



# Article Genuine Dilatons in Gauge Theories

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**Abstract:** A genuine dilaton  $\sigma$  allows scales to exist even in the limit of *exact* conformal invariance. In gauge theories, these may occur at an infrared fixed point (IRFP)  $\alpha_{IR}$  through dimensional transmutation. These large scales at  $\alpha_{IR}$  can be separated from small scales produced by  $\theta^{\mu}_{\mu}$ , the trace of the energy-momentum tensor. For quantum chromodynamics (QCD), the conformal limit can be combined with chiral  $SU(3) \times SU(3)$  symmetry to produce chiral-scale perturbation theory  $\chi PT_{\sigma}$ , with  $f_0(500)$  as the dilaton. The technicolor (TC) analogue of this is crawling TC: at low energies, the gauge coupling  $\alpha$  goes directly to (but does not walk past)  $\alpha_{IR}$ , and the massless dilaton at  $\alpha_{IR}$  corresponds to a light Higgs boson at  $\alpha \leq \alpha_{IR}$ . It is suggested that the  $W^{\pm}$  and  $Z^0$  bosons set the scale of the Higgs boson mass. Unlike crawling TC, in walking TC,  $\theta^{\mu}_{\mu}$  produces *all* scales, large and small, so it is hard to argue that its "dilatonic" candidate for the Higgs boson is not heavy.

**Keywords:** quantum chromodynamics; technicolor; conformal; Nambu–Goldstone; Wigner–Weyl; dilaton; scalon; renormalization; fixed point

# 1. Introduction

It is surprising how far the notion of a "dilaton" has strayed from the original version of 1968–1970. Now it can be any of the following:

- I. a scalar Nambu–Goldstone (NG) boson for *exact* conformal invariance of a Hamiltonian  $\mathcal{H}$  which has a *scale-dependent* ground state  $|vac\rangle$  and hence scale-dependent amplitudes in the limit of scale invariance; or
- II. a scalar component of the gravitational field; or
- III. a scalar particle in a theory where conformal invariance is permitted only in the Wigner–Weyl (WW) mode (scale-invariant amplitudes). In terms of a Hamiltonian  $\mathcal{H}$ , scale-dependent effects such as fermion condensation exist *only* in the presence of a term  $\delta \mathcal{H}$  which breaks scale invariance explicitly in  $\mathcal{H} = \mathcal{H}_0 + \delta \mathcal{H}$ . Both  $\mathcal{H}_0$  and its ground state  $|vac\rangle_0$  are conformal invariant.

Evidently, I and III contradict each other and may have little to do with II. Typical of III are (a) deformed conformal Lagrangians and (b) walking TC, which have been promoted as ways of explaining why the Higgs boson is so light. I observe that these theories are very unlikely to achieve this because they apparently involve just one scale, set by  $\theta^{\mu}_{\mu}$ . Two scales are needed, as in crawling TC [1], which is a type-I theory.

Most papers on "dilatons" consider type-II or type-III. The problem is with assertions that type-III theories can be matched to type-I effective Lagrangians. So I begin with a quick summary of the fundamental type-I theory in its original setting, strong interactions (Section 2).

Dilatonic versions of gauge theories are considered in Section 3. For quantum chromodynamics (QCD), only a type-I theory is possible, chiral-scale perturbation theory  $\chi PT_{\sigma}$  [2–4], where the dilaton at the IRFP corresponds to the *light* resonance  $f_0(500)$ . The TC analogue of this is crawling TC, where the

Higgs boson is the type-I TC analogue of  $f_0$ . The scale-dependent IRFP lies *outside* the conformal window [5].

Section 4 compares crawling TC with type-III theories for the Higgs boson. The type-III concept is due to Gildener and Weinberg [6]. They called their spin-0<sup>+</sup> particle a "scalon"—a good name—but that morphed into the term "dilaton" in "dilatonic" walking TC [7–15] and deformed conformal potential theory [16–23]. I will reserve the term "genuine dilaton" for type-I dilatons.

The key observation is that in type-III theories, it is hard to distinguish small scales from large scales because they are all generated by the trace  $\theta^{\mu}_{\mu}$ . For example, in walking TC, the sill of the conformal window produces the fermion condensate  $\langle \overline{\psi}\psi \rangle_{\text{vac}}$ . That indicates a *large* mass

$$m_{h_{\rm III}} = \langle h_{\rm III} | \theta^{\mu}_{\mu} | h_{\rm III} \rangle \sim \text{ a few TeV}$$
(1)

for the would-be Higgs boson  $h_{\text{III}}$ , no matter how slowly  $\alpha$  walks. A similar conclusion was drawn in early work on walking TC [7,9,10]. It cannot be undone by assuming [19] an equivalence to a type-I dilaton Lagrangian below the sill.

By contrast, in a type-I theory, all large scales arise from the scale dependence of amplitudes in the exact conformal limit: that is what is meant by the NG mode for conformal invariance. In crawling TC, the fermion condensate  $\langle \bar{\psi}\psi \rangle_{\text{vac}}$  sets the scale *at* the IRFP  $\alpha_{\text{IR}}$ . For values of  $\alpha$  just below  $\alpha_{\text{IR}}$ ,  $\theta^{\mu}_{\mu}$  appears as a small perturbation which produces small scales, such as the mass acquired by the type-I dilaton: a light Higgs boson.

Why has the concept that there can be scale dependence in the conformal limit, which seemed so simple in 1968–70, been so systematically overlooked since then? In particular, why must all IRFP's be in WW mode? An IRFP cannot appear outside the conformal window, by definition? I offer possible reasons for these points of view in Section 5, with a separate Section 6 specifically for IRFP's.

Possible tests of these proposals are considered in Section 7. A light scalar boson has been observed in lattice data for SU(3) gauge theory with  $N_f = 8$  triplet fermions [24–27] and two sextet fermions [28,29]. In each case, this is being interpreted as a type-III dilaton for walking TC, but it is more likely to be a genuine dilaton, and hence evidence for an IRFP just outside the conformal window.

### 2. Hadronic Physics

The idea that scale and conformal invariance may be spontaneously (i.e., *not* explicitly) broken dates from 1962 (footnote 38 of [30]). An analogy was drawn with the partial conservation of the axial-vector currents  $\vec{\mathcal{F}}_{\mu 5}$  for chiral  $SU(3)_L \times SU(3)_R$  symmetry, where  $\pi$ , K,  $\bar{K}$ ,  $\eta$  pole dominance of amplitudes of the divergences  $\partial^{\mu}\vec{\mathcal{F}}_{\mu 5}$  yields soft-meson relations such as the Goldberger–Treiman relation for the pion-nucleon coupling constant  $g_{\pi NN}$ . Similarly, a spin-0<sup>+</sup> particle  $\sigma$  tied to the trace  $\theta^{\mu}_{\mu}$  of the energy-momentum tensor  $\theta_{\mu\nu}$  couples universally to particle mass. For a nucleon N with mass  $M_N$ , the  $\sigma$ -nucleon coupling constant  $g_{\sigma NN}$  is given by

$$f_{\sigma}g_{\sigma NN} \simeq M_N$$
, (2)

where  $f_{\sigma}$  is the scalar analogue of the pion decay constant  $f_{\pi}$ :

$$\langle \sigma | \theta_{\mu\nu} | \text{vac} \rangle = (f_{\sigma}/3)(q_{\mu}q_{\nu} - g_{\mu\nu}q^2).$$
(3)

The currents for scale and conformal transformations can be written in terms of  $\theta_{\mu\nu}$  (improved [31] if spin-0 fields are present) as follows:

$$\mathcal{D}_{\nu}(x) = x^{\mu}\theta_{\mu\nu}(x) \quad \text{and} \quad \mathcal{K}_{\mu\nu} = (2x_{\mu}x_{\lambda} - x^{2}g_{\mu\lambda})\theta_{\nu}^{\lambda}.$$
 (4)

As for chiral  $SU(3)_L \times SU(3)_R$  currents, explicit breaking of the symmetry is measured by current divergences:

$$\partial^{\nu} \mathcal{D}_{\nu} = \theta^{\lambda}_{\lambda} \quad \text{and} \quad \partial^{\nu} \mathcal{K}_{\mu\nu} = 2x_{\mu}\theta^{\lambda}_{\lambda}.$$
 (5)

Therefore, scale and conformal invariance correspond to the limit

$$\theta_{\lambda}^{\lambda} \to 0.$$
 (6)

The question [32–35] is, does the vacuum state respect the symmetry, or break it? Are scale and conformal invariance realized in the WW mode or the "spontaneous" NG mode?

