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Singularity Theorems in the Effective Field Theory for Quantum Gravity at Second Order in Curvature

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Abstract: In this paper, we discuss singularity theorems in quantum gravity using effective field theory methods. To second order in curvature, the effective field theory contains two new degrees of freedom which have important implications for the derivation of these theorems: a massive spin-2 field and a massive spin-0 field. Using an explicit mapping of this theory from the Jordan frame to the Einstein frame, we show that the massive spin-2 field violates the null energy condition, while the massive spin-0 field satisfies the null energy condition, but may violate the strong energy condition. Due to this violation, classical singularity theorems are no longer applicable, indicating that singularities can be avoided, if the leading quantum corrections are taken into account.

Keywords: quantum gravity; singularity theorems

1. Introduction

The significance of singularity theorems in general relativity first presented in the seminal papers of Penrose and Hawking [1,2] cannot be overemphasised. Since these foundational works, several adaptations and refinements of the singularity theorems have been developed (see, e.g., [3–7]). In general, all these theorems boil down to the same principle: the assumption of some energy condition together with some global statement about space-time leads to the prediction of geodesic incompleteness somewhere in the space-time. Geodesic incompleteness is then often taken as equivalent to the existence of a singularity, although the latter is a slightly stronger statement (see, e.g., [8]).

A crucial ingredient for the proof of most of singularity theorems is the Raychaudhuri equation¹ that can be derived from the Einstein field equations. It is therefore crucial to assume classical general relativity for singularity theorems to hold, and for any deviations of general relativity, one would have to reassess the derivation of singularity theorems, as was done, for example, for $f(R)$ gravity [10].

It is clear that general relativity needs to be embedded in a gravitational theory which can be quantised, i.e., a theory of quantum gravity, if one accounts for the quantum properties of matter and space-time. Such a theory of quantum gravity is not known yet, but many different approaches to such a theory have been formulated. Furthermore, any theory of quantum gravity should in the infrared limit reduce to general relativity. Despite the lack of a unique theory of quantum gravity, quantum corrections to general relativity solutions can be calculated using effective field theory methods [11–17]. Calculations done in this framework apply to any ultraviolet complete theory of quantum gravity and are valid at energies scales up to the Planck mass, and thus in the entire spectrum that can potentially be probed experimentally.

¹ However, see [9] for a recent example that doesn't make use of this equation.

It is expected that in a theory for quantum gravity, singularities will be resolved, since singularities lead to pathologies both in general relativity and quantum field theory. However, singularities cannot be avoided as long as singularity theorems hold. Therefore, an important question is whether the assumptions of the singularity theorems break down in a theory for quantum gravity. A discussion of possible quantum loop holes for the singularity theorems can, for example, be found in [18].

In this work, we discuss the validity of the singularity theorems in the framework of the effective field theory approach to quantum gravity. A drawback of this approach is that the theory is not valid at energy scales larger than the Planck mass which corresponds to regions of large curvature, where singularities are expected to form. We shall assume that the physics responsible for the avoidance of singularities becomes relevant at energies below the Planck scale and can thus be described within our mathematical framework; an example would be, e.g., a bounce solution in a stellar collapse to a black hole [19] or in Friedmann–Lemaître–Robertson–Walker (FLRW) cosmology which would avoid a Big Crunch solution [20]. We note that this approach goes beyond general relativity and it is applicable to any theory of quantum gravity that does not break diffeomorphism invariance.

This paper is organised as follows: in the next section, we derive the action for effective quantum gravity in the Einstein frame. In section 4, we discuss singularity theorems in effective quantum gravity using this action. In section 4, we then conclude. Furthermore, in appendix A, we discuss the classical Hawking and Penrose singularity theorems, and in appendix B, we discuss a refined statement of Hawking’s theorem using weakened energy conditions.

In this paper, we work in the $(+ - - -)$ metric and use the conventions $R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \dots$, $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$, $T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S_m}{\delta g^{\mu\nu}}$. Furthermore, $\kappa^2 = 8\pi G_N$.

