:

$$M_u = \begin{pmatrix} y_2 \theta_{13}^2 v_u & y_2 \theta_{13}^2 v_u & y_1 \theta_{13} v_u & y_4 \theta_{13} \theta_{14} v_u & y_6 \theta_{43} \\ y_1 \theta_{13} v_u & y_1 \theta_{13} v_u & \epsilon y_t v_u & y_3 \theta_{14} v_u & y_7 \theta_{51} \\ y_1 \theta_{13} v_u & y_1 \theta_{13} v_u & y_t v_u & y_3 \theta_{14} v_u & y_7 \theta_{51} \\ y_4 \theta_{13} \theta_{14} v_u & y_4 \theta_{13} \theta_{14} v_u & y_3 \theta_{14} v_u & y_5 \theta_{14}^2 v_u & y_8 \theta_{54} \\ y_6 \theta_{53} & y_6 \theta_{53} & y_7 \theta_{51} & y_8 \theta_{54} & y_9 \theta_{51} v_d \end{pmatrix}, \tag{14}$$

where $v_u = \langle H_u \rangle$, $v_d = \langle H_d \rangle$, $\theta_{ij} = \langle \theta_{ij} \rangle$, $\vartheta_{ij} = \langle \theta_{ij} \rangle / \Lambda$ and $\epsilon \ll 1$ is a suppression factor introduced here to capture the local effects of Yukawa couplings descending from a common operator [20,21].

Next, we analyze the couplings of the down-quark and charged lepton sectors. In the vector-like part of the model, up and down quark sectors share some common superpotential operators. These are given in Equation (13) with couplings y_6, y_7 and y_8 . The remaining dominant terms contributing to the down-type quarks are:

$$W \supset \frac{\kappa_0}{\Lambda} 10_1 \overline{5}_2 \overline{5}_4 \theta_{41} + \frac{\kappa_1}{\Lambda} 10_1 \overline{5}_6 \overline{5}_4 \theta_{45} + \frac{\kappa_2}{\Lambda} 10_2 \overline{5}_2 \overline{5}_4 \theta_{43} + \frac{\kappa_3}{\Lambda^2} 10_2 \overline{5}_6 \overline{5}_4 \theta_{13} \theta_{45} + \frac{\kappa_4}{\Lambda^2} 10_2 \overline{5}_6 \overline{5}_4 \theta_{43} \theta_{15} \\ + \kappa_5 10_3 \overline{5}_2 \overline{5}_4 + \frac{\kappa_6}{\Lambda} 10_3 \overline{5}_6 \overline{5}_4 \theta_{15} + \kappa_7 10_1 \overline{5}_5 \overline{5}_4 + \frac{\kappa_8}{\Lambda} 10_2 \overline{5}_5 \overline{5}_4 \theta_{13} + \frac{\kappa_9}{\Lambda} 10_3 \overline{5}_5 \overline{5}_4 \theta_{14} \\ + \frac{\kappa_{10}}{\Lambda} 5_7 \overline{5}_2 \theta_{41} \theta_{53} + \frac{\kappa_{11}}{\Lambda} 5_7 \overline{5}_2 \theta_{51} \theta_{43} + \kappa_{12} 5_7 \overline{5}_6 \theta_{43} + \kappa_{13} 5_7 \overline{5}_5 \theta_{53} + \frac{\kappa_{14}}{\Lambda} \overline{10}_4 5_7 \overline{5}_1 \theta_{53}, \tag{15}$$

with κ_i being coupling constant coefficients.

As regards the charged lepton sector, we start with the common operators between the bottom quark and charged leptons, which are given in Equation (15) with couplings $\kappa_2, \kappa_3, \kappa_4, \kappa_5$ and κ_6 . There are also common operators between the up quark and the charged lepton sector, which are given in Equation (13) with couplings y_6 and y_8 . All the other contributions for the charged lepton mass matrix descend from the operators

$$\lambda_1 10_2 \overline{5}_3 \overline{5}_4 + \frac{\lambda_2}{\Lambda} 10_3 \overline{5}_3 \overline{5}_4 \theta_{34} + \lambda_3 5_7 \overline{5}_6 \theta_{43} + \frac{\lambda_4}{\Lambda} 5_7 \overline{5}_2 \theta_{41} \theta_{53} + \frac{\lambda_5}{\Lambda} 5_7 \overline{5}_2 \theta_{51} \theta_{43} + \lambda_6 5_7 \overline{5}_3 \theta_{51} + \frac{\lambda_7}{\Lambda} \overline{10}_4 5_7 \overline{5}_1 \theta_{53}. \tag{16}$$

where λ_i denotes coupling constant coefficients.

When the various singlet fields θ_{ij} acquire vacuum expectation values (VEV), $\langle \theta_{ij} \rangle \neq 0$, they generate hierarchical non-zero entries in the mass matrices of quarks and charged leptons. These VEVs, however, are subject to phenomenological requirements. Such an important constraint comes from the μ -term, which, in principle, can be materialized through the coupling $\theta_{15} 5_1 \overline{5}_4$. Clearly, to avoid decoupling of the Higgs doublets from the light spectrum, we must require $\langle \theta_{15} \rangle \approx 0$. Consequently, the mass terms involving θ_{15} of the down and charged leptons can be ignored.

We obtain the following down quark mass matrix:

$$M_d = \begin{pmatrix} \kappa_3 \vartheta_{13} \vartheta_{45} v_d & \kappa_3 \vartheta_{13} \vartheta_{45} v_d & \kappa_1 \vartheta_{45} v_d & 0 & \kappa_{12} \vartheta_{43} \\ \kappa_3 \vartheta_{13} \vartheta_{45} v_d & \kappa_3 \vartheta_{13} \vartheta_{45} v_d & \kappa_1 \vartheta_{45} v_d & 0 & \kappa_{12} \vartheta_{43} \\ \kappa_2 \vartheta_{43} v_d & \kappa_2 \vartheta_{43} v_d & \kappa_0 \vartheta_{41} v_d & \kappa_5 v_d & \kappa_{10} \vartheta_{41} \vartheta_{53} + \kappa_{11} \vartheta_{51} \vartheta_{43} \\ \kappa_8 \vartheta_{13} v_d & \kappa_8 \vartheta_{13} v_d & \kappa_7 v_d & \kappa_9 \vartheta_{14} v_d & \kappa_{13} \vartheta_{53} \\ y_6 \vartheta_{53} & y_6 \vartheta_{53} & y_7 \vartheta_{51} & y_8 \vartheta_{54} & \kappa_{14} \vartheta_{53} v_u \end{pmatrix}. \tag{17}$$

The mass texture for the charged leptons has the following form:

$$M_e = \begin{pmatrix} \kappa_3 \vartheta_{13} \vartheta_{45} v_d & \kappa_2 \vartheta_{43} v_d & \kappa_2 \vartheta_{43} v_d & \lambda_1 v_d & y_6 \vartheta_{53} \\ \kappa_3 \vartheta_{13} \vartheta_{45} v_d & \kappa_2 \vartheta_{43} v_d & \kappa_2 \vartheta_{43} v_d & \lambda_1 v_d & y_6 \vartheta_{53} \\ \kappa_3 \vartheta_{13} \vartheta_{45} v_d & \kappa_2 \vartheta_{43} v_d & \kappa_2 \vartheta_{43} v_d & \lambda_1 v_d & y_6 \vartheta_{53} \\ 0 & \kappa_5 v_d & \kappa_5 v_d & \lambda_2 \vartheta_{34} v_d & y_8 \vartheta_{54} \\ \lambda_3 \vartheta_{43} & \lambda_4 \vartheta_{41} \vartheta_{53} + \lambda_5 \vartheta_{51} \vartheta_{43} & \lambda_4 \vartheta_{41} \vartheta_{53} + \lambda_5 \vartheta_{51} \vartheta_{43} & \lambda_6 \vartheta_{51} & \lambda_7 \vartheta_{53} v_u \end{pmatrix} \tag{18}$$

4.2. Model B

The second model is obtained using the following set of flux parameters:

$$m_1 = -m_3 = m_4 = 1, \quad m_2 = 2, \quad M_1 = M_2 = M_3 = M_4 = 0, \quad M_7 = -M_5 = 1, \quad M_6 = -3.$$