A comparison with the chiral  $SU(3)_L \times SU(3)_R$  group in hadronic physics is instructive [35]. In that case, both modes occur. The subgroup  $SU(3)_{L+R}$  associated with vector currents  $\vec{\mathcal{F}}_{\mu}$  has a symmetry limit in the WW mode—its generators  $\vec{\mathcal{F}}$  annihilate the vacuum state:

$$\vec{F} |\mathrm{vac}\rangle \to 0$$
,  $\vec{F} = \int d^3x \, \vec{\mathcal{F}}_0$ . (7)

The symmetry is manifest: its representations can be seen in the particle spectrum. The rest of the group, represented by cosets  $SU(3)_L \times SU(3)_R / SU(3)_{L+R}$  and generated by axial charges  $\vec{F}_5$ , has its symmetry realized in the NG mode:

$$\vec{F}_5 |\mathrm{vac}\rangle \not\to 0$$
,  $\vec{F}_5 = \int d^3x \, \vec{\mathcal{F}}_{05}$ . (8)

As a result, the axial part of the symmetry is hidden,  $|vac\rangle$  becomes a member of a degenerate set of physically equivalent vacua

$$|\operatorname{vac}\rangle_{\vec{\alpha}} = \exp\{i\vec{\alpha}\cdot\vec{F}_5\}|\operatorname{vac}\rangle_{\vec{\alpha}=0},$$
(9)

and there is a massless NG boson for each independent direction in  $\vec{\alpha}$  space: for  $SU(3)_L \times SU(3)_R$ , eight 0<sup>-</sup> NG bosons  $\pi$ , K,  $\bar{K}$ ,  $\eta$ . A unique vacuum state can be picked out by perturbing the Hamiltonian with a term which breaks the axial part of the symmetry and gives the NG bosons mass.

Similarly, for the limit of scale and conformal invariance, "there are two possibilities: either all particle masses go to zero, or there is a massless scalar boson of the NG type that allows other masses to be non-zero" [35].

The first possibility refers to the WW scaling mode, where scale and conformal invariance are manifest. Let

$$D(t) = \int d^3x \, \mathcal{D}_0(t, \mathbf{x}) \quad \text{and} \quad K_\mu(t) = \int d^3x \, \mathcal{K}_{\mu 0}(t, \mathbf{x}) \tag{10}$$

generate scale and special conformal transformations. In the symmetry limit, *D* and  $K_{\mu}$  become time independent, and their commutators with the translation and Lorentz generators  $P_{\mu}$  and  $M_{\mu\nu}$  simplify, e.g.,

$$[K_{\mu}, P_{\nu}] = -2i(g_{\mu\nu}D + M_{\mu\nu}), \quad \theta^{\lambda}_{\lambda} \to 0.$$
<sup>(11)</sup>

Given that  $|vac\rangle$  is the only state annihilated by both  $P_{\mu}$  and  $M_{\mu\nu}$ , it follows from (11) that  $K_{\mu}|vac\rangle = 0$  implies  $D|vac\rangle = 0$  and vice versa. Conformal invariance of the vacuum state implies that the theory lies *within* the conformal window: Green's functions exhibit power-law behavior characteristic of representations of the conformal group SO(4, 2). Dimensional couplings vanish, e.g., scalar particles decouple from  $\theta_{\mu\nu}$ :

$$\langle \varphi | \theta_{\mu\nu} | \mathrm{vac} \rangle \to 0.$$
 (12)

Particles are massless or do not exist [36], and the rest of the mass spectrum is empty or continuous. Consequently, the WW-mode scaling limit is nothing like the real world. Key physical properties such as a massive spectrum can arise only as *dominant* contributions from terms in the Hamiltonian which break scale invariance *explicitly*.

The other possibility is the NG scaling mode, where there is a non-compact degeneracy of Poincaré invariant vacua

$$|\mathrm{vac}\rangle_{\rho} = \exp\{i\rho D\} |\mathrm{vac}\rangle_{\rho=0}$$
. (13)

As for the chiral case (9), these vacua are physically equivalent; one of them is picked out if a small symmetry breaking term is added to the Hamiltonian. Equation (11) remains valid, so  $D|vac\rangle \neq 0$  implies  $K_{\mu}|vac\rangle \neq 0$ . Conformal symmetry is hidden by the dependence of amplitudes on dimensional constants such as masses. This is allowed if there is a massless 0<sup>+</sup> NG boson  $\sigma$  for scale and conformal invariance: the *dilaton*<sup>1</sup>.

The key property of a dilaton is that the decay constant  $f_{\sigma}$  in (3) remains *non-zero* in the scale-symmetric limit:

$$f_{\sigma} \not\to 0$$
,  $\theta^{\mu}_{\mu} \to 0$ . (14)

Since  $f_{\sigma}$  has dimensions of mass, amplitudes can depend on scales in the limit (14). Scalar Goldberger–Treiman relations of the form (2) become exact, so particles such as nucleons *N* can remain massive in a theory with NG-mode scale invariance. The pion decay constant  $f_{\pi}$  can also remain non-zero: the NG mode for conformal invariance is compatible with the NG mode for chiral invariance [37–42].

Evidently, compared with the WW mode, the NG scaling mode offers the great advantage that there is a chance that it approximates the real world. A *small* scale-violating perturbation of the Hamiltonian may be sufficient to give the dilaton and other NG bosons their observed small masses and make small corrections to large masses in the non-NG particle sector. The consistency of assuming an NG mode for scale invariance was confirmed via effective Lagrangians [32,34,37,38,40], and by the end of 1970, a complete understanding had been achieved [40,41,43]. However, dilaton phenomenology at that time did not go far: the only candidate for  $\sigma$  was a vague 0<sup>+</sup> resonance  $\epsilon$ (700) which was last listed in the Particle Data Tables in 1974. The main results were for the  $\sigma \rightarrow \pi\pi$  coupling [38,39],

$$f_{\sigma}g_{\sigma\pi\pi} \simeq -m_{\sigma}^2$$
 i.e.,  $\sigma$ -width  $\sim \sigma$ -mass, (15)

and for the  $\sigma \rightarrow \gamma \gamma$  coupling due to the electromagnetic trace anomaly [44–46]:

$$f_{\sigma}g_{\sigma\gamma\gamma} \simeq \frac{Re^2}{6\pi^2}, \quad R = \left. \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \right|_{\text{high energy}}.$$
 (16)

#### 3. Gauge Theories

This line of investigation was resumed almost 40 years later, prompted by a partial-wave analysis [47] which isolated the broad low-mass 0<sup>+</sup> resonance<sup>2</sup>  $f_0(500)$  at 441 – 272*i* MeV, with small experimental and theoretical uncertainties—unlike the  $\epsilon$ (700). A perfect candidate for the hadronic dilaton of 1968–72 had appeared:

$$\sigma|_{\text{hadronic}} = f_0 \,. \tag{17}$$

The mass of  $f_0$  is close to K(495) and  $\eta(549)$ , so it makes sense to extend standard chiral  $SU(3) \times SU(3)$  perturbation theory  $\chi$ PT<sub>3</sub> to chiral-scale perturbation theory [2–4]  $\chi$ PT<sub> $\sigma$ </sub>, with NG bosons  $\pi$ , K,  $\bar{K}$ ,  $\eta$ ,  $\sigma$  in the combined limit of chiral and conformal symmetry.