2. Effective Quantum Gravity in the Einstein Frame

While an ultraviolet complete theory of quantum gravity is still elusive, it has been shown [11–17] that quantum gravity can be well described by an effective field theory as long as one considers physical effects taking place at energies below the Planck scale. The effective field theory is obtained by integrating out the graviton fluctuations and potentially other massless degrees of freedom. After the various low energy fields have been integrated out, one finds the following action

$$S = \int d^4x \sqrt{|g|} \left\{ -\frac{R}{2\kappa^2} + c_1(\mu)R^2 + c_2(\mu)R_{\mu\nu}R^{\mu\nu} + c_3(\mu)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \alpha R \ln\left(\frac{\square}{\mu^2}\right) R + \beta R_{\mu\nu} \ln\left(\frac{\square}{\mu^2}\right) R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma} \ln\left(\frac{\square}{\mu^2}\right) R^{\mu\nu\rho\sigma} + \mathcal{O}(\kappa^2) \right\} + S_m, \tag{1}$$

where μ is the renormalisation scale. The action is given up to second order in curvature and higher order corrections are suppressed in the $\mathcal{O}(\kappa^2)$ term. The effective action of any ultraviolet complete theory of quantum gravity that respects diffeomorphism invariance can be written in this form when expanded to second order in curvature. We emphasise that the Wilson coefficients c_i depend on the UV completion of the theory and are only calculable within a specific UV-complete model of quantum gravity. Nevertheless, it is expected that these coefficients are non-zero unless some undiscovered symmetry protects them or if fine tuning occurs. Moreover, the values are bounded by the Eöt-Wash experiment [21] to $c_i \lesssim 10^{61}$. The non-local Wilson coefficients α, β, γ are calculable and independent of such a specific UV-completion. The values of these coefficients are given in Table 1.

We will now map this theory to the Einstein frame, in which the theory is represented as standard general relativity with additional matter fields. After this frame transformation, the usual singularity theorems are applicable, if the new fields satisfy the given energy conditions. Mappings to the Einstein frame for R and $R_{\mu\nu}$ theories have been discussed in [25–30]. Furthermore, the case of higher derivative gravity without non-local interactions has been discussed in [31]. Here we follow the same approach but include the non-local terms in the effective quantum gravity formalism.

Table 1. Non-local Wilson coefficients of various fields. All numbers should be divided by $11,520\pi^2$. ξ denotes the value of the non-minimal coupling for a scalar theory. We refer to [22,23] for the calculation of the values for the scalar, fermion and vector field. It is known that the graviton self interactions [24] make the form factors ill-defined, as the Wilson coefficients become gauge dependent. However, there is a well-defined procedure to resolve these ambiguities [12,13].

	α	β	γ
Scalar	$5(6\xi - 1)^2$	-2	2
Fermion	-5	8	7
Vector	-50	176	-26
Graviton	250	-244	424

Using the Gauss–Bonnet theorem, the effective action can be rewritten as²

$$S = -\frac{1}{2\kappa^2} \int d^4x \sqrt{|g|} \left\{ R - \kappa^2 R \hat{L}_1 R - \kappa^2 C_{\mu\nu\rho\sigma} \hat{L}_2 C^{\mu\nu\rho\sigma} + \mathcal{O}(\kappa^4) \right\} + S_m, \tag{2}$$

where C is the Weyl tensor and

$$\hat{L}_1 = \frac{2}{3} \left[3c_1(\mu) + c_2(\mu) + c_3(\mu) + (3\alpha + \beta + \gamma) \ln \left(\frac{\square}{\mu^2} \right) \right], \tag{3}$$

$$\hat{L}_2 = \left[c_2(\mu) + 4c_3(\mu) + (\beta + 4\gamma) \ln \left(\frac{\square}{\mu^2} \right) \right]. \tag{4}$$

We apply a Legendre transform to the function

$$f_1(R) = R - \kappa^2 R \hat{L}_1 R, \tag{5}$$

and find

$$S = -\frac{1}{2\kappa^2} \int d^4x \sqrt{|g|} \left\{ \phi R - V_1(\phi) - \kappa^2 C_{\mu\nu\rho\sigma} \hat{L}_2 C^{\mu\nu\rho\sigma} + \mathcal{O}(\kappa^4) \right\} + S_m, \tag{6}$$

where

$$R = \frac{\partial V_1(\phi)}{\partial \phi}, \tag{7}$$

$$\phi = \frac{\partial f_1(R)}{\partial R}. \tag{8}$$

We integrate the first equation and fix the integration constant such that

$$V_1(\phi) = -\frac{1}{4\kappa^2} (\phi - 1) \hat{L}_1^{-1} (\phi - 1), \tag{9}$$

where we use the notation \hat{L}_1^{-1} to denote the Green’s function of the operator \hat{L}_1 . If we apply a conformal transformation to the metric

$$g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} = |\phi| g_{\mu\nu} = \exp \left(\sqrt{\frac{2\kappa^2}{3}} \chi \right) g_{\mu\nu}, \tag{10}$$