The corresponding $U(1)'$ for the various matter curves are:

$$10_1 : -2, \quad 10_2 : -2, \quad 10_3 : -\frac{1}{2}, \quad 10_4 : \frac{11}{2}, \\ 5_1 : 4, \quad 5_2 : 4, \quad 5_3 : \frac{3}{2}, \quad 5_4 : -\frac{7}{2}, \quad 5_5 : \frac{3}{2}, \quad 5_6 : -\frac{7}{2}, \quad 5_7 : 6.$$

A workable distribution of the fermion generations and Higgs fields into matter curves is as follows:

$$10_1 \longrightarrow Q_3 + u_{2,3}^c, \quad 10_2 \longrightarrow Q_{1,2} + u_1^c + e_{1,2,3}^c, \quad \bar{10}_3 \longrightarrow \bar{Q}_4 + \bar{u}_4^c + \bar{e}_4^c, \quad 10_4 \longrightarrow Q_4 + u_4^c + e_4^c, \\ 5_1 \longrightarrow H_u, \quad \bar{5}_2 \longrightarrow H_d, \quad \bar{5}_3 \longrightarrow L_4, \quad \bar{5}_4 \longrightarrow L_3, \quad \bar{5}_5 \longrightarrow d_4^c, \quad \bar{5}_6 \longrightarrow d_{1,2,3}^c + L_{1,2}, \quad 5_7 \longrightarrow \bar{d}_4^c + \bar{L}_4.$$

The μ -term here is realized through the coupling $\theta_{13} 5_1 \bar{5}_2$, so we require that $\langle \theta_{13} \rangle$ is very small compared to the other singlet VEVs. This restriction obligates us to take high order terms into account for some couplings. We write down the various terms that construct the fermion mass matrices starting from the up-quark sector.

The dominant contributions to the up-type quarks descend from the following superpotential terms:

$$W \supset y_t 10_1 10_1 5_1 + \frac{y_1}{\Lambda^2} 10_1 10_2 5_1 \vartheta_{14} \vartheta_{43} + \frac{y_2}{\Lambda^4} 10_2 10_2 5_1 \vartheta_{14}^2 \vartheta_{43}^2 + \frac{y_3}{\Lambda} 10_1 10_4 5_1 \vartheta_{15} + \frac{y_4}{\Lambda^2} 10_2 10_4 5_1 \vartheta_{13} \vartheta_{15} \\ + \frac{y_5}{\Lambda^2} 10_4 10_4 5_1 \vartheta_{15}^2 + y_6 10_1 \bar{10}_3 \vartheta_{41} + y_7 10_2 \bar{10}_3 \vartheta_{43} + y_8 10_4 \bar{10}_3 \vartheta_{45} + \frac{y_9}{\Lambda^2} \bar{10}_3 \bar{10}_3 \bar{5}_2 \vartheta_{41} \vartheta_{43}. \tag{19}$$

The operators yield the following mass texture:

$$M_u = \begin{pmatrix} y_2 \vartheta_{14}^2 \vartheta_{43}^2 v_u & y_2 \vartheta_{14}^2 \vartheta_{43}^2 v_u & y_1 \vartheta_{14} \vartheta_{43} v_u & 0 & y_7 \vartheta_{53} \\ y_1 \vartheta_{14} \vartheta_{43} v_u & y_1 \vartheta_{14} \vartheta_{43} v_u & \epsilon y_t v_u & y_3 \vartheta_{15} v_u & y_6 \vartheta_{41} \\ y_1 \vartheta_{14} \vartheta_{43} v_u & y_1 \vartheta_{14} \vartheta_{43} v_u & y_t v_u & y_3 \vartheta_{15} v_u & y_6 \vartheta_{41} \\ 0 & 0 & y_3 \vartheta_{15} v_u & y_5 \vartheta_{15} v_u & y_8 \vartheta_{45} \\ y_7 \vartheta_{43} & y_7 \vartheta_{43} & y_6 \vartheta_{41} & y_8 \vartheta_{45} & y_9 \vartheta_{41} \vartheta_{43} v_b \end{pmatrix} \tag{20}$$

We continue with the bottom sector of the model. There are common operators between the top and bottom sectors, which are given in Equation (19) with couplings y_6, y_7 and y_8 . The remaining dominant terms contributing to the down-type quarks are:

$$\begin{aligned}
 W \supset & \frac{\kappa_0}{\Lambda} 10_1 \bar{5}_6 \bar{5}_2 \theta_{43} + \frac{\kappa_1}{\Lambda^3} 10_2 \bar{5}_6 \bar{5}_2 \theta_{14} \theta_{43}^2 + \frac{\kappa_2}{\Lambda} 10_1 \bar{5}_5 \bar{5}_2 \theta_{53} + \frac{\kappa_3}{\Lambda^4} 10_2 \bar{5}_5 \bar{5}_2 \theta_{14} \theta_{43}^2 \theta_{54} + \frac{\kappa_4}{\Lambda^2} 10_4 \bar{5}_5 \bar{5}_2 \theta_{14} \theta_{34} \\
 & + \frac{\kappa_5}{\Lambda^2} 10_4 \bar{5}_6 \bar{5}_2 \theta_{43} \theta_{15} + \kappa_6 \bar{5}_7 \bar{5}_6 \theta_{43} + \kappa_7 \bar{5}_7 \bar{5}_5 \theta_{53} + \frac{\kappa_8}{\Lambda} \overline{10}_3 \bar{5}_7 \bar{5}_1 \theta_{43} .
 \end{aligned} \tag{21}$$

From these operators, we obtain the following mass matrix describing the down quark sector:

$$M_d = \begin{pmatrix} \kappa_1 \vartheta_{14} \vartheta_{43}^2 v_d & \kappa_1 \vartheta_{14} \vartheta_{43}^2 v_d & \epsilon^2 \kappa_0 v_d & \kappa_5 \vartheta_{43} \vartheta_{15} & \kappa_6 \theta_{43} \\ \kappa_1 \vartheta_{14} \vartheta_{43}^2 v_d & \kappa_1 \vartheta_{14} \vartheta_{43}^2 v_d & \epsilon \kappa_0 v_d & \kappa_5 \vartheta_{43} \vartheta_{15} & \kappa_6 \theta_{43} \\ \kappa_1 \vartheta_{14} \vartheta_{43}^2 v_d & \kappa_1 \vartheta_{14} \vartheta_{43}^2 v_d & \kappa_0 v_d & \kappa_5 \vartheta_{43} \vartheta_{15} & \kappa_6 \theta_{43} \\ \kappa_3 \vartheta_{14} \vartheta_{43}^2 \vartheta_{54} v_d & \kappa_3 \vartheta_{14} \vartheta_{43}^2 \vartheta_{54} v_d & \kappa_2 \vartheta_{53} v_d & \kappa_4 \vartheta_{14} \vartheta_{34} v_d & \kappa_7 \theta_{53} \\ y_7 \theta_{43} & y_7 \theta_{43} & y_6 \theta_{41} & y_8 \theta_{45} & \kappa_8 \vartheta_{43} v_u \end{pmatrix} . \tag{22}$$

Regarding the charged lepton sector, the common operators between the bottom sector and charged leptons are those in Equation (21) with couplings $\kappa_1, \kappa_5, \kappa_6$ and κ_7 . We also have common operators between the top and charged lepton sector, which are given in Equation (19) with couplings y_7 and y_8 . Pure charged lepton contributions descend from the operators

$$\frac{\lambda_1}{\Lambda} 10_2 \bar{5}_4 \bar{5}_2 \theta_{43} + \frac{\lambda_2}{\Lambda} 10_4 \bar{5}_4 \bar{5}_2 \theta_{45} + \frac{\lambda_3}{\Lambda} 10_2 \bar{5}_3 \bar{5}_2 \theta_{53} + \lambda_4 10_4 \bar{5}_3 \bar{5}_2 + \lambda_5 \bar{5}_7 \bar{5}_4 \theta_{41} + \lambda_6 \bar{5}_7 \bar{5}_3 \theta_{51} + \frac{\lambda_7}{\Lambda} \overline{10}_3 \bar{5}_7 \bar{5}_1 \theta_{43} . \tag{23}$$