A common question here is: what is so special about the case of  $N_f = 3$  flavors?

<sup>&</sup>lt;sup>1</sup> This term was coined in 1969, and first appeared in print in [37].

<sup>&</sup>lt;sup>2</sup> A successor  $f_0(400-900)$  to the dormant  $\epsilon(700)$  resonance was first identified in 1996 [48] in the context of the linear sigma model. The key features of the 2006 analysis [47] were its model independence and precision, which led to the inclusion of  $f_0(500)$  in the 2008 Particle Data Tables. See [49] for an extensive review. Our symbol  $\sigma$  for the dilaton does *not* mean that we rely on the sigma model.

This has to do with whether good phenomenology results, in a first approximation, when a given quark flavor is considered to be

- 1. heavy enough to be decoupled, or
- 2. light enough to be part of a chiral perturbation theory, or
- 3. neither.

The quarks *t*, *b* and *c* are far too heavy to belong to category 2. However, they certainly belong to category 1 if we restrict ourselves to particle states and operators constructed from *u*, *d*, *s* quarks and consider amplitudes at energies  $\ll m_c$ . So, let us decouple the heavy quarks:

$$m_Q \to \infty$$
,  $Q = t, b, c.$  (18)

Then, since u and d are much lighter than s, it is tempting to try chiral perturbation theory  $\chi PT_2$  based on approximate  $SU(2)_L \times SU(2)_R$  symmetry. That gives relatively precise results (5–10% accuracy), but only if the *s*-quark mass  $m_s$  is held fixed. That becomes a problem if we want to make a connection with scale invariance, because  $\theta^{\mu}_{\mu}$  contains a renormalized version of the mass term  $m_s \bar{s}s$ . Any attempt to decouple *s* by taking the limit  $m_s \to \infty$  would be a terrible approximation, given e.g., the observed  $SU(3)_{L+R}$  multiplet structure of particle states.

The remaining possibility is that u, d and s all belong to category 2, which corresponds to approximate  $SU(3)_L \times SU(3)_R$  symmetry. The result is  $\chi PT_3$ , a less precise but adequate theory  $\sim 30\%$  accuracy, apart from difficulties in 0<sup>+</sup> channels due to the low-lying  $f_0(500)$  resonance. Unitarized chiral perturbation theory  $U\chi PT$  [49,50] is a general dispersive method for dealing with this, while  $\chi PT_\sigma$  is a QCD-based effective theory with  $f_0$  treated as a NG boson. These technicalities do not detract from the requirement that QCD must make sense in the IR limit that category-2 masses and all momenta  $p_{Operators/NG}$  for physical operators and NG bosons tend to zero:

$$p_{\text{Operators/NG}} \to 0$$
,  $m_q \to 0$ ,  $q = u, d, s.$  (19)

The key observation is that there is no way of distinguishing this limit from the infrared limit of the renormalization group (RG) for gauge theories with massless fermions. We are therefore able to determine to some extent how the gauge coupling runs in that limit. That is the benefit of having *no* quarks in category 3.

The result of the decoupling (18) is QCD for  $N_f = 3$  quark flavors q = u, d, s. Since QCD is renormalizable, there is a trace anomaly [51–54] proportional to the  $N_f = 3$  Callan–Symanzik function  $\beta(\alpha_s)$ ,

$$\theta^{\mu}_{\mu}\Big|_{\rm QCD} = \frac{\beta(\alpha_s)}{4\alpha_s} G^{a\mu\nu} G^a_{\mu\nu} + \left(1 + \gamma_m(\alpha_s)\right) \sum_{q=u,d,s} m_q \bar{q}q \,, \tag{20}$$

where  $\alpha_s$  is the gauge coupling for strong interactions and  $G^a_{\mu\nu}$  is the field-strength tensor. The theory is asymptotically free, so  $\alpha_s$  increases as low-momentum scales are approached. In the infrared limit (19), there are two main possibilities:

- (A) This is the conventional alternative. The result is the green curve labelled  $\chi PT_3$  in Figure 1, where  $\beta(\alpha_s)$  remains negative and  $\alpha_s$  runs to  $+\infty$ . In that limit, the gluonic part  $(\beta/4\alpha_s)G^2$  of the trace is still present and breaks conformal invariance explicitly. Apart from the massless  $0^-$  bosons  $\{\pi, K, \eta\}$ , all hadrons, *including*  $f_0(500)$ , acquire their mass through this mechanism. Please note that the chiral condensate  $\langle \bar{q}q \rangle_{vac} \neq 0$  survives in this limit.
- (B) There is an IRFP  $\alpha_{IRs}$  at which  $\beta$  vanishes and beyond which  $\alpha_s$  cannot go (the red curve in Figure 1 labelled  $\chi PT_{\sigma}$ ):

$$\alpha_s \rightarrow \alpha_{\rm IRs}$$
,  $\beta(\alpha_{\rm IRs}) = 0$ . (21)

As a result, both the gluonic and quark-mass terms in Equation (20) vanish and the theory becomes conformal invariant:

$$\theta^{\mu}_{\mu}\Big|_{\rm QCD} \to 0.$$
 (22)

Since this is equivalent to the chiral limit (19), the 0<sup>-</sup> NG bosons  $\pi$ , K,  $\eta$  and hence the chiral condensate  $\langle \bar{q}q \rangle_{\text{vac}}$  survive, and so  $\langle \bar{q}q \rangle_{\text{vac}}$  acts as a scale condensate. That implies the presence of a massless dilaton  $\sigma$  at the IRFP, which permits all non-NG hadrons to be *massive* in the conformal limit (22). Chiral-scale perturbation theory  $\chi$ PT $_{\sigma}$  is then a simultaneous expansion about  $\alpha_{\text{IR}s}$  and in the u, d, s masses. Note the desirable scale separation between the NG-boson sector { $\pi$ , K,  $\eta$ ,  $f_0$ } and heavy hadrons in Figure 2.



**Figure 1.** Alternatives for the  $N_f = 3 \text{ QCD } \beta$ -function. (A) Conventional  $N_f = 3$  soft-meson theory  $\chi \text{PT}_3$  (green curve) involves a large breaking of scale invariance at  $\alpha_s \sim \infty$  to ensure that heavy hadrons such as nucleons acquire sufficient mass, but then that mechanism also generates the mass of the  $f_0$ , which is *not* heavy:  $m_{f_0} \sim m_K \ll m_N$ . (B) That problem is solved in  $\chi \text{PT}_{\sigma}$  (red curve): (a) the massless dilaton  $\sigma$  at  $\alpha_{\text{IR}s}$  allows nucleons to be heavy (Equation (2)), and (b)  $\sigma$  becomes the pseudodilaton  $f_o$  as it acquires a small (mass)<sup>2</sup> to first order in  $\epsilon_s = \alpha_{\text{IR}s} - \alpha_s$ . Both curves are consistent with model-independent U $\chi$ PT.

$$\chi PT_{\sigma} \xrightarrow[\pi]{K^*} f_0 K \eta \xrightarrow{\text{Scale}} \rho \omega \xrightarrow{\text{Non-NG sector}} (\text{mass})^2$$

**Figure 2.** The hadronic spectrum below 1 GeV, seen from the point of view of  $\chi PT_{\sigma}$ . Masses and momenta *p* of NG bosons, including  $f_0(500)$ , are small relative to scales of the non-NG sector. Please note that  $\chi PT_{\sigma}$  works only for  $N_f = 3$  light flavors; there is no analogue of it for  $N_f = 2$  because of the presence of the *s* quark:  $m_s \gg m_{u,d}$ .