² Due to the presence of a $\ln(\square)$ term in \hat{L}_2 , the Gauss–Bonnet theorem does not hold in full generality. However, it is valid up to this order in κ [15,32–34]

where we have introduced a new field χ , we can rewrite the action as

$$S = -\frac{1}{2\kappa^2} \int d^4x \sqrt{|\bar{g}|} \left\{ \bar{R} + \sqrt{6\kappa} \bar{\square} \chi - \kappa^2 \bar{\nabla}^\mu \chi \bar{\nabla}_\mu \chi - \frac{V_1[\phi(\chi)]}{\phi(\chi)^2} - \kappa^2 \bar{C}_{\mu\nu\rho\sigma} \hat{L}_2 \bar{C}^{\mu\nu\rho\sigma} - \frac{2\kappa^2 \mathcal{L}_m(X, g^{\mu\nu})}{\phi(\chi)^2} + \mathcal{O}(\kappa^4) \right\}, \tag{11}$$

where we have used that the Weyl tensor does not transform under a conformal rescaling of the metric. Furthermore, X represents all matter fields.

We can drop the total divergence term, since it does not affect the equations of motion, and apply the Gauss–Bonnet theorem to rewrite the Weyl tensor. We then find

$$S = -\frac{1}{2\kappa^2} \int d^4x \sqrt{|\bar{g}|} \left\{ \bar{R} - \kappa^2 \bar{\nabla}^\mu \chi \bar{\nabla}_\mu \chi - \frac{V_1[\phi(\chi)]}{\phi(\chi)^2} - 2\kappa^2 \bar{R}_{\mu\nu} \hat{L}_2 \bar{R}^{\mu\nu} + \frac{2\kappa^2}{3} \bar{R} \hat{L}_2 \bar{R} - \frac{2\kappa^2 \mathcal{L}_m(X, g^{\mu\nu})}{\phi(\chi)^2} + \mathcal{O}(\kappa^4) \right\}. \tag{12}$$

We consider the function

$$f_2(\bar{R}_{\mu\nu}) = \bar{R} - 2\kappa^2 \bar{R}_{\mu\nu} \hat{L}_2 \bar{R}^{\mu\nu} + \frac{2\kappa^2}{3} \bar{R} \hat{L}_2 \bar{R}, \tag{13}$$

and apply a Legendre transform to this part of the action, which results in

$$S = -\frac{1}{2\kappa^2} \int d^4x \sqrt{|\bar{g}|} \left\{ \psi^{\mu\nu} \bar{R}_{\mu\nu} - V_2(\psi^{\mu\nu}) - \kappa^2 \bar{\nabla}^\mu \chi \bar{\nabla}_\mu \chi - \frac{V_1[\phi(\chi)]}{\phi(\chi)^2} - \frac{2\kappa^2 \mathcal{L}_m(X, g^{\mu\nu})}{\phi(\chi)^2} + \mathcal{O}(\kappa^4) \right\}, \tag{14}$$

where³

$$\bar{R}_{\mu\nu} = \frac{\partial V_2(\psi^{\mu\nu})}{\partial \psi^{\mu\nu}}, \tag{15}$$

$$\psi^{\mu\nu} = \frac{\partial f_2(\bar{R}_{\mu\nu})}{\partial \bar{R}_{\mu\nu}}. \tag{16}$$

We integrate the first equation and fix the integration constant such that⁴

$$V_2(\psi^{\mu\nu}) = -\frac{1}{8\kappa^2} \left(\psi_{\mu\nu} - \frac{1 \mp i\sqrt{3}}{4} \psi \bar{g}_{\mu\nu} \mp i\sqrt{3} \bar{g}_{\mu\nu} \right) \hat{L}_2^{-1} \left(\psi^{\mu\nu} - \frac{1 \mp i\sqrt{3}}{4} \psi \bar{g}^{\mu\nu} \mp i\sqrt{3} \bar{g}^{\mu\nu} \right). \tag{17}$$

We perform another metric transformation such that

$$\bar{g}_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \sqrt{|\psi|} \bar{g}_{\mu\rho} \left(\psi^{-1} \right)^\rho{}_\nu, \tag{18}$$

³ Note that the spin-2 field is symmetric in its indices, since $R_{\mu\nu}$ is symmetric.