Collectively, all the contributions lead to the following mass matrix:

$$M_e = \begin{pmatrix} \kappa_1 \vartheta_{14} \vartheta_{43}^2 v_d & \kappa_1 \vartheta_{14} \vartheta_{43}^2 v_d & \lambda_1 \vartheta_{43} v_d & \lambda_3 \vartheta_{53} v_d & y_7 \theta_{43} \\ \kappa_1 \vartheta_{14} \vartheta_{43}^2 v_d & \kappa_1 \vartheta_{14} \vartheta_{43}^2 v_d & \lambda_1 \vartheta_{43} v_d & \lambda_3 \vartheta_{53} v_d & y_7 \theta_{43} \\ \kappa_1 \vartheta_{14} \vartheta_{43}^2 v_d & \kappa_1 \vartheta_{14} \vartheta_{43}^2 v_d & \lambda_1 \vartheta_{43} v_d & \lambda_3 \vartheta_{53} v_d & y_7 \theta_{43} \\ \kappa_5 \vartheta_{13} \vartheta_{45} v_d + \kappa_6 \vartheta_{43} \vartheta_{15} v_d & \kappa_5 \vartheta_{13} \vartheta_{45} v_d + \kappa_6 \vartheta_{43} \vartheta_{15} v_d & \lambda_2 \vartheta_{45} v_d & \lambda_4 v_d & y_8 \theta_{45} \\ \kappa_7 \theta_{43} & \kappa_7 \theta_{43} & \lambda_5 \theta_{41} & \lambda_6 \theta_{51} & \lambda_7 \vartheta_{43} v_u \end{pmatrix} . \tag{24}$$

4.3. Model C

Next, a representative model of the class C is analyzed. The flux integers along with the corresponding c_i coefficients are given in Table 2. The resulting $U(1)'$ charges for the various matter curves are

$$\begin{aligned}
 10_1 : \frac{1}{4}, \quad 10_2 : -\frac{1}{2}, \quad 10_3 : \frac{1}{4}, \quad 10_4 : -\frac{1}{4}, \\
 \bar{5}_1 : -\frac{1}{2}, \quad \bar{5}_2 : \frac{1}{4}, \quad \bar{5}_3 : -\frac{1}{2}, \quad \bar{5}_4 : 0, \quad \bar{5}_5 : \frac{1}{4}, \quad \bar{5}_6 : \frac{3}{4}, \quad \bar{5}_7 : 0 .
 \end{aligned}$$

and we assume the following distribution of the various fermion and Higgs fields into matter curves

$$\begin{aligned}
 10_1 \longrightarrow Q_{2,3} + u_{1,2,3}^c + e_3^c, \quad 10_2 \longrightarrow Q_4 + u_4^c + e_4^c, \quad 10_3 \longrightarrow Q_1 + e_{1,2}^c, \quad \overline{10}_4 \longrightarrow \overline{Q}_4 + \overline{u}_4^c + \overline{e}_4^c, \\
 \bar{5}_1 \longrightarrow H_u, \quad \bar{5}_2 \longrightarrow L_1, \quad \bar{5}_3 \longrightarrow H_d, \quad \bar{5}_4 \longrightarrow \overline{d}_4^c, \quad \bar{5}_5 \longrightarrow d_{1,2,3}^c + L_{2,3}, \quad \bar{5}_6 \longrightarrow d_4^c + L_4, \quad \bar{5}_7 \longrightarrow \overline{L}_4 .
 \end{aligned}$$

The μ -term here comes through the coupling $\theta_{14} \bar{5}_1 \bar{5}_3$, and so, we require $\langle \theta_{14} \rangle \approx 0$. Once again, we will consider high order terms for some couplings. Next, we write down the various superpotential terms, leading to the the mass matrices for the up, bottom and charged lepton sectors.

The dominant contributions to the up-type quarks descend from the following superpotential terms:

$$\begin{aligned}
 W \supset & y_t 10_1 10_1 5_1 + \frac{y_1}{\Lambda^2} 10_1 10_3 5_1 \theta_{13} \theta_{34} + \frac{y_2}{\Lambda} 10_1 10_2 5_1 \theta_{13} + \frac{y_3}{\Lambda^2} 10_3 10_2 5_1 \theta_{13} \theta_{14} + \frac{y_4}{\Lambda^2} 10_2 10_2 5_1 \theta_{13}^2 \\
 & + y_5 10_1 \bar{10}_4 \theta_{51} + y_6 10_2 \bar{10}_4 \theta_{53} + y_7 10_3 \bar{10}_4 \theta_{54} + \frac{y_8}{\Lambda^2} \bar{10}_4 \bar{10}_4 \bar{5}_3 \theta_{51} \theta_{54} ,
 \end{aligned} \tag{25}$$

which yield the following mass texture:

$$M_u = \begin{pmatrix} y_1 \vartheta_{13} \vartheta_{34} v_u & \eta^3 y_t v_u & \eta^2 y_t v_u & y_2 \vartheta_{13} v_u & y_5 \theta_{51} \\ y_1 \vartheta_{13} \vartheta_{34} v_u & \eta^2 y_t v_u & \eta y_t v_u & y_2 \vartheta_{13} v_u & y_5 \theta_{51} \\ y_1 \vartheta_{13} \vartheta_{34} v_u & \eta y_t v_u & y_t v_u & y_2 \vartheta_{13} v_u & y_5 \theta_{51} \\ 0 & y_2 \vartheta_{13} v_u & y_2 \vartheta_{13} v_u & y_4 \vartheta_{13}^2 v_u & y_6 \theta_{53} \\ y_7 \theta_{54} & y_5 \theta_{51} & y_5 \theta_{51} & y_6 \theta_{53} & y_8 \vartheta_{51} \vartheta_{54} v_b \end{pmatrix} , \tag{26}$$

where η is a small constant parameter describing local Yukawa effects.

There are common operators between the top and bottom sectors. These are given in Equation (25) with couplings y_5 , y_6 and y_7 . The remaining operators contributing to the down-type quarks are:

$$\begin{aligned}
 W \supset & \frac{k}{\Lambda} 10_1 \bar{5}_5 \bar{5}_3 \theta_{54} + \frac{k_0}{\Lambda^3} 10_3 \bar{5}_5 \bar{5}_3 \theta_{13} \theta_{34} \theta_{54} + k_1 10_1 \bar{5}_6 \bar{5}_3 + \frac{k_2}{\Lambda} 10_3 \bar{5}_6 \bar{5}_3 \theta_{14} + \frac{k_3}{\Lambda} 5_4 \bar{5}_5 \theta_{14} \theta_{53} \\
 & + \frac{k_4}{\Lambda^2} 10_2 \bar{5}_5 \bar{5}_3 \theta_{14} \theta_{53} + \frac{k_7}{\Lambda^2} \bar{10}_4 5_4 5_1 \theta_{14} \theta_{53} + \frac{k_9}{\Lambda} 10_2 \bar{5}_6 \bar{5}_3 \theta_{13} + k_{10} 5_4 \bar{5}_6 \theta_{13} .
 \end{aligned} \tag{27}$$

Combining all the terms, we obtain the following mass matrix describing the down quark sector:

$$M_d = \begin{pmatrix} k_0 \vartheta_{13} \vartheta_{34} \vartheta_{54} v_d & k \epsilon^3 \vartheta_{54} v_d & k \epsilon^2 \vartheta_{54} v_d & 0 & 0 \\ k_0 \vartheta_{13} \vartheta_{34} \vartheta_{54} v_d & k \epsilon^2 \vartheta_{54} v_d & k \epsilon \vartheta_{54} v_d & 0 & 0 \\ k_0 \vartheta_{13} \vartheta_{34} \vartheta_{54} v_d & k \epsilon \vartheta_{54} v_d & k \vartheta_{54} v_d & 0 & 0 \\ 0 & k_1 \zeta v_d & k_1 v_d & k_9 \vartheta_{13} v_d & k_{10} \theta_{13} \\ y_7 \theta_{54} & y_5 \zeta \theta_{51} & y_5 \theta_{51} & y_6 \theta_{53} & k_7 \vartheta_{14} \vartheta_{53} v_u \end{pmatrix} , \tag{28}$$

where ϵ and ζ are small constant parameters describing local Yukawa effects.