Alternative (A) is possible, but from the point of view of QCD, the small mass of  $f_0(500)$  is an unexplained accident. In a nonperturbative setting, the simplest (and perhaps only) argument for a small mass is that the theory approximates a symmetry in NG mode. If  $f_0$  is not an NG boson, then we have no symmetry to force its constituents,  $q\bar{q}$  or [49,55]  $q\bar{q}q\bar{q}$ , to be bound together so strongly compared with other heavy hadrons.

In alternative (B), the small  $f_0$  mass is due to the approximate conformal symmetry of  $N_f = 3$  QCD, together with the small values of the current-quark masses  $m_{u,d,s}$ . The  $q\bar{q}$  binding of  $f_0$  is similar to that of  $\pi$ , K,  $\eta$  but in *P*-wave instead of *S*-wave.

The scale separation shown in Figure 2 means that an effective chiral-scale Lagrangian for  $\chi PT_{\sigma}$  can be set up with leading-order (LO) terms given entirely by the tree approximation. That should be contrasted with conventional  $\chi PT_3$ , where the tree approximation fails in 0<sup>+</sup> channels and the LO

must be patched up with unitarized  $\pi$ , *K*,  $\eta$  loops. In  $\chi$ PT $_{\sigma}$ , the role of unitarization is to patch up *next-to-LO*  $\pi$ , *K*,  $\eta$ ,  $\sigma$  loop diagrams; so far, little has been done in that regard.

Concerns about this scheme typically run along the following lines:

- (a) *IRFP's outside the conformal window are not taken seriously in the literature: they do not exist either in principle or on the lattice.* Questions of principle and evidence from the lattice are analyzed in Sections 6 and 7 respectively.
- (b) There are no light dilatons in gauge theories [9,10]. These claims are made for a type-III definition of "dilaton", to be discussed further in Section 4 below. They do not affect the identification above of the  $f_0(500)$  as a genuine dilaton  $\sigma$  for QCD.
- (c) There is no light dilaton in hadronic physics because there is no scalar particle nearly degenerate with pions. This overlooks the role of  $m_s \gg m_{u,d}$ , and comes from TC literature, where all chiral NG bosons are called "technipions" and none "technikaons" or "technietas".
- (d) A light dilaton is not seen for  $N_f = 4$ . This refers to the lattice study [27] closest to the relevant case  $N_f = 3$ . See Section 7.
- (e) There may be an IRFP for  $N_f = 2$  which would, in analogy with the case  $N_f = 3$ , produce a spin-0<sup>+</sup> particle with mass  $O(m_{\pi})$ , contrary to experiment. An IRFP at  $N_f = 2$  is not excluded but, as noted above, a connection with scale invariance can be obtained only by decoupling the *s* quark, and  $m_s \sim \infty$  is a very bad approximation. The argument works only for  $N_f = 3$ .

Continuing with the case  $N_f = 3$ , let us consider the LO approximation for

$$2m_{\sigma}^{2} = \langle \sigma | \theta_{\mu}^{\mu} | \sigma \rangle \simeq \frac{\alpha_{\rm IRs} - \alpha_{s}}{4\alpha_{\rm IRs}} \beta_{\rm QCD}^{\prime} \langle \sigma | G^{2} | \sigma \rangle + \left(1 + \gamma_{m}(\alpha_{\rm IRs})\right) \sum_{q=u,d,s} m_{q} \langle \sigma | \bar{q}q | \sigma \rangle \,. \tag{23}$$

Here  $\beta'_{QCD} \ge 0$  is the slope at the IRFP of the red curve in Figure 1. The optimal space-like scale  $-m^2$  at which  $\alpha_s = \alpha_s(-m^2)$  should be evaluated is determined by how close to the limit (19) it is possible to go in the real world. Soft-meson theorems for approximate  $SU(3) \times SU(3)$  involve  $O(m_K)$  extrapolations in NG-boson masses and momenta<sup>3</sup>, so we take  $m \sim m_K$ . Therefore, relative to a QCD large scale such as  $M_N$ , effects due to

$$\epsilon_s = \alpha_{\rm IRs} - \alpha_s \simeq \alpha_s(0) - \alpha_s(-m_K^2) = O(m_K^2/m_N^2)$$
(24)

are similar in magnitude to those of  $m_s$ . That is why  $f_0(500)$  is as *light* as *K* and  $\eta$  (Figure 2).

The physical consequences of having an IRFP at  $\alpha_{\text{IRS}}$  can be seen in low-energy mesonic processes. The results are encoded in a chiral-scale Lagrangian  $\mathcal{L}_{\text{eff}}|_{\text{QCD}}$  for  $\chi \text{PT}_{\sigma}$  [2,3]. The formalism is entirely standard, having been invented in 1969–1970 [37,38,40,56], so will not be repeated in full here. Under global conformal transformations  $x \to x'$ , the Goldstone field  $\sigma$  is translated by a constant:

$$\sigma \to \sigma - (f_{\sigma}/4) \ln |\det(\partial x'/\partial x)|.$$
(25)

Then  $\exp(\sigma d / f_{\sigma})$  transforms covariantly with dimension *d* and can be used to adjust the dimensions of terms in an effective Lagrangian. For example, the mass term  $M_N \overline{N} N$  for a nucleon can be converted into a scale-invariant potential  $V_{inv}$ ,

$$M_N \overline{N}N \longrightarrow V_{\rm inv} = M_N e^{\sigma/f_\sigma} \overline{N}N = M_N \overline{N}N \{1 + \sigma/f_\sigma + \dots\}, \qquad (26)$$

from which the scalar Goldberger–Treiman relation (2) may be deduced. So, any effective Lagrangian can be made conformal invariant by introducing a suitable dependence on  $\sigma$ . The key point is that this

<sup>&</sup>lt;sup>3</sup> Including pion momenta, as in  $\eta \to 3\pi$  decay. Distinguish  $\pi\pi \to \pi\pi$  for  $O(m_K)$  momenta from the same process for  $O(m_\pi)$  momenta, where  $\chi$ PT<sub>2</sub>, a different theory, is applicable. See Footnote 7 and Figure 4 in [3].

produces amplitudes which depend on scales such as  $M_N$ ,  $f_{\pi}$  and  $\Lambda_{QCD}$  in the limit of *exact* conformal invariance, i.e., *at the IRFP*  $\alpha_{IR}$ .

The same technique is applied to a chiral Lagrangian by noting that the unitary matrix field U for chiral NG bosons has dimension 0. For example, the Lagrangian

$$\mathcal{L}_{\rm inv} = \frac{1}{2} \partial^{\mu} \sigma \partial_{\mu} \sigma e^{2\sigma/f_{\sigma}} + \frac{1}{4} f_{\pi}^2 \text{Tr} \, \partial^{\mu} U \partial_{\mu} U^{\dagger} e^{2\sigma/f_{\sigma}}$$
(27)

has dimension-4: both chiral and scale invariance are preserved. The term  $|\partial \vec{\pi}|^2 \sigma / f_{\sigma}$  obtained from Equation (27) corresponds to the result (15) for the dilaton width.

The effective Lagrangian for  $\chi PT_{\sigma}$  generalizes (27) to include all possible LO terms consistent with the conformal and chiral  $SU(3)_L \times SU(3)_R$  properties of QCD for small values of  $m_{u,d,s}$  and  $\epsilon_s$ . The most important result is that the  $\Delta I = 1/2$  rule for nonleptonic kaon decays is a consequence of broken scale and chiral invariance. Equation (15) remains valid [4], while *R* in Equation (16) is replaced by the high-energy ratio  $R_{IR}$  for the scale-invariant theory at  $\alpha_{IR}$ .