⁴ The potential V_2 is real, which can easily be shown by evaluating the expression.

where we define the determinants

$$|g| = \det(g_{\mu\nu}), \tag{19}$$

$$|\psi| = \det(\pi^\mu_\nu), \tag{20}$$

and we write

$$\tilde{\psi}^\mu_\nu = \psi^\mu_\nu, \tag{21}$$

$$\tilde{\psi}^{\mu\nu} = \tilde{\psi}^\mu_\rho \tilde{g}^{\rho\nu}, \tag{22}$$

$$\tilde{\psi}_{\mu\nu} = \tilde{g}_{\mu\rho} \tilde{\psi}^\rho_\nu. \tag{23}$$

We obtain the transformed action

$$S = -\frac{1}{2\kappa^2} \int d^4x \sqrt{|\tilde{g}|} \left\{ \tilde{R} - \kappa^2 (\psi^{-1})^\mu_\nu \tilde{\nabla}^\nu \chi \tilde{\nabla}_\mu \chi + \tilde{g}^{\mu\nu} (\tilde{\nabla}_\rho Q^\rho_{\mu\nu} - \tilde{\nabla}_\nu Q^\rho_{\rho\mu} + Q^\rho_{\rho\sigma} Q^\sigma_{\mu\nu} - Q^\rho_{\sigma\mu} Q^\sigma_{\rho\nu}) - \frac{V_1[\phi(\chi)]}{\phi(\chi)^2 \sqrt{|\psi|}} - \frac{V_2(\psi^{\mu\nu})}{\sqrt{|\psi|}} - \frac{2\kappa^2 \mathcal{L}_m(X, g^{\mu\nu})}{\phi(\chi)^2 \sqrt{|\psi|}} + \mathcal{O}(\kappa^4) \right\}, \tag{24}$$

where

$$Q^\rho_{\mu\nu}(\psi^\alpha_\beta) = \frac{1}{2} \tilde{g}^{\rho\sigma}(\psi^\alpha_\beta) (\tilde{\nabla}_\mu \tilde{g}_{\nu\sigma}(\psi^\alpha_\beta) + \tilde{\nabla}_\nu \tilde{g}_{\sigma\mu}(\psi^\alpha_\beta) - \tilde{\nabla}_\sigma \tilde{g}_{\mu\nu}(\psi^\alpha_\beta)). \tag{25}$$

We again drop the total derivative terms, and we define a new spin-2 field ζ such that

$$\psi^\mu_\nu = \left(1 + \frac{\kappa}{2} \zeta\right) \delta^\mu_\nu - \kappa \tilde{\zeta}^\mu_\nu \tag{26}$$

with $\tilde{\zeta} = \zeta^\mu_\mu$. We find

$$V_2(\psi^{\mu\nu}) = -\frac{1}{8} (\tilde{\zeta}^\mu_\nu \hat{L}_2^{-1} \tilde{\zeta}^\nu_\mu - \tilde{\zeta} \hat{L}_2^{-1} \zeta). \tag{27}$$

After this transformation, the action becomes

$$S = \int d^4x \sqrt{|\tilde{g}|} \left\{ -\frac{\tilde{R}}{2\kappa^2} + \frac{1}{2} \tilde{\nabla}^\nu \chi \tilde{\nabla}_\nu \chi + \frac{V_1[\phi(\chi)]}{2\kappa^2 \phi(\chi)^2 \sqrt{|\psi(\tilde{\zeta})|}} - \left[\frac{1}{2} \tilde{\zeta} \tilde{\square} \tilde{\zeta} - \frac{1}{2} \tilde{\zeta}^{\mu\nu} \tilde{\square} \tilde{\zeta}_{\mu\nu} - \tilde{\zeta}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \tilde{\zeta} + \tilde{\zeta}^{\mu\nu} \tilde{\nabla}_\rho \tilde{\nabla}_\nu \tilde{\zeta}^\rho_\mu \right] + \frac{V_2(\psi^{\mu\nu}(\tilde{\zeta}))}{2\kappa^2 \sqrt{|\psi(\tilde{\zeta})|}} + \mathcal{L}_m(X, g^{\mu\nu}) \right\} + \mathcal{O}(\kappa), \tag{28}$$

where we used that $\phi(\chi) = 1 + \mathcal{O}(\kappa)$, $\psi^\mu_\nu = \delta^\mu_\nu + \mathcal{O}(\kappa)$. In addition, we expand the terms containing a potential using $\hat{L} = \tilde{L} + \mathcal{O}(\kappa)$ and find