In the charged lepton sector, we have some contributions descending from terms in Equation (27). These are the operators with couplings k , k_0 , k_1 , k_2 , k_4 and k_9 . We also have common operators between the top and charged lepton sectors, which are given in Equation (25) with couplings y_5 , y_6 and y_7 . All the other leptonic contributions descend from the following operators

$$\begin{aligned}
 W \supset & \frac{\lambda_1}{\Lambda} 10_3 \bar{5}_2 \bar{5}_3 \theta_{54} + \frac{\lambda_2}{\Lambda} 10_1 \bar{5}_2 \bar{5}_3 \theta_{51} + \frac{\lambda_3}{\Lambda} 10_2 \bar{5}_2 \bar{5}_3 \theta_{53} \\
 & + \frac{\lambda_4}{\Lambda} \bar{5}_7 \bar{5}_2 \theta_{41} \theta_{53} + \lambda_5 \bar{5}_7 \bar{5}_5 \theta_{53} + \lambda_6 \bar{5}_7 \bar{5}_6 \theta_{43} + \frac{\lambda_7}{\Lambda} \bar{10}_4 \bar{5}_7 \bar{5}_1 \theta_{53} .
 \end{aligned} \tag{29}$$

Hence, the mass texture for the charged leptons has the following form:

$$M_e = \begin{pmatrix} \lambda_1 \vartheta_{54} v_d & k_0 \vartheta_{13} \vartheta_{34} \vartheta_{54} v_d & k_0 \vartheta_{13} \vartheta_{34} \vartheta_{54} v_d & 0 & y_7 \theta_{54} \\ \lambda_1 \vartheta_{54} v_d & k_0 \vartheta_{13} \vartheta_{34} \vartheta_{54} v_d & k_0 \vartheta_{13} \vartheta_{34} \vartheta_{54} v_d & 0 & y_7 \theta_{54} \\ \lambda_2 \vartheta_{51} v_d & k \vartheta_{54} v_d & k \vartheta_{54} v_d & k_1 v_d & y_5 \theta_{51} \\ \lambda_3 \vartheta_{53} v_d & 0 & 0 & k_9 \vartheta_{13} v_d & y_6 \theta_{53} \\ \lambda_4 \vartheta_{41} \theta_{53} & \lambda_5 \theta_{53} & \lambda_5 \theta_{53} & \lambda_6 \theta_{43} & \lambda_7 \vartheta_{53} v_u \end{pmatrix} . \tag{30}$$

4.4. Model D

We now pick out a model belonging to the fourth class. The $U(1)'$ charges for the various matter curves are:

$$10_1 : -\frac{3}{4}, \quad 10_2 : -\frac{1}{2}, \quad 10_3 : \frac{7}{4}, \quad 10_4 : -\frac{3}{4},$$

$$5_1 : \frac{3}{2}, \quad 5_2 : \frac{1}{4}, \quad 5_3 : -1, \quad 5_4 : -\frac{3}{2}, \quad 5_5 : -\frac{9}{4}, \quad 5_6 : \frac{1}{4}, \quad 5_7 : 1.$$

A promising distribution of the fermion generations and Higgs fields into the various matter curves is as follows:

$$10_1 \longrightarrow Q_{2,3} + u_{1,2,3}^c + e_3^c, \quad \bar{10}_2 \longrightarrow \bar{Q}_4 + \bar{u}_4^c + \bar{e}_4^c, \quad 10_3 \longrightarrow Q_4 + u_4^c + e_4^c, \quad 10_4 \longrightarrow Q_1 + e_{1,2}^c,$$

$$5_1 \longrightarrow H_u, \quad \bar{5}_2 \longrightarrow d_{1,2}^c + L_{1,2}, \quad \bar{5}_3 \longrightarrow H_d, \quad 5_4 \longrightarrow \bar{d}_4^c, \quad \bar{5}_5 \longrightarrow d_4^c + L_4, \quad \bar{5}_6 \longrightarrow d_{2,3}^c + L_3, \quad 5_7 \longrightarrow \bar{L}_4.$$

In this case, the μ -term is realized through the coupling $\theta_{14} 5_1 \bar{5}_3$, and therefore, as in the previous models, we require that the singlet VEV is negligibly small: $\langle \theta_{14} \rangle \approx 0$.

Now, we write down the various superpotential terms of the model that lead to the mass matrices for the top and bottom sectors.

The dominant contributions to the up-type quarks descend from the following superpotential operators:

$$W \supset y_t 10_1 10_1 5_1 + \frac{y_1}{\Lambda} 10_1 10_4 5_1 \theta_{15} + \frac{y_2}{\Lambda^2} 10_3 10_1 5_1 \theta_{13} \theta_{34} + \frac{y_3}{\Lambda^2} 10_3 10_4 5_1 \theta_{14} \theta_{15} + \frac{y_4}{\Lambda^4} 10_3 10_3 5_1 \theta_{13}^2 \theta_{34}^2$$

$$+ y_5 10_1 \bar{10}_2 \theta_{31} + y_6 10_4 \bar{10}_2 \theta_{35} + y_7 10_3 \bar{10}_2 \theta_{34} + \frac{y_8}{\Lambda^2} \bar{10}_2 \bar{10}_2 \bar{5}_3 \theta_{31} \theta_{34}. \tag{31}$$

These operators yield the following mass texture:

$$M_u = \begin{pmatrix} y_1 \theta_{15} v_u & \epsilon^3 y_t v_u & \epsilon^2 y_t v_u & y_2 \theta_{13} \theta_{34} v_u & y_5 \theta_{31} \\ y_1 \theta_{15} v_u & \epsilon^2 y_t v_u & \epsilon y_t v_u & y_2 \theta_{13} \theta_{34} v_u & y_5 \theta_{31} \\ y_1 \theta_{15} v_u & \epsilon y_t v_u & y_t v_u & y_2 \theta_{13} \theta_{34} v_u & y_5 \theta_{31} \\ 0 & y_2 \theta_{13} \theta_{34} v_u & y_2 \theta_{13} \theta_{34} v_u & y_4 \theta_{13}^2 \theta_{34}^2 v_u & y_7 \theta_{34} \\ y_6 \theta_{35} & y_5 \theta_{31} & y_5 \theta_{31} & y_7 \theta_{34} & y_8 \theta_{31} \theta_{34} v_b \end{pmatrix}. \tag{32}$$

In close analogy with the previous cases, the operators with couplings y_5, y_6 and y_7 in Equation (31) also contribute to the bottom sector. The remaining dominant terms contributing to the down-type quarks are:

$$W \supset y_b 10_1 \bar{5}_6 \bar{5}_3 + \frac{\kappa_1}{\Lambda} 10_1 \bar{5}_2 \bar{5}_3 \theta_{51} + \frac{\kappa_2}{\Lambda} 10_4 \bar{5}_6 \bar{5}_3 \theta_{15} + \kappa_3 10_4 \bar{5}_2 \bar{5}_3 + \frac{\kappa_4}{\Lambda^2} 10_3 \bar{5}_6 \bar{5}_3 \theta_{13} \theta_{34}$$

$$+ \frac{\kappa_5}{\Lambda} 10_3 \bar{5}_2 \bar{5}_3 \theta_{54} + \frac{\kappa_6}{\Lambda} 10_1 \bar{5}_5 \bar{5}_3 \theta_{54} + \frac{\kappa_7}{\Lambda} 10_4 \bar{5}_5 \bar{5}_3 \theta_{13} + \frac{\kappa_8}{\Lambda^3} 10_3 \bar{5}_5 \bar{5}_3 \theta_{13} \theta_{34} \theta_{54} + \kappa_9 5_4 \bar{5}_2 \theta_{53}$$

$$+ \kappa_{10} 5_4 \bar{5}_6 \theta_{13} + \frac{\kappa_{11}}{\Lambda} 5_4 \bar{5}_5 \theta_{13} \theta_{54} + \frac{\kappa_{12}}{\Lambda^2} \bar{10}_2 5_4 5_1 \theta_{13} \theta_{34}. \tag{33}$$