Crawling TC [1] is the most recent application of the idea of an NG mode at the IRFP of a gauge theory. It adopts the standard TC viewpoint [57–59] that the Higgs mechanism is the dynamical effect of a gauge theory which resembles QCD, with a TC coupling  $\alpha$  which is nonperturbative at scales of a few TeV. Where it differs from other TC theories is that, in analogy with (21), the TC gauge coupling runs to an infrared fixed point  $\alpha_{IR}$  with conformal invariance in NG mode. So, at  $\alpha_{IR}$ , there is a massless dilaton—a feature unique to crawling TC. At energies much less than a TeV,  $\alpha$  sits just below  $\alpha_{IR}$  and the dilaton acquires a mass  $\ll$  TeV. It makes sense to identify this massive  $\sigma$  particle with the mass 125 GeV Higgs boson.

The characteristic feature of crawling TC is its dependence on the slope  $\beta'$  of the TC  $\beta$  function at the fixed point:

$$\beta' = \left. \frac{d\beta}{d\alpha} \right|_{\alpha = \alpha_{\rm IR}} > 0 \,. \tag{28}$$

In particular, the Higgs potential in leading order is a nonpolynomial function

$$V(h) = \frac{M_{\sigma}^2 F_{\sigma}^2}{\beta'} \left[ -\frac{1}{4} \left( 1 + \frac{h}{F_{\sigma}} \right)^4 + \frac{1}{4 + \beta'} \left( 1 + \frac{h}{F_{\sigma}} \right)^{4 + \beta'} + \frac{\beta'}{4(4 + \beta')} \right],$$
(29)

where h = h(x) is a fluctuating Higgs field,  $F_{\sigma}$  is the TC analogue of  $f_{\sigma}$ , and  $M_{\sigma}$  is identified as the Higgs boson mass  $m_h$ .

## 4. Comparison of Crawling and Walking TC

While writing [1], we became aware that the 1968–70 concepts of "dilaton" and "spontaneous breaking of conformal invariance", on which our work relies, have lost their original meaning (Section 1). Most of the thousands of papers on the subject written since 1972 do not recognize the type-I concept that *scale-dependent* amplitudes can occur in the limit of conformal invariance. For most authors, the label "dilaton" is just a fancy name for a scalar field appearing in a conformal theory. That has led to a lack of clarity between competing concepts.

Most definitions of "dilaton" on the Internet are of type-II: they refer to a scalar component of the gravitational field. There is now a vast literature on this. The term was first used in that context in 1971 [60]. At the time, it drew the remark [61] (quoted in [62]) that "Brans–Dickeon" would be a better name.

There is a third meaning for "dilaton" (type-III), also with an extensive literature, which unfortunately contradicts the 1968–70 definition reviewed above. This re-working of the subject started in 1976:

- 1. Fubini [63] noted problems with the conformal NG mode for  $\lambda \phi^4$  theory which subsequent authors incorrectly interpreted as an inconsistency of type-I theories in general; see Section 5 below.
- 2. Gildener and Weinberg (GW) [6] introduced the concept of a spin-0<sup>+</sup> "scalon" associated with a flat direction of the potential of a massless gauge theory in the tree approximation. Scale invariance is broken *explicitly* by one-loop corrections of the Coleman–Weinberg (CW) [64] type. The analysis is entirely consistent, except for a remark that the result is an example of a "spontaneous breaking" of scale invariance<sup>4</sup>. That is not so: the tree approximation is scale-free by construction, so the invariance is realized in the WW mode. In that limit, the "scalon" is massless but is not a genuine dilaton because it lacks a decay constant connecting  $\theta_{\mu\nu}$  to the vacuum. All breaking of conformal invariance is *explicit*: the one-loop corrections violate scale invariance of the Hamiltonian.

As reported in Section 1, the GW scalon became the type-III "dilaton" of walking TC [7–15] and conformal potentials deformed by the CW mechanism [16–23]. Exact conformal invariance is clearly in the WW mode, e.g., within the conformal window for walking TC, yet the "breaking" of the symmetry is said to be both "explicit" and "spontaneous". Other versions of this contradiction are that "approximate conformal invariance is spontaneously broken", or that the breaking "triggers" scale generation "spontaneously".

A general definition for the type-III dilaton  $\varphi$  for walking TC and CW-deformed potentials is as follows. It is a 0<sup>+</sup> particle in a theory which approximates a system with *exact* conformal invariance in the WW mode,

$$D|\mathrm{vac}\rangle = 0 = K_{\mu}|\mathrm{vac}\rangle, \quad \theta^{\mu}_{\mu} \to 0,$$
(30)

and obeys Equation (12). All scales, large and small, are "triggered" when the Hamiltonian<sup>5</sup> is perturbed by a term which breaks conformal invariance *explicitly*. These large scales include a fermion condensate  $\langle \bar{\psi}\psi \rangle_{vac}$  and hence chiral NG bosons.

Walking TC assumes that all infrared fixed points lie within the conformal window, where deep infrared dynamics is scale-free and Green's functions exhibit the power-law scaling expected for the WW mode. The gauge coupling  $\alpha$  for a theory just outside the conformal window is supposed to walk slowly when it passes the IRFP  $\alpha_{WW}$  of a theory just inside the window. The result is then a small  $\beta$ -function which (it is hoped) can be held responsible for small-scale effects such as the mass of the Higgs boson. Physics outside the conformal window is vastly different from physics inside, so there must be a discontinuity or phase transition in  $N_f$  at a sill [65–67] produced by a term  $\delta \mathcal{H}$  in  $\theta_{00}$  which breaks conformal symmetry explicitly. Despite being proportional to  $\beta$ ,  $\delta \mathcal{H}$  must produce effects ~ several TeV, such as  $\langle \overline{\psi}\psi \rangle_{vac}$ .

Therefore, even though  $\theta^{\mu}_{\mu}$  is formally small, its effects are ~ a few TeV, as foreshadowed in Section 1 and in general remarks below Equation (12). So it is hard to argue that the sill produces a small-mass Higgs boson. That can be done only if an explicit model for the sill can be formulated with unusual properties. The model would have to specify a large-scale mechanism for the gauge theory to produce the chiral condensate  $\langle \bar{\psi}\psi \rangle_{\text{vac}}$  without affecting the mass of the 0<sup>+</sup> boson. Early attempts in that direction [7,9,10] came to the conclusion that this is not possible: type-III dilatons are heavy<sup>6</sup>. Of course, the self-consistency of these gauge-theory models for fermion condensation is far from obvious, but that does not mean that the difficulty can be circumvented by assuming (as in [19]) that a type-I dilaton Lagrangian from 1970 [37,38,40] may be valid below the sill but not above it. For that to be convincing, the self-consistent model for the sill would have to produce a dilaton-like Lagrangian.

<sup>&</sup>lt;sup>4</sup> Coleman and E. Weinberg [64] stick to the textbook definition of the term "spontaneous", i.e., for breaking which is *not* explicit, and apply it *only* to the breaking of chiral invariance. In footnote 8, they note that scale invariance is broken explicitly by the one-loop trace anomaly.

<sup>&</sup>lt;sup>5</sup> In walking TC, the decomposition  $\mathcal{H} = \mathcal{H}_0 + \delta \mathcal{H}$  is often not considered explicitly. Such theories involve extrapolations in  $N_f$ , with an understanding that the extra flavor fields are almost decoupled.