$$S = \int d^4x \sqrt{|\tilde{g}|} \left\{ -\frac{\tilde{R}}{2\kappa^2} + \frac{1}{2} \tilde{\nabla}^\mu \chi \tilde{\nabla}_\mu \chi - \chi (12\kappa^2 \hat{L}_1)^{-1} \chi - \left[\frac{1}{2} \tilde{\zeta} \tilde{\square} \tilde{\zeta} - \frac{1}{2} \tilde{\zeta}^{\mu\nu} \tilde{\square} \tilde{\zeta}_{\mu\nu} - \tilde{\zeta}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \tilde{\zeta} + \tilde{\zeta}^{\mu\nu} \tilde{\nabla}_\rho \tilde{\nabla}_\nu \tilde{\zeta}^\rho_\mu \right] - \left[\tilde{\zeta}^{\mu\nu} (16\kappa^2 \hat{L}_2)^{-1} \tilde{\zeta}_{\mu\nu} - \tilde{\zeta} (16\kappa^2 \hat{L}_2)^{-1} \tilde{\zeta} \right] + \mathcal{L}_m(X, g^{\mu\nu}) \right\} + \mathcal{O}(\kappa), \tag{29}$$

where indices on $\tilde{\zeta}$ are raised and lowered with \tilde{g} . We then find the equations of motion for the scalar field:

$$\tilde{\square} \chi = -(6\kappa^2 \hat{L}_1)^{-1} \chi + \mathcal{O}(\kappa). \tag{30}$$

We can solve the equation of motion for the Green’s function $(6\kappa^2\hat{L}_1)^{-1}$ by Fourier transformation:

$$\int d^4k \left\{ -k^2 + \frac{1}{4\kappa^2 [3c_1(\mu) + c_2(\mu) + c_3(\mu) + (3\alpha + \beta + \gamma) \ln \left(\frac{-k^2}{\mu^2} \right)]} \right\} \chi(k) = \mathcal{O}(\kappa). \tag{31}$$

This results in the mass of the scalar field given by

$$m_0^2 = \frac{1}{4\kappa^2(3\alpha + \beta + \gamma)W \left(-\frac{1}{4\mu^2\kappa^2(3\alpha + \beta + \gamma)} \exp \left[\frac{3c_1(\mu) + c_2(\mu) + c_3(\mu)}{3\alpha + \beta + \gamma} \right] \right)}, \tag{32}$$

which corresponds to earlier results (see, e.g., [35]). We can do a similar analysis for the tensor field, which yields (cf. [35])

$$m_2^2 = \frac{1}{2\kappa^2(\beta + 4\gamma)W \left(-\frac{1}{2\mu^2\kappa^2(\beta + 4\gamma)} \exp \left[\frac{c_2(\mu) + 4c_3(\mu)}{\beta + 4\gamma} \right] \right)}. \tag{33}$$

This resulting action is

$$\begin{aligned} S = \int d^4x \sqrt{|\tilde{g}|} \left\{ -\frac{\tilde{R}}{2\kappa^2} + \frac{1}{2} \tilde{\nabla}^\mu \chi \tilde{\nabla}_\mu \chi - \frac{1}{2} m_0^2 \chi^2 \right. \\ \left. - \left[\frac{1}{2} \tilde{\xi} \tilde{\square} \tilde{\xi} - \frac{1}{2} \tilde{\xi}^{\mu\nu} \tilde{\square} \tilde{\xi}_{\mu\nu} - \tilde{\xi}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \tilde{\xi} + \tilde{\xi}^{\mu\nu} \tilde{\nabla}_\rho \tilde{\nabla}_\nu \tilde{\xi}^\rho_\mu \right] \right. \\ \left. - \frac{1}{2} m_2^2 [\tilde{\xi}^{\mu\nu} \tilde{\xi}_{\mu\nu} - \tilde{\xi} \tilde{\xi}] + \mathcal{L}_m(X, g^{\mu\nu}) \right\} + \mathcal{O}(\kappa). \tag{34} \end{aligned}$$