Collecting all the terms, we obtain the following mass matrix for the down-quark sector of the model:

$$M_d = \begin{pmatrix} \kappa_3 v_d & \kappa_1 \theta_{51} v_d & \kappa_1 \theta_{51} v_d & \kappa_5 \theta_{54} v_d & \kappa_9 \theta_{53} \\ \kappa_2 \theta_{15} v_d & \epsilon^2 y_b v_d & \epsilon y_b v_d & \kappa_4 \theta_{13} \theta_{34} v_d & \kappa_{10} \theta_{13} \\ \kappa_2 \theta_{15} v_d & \epsilon y_b v_d & y_b v_d & \kappa_4 \theta_{13} \theta_{34} v_d & \kappa_{10} \theta_{13} \\ \kappa_7 \theta_{13} v_d & \kappa_6 \theta_{54} v_d & \kappa_6 \theta_{54} v_d & \kappa_8 \theta_{13} \theta_{34} \theta_{54} v_d & \kappa_{11} \theta_{13} \theta_{54} \\ y_6 \theta_{35} & y_5 \theta_{31} & y_5 \theta_{31} & y_7 \theta_{34} & \kappa_{12} \theta_{13} \theta_{34} v_u \end{pmatrix}. \tag{34}$$

We turn now to the charged lepton sector of the model. The operators with couplings $\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_6, \kappa_7$ and κ_8 in Equation (33) also contribute to the charged lepton mass matrix. We also have common operators between the top and charged lepton sectors, which

are given in Equation (31) with couplings y_5, y_6 and y_7 . Additional contributions descend from the operators:

$$\lambda_1 \bar{5}_7 \bar{5}_2 \theta_{41} \theta_{53} + \lambda_2 \bar{5}_7 \bar{5}_2 \theta_{51} \theta_{43} + \lambda_3 \bar{5}_7 \bar{5}_6 \theta_{43} + \lambda_4 \bar{5}_7 \bar{5}_5 \theta_{53} + \lambda_5 \bar{10}_2 \bar{5}_7 \bar{5}_1. \tag{35}$$

Combining all the contributions, we receive the following mass matrix for the charged leptons of the model

$$M_e = \begin{pmatrix} \kappa_3 v_d & \kappa_3 v_d & \kappa_2 \vartheta_{15} v_d & \kappa_7 \vartheta_{13} v_d & y_6 \theta_{35} \\ \kappa_3 v_d & \kappa_3 v_d & \kappa_2 \vartheta_{15} v_d & \kappa_7 \vartheta_{13} v_d & y_6 \theta_{35} \\ \kappa_1 \vartheta_{51} v_d & \kappa_1 \vartheta_{51} v_d & y_\tau v_d & \kappa_6 \vartheta_{54} v_d & y_5 \theta_{31} \\ \kappa_5 \vartheta_{54} v_d & \kappa_5 \vartheta_{54} v_d & \kappa_4 \vartheta_{14} v_d & \kappa_8 \vartheta_{13} \vartheta_{34} \vartheta_{54} v_d & y_7 \theta_{34} \\ \lambda_1 \theta_{41} \theta_{53} + \lambda_2 \theta_{51} \theta_{43} & \lambda_1 \theta_{41} \theta_{53} + \lambda_2 \theta_{51} \theta_{43} & \lambda_3 \theta_{43} & \lambda_4 \theta_{53} & \lambda_5 v_u \end{pmatrix}. \tag{36}$$

4.5. Model E

For the fifth and final model, we have the following $U(1)'$ charges for the various matter curves:

$$10_1 : \frac{9}{2}, \quad 10_2 : \frac{11}{2}, \quad 10_3 : \frac{9}{2}, \quad 10_4 : -8, \\ 5_1 : -9, \quad 5_2 : 1, \quad 5_3 : 9, \quad 5_4 : \frac{7}{2}, \quad 5_5 : -1, \quad 5_6 : \frac{27}{2}, \quad 5_7 : \frac{7}{2}.$$

In order to receive realistic mass hierarchies, we choose the following distribution of the fermion generations and Higgs fields into matter curves:

$$10_1 \longrightarrow Q_3 + u_{2,3}^c, \quad \bar{10}_2 \longrightarrow \bar{Q}_4 + \bar{u}_4^c + \bar{e}_4^c, \quad 10_3 \longrightarrow Q_{1,2} + u_1^c + e_{1,2,3}^c, \quad 10_4 \longrightarrow Q_4 + u_4^c + e_4^c, \\ 5_1 \longrightarrow H_u, \quad \bar{5}_2 \longrightarrow H_d, \quad 5_3 \longrightarrow \bar{d}_4^c, \quad \bar{5}_4 \longrightarrow L_3, \quad 5_5 \longrightarrow \bar{L}_4, \quad \bar{5}_6 \longrightarrow d_4^c + L_4, \quad \bar{5}_7 \longrightarrow d_{1,2,3}^c + L_{1,2}.$$

With this choice, the μ -term is realized through the coupling $\theta_{13} \bar{5}_1 \bar{5}_2$, which implies that $\langle \theta_{13} \rangle \approx 0$. With this constraint, we write down the various operators for the top and bottom quark sectors.

We start again with dominant contributions to the up-type quarks. These are

$$W \supset y_t 10_1 10_1 \bar{5}_1 + \frac{y_1}{\Lambda} 10_1 10_3 \bar{5}_1 \theta_{14} + \frac{y_2}{\Lambda^2} 10_3 10_3 \bar{5}_1 \theta_{14}^2 + \frac{y_3}{\Lambda^2} 10_3 10_4 \bar{5}_1 \theta_{14} \theta_{15} + \frac{y_4}{\Lambda} 10_1 10_4 \bar{5}_1 \theta_{15} \\ + \frac{y_5}{\Lambda^2} 10_4 10_4 \bar{5}_1 \theta_{15}^2 + y_6 10_3 \bar{10}_2 \theta_{34} + y_7 10_1 \bar{10}_2 \theta_{31} + y_8 10_4 \bar{10}_2 \theta_{35} + \frac{y_9}{\Lambda^2} \bar{10}_2 \bar{10}_2 \bar{5}_2 \theta_{31}^2 \tag{37}$$

and generate the following mass texture:

$$M_u = \begin{pmatrix} y_2 \vartheta_{14}^2 v_u & y_2 \vartheta_{14}^2 v_u & y_1 \vartheta_{14} v_u & y_3 \vartheta_{14} \vartheta_{15} v_u & y_6 \theta_{34} \\ y_1 \vartheta_{14} v_u & y_1 \vartheta_{14} v_u & \epsilon y_t v_u & y_4 \vartheta_{15} v_u & y_7 \theta_{31} \\ y_1 \vartheta_{14} v_u & y_1 \vartheta_{14} v_u & y_t v_u & y_4 \vartheta_{15} v_u & y_7 \theta_{31} \\ y_3 \vartheta_{14} \vartheta_{15} v_u & y_4 \vartheta_{15} v_u & y_4 \vartheta_{15} v_u & y_5 \vartheta_{15}^2 v_u & y_8 \theta_{35} \\ y_6 \theta_{34} & y_6 \theta_{34} & y_7 \theta_{31} & y_8 \theta_{35} & y_9 \vartheta_{31}^2 v_b \end{pmatrix}. \tag{38}$$

The operators in Equation (37) with couplings y_6, y_7 and y_8 also contribute to the bottom sector of the model. In addition, we have the following superpotential terms for the bottom sector:

$$W \supset y_b 10_1 \bar{5}_7 \bar{5}_2 + \frac{\kappa_1}{\Lambda} 10_3 \bar{5}_7 \bar{5}_2 \theta_{14} + \frac{\kappa_2}{\Lambda} 10_4 \bar{5}_7 \bar{5}_2 \theta_{15} + \frac{\kappa_3}{\Lambda} 10_3 \bar{5}_6 \bar{5}_2 \theta_{13} + \frac{\kappa_4}{\Lambda} 10_1 \bar{5}_6 \bar{5}_2 \theta_{43} \\ + \kappa_5 \bar{5}_7 \bar{5}_3 \theta_{15} + \frac{\kappa_6}{\Lambda} \bar{5}_6 \bar{5}_3 \theta_{13} \theta_{45} + \frac{\kappa_7}{\Lambda} \bar{10}_2 \bar{5}_3 \bar{5}_1 \theta_{15} + \frac{\kappa_8}{\Lambda^2} 10_4 \bar{5}_6 \bar{5}_2 \theta_{13} \theta_{45}. \tag{39}$$