<sup>&</sup>lt;sup>6</sup> The analysis does not appear to depend on their having an ultraviolet (UV) fixed point instead of an IRFP.

This should be contrasted with crawling TC, which relies on a single assumption that there is an IRFP  $\alpha_{IR}$  in the NG mode for conformal invariance. Support for this comes from evidence noted in Section 3 for the analogue theory [2–4] for QCD. Assumptions about the detailed dynamics of the sill are not needed; indeed, the sill plays no role in the extrapolation from  $\alpha_{IR}$  to  $\alpha$ . That extrapolation accounts for small-scale corrections to the scale set at  $\alpha_{IR}$ , including the mass of the Higgs boson

$$m_h = O(\epsilon)$$
,  $\epsilon = \alpha_{\rm IR} - \alpha \gtrsim 0$ . (31)

See [1] for an explicit  $O(\epsilon)$  formula for  $m_h$  in terms of the gluon condensate at  $\alpha_{IR}$ . A diagrammatic comparison of the two theories is shown in Figure 3.



**Figure 3.** Graphs of the TC  $\beta$  function in SU(3) gauge theories with  $N_f$  Dirac flavors for (**a**) crawling TC and (**b**) walking TC. On each graph, the sill of the conformal window at  $N_f^c$  flavors is shown as a thick gray line. In (**a**), there is a genuine massless dilaton at  $\alpha_{IR}$  outside the conformal window  $(N_f \leq N_f^c)$ , so large scales are possible at that point. Small-scale corrections such as  $m_h = 125$  GeV for the Higgs boson mass occur for  $\alpha \leq \alpha_{IR}$ . In (**b**), the physical theory below the sill (red line,  $N_f \leq N_f^c$ ) is scale-dependent and lacks an IRFP. The theory above the sill and hence inside the conformal window (blue line,  $N_f \leq 16$ ) is scale-free and has an IRFP  $\alpha_{ww}$  with conformal invariance in WW mode. The sill generates all scales in the physical theory, both large and small, no matter how closely the red line in the walking region approaches  $\alpha_{ww}$ , with the walking  $\alpha$  evaluated at a space-like scale  $\sim -\Lambda_{TC}^2$ . Therefore, a type-III "dilaton" is unlikely to be light.

At what scale should  $\alpha$  be evaluated? Unlike QCD, where the relevant scale in (24) was found to be set by the heaviest light quark *s*, TC theory is a gauge theory with massless fermions with no obvious analogue of *approximate* chiral  $SU(3)_L \times SU(3)_R$  symmetry. However, TC is in some way perturbed by the electroweak theory responsible for the  $W^{\pm}$  and  $Z^0$  bosons and lighter non-TC particles. So, it is tempting to suppose that the optimal scale for  $\alpha$  is set by the largest small (non-TC) scale available, i.e.,  $M_{W,Z}$ :

$$\epsilon \simeq \alpha(0) - \alpha(-M_{W,Z}^2) = O(M_{W,Z}^2/\Lambda_{\rm TC}^2).$$
(32)

Here  $\Lambda_{TC}$  is a typical nonperturbative TC scale  $\simeq$  a few TeV. This would explain why the Higgs boson is almost as *light* as  $W^{\pm}$  and Z.

Theoretical support for this outcome requires the construction of a fully unified gauge theory which combines the Standard Model with TC. That deserves further investigation.

# 5. Scale Dependence in the Conformal Limit

Evidently the proposition that amplitudes at  $\alpha_{IR}$  can be scale-dependent requires further explanation. In the limit of exact conformal invariance, (a) is scale dependence of the ground state generally possible, and (b) can it occur at an IRFP of a massless gauge theory (Section 6 below)?

The argument against (a) typically refers to Fubini [63] and runs as follows [22]: "if a theory is exactly conformal, it either does not break scale invariance, or the breaking scale is arbitrary (a flat direction)." In effect, it is being argued that the NG mode for *exact* conformal invariance with a type-I dilaton is absolutely impossible. That cannot be so:

1. Fubini's analysis is restricted to  $\lambda \phi^4$  theories and therefore does not constitute a general proof that strict conformal invariance must be manifest, i.e., in WW mode. To obtain the NG mode for conformal invariance, simply omit the  $\phi^4$  term and add other invariants to  $\frac{1}{2}(\partial \phi)^2$  such as couplings to chiral NG bosons or (say) the 4-point self-interaction

$$\mathcal{L}_{4-\text{pt.}} = \kappa \left( \partial \phi / \phi \right)^4, \quad \kappa = \text{const.},$$
 (33)

and [1] *constrain*  $\phi$ , e.g., to a half line

$$\phi$$
 > const. (34)

2. As noted in [1], Fubini's conclusion was anticipated in 1970 by Zumino (page 472 of [40]), who observed that a dilaton Lagrangian is consistent only if the quartic term *vanishes* in the conformal limit:

$$\lambda = O(\epsilon), \quad \epsilon \to 0.$$
 (35)

Here  $\epsilon$  is a measure of the explicit breaking of conformal symmetry. Equation (35) reflects the fact that, like other genuine NG bosons, type-I dilatons for  $\epsilon \rightarrow 0$  are massless and cannot self-interact at zero momentum: they correspond to a flat direction of the dilaton potential.

- 3. All dilaton Lagrangians from 1968–1670 which obey Zumino's rule (35) are counterexamples [32–34,37,38,40]: they exist in the limit of exact conformal symmetry and produce amplitudes which depend on a *non*-arbitrary scale, the dilaton decay constant  $f_{\sigma}$  of Equations (3) and (14). All except [32] allow chiral condensates to exist in the conformal limit  $\epsilon \rightarrow 0$ .
- 4. The "flat direction" is *not* associated with a continuum of scales. Instead, it corresponds to the continuum of degenerate vacuum states (13).

The quote continues: "Thus an explicit breaking must be present to trigger and stabilize the spontaneous breaking of scale invariance."

- 5. Again, the effective Lagrangians above are counterexamples. A tiny  $0(\epsilon)$  scale-violating perturbation  $\delta \mathcal{H}_{\text{tiny}}$  can pick out one of the degenerate vacua (stabilization) and produce *tiny* corrections to the scale-dependent amplitudes and masses of the type-I theory at  $\epsilon = 0$ .
- 6. Implicit in this quote is the type-III assumption that there are no scales in the  $\epsilon = 0$  theory, so it is necessary to have a large discontinuity appear "spontaneously" at a small or infinitesimal value of  $\epsilon \neq 0$  to produce large scales. If the  $\epsilon = 0$  theory is in the WW scaling mode, it does not have scale-degenerate vacua, so there is nothing to stabilize.
- 7. The large discontinuity is a problem for type-III phenomenology, because  $\theta^{\mu}_{\mu} \sim 0$  is such a bad approximation.

A formal argument that "the breaking scale is arbitrary" in a conformal invariant theory was first given by Wess [68]. It is most simply derived from the identity [34]

$$e^{iD\rho}P^2e^{-iD\rho} = e^{2\rho}P^2, \quad \theta^{\mu}_u \to 0,$$
 (36)

which implies that mass- $\mathcal{M}$  eigenstates  $|\mathcal{M}\rangle$  obey the relation

$$|e^{\rho}\mathcal{M}\rangle = e^{iD\rho}|\mathcal{M}\rangle. \tag{37}$$

That implies a spectrum of zero-mass particles, or a continuum  $0 \leq M < \infty$ , or both—provided that the ground state is unique (WW mode of conformal invariance).

However, for vacua (13) degenerate under scale transformations, this conclusion is not valid because states related by  $e^{iD\rho}$  belong to different worlds, W and W'. A discrete scale  $\mathcal{M}$  can exist in W and correspond to a discrete scale  $\mathcal{M}'$  in W'. Since dimensional units are also scaled up or down in the same way, e.g.,

$$\text{GeV} \to \text{GeV}' = e^{\rho} \text{GeV},$$
 (38)

experimental data in W and W' are identical. Therefore these worlds are physically equivalent, as for any other symmetry in the NG mode. See Appendix D of [1] for details.