We can then find the equation of motion for the metric

$$\begin{aligned} \left(\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{R} \tilde{g}_{\mu\nu} \right) = \kappa^2 \left\{ \tilde{T}_{\mu\nu} + \tilde{\nabla}_\mu \chi \tilde{\nabla}_\nu \chi - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{\nabla}^\rho \chi \tilde{\nabla}_\rho \chi + \frac{1}{2} m_0^2 \tilde{g}_{\mu\nu} \chi^2 \right. \\ - 2\tilde{\xi}_{\mu\nu} \tilde{\square} \tilde{\xi} - \tilde{\xi} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \tilde{\xi} + 2\tilde{\xi}_{\mu\rho} \tilde{\square} \tilde{\xi}^\rho_\nu + \tilde{\xi}^{\rho\sigma} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \tilde{\xi}_{\rho\sigma} \\ + 2\tilde{\xi}^\rho_\mu \tilde{\nabla}_\nu \tilde{\nabla}_\rho \tilde{\xi} + 2\tilde{\xi}^\rho_\mu \tilde{\nabla}_\rho \tilde{\nabla}_\nu \tilde{\xi} + 2\tilde{\xi}^{\rho\sigma} \tilde{\nabla}_\rho \tilde{\nabla}_\sigma \tilde{\xi}_{\mu\nu} \\ - 2\tilde{\xi}^\rho_\mu \tilde{\nabla}_\sigma \tilde{\nabla}_\rho \tilde{\xi}^\sigma_\nu - 2\tilde{\xi}^\rho_\mu \tilde{\nabla}_\sigma \tilde{\nabla}_\nu \tilde{\xi}^\sigma_\rho - 2\tilde{\xi}^{\rho\sigma} \tilde{\nabla}_\mu \tilde{\nabla}_\sigma \tilde{\xi}_{\nu\rho} \\ \left. + \tilde{g}_{\mu\nu} \left[\frac{1}{2} \tilde{\xi} \tilde{\square} \tilde{\xi} - \frac{1}{2} \tilde{\xi}^{\rho\sigma} \tilde{\square} \tilde{\xi}_{\rho\sigma} - \tilde{\xi}^{\rho\sigma} \tilde{\nabla}_\rho \tilde{\nabla}_\sigma \tilde{\xi} + \tilde{\xi}^{\rho\sigma} \tilde{\nabla}_\lambda \tilde{\nabla}_\sigma \tilde{\xi}^\lambda_\rho \right] \right. \\ \left. - 2m_2^2 [\tilde{\xi}^\rho_\mu \tilde{\xi}_{\nu\rho} - \tilde{\xi}_{\mu\nu} \tilde{\xi}] + \frac{1}{2} m_2^2 \tilde{g}_{\mu\nu} [\tilde{\xi}^{\rho\sigma} \tilde{\xi}_{\rho\sigma} - \tilde{\xi} \tilde{\xi}] \right\} \\ + \mathcal{O}(\kappa^3). \tag{35} \end{aligned}$$

This can be rewritten in the form

$$\begin{aligned} \tilde{R}_{\mu\nu} = \kappa^2 \left\{ \tilde{T}_{\mu\nu} - \frac{1}{2} \tilde{T} \tilde{g}_{\mu\nu} + \tilde{\nabla}_\mu \chi \tilde{\nabla}_\nu \chi - \frac{1}{2} m_0^2 \tilde{g}_{\mu\nu} \chi^2 \right. \\ - 2\tilde{\xi}_{\mu\nu} \tilde{\square} \tilde{\xi} - \tilde{\xi} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \tilde{\xi} + 2\tilde{\xi}_{\mu\rho} \tilde{\square} \tilde{\xi}^\rho_\nu + \tilde{\xi}^{\rho\sigma} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \tilde{\xi}_{\rho\sigma} \\ + 2\tilde{\xi}^\rho_\mu \tilde{\nabla}_\nu \tilde{\nabla}_\rho \tilde{\xi} + 2\tilde{\xi}^\rho_\mu \tilde{\nabla}_\rho \tilde{\nabla}_\nu \tilde{\xi} + 2\tilde{\xi}^{\rho\sigma} \tilde{\nabla}_\rho \tilde{\nabla}_\sigma \tilde{\xi}_{\mu\nu} \\ - 2\tilde{\xi}^\rho_\mu \tilde{\nabla}_\sigma \tilde{\nabla}_\rho \tilde{\xi}^\sigma_\nu - 2\tilde{\xi}^\rho_\mu \tilde{\nabla}_\sigma \tilde{\nabla}_\nu \tilde{\xi}^\sigma_\rho - 2\tilde{\xi}^{\rho\sigma} \tilde{\nabla}_\mu \tilde{\nabla}_\sigma \tilde{\xi}_{\nu\rho} \\ \left. + \tilde{g}_{\mu\nu} [\tilde{\xi} \tilde{\square} \tilde{\xi} - \tilde{\xi}^{\rho\sigma} \tilde{\square} \tilde{\xi}_{\rho\sigma} - 2\tilde{\xi}^{\rho\sigma} \tilde{\nabla}_\rho \tilde{\nabla}_\sigma \tilde{\xi} + 2\tilde{\xi}^{\rho\sigma} \tilde{\nabla}_\lambda \tilde{\nabla}_\sigma \tilde{\xi}^\lambda_\rho] \right. \\ \left. - 2m_2^2 [\tilde{\xi}^\rho_\mu \tilde{\xi}_{\nu\rho} - \tilde{\xi}_{\mu\nu} \tilde{\xi}] + \frac{1}{2} m_2^2 \tilde{g}_{\mu\nu} [\tilde{\xi}^{\rho\sigma} \tilde{\xi}_{\rho\sigma} - \tilde{\xi} \tilde{\xi}] \right\} + \mathcal{O}(\kappa^3). \tag{36} \end{aligned}$$