Collectively, we obtain the following down quark mass matrix:

$$M_d = \begin{pmatrix} \kappa_1 \vartheta_{14} v_d & \kappa_1 \vartheta_{14} v_d & \epsilon^2 y_b v_d & \kappa_2 \vartheta_{15} v_d & \kappa_5 \vartheta_{15} \\ \kappa_1 \vartheta_{14} v_d & \kappa_1 \vartheta_{14} v_d & \epsilon y_b v_d & \kappa_2 \vartheta_{15} v_d & \kappa_5 \vartheta_{15} \\ \kappa_1 \vartheta_{14} v_d & \kappa_1 \vartheta_{14} v_d & y_b v_d & \kappa_2 \vartheta_{15} v_d & \kappa_5 \vartheta_{15} \\ \kappa_3 \vartheta_{13} v_d & \kappa_3 \vartheta_{13} v_d & \kappa_4 \vartheta_{43} v_d & \kappa_8 \vartheta_{43} \vartheta_{15} v_d & \kappa_6 \vartheta_{13} \vartheta_{45} \\ y_6 \vartheta_{34} & y_6 \vartheta_{34} & y_7 \vartheta_{31} & y_8 \vartheta_{35} & \kappa_7 \vartheta_{15} v_u \end{pmatrix} \quad (40)$$

The bottom sector shares common operators between with the charged lepton sector of the model. These are given in Equation (39) with couplings κ_1, κ_2 and κ_8 . We also have common operators between top and charged lepton sectors, which are given in Equation (37) with couplings y_6 and y_8 . All the other contributions descend from the operators:

$$W \supset y_\tau 10_3 \bar{5}_4 \bar{5}_2 + \frac{\lambda_1}{\Lambda} 10_4 \bar{5}_4 \bar{5}_2 \vartheta_{45} + \frac{\lambda_2}{\Lambda} 10_3 \bar{5}_6 \bar{5}_2 \vartheta_{13} + \lambda_3 5_5 \bar{5}_7 \vartheta_{35} + \frac{\lambda_4}{\Lambda} 5_5 \bar{5}_4 \vartheta_{45} \vartheta_{31} + \lambda_5 5_5 \bar{5}_6 \vartheta_{45} + \frac{\lambda_6}{\Lambda} \bar{10}_2 5_5 5_1 \vartheta_{35} . \quad (41)$$

Combining the various contributions described so far, we end up with the following mass matrix for the charged lepton sector of the model:

$$M_e = \begin{pmatrix} \kappa_1 \vartheta_{14} v_d & \kappa_1 \vartheta_{14} v_d & \epsilon^2 y_\tau v_d & \lambda_2 \vartheta_{13} & y_6 \vartheta_{34} \\ \kappa_1 \vartheta_{14} v_d & \kappa_1 \vartheta_{14} v_d & \epsilon y_\tau v_d & \lambda_2 \vartheta_{13} & y_6 \vartheta_{34} \\ \kappa_1 \vartheta_{14} v_d & \kappa_1 \vartheta_{14} v_d & y_\tau v_d & \lambda_2 \vartheta_{13} & y_6 \vartheta_{34} \\ \kappa_2 \vartheta_{15} v_d & \kappa_2 \vartheta_{15} v_d & \lambda_1 \vartheta_{45} v_d & \kappa_8 \vartheta_{15} \vartheta_{43} & y_8 \vartheta_{35} \\ \lambda_3 \vartheta_{35} & \lambda_3 \vartheta_{35} & \lambda_4 \vartheta_{31} \vartheta_{45} & \lambda_5 \vartheta_{45} & \lambda_6 \vartheta_{35} v_u \end{pmatrix} . \quad (42)$$

5. Flavor Violation Observables

Since the Z' gauge boson couples differently with the vector-like fields, new flavor violation phenomena might emerge, and other rare processes could be amplified provided there is sufficient mixing of the vector-like fields with the SM matter ones [13,14]. In order to examine whether the present models can account for the observed LHCb-anomalies, we need to determine the unitary transformations that diagonalize the mass matrices of the models described in the previous section.

Due to the complicated form of the various matrices, we diagonalize them perturbatively around some small mixing parameter. We perform this procedure for model A, while the analysis for the rest of the models is very similar. A detailed phenomenological investigation will follow in a future publication.

5.1. Some Phenomenological Predictions of Model A

To proceed with the analysis and discuss some phenomenological implications, first, we work out the mass matrices and the mixing for quarks and leptons.

Quarks: We start with the quark sector of model A and the matrix for the down quarks. In order to simplify the down quark mass matrix from Equation (17), we assume that some terms are very small and that approximately vanish. In particular, we consider that $\kappa_5 = \kappa_{10} = \kappa_{11} = \kappa_{12} = \kappa_{14} = y_6 = y_7 \approx 0$. We make the following simplifications

$$\kappa_0 \vartheta_{41} v_d = \kappa_1 \vartheta_{45} v_d = m, \quad \kappa_2 \vartheta_{43} v_d = \alpha m, \quad \kappa_3 \vartheta_{13} \vartheta_{45} v_d = \theta m, \quad \kappa_9 \vartheta_{14} v_d = c \mu, \quad \kappa_8 \vartheta_{13} v_d = b m, \\ \kappa_{13} \vartheta_{53} \simeq y_8 \vartheta_{54} = M,$$

where the mass parameter M characterizes the mass scale of the extra vector-like states, whilst $m \sim v_d$ is related to the electroweak scale. We have also assumed that the small Yukawa parameters are identical ($\epsilon \approx \zeta$). With the above assumptions, the matrix receives the simplified form

$$M_d = \begin{pmatrix} \theta m \zeta^4 & \theta m \zeta & m \zeta^4 & 0 & 0 \\ \theta m \zeta & \theta m & m \zeta^2 & 0 & 0 \\ \alpha m \zeta^2 & \alpha m & m & 0 & 0 \\ bm & bm \zeta & 0 & c\mu & M \\ 0 & 0 & 0 & M & 0 \end{pmatrix}. \tag{43}$$

In summary, m , M and μ represent mass parameters, while α , θ , c , b and ζ are dimensionless coefficients. Keeping terms up to first order in ζ , the mass matrix $M_d M_d^T$ can be written as

$$M_d M_d^T \approx \begin{pmatrix} 0 & \theta^2 m^2 \zeta & \alpha \theta m^2 \zeta & 0 & 0 \\ \theta^2 m^2 \zeta & \theta^2 m^2 & \alpha \theta m^2 & 2b\theta m^2 \zeta & 0 \\ \alpha \theta m^2 \zeta & \alpha \theta m^2 & \alpha^2 m^2 + m^2 & \alpha b m^2 \zeta & 0 \\ 0 & 2b\theta m^2 \zeta & \alpha b m^2 \zeta & b^2 m^2 + c^2 \mu^2 + M^2 & c\mu M \\ 0 & 0 & 0 & c\mu M & M^2 \end{pmatrix}. \tag{44}$$

We observe that Equation (44) can be cast in the form $M_d^2 \approx \mathbf{A} + \zeta \mathbf{B}$ where:

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \theta^2 m^2 & \alpha \theta m^2 & 0 & 0 \\ 0 & \alpha \theta m^2 & \alpha^2 m^2 + m^2 & 0 & 0 \\ 0 & 0 & 0 & b^2 m^2 + c^2 \mu^2 + M^2 & c\mu M \\ 0 & 0 & 0 & c\mu M & M^2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & \theta^2 m^2 & \alpha \theta m^2 & 0 & 0 \\ \theta^2 m^2 & 0 & 0 & 2b\theta m^2 & 0 \\ \alpha \theta m^2 & 0 & 0 & \alpha b m^2 & 0 \\ 0 & 2b\theta m^2 & \alpha b m^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{45}$$