Sometimes type-I dilaton Lagrangians are written in a form such that scales do not appear explicitly in the conformal limit. That happens when all fields are chosen to transform homogeneously under scale transformations. The result is a polynomial Lagrangian with dimensionless coupling constants which is easily confused with the conformal WW mode considered by Gildener and Weinberg [6]. The difference for the NG mode is that the scale may be hidden in a constraint like (34) which must be implemented nonlinearly. The simplest example is the constraint  $\phi > 0$  which is not changed by scale transformations and seems to have no scale dependence. However, to implement it, a scale must be introduced, as is evident from the mapping [56]

$$\phi = f_{\sigma} \exp(\sigma/f_{\sigma}) > 0 \tag{39}$$

from the unconstrained Goldstone field  $\sigma$ .

Flat directions for conformal invariant Lagrangians are also possible for type-III theories, as noted by Gildener and Weinberg [6], but they do *not* correspond to the vacuum degeneracy (13) because a type-III vacuum state is conformal invariant. Instead, the flat direction corresponds to field-translation invariance

$$\phi(x) \to \phi(x) + c \quad \text{for} \quad -\infty < c < \infty, \ c = \text{ const.},$$
(40)

which forbids definitions like (39) that introduce a scale. In particular, Equation (39) cannot be used above the sill of the conformal window.

#### 6. Scale Dependence at an IRFP

There is an extensive literature on IRFP's, but in almost all of it, "conformality" (a lack of scale dependence at IRFP's) is accepted without question. This may be because:

- 1. The initial work [69] was perturbative with a scale-free IRFP. That implied manifest chiral symmetry, so an IRFP of that type would presumably be close to a discontinuous transition to a phase where fermions can condense [70]. That became the model for walking TC.
- 2. It is relatively easy to find scale-free IRFPs on the lattice: Green's functions exhibit power-law behavior in the conformal window. That does *not* test the possibility of IRFP's outside the conformal window (Section 7).
- 3. There is a belief that dimensional transmutation, which produces nonperturbative scales like  $\Lambda_{QCD}$  or  $\Lambda_{TC}$ , implies  $\theta^{\mu}_{\mu} \neq 0$ . If true, that would exclude scale dependence at IRFPs.

When using the term "dimensional transmutation", care must be exercised not to conflate two distinct concepts:

(a) RG-invariant scales  $\mathcal{M}$  induced by the renormalization scale  $\mu$  of  $\alpha$ ,

$$\mathcal{M} = \mu \exp\left\{-\int_{\kappa_{\mathcal{M}}}^{\alpha} dx / \beta(x)\right\}, \quad 0 < \kappa_{\mathcal{M}} < \alpha_{\mathrm{IR}},$$
(41)

where  $\kappa_{\mathcal{M}}$  is a dimensionless constant that depends on  $\mathcal{M}$  but not on  $\alpha$  or  $\mu$ . Examples of  $\mathcal{M}$  for massless  $N_f = 3$  QCD are non-NG hadron masses such as  $M_N$  and dimensional constants like  $f_{\sigma}$ ,  $f_{\pi}$  and  $\Lambda_{\text{QCD}}$ , and for TC, their counterparts such as  $F_{\sigma}$ ,  $F_{\pi}$  and  $\Lambda_{\text{TC}}$ .

(b) The trace anomaly which, if present, is also induced by  $\mu$ .

If (a) and (b) are conflated, the idea that dimensional transmutation and hence scale dependence may occur at an IRFP looks like an absolute contradiction. This confusion in terminology has arisen because the original CW analysis was performed at *one-loop* order<sup>4</sup>. In that order, an IRFP cannot occur, so there was no need to distinguish (a) and (b).

There is no proof<sup>7</sup> that  $\mathcal{M} \neq 0$  implies  $\beta \neq 0$ . As  $\alpha$  moves from the perturbative region into the hadronization region and then beyond into the infrared region, it is hard to argue that all RG-invariants  $\mathcal{M}$  suddenly turn themselves off when a nonperturbative IRFP is encountered. So there is a theoretical possibility that dimensional transmutation *in the sense of (a)* may occur at an IRFP  $\alpha_{IR}$ . It should be investigated.

Fermion condensation is a special case of this. Figure 4 shows the standard condition for the self-energy  $\Sigma$  implied by the Schwinger-Dyson equation for the fermion propagator. If a non-zero solution for  $\Sigma$  and hence  $\langle \bar{\psi}\psi \rangle_{\text{vac}}$  exists for finite values of the fermion-gluon coupling constant *g*, why should this result not be valid at the value  $g_{\text{IR}}$  corresponding to the IRFP  $\alpha_{\text{IR}} = g_{\text{IR}}^2/(4\pi)$ ?

**Figure 4.** Self-consistent condition for the fermion self-energy  $\Sigma$ , with  $\alpha = g^2/(4\pi)$ . The gauge-boson and fermion propagators are fully dressed, while  $\Sigma$  and the vertex labelled 1PI are one-particle-irreducible.

To my knowledge, the only argument against this conclusion is a claim that the dynamical mass acquired by fermions due to the condensate would cause them to become relatively heavy at low energies and so decouple in the infrared limit. The trouble with that is evident from Figure 1 for QCD with  $N_f = 3$  flavors. Decoupling of u, d, s would imply that the quark condensate  $\langle \bar{q}q \rangle_{\text{vac}}$  and hence  $\pi, K, \eta$  decouple in the infrared limit (19). That would destroy chiral  $SU(3)_L \times SU(3)_R$  perturbation theory and spell the end of QCD, irrespective of whether an IRFP exists or not.

The hole in this decoupling argument was examined at length in Appendix A of [1]. It has to do with the distinction between current and constituent quarks. Current quarks refer to the u, d, s fields in the QCD Lagrangian, with small "current-quark" masses  $m_{u,d,s}$  which govern the masses of  $\pi, K, \eta$ . Constituent or "dressed" quarks have large masses  $M_{u,d} \sim 300$  MeV and  $M_s \sim 450$  MeV in a quantum-mechanical Hamiltonian H which reproduces the spectrum of non-NG hadrons, e.g.,  $M_N = 2M_u + M_d$ . The constituent masses are "dynamical" because H is (presumably) the result of integrating out the NG-boson sector and so has its scale set by  $\langle \bar{q}q \rangle_{\text{vac}}$ .

Obviously, the constituent masses cannot be regarded as masses in the QCD Lagrangian because that would prevent the chiral limit being taken, and pions would have mass ~  $2M_{u,d}$ . So, one would expect the Appelquist–Carazzone theorem [71] to apply only to heavy current-quark masses such as  $m_{t,b,c}$ , and not to  $M_{u,d,s}$ . That is the result found in [1].

Evidently items 1–3 are assumptions characteristic of type-III theories. For the type-I theories  $\chi PT_{\sigma}$  and crawling TC, the problem is to find a satisfactory replacement for item 2. If scales are present at  $\alpha_{IR}$ , Green's functions do *not* exhibit power-law behavior; rather, they behave much like amplitudes observed in the real world.