3. Singularity Theorems in Effective Quantum Gravity

3.1. Massive Scalar Field

It is known that a massive scalar field always satisfies the null energy condition, but can easily violate the strong condition (cf. [36,37]). The energy momentum tensor is given by

$$T_{\mu\nu} = \nabla_\mu \chi \nabla_\nu \chi - \frac{1}{2} g_{\mu\nu} \left(\nabla^\rho \chi \nabla_\rho \chi + m^2 \chi^2 \right). \tag{37}$$

Hence,

$$T_{\mu\nu} v^\mu v^\nu = (v^\mu \nabla_\mu \chi)^2 \geq 0, \tag{38}$$

where v is an arbitrary null vector. We conclude that the null energy condition is satisfied. However,

$$T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T = \nabla_\mu \chi \nabla_\nu \chi - \frac{1}{2} g_{\mu\nu} m_0^2 \chi^2 \tag{39}$$

which leads to

$$\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) t^\mu t^\nu = (t^\mu \nabla_\mu \chi)^2 - \frac{1}{2} m_0^2 \chi^2, \tag{40}$$

where t is an arbitrary normalised time-like vector. We see that this expression could be both larger and smaller than 0. Consequently, the strong energy condition does not necessarily hold. We conclude that the scalar field arising in effective quantum gravity could resolve cosmological singularities, but not black hole singularities.

3.2. Bounds on the Mass of the Massive Scalar Field

Using the results from Appendix B, we can derive a bound on the mass of the scalar field for which the cosmological singularity theorem still holds. First, consider the action (34) containing only the massive scalar. Equation (36) then reduces to

$$\tilde{R}_{\mu\nu} = \kappa^2 \left\{ \tilde{T}_{\mu\nu} - \frac{1}{2} \tilde{T} \tilde{g}_{\mu\nu} + \tilde{\nabla}_\mu \chi \tilde{\nabla}_\nu \chi - \frac{1}{2} m_0^2 \tilde{g}_{\mu\nu} \chi^2 \right\} + \mathcal{O}(\kappa^3). \tag{41}$$

Let us consider a globally hyperbolic four-dimensional space-time with compact Cauchy hypersurface S , and assume $|\chi| < \chi_{\max}$ is bounded towards the past of S . Then,

$$\begin{aligned} \int_0^T e^{-\frac{2C\tau}{n-1}} R_{\mu\nu}(\tau) \hat{\gamma}^\mu \hat{\gamma}^\nu(\tau) d\tau &\geq -\frac{1}{2} \kappa^2 m_0^2 \chi_{\max}^2 \int_0^T e^{-\frac{2C\tau}{n-1}} d\tau \\ &\geq -\frac{3\kappa^2}{4C} m_0^2 \chi_{\max}^2, \end{aligned} \tag{42}$$

where $\hat{\gamma}$ is a normalised tangent vector to a past directed time-like geodesic and where we have used the strong energy condition in the first line. We find that

$$-\frac{C}{2} + \int_{-T}^0 e^{\frac{2C\tau}{n-1}} R_{\mu\nu}(\tau) \hat{\gamma}^\mu(\tau) \hat{\gamma}^\nu(\tau) d\tau \geq -\frac{C}{2} - \frac{3\kappa^2}{4C} m_0^2 \chi_{\max}^2 \tag{43}$$

for any $C > 0$. The right hand side is maximised for $C = \sqrt{\frac{3}{2}} \kappa m_0 \chi_{\max}$. By Theorem A3, we then find that \mathcal{M} is past geodesically incomplete, if

$$\theta > \sqrt{\frac{3}{2}} \kappa m_0 \chi_{\max} \tag{44}$$

everywhere on S . Hence, for

$$m_0 < \sqrt{\frac{2}{3}} \frac{\theta_{\min}}{\kappa \chi_{\max}}, \tag{45}$$

the singularity theorem still holds.