The local Yukawa parameter ζ couples the electroweak sector with the heavy vector-like part and can be used as a perturbative mixing parameter. The block-diagonal matrix, \mathbf{A} , is the leading order part of the matrix and can be diagonalized by a unitary matrix $V_{b_L}^0$ as $V_{b_L}^0 \mathbf{A} V_{b_L}^{0T}$. For small values of the parameter α , the eigenvalues of this matrix are written as

$$x_1 = 0, \quad x_2 \approx \theta^2 (m^2 - \alpha^2 m^2), \quad x_3 \approx \alpha^2 \theta^2 m^2 + \alpha^2 m^2 + m^2, \quad x_4 \approx M^2, \quad x_5 \approx b^2 m^2 + M^2. \tag{46}$$

We observe here that the eigenvalues appear with the desired hierarchy. The corresponding unitary matrix, which diagonalizes the matrix \mathbf{A} and returns the eigenvalues in Equation (46), is

$$V_{b_L}^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha^2 \theta^2}{2} - 1 & \alpha \theta & 0 & 0 \\ 0 & \alpha \theta & 1 - \frac{\alpha^2 \theta^2}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{c\mu M}{b^2 m^2} & 1 \\ 0 & 0 & 0 & 1 & \frac{c\mu M}{b^2 m^2} \end{pmatrix}. \tag{47}$$

The columns of this matrix are the unperturbed eigenvectors $v_{b_i}^0$ of the initial matrix.

Now we focus on the corrections to the eigenvectors due to the perturbative part $\zeta \mathbf{B}$, which are given by the relation:

$$v_{b_i} \approx v_{b_i}^0 + \zeta \sum_{j \neq i} \frac{(V_{b_L}^0 \mathbf{B} V_{b_L}^{0+})_{ji}}{x_i - x_j} v_{b_j}^0, \tag{48}$$

where the second term displays the $\mathcal{O}(\zeta)$ corrections to the basic eigenvectors of the leading order matrix \mathbf{A} . The corrected diagonalizing matrices schematically receive the form $V_{b_L} = V_{b_L}^0 + \zeta V_{b_L}^1$ and similarly for the up quarks and leptons. This way, the mixing parameter ζ enters in the expressions associated with the various flavor violation observables.

Computing the eigenvectors using Equation (48), the $\mathcal{O}(\zeta)$ corrected unitary matrix is:

$$V_{b_L} \approx \begin{pmatrix} 1 & -\zeta & 0 & 0 & 0 \\ -\zeta & \frac{\alpha^2\theta^2}{2} - 1 & \alpha\theta & \frac{2b\theta m^2\zeta}{M^2} & -\frac{2bc\theta\mu m^2\zeta}{M^3} \\ \alpha\theta\zeta & \alpha\theta & 1 - \frac{\alpha^2\theta^2}{2} & -\frac{\alpha b m^2\zeta}{M^2} & -\frac{\alpha bc\mu m^2\zeta}{M^3} \\ 0 & -\frac{2c\theta\mu\zeta}{bM} & \frac{c\mu\zeta}{bM} & -\frac{c\mu M}{b^2 m^2} & 1 \\ 0 & \frac{2b\theta m^2\zeta}{M^2} & \frac{\alpha b m^2\zeta}{M^2} & 1 & \frac{c\mu M}{b^2 m^2} \end{pmatrix}. \tag{49}$$

We assume here that the mixing in the top sector is small and that the main mixing descends from the bottom quark sector.

Charged Leptons: We turn now to the charged lepton mass matrix in Equation (18). We notice that some parameters from the top and bottom sectors also contribute here, so the same assumptions for these parameters will be considered here too. Additionally, we assume that $\lambda_1 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_7 \approx 0$, and we make the following simplifications

$$\lambda_2\vartheta_{34}v_d = c\mu, \lambda_5\vartheta_{51}\vartheta_{43} = qm, \lambda_6\vartheta_{51} \approx y_8\vartheta_{54} = M,$$

where the mass parameter M characterizes the vector-like scale, and q, c are dimensionless parameters.

With these approximations, the matrix receives the following form:

$$M_e \approx \begin{pmatrix} \theta m & \alpha m\zeta^4 & \alpha m\zeta^4 & 0 & 0 \\ \theta m\zeta^3 & \alpha m\zeta & \alpha m\zeta^2 & 0 & 0 \\ \theta m\zeta & \alpha m\zeta^2 & \alpha m & 0 & 0 \\ 0 & 0 & 0 & c\mu & M \\ 0 & mq & m\zeta q & M & 0 \end{pmatrix}. \tag{50}$$

We proceed by perturbatively diagonalizing the lepton square mass matrix $M_e M_e^T$ (M_e^2 for short) using ζ as the expansion parameter. Keeping up to $\mathcal{O}(\zeta)$ terms, we write the mass square matrix in the form $M_e^2 \approx \mathbf{A} + \zeta \mathbf{B}$, where:

$$\mathbf{A} = \begin{pmatrix} \theta^2 m^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha^2 m^2 & 0 & 0 \\ 0 & 0 & 0 & c^2 \mu^2 + M^2 & c\mu M \\ 0 & 0 & 0 & c\mu M & m^2 q^2 + M^2 \end{pmatrix}, \tag{51}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & \theta^2 m^2 \zeta & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha m^2 \zeta q \\ \theta^2 m^2 \zeta & 0 & 0 & 0 & \alpha m^2 \zeta q \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha m^2 \zeta q & \alpha m^2 \zeta q & 0 & 0 \end{pmatrix}. \tag{52}$$

The eigenvalues of the dominant part are:

$$x_1 = 0, \quad x_2 = \alpha^2 m^2, \quad x_3 = \theta^2 m^2, \quad x_4 = M^2, \quad x_5 = M^2 + m^2 q^2 \tag{53}$$

and the unitary matrix $V_{e_L}^0$, which diagonalizes the dominant matrix \mathbf{A} , is:

$$V_{e_L}^0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \frac{c\mu M}{m^2 q^2} \\ 0 & 0 & 0 & \frac{c\mu M}{m^2 q^2} & 1 \end{pmatrix}. \tag{54}$$

The $\mathcal{O}(\xi)$ corrections to the eigenvectors due to the perturbative part $\xi \mathbf{B}$ can be found by applying the relation in Equation (48). Then, for the final unitary matrix, we obtain

$$V_{e_L} \approx \begin{pmatrix} 0 & 1 & 0 & \frac{\alpha c \mu m^2 \xi q}{M^3} & \alpha \left(-\frac{c^2 \mu^2 \xi}{m^2 q^3} - \frac{m^2 \xi q}{M^2} \right) \\ -\frac{\xi(\alpha^2 + \theta^2)}{\theta^2} & 0 & 1 & \frac{\alpha c \mu m^2 \xi q}{M^3} & -\frac{\alpha m^2 \xi q}{M^2} \\ 1 & 0 & \xi & 0 & 0 \\ 0 & \frac{\alpha c \mu \xi}{Mq} & \frac{\alpha c \mu \xi}{Mq} & -1 & \frac{c \mu M}{m^2 q^2} \\ 0 & \frac{\alpha m^2 \xi q}{M^2} & \frac{\alpha m^2 \xi q}{M^2} & \frac{c \mu M}{m^2 q^2} & 1 \end{pmatrix} \quad (55)$$

where, in order to simplify the final result, we have assumed the series expansions for small α , θ and c , keeping only the dominant terms.