#### 7. Nonperturbative Tests of Type-I Theories

The obvious tactic is to define  $\alpha$  non-perturbatively outside the conformal window and see if it stops increasing as the infrared limit is approached. There are two difficulties:

1. A true analogue of the Gell-Mann–Low function  $\psi(x)$  for quantum electrodynamics [72] is assumed to exist for non-Abelian gauge theories but is yet to be identified. Prescriptions for

<sup>&</sup>lt;sup>7</sup> See Section 2 of [1], especially the text below Equation (27).

the running coupling exist beyond perturbation theory [73], but there is a danger that their properties are artefacts of their definition. We have no analytic proof that any of them runs monotonically and provides an unbiased test of whether the dynamics chooses to have an IRFP or not. The method of effective charges [74–76] is nonperturbative, but there are as many definitions as there are physical processes, and it is not obvious which of them has the desired properties all the way to the far infrared.

2. Lattice studies [77,78] feature precise measurements of  $\Lambda_{QCD}$  in UV logarithms ( $\beta \sim$  perturbative) and clear evidence for hadronization and quark condensation at intermediate energies. However, for small  $N_f$  values such as  $N_f = 3$ , it is hard to reach the infrared region far below the non-NG hadronic spectrum.

A less ambitious procedure is to look for a light scalar particle in the particle spectrum. The problem then is to decide whether this is evidence for a type-III or a type-I theory.

In the context of walking TC, the most interesting cases are those just under the sill of the conformal window. Evidence for a light scalar particle almost degenerate with technipions has been found in lattice data for SU(3) with  $N_f = 8$  Dirac fermions in the fundamental representation [24–27] and two Dirac fermions in the sextet representation [28]. In each case, the particle is identified as a "dilaton". In this type-III interpretation, the small mass is considered to be due to  $\alpha$  being close to a WW-mode fixed point  $\alpha_{WW}$  just inside the conformal window.

However, as explained above, that involves unlikely assumptions about the dual character of  $\delta \mathcal{H}$ , the term in  $\mathcal{H}$  which breaks scale invariance explicitly. Type-III theories require  $\delta \mathcal{H}$  to generate large-scale effects such as  $\Lambda_{TC}$  and the fermion condensate at the sill, but to *desist* in cases where that is inconvenient, e.g., the scalar-boson mass.

The most likely explanation [1] of the light scalar particle is that there exists an IRFP  $\alpha_{NG}$  just outside the conformal window. Since there is scale dependence at  $\alpha_{NG}$ , a genuine type-I dilaton and hence all large-scale effects exist at that point. At  $\alpha_{NG}$ , both conformal and chiral invariance are in the NG mode, so massless technipions exist there as well as the type-I dilaton. Large-scale effects cannot be due to  $\delta \mathcal{H}$ , because  $\alpha$  can run smoothly to  $\alpha_{NG}$  in the conformal limit  $\delta \mathcal{H} \rightarrow 0$ . The sill does not get in the way, so there is no need to assume anything about its dynamics. The small scale of the scalar-particle mass  $M_{\sigma}$  corresponds to  $\alpha$  being in the infrared region close to  $\alpha_{NG}$  (Figure 5).



**Figure 5.** Competing explanations for the appearance of a scalar particle in lattice data for SU(3) gauge theory with  $N_f = 8$  triplet fermions. Two IRFPs are shown: (a) the closest scale-free IRFP  $\alpha_{WW}$  just inside the conformal window,  $N_f^c < N_f \le 16$ , and (b) a scale-dependent IRFP  $\alpha_{NG}$  for  $N_f = 8 \le N_f^c$ . Walking TC assumes that  $\alpha_{NG}$  is not present. Instead, the small scalar mass is supposed to arise at an intermediate energy where the curve is closest to the axis, and then the theory chooses the blue line labelled "walking" to approach the infrared region. In crawling TC, the  $N_f = 8$  theory enters the infrared region as it approaches the axis and chooses the red line labelled "crawling". The short length of the red line accounts for the small mass acquired by the type-I dilaton.

In walking TC, IRFPs outside the conformal window are thought to be forbidden. Instead, an explanation for the light scalar particle is sought by appending dilaton Lagrangians to the type-III framework [29,79–82]. In fact, these effective Lagrangians are type-I theories developed in 1970 [37,38,40]: they generate asymptotic expansions in  $\delta \mathcal{H}_{eff} \sim 0$  about a conformal limit with scale-dependent amplitudes depending on the decay constant  $f_{\sigma}$  or its TC analogue  $F_{\sigma}$ . The question is: if  $\alpha_{NG}$  is not available, about what point is the expansion to be performed?

Since the emphasis in walking TC has been to minimize  $|\beta|$ , presumably the understanding has been that the expansion should be carried out about  $\alpha_{WW}$ . The trouble with that is the lack of scale dependence at  $\alpha_{WW}$ . An alternative has just been suggested [83], that the dilaton Lagrangian expansion should correspond to expanding in "the distance to the conformal window", i.e., about the sill, where there is certainly a large scale. In a type-III theory, the sill acquires its scale dependence from a large-scale violation  $\delta \mathcal{H}$  in the Hamiltonian. The problem is then that the expansion of  $\mathcal{L}_{dil}$  is about a point where the corresponding effective Hamiltonian is conformally *invariant*.

The conclusion is that type-I and type-III theories should not be mixed—they are based on contradictory assumptions. A type-I effective Lagrangian introduced to discuss the light scalar boson should be given a type-I point about which it can be expanded:  $\alpha_{NG}$ . Why this should be such a fearsome prospect is puzzling.

Finally, I should comment on the perception that lattice data for  $N_f = 4$  implies that the  $f_0(500)$  is heavy, contrary to my remarks below Figure 2. A comparison of data for  $N_f = 4$  and  $N_f = 8$  (Figure 1 of [27]) shows that as the fermion mass  $m_{\psi}$  becomes small, the light scalar particle is almost degenerate with (techni-)pions for  $N_f = 8$  but not for  $N_f = 4$ . This indicates that the gluonic contribution to the scalar mass is negligible for  $N_f = 8$  but not for  $N_f = 4$ . A type-III interpretation of this is that the gluonic contribution is a large-scale effect due to  $\delta \mathcal{H}$ —a point of view similar to that of [7,9,10].

In type-I theories, there is no problem. A gluonic contribution to the scalar mass is a small-scale effect due to the coupling being close to but not at the NG-mode IRFP. For  $N_f = 8$ , apparently  $\epsilon = \alpha_{\text{NG}} - \alpha$  is so small that the scalar mass is dominated by the masses  $m_{\psi}$  used in the lattice analysis. For  $N_f = 4$  (a lattice-friendly approximation to the physical case of  $N_f = 3$  light flavors), the effects of  $\epsilon$  appear similar in magnitude to those of  $m_{\psi}$ , within fairly large errors.

So, I maintain that the Higgs boson is the direct TC analogue of  $f_0(500)$  for QCD: both are derived from type-I dilatons at scale-dependent IRFPs. The main difference is in the ratio r of small-scale to large-scale effects,

$$r_{\rm QCD} \approx \frac{500 \,\text{MeV}}{4\pi (92 \,\text{MeV})} = 0.4 \quad \text{and} \quad r_{\rm TC} \approx \frac{125 \,\text{GeV}}{4\pi (246 \,\text{GeV})} = 0.04 \,.$$
 (42)

This is permissible in type-I theories because large- and small-scale effects have separate origins.

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#### Abbreviations

The following abbreviations are used in this manuscript:

IRFP	infrared fixed point
QCD	Quantum Chromodynamics
$\chi \mathrm{PT}_{\sigma}$	chiral-scale perturbation theory
TC	Technicolor
NG	Nambu–Goldstone
WW	Wigner–Weyl
$\chi PT_3$	chiral $SU(3)_L \times SU(3)_R$ perturbation theory
$\chi PT_2$	chiral $SU(2)_L \times SU(2)_R$ perturbation theory

 $U\chi PT$  unitarized chiral perturbation theory

- RG renormalization group
- LO leading order
- GW Gildener-Weinberg
- CW Coleman-Weinberg
- UV ultraviolet

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