5.2. B-Meson Anomalies at LHCb

In the presence of a fourth generation, where the $U(1)'$ charge assignments of its constituents differ from those of the SM families, many interesting rear flavor processes are expected to be enhanced, and a detailed consideration will appear in a forthcoming publication. Here, we shall focus only on the B-meson anomaly associated with the $b \rightarrow s \ell \ell$ decay and, in particular, the ratio $R_{K^{(*)}} = \text{BR}(B \rightarrow K^{(*)} \mu \mu) / \text{BR}(K^{(*)} e e)$. Due to the non-universal coupling of the Z' gauge boson with the vector-like fermions, the $C_9^{\mu\mu}$ Wilson coefficient, which contributes to the flavor violation transition $b \rightarrow s \ell \ell$, is given by:

$$C_9^{\mu\mu} = -\frac{\sqrt{2}}{4G_F} \frac{16\pi^2}{e^2} \left(\frac{g'}{M_{Z'}} \right)^2 \frac{(Q'_{d_L})_{23} (Q'_{e_L})_{22}}{V_{tb} V_{ts}^*}, \quad (56)$$

where the matrices Q'_{d_L} and Q'_{e_L} are defined as [22]

$$Q'_{f_L} \equiv V_{f_L} q'_{f_L} V_{f_L}^\dagger, \quad (57)$$

with q'_{f_L} being 5×5 diagonal matrices of $U(1)'$ charges³.

The elements $(Q'_{d_L})_{23}$ and $(Q'_{e_L})_{22}$ participating in the $C_9^{\mu\mu}$ coefficient can be obtained from Equation (57) using the diagonalization matrices V_{f_L} computed above. We have that

$$(Q'_{d_L})_{23} \approx -\frac{1}{2} \alpha \theta \xi^2 \quad \text{and} \quad (Q'_{e_L})_{22} \approx -1 - \xi^2. \quad (58)$$

Finally, using the set of values $G_F \approx 11.66 \text{ TeV}^{-2}$, $e \approx 0.303$, $V_{tb} \approx 0.99$ and $V_{ts} \approx 0.0404$, we estimate that :

$$C_9^{\mu\mu} \approx -652.5 \alpha \theta \xi^2 \left(\frac{g'}{M_{Z'}} \right)^2 + 5220 \alpha b^2 \theta \xi^2 \left(\frac{g'}{M_{Z'}} \right)^2 \left(\frac{m}{M} \right)^4 \quad (59)$$

where the mass parameters m , M and $M_{Z'}$ are displayed in TeV units.

Using an indicative set of values $\alpha \rightarrow 0.06$, $\theta \rightarrow 0.27$, $\xi \rightarrow 0.5$, $m \rightarrow 0.1$, $b \rightarrow 0.1$, $M \rightarrow 1.2$ in Equation (59), we obtain

$$C_9^{\mu\mu} \approx -2.64 \left(\frac{g'}{M_{Z'}} \right)^2. \quad (60)$$

According to the most recent global fits [24], an explanation of the current experimental data requires $C_9^{\mu\mu} \approx -0.82$, so in this model, the Z' gauge coupling-mass ratio should be of the order $\frac{g'}{(M_{Z'}/TeV)} \approx \frac{1}{2}$ in order for the model to explain the observed R_K anomalies. This implies a rather small Z' mass [25], unless g' is associated with some strong coupling regime. Of course, the computation of $C_9^{\mu\mu}$ is very sensitive to the mass and mixing details of the representative model chosen in this example, and a comprehensive analysis of the

whole set of models will determine whether sufficient mixing effects can predict the various deviations observed in B -meson decays; however, this analysis is beyond the goal of the present work.

6. R-parity Violation Terms

A remarkable observation is that particular R-parity violating (RPV) terms, such as $\lambda'_{ijk} L_i Q_j d_k^c$, could explain the anomalies related to the $b \rightarrow s\ell\ell$ flavor-violating process [26–29].

In this section, we look for possible R-parity violating terms (RPV) in the tree-level superpotential (dubbed here W_{tree}^{RPV}) for the models A, B, C, D, E presented in Table 3 and briefly discuss their consequences. We distinguish the RPV terms in those which couple only the MSSM fields and those which share Yukawa couplings with extra vector-like families. If the former are present in W_{tree}^{RPV} , they lead to hard violations of baryon and/or lepton numbers and must be suppressed. In F-theory constructions, such terms can be eliminated either by judicious flux restrictions piercing the various matter curves [30] or by additional (discrete) symmetries emanating from the background geometry of the theory [31–34]. In Section 4 of [32], there are examples of how this R-parity can be built. On the other hand, provided certain restrictions and conditions are fulfilled [26], such couplings may contribute to the B -meson anomalies and other interesting effects, such as the $(g - 2)_\mu$ anomaly [35,36], without exceeding baryon and/or lepton number violating bounds.

Below, for each one of the five classes of models, among all possible superpotential couplings, we single out the RPV terms. Taking into account that in all the models presented so far, the up-Higgs doublet is isolated at the 5_1 matter curve, the possible RPV operators of the $10 \times \bar{5} \times \bar{5}$ form are:

$$10_1(\bar{5}_2\bar{5}_7 + \bar{5}_3\bar{5}_6 + \bar{5}_4\bar{5}_5), \quad 10_2\bar{5}_3\bar{5}_4, \quad 10_3\bar{5}_2\bar{5}_4, \quad 10_4\bar{5}_2\bar{5}_3. \tag{61}$$

Next, we discuss each model separately.

Model A. Using Table 3 and taking into account Equation (61), we find that the only RPV term of the model is:

$$10_1\bar{5}_3\bar{5}_6 \longrightarrow L_4 Q_3 d_{1,2}^c. \tag{62}$$

We observe that R-parity violation occurs with terms that involve the vector-like family, and there are no terms that have only the three quark and lepton families of the MSSM. However, as recently shown in [28,29], the coupling $L_4 Q_3 d_2^c$ can have significant contributions to the $b \rightarrow s\mu\mu$ process through photonic penguin diagrams.

Model B. Following the same procedure, we found that this model contains the following RPV terms:

$$W_{tree}^{RPV} \supset 10_1\bar{5}_4\bar{5}_5 + 10_2\bar{5}_3\bar{5}_4 \longrightarrow L_3 Q_3 d_4^c + L_3 L_4 e_{1,2,3}^c. \tag{63}$$

The first operator here does not contribute to the $b \rightarrow sll$ process due to the absence of the second generation quark in the coupling. On the other hand, the term $L_3 L_4 e_2^c$, which descends from the second operator, leads to non-negligible contributions to the anomalous magnetic moment of the muon [35]. Combining this with non-zero Z' contributions may lead to a sufficient explanation of the $(g - 2)_\mu$ anomaly.

Model C, D and E. There are no renormalizable RPV terms in these models. Therefore, in this case, an explanation of the observed experimental discrepancies is expected from Z' interactions and through the mixing of SM fermions with the extra vector-like states.

7. Conclusions

In this article, we have expanded our previous work on F-theory-motivated models by performing a scan of all the possible $SU(5) \times U(1)'$ semi-local constructions, predicting a complete family of vector-like exotics. We use $U(1)'$ hypercharge flux to obtain the symmetry breaking of the non-abelian part and a Z_2 monodromy to guarantee a tree-

level top Yukawa coupling. Moreover, we have imposed phenomenological restrictions on the various flux parameters in order to obtain exactly three chiral generations and one vector-like complete family of quarks and leptons. In addition, demanding anomaly cancellation, we have found that there exist 192 models with universal $U(1)'$ charges for the MSSM families and non-universal for the vector-like states. These 192 models fall into five distinct classes with respect to their $SU(5) \times U(1)'$ properties, classified as A, B, C, D and E in the analysis. We have presented one illustrative model for each class, exploring the basic properties, computing the superpotential terms and constructing the fermion mass matrices. For the models derived in the context of class A , in particular, we have exemplified how these types of models can explain the observed R_K anomalies through the mixing with the vector-like states without violating other flavor violation observables by virtue of the universal nature of the three SM families. We also discussed the presence of RPV couplings in these type of models and their possible contribution in the observed experimental deviations from the SM predictions. It is worth emphasizing that due to the flux restrictions and the symmetries of the theory, only a restricted number of the possible RPV terms appear in the models. This way, with a careful choice of the flux parameters, it is possible to interpret such deviation effects while avoiding significant contributions to dangerous proton decay effects. A detailed account of such new physics phenomena is beyond the present work and is left for a future publication.

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Notes

- 1 Note that a ‘mirror’ solution subject to $c_i \rightarrow -c_i$ also exists.
- 2 For a detailed definition of the singlet spectrum of the theory, see [10].
- 3 For a discussion on the effects of complex valued contributions to the Wilson coefficients due to large CP-violation effects, see [23].

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