We can use the expression for the mass of the scalar (32) to find a condition for the Wilson coefficients. Let us first ignore the non-local terms α, β, γ . We then find

$$m_0^2 = \frac{1}{4\kappa^2 [3c_1(\mu) + c_2(\mu) + c_3(\mu)]}. \tag{46}$$

We thus find that the singularity theorem holds for

$$3c_1(\mu) + c_2(\mu) + c_3(\mu) > \frac{3\chi_{\max}^2}{8\theta_{\min}^2}, \tag{47}$$

where we have assumed $3c_1(\mu) + c_2(\mu) + c_3(\mu) > 0$, as the opposite would imply that the scalar field is tachyonic⁵. If we include the non-local contributions, we find instead

$$3c_1(\mu) + c_2(\mu) + c_3(\mu) > \text{Re} \left(\frac{3\chi_{\max}^2}{8\theta_{\min}^2} + (3\alpha + \beta + \gamma) \ln \left[-\frac{3\mu^2 \kappa^2 \chi_{\max}^2}{2\theta_{\min}^2} \right] \right), \tag{48}$$

where only the logarithm has a complex part that accounts for the decay width of the field [38–40].

We can make an estimate of the expansion parameter for our universe by assuming the FLRW metric and by assuming that we live on a compact Cauchy hypersurface with a Hubble parameter that is constant along the surface. We find

$$\theta_{\min} = \frac{1}{3} H \approx 10^{-18} \text{ s}^{-1}, \tag{49}$$

where the Hubble parameter is fixed by experiment⁶. In addition, we require an estimate for χ_{\max} , which will rely on theoretical prejudice. However, for the effective action to be consistent, one would expect that both the scalar and tensor fields arising in the Einstein frame do not exceed the Planck scale. We thus make the rough estimate

$$\chi_{\max} = \sqrt{\frac{c^5}{8\pi G_N \hbar}} = 10^{42} \text{ s}^{-1}. \tag{50}$$

Hence,

$$\frac{3\chi_{\max}^2}{8\theta_{\min}^2} = 10^{121}. \tag{51}$$

Furthermore, the non-local part leads to a correction given by

$$(3\alpha + \beta + \gamma) \ln \left[-\frac{3\mu^2 \kappa^2 \chi_{\max}^2}{2\theta_{\min}^2} \right] \approx 10^2, \tag{52}$$

where we have used the known values for α, β, γ assuming only standard model fields. Furthermore, we have set the cutoff scale $\mu \approx \kappa^{-1}$. These non-local corrections are thus negligible compared to the local contributions.

⁵ We do not consider the tachyonic case, as it is unphysical. It can be shown, however, using Equation (40) that in this case, the strong energy condition is satisfied.

⁶ We take $H_0 \approx 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$.

We conclude that the singularity theorem holds, if the local Wilson coefficients satisfy the condition

$$3c_1(\mu) + c_2(\mu) + c_3(\mu) \gtrsim 10^{121} \tag{53}$$

or, equivalently,

$$m_0 \lesssim 10^{-34} \text{ eV}/c^2. \tag{54}$$

The singularity theorem can thus be violated for a large range of values.

The scalar and spin-2 particles give rise to corrections to the Newtonian potential according to the formula

$$\Phi(r) = -\frac{G_N m}{r} \left(1 + \frac{1}{3} e^{-\text{Re}(m_0)r} - \frac{4}{3} e^{-\text{Re}(m_2)r} \right). \tag{55}$$

The Eöt-Wash experiment [21] sets bounds on deviations from this potential. Assuming that the corrections do not cancel each other, both corrections should satisfy these experimental bounds, i.e.,

$$m_0, m_2 \geq 10^{-3} \text{ eV}/c^2 \tag{56}$$

Hence, the singularity theorem can be violated for all feasible values of the Wilson coefficients.

It might seem counterintuitive that tiny Wilson coefficients already lead to a breakdown of the assumptions of the singularity theorems, while large Wilson coefficients do not. In particular, the smaller the Wilson coefficients, the closer the action is to the Einstein–Hilbert action. However, small Wilson coefficients lead to very massive scalar fields, which can violate the strong energy condition, as can be seen in Equation (40). Furthermore, the Einstein equation is a second order differential equation, while the introduction of the terms quadratic in the Ricci scalar and tensor make it a fourth order equation. As is well known, solutions of differential equations are generically not stable against perturbations that change the class of the differential equation (cf. [41] for a discussion of this fact in the context of general relativity).

3.3. Spin-2 Massive Ghost

Let us now turn to the massive spin-2 field. Since this field is a ghost, one would expect it to violate the null energy condition. Indeed, we can write the energy momentum tensor explicitly

$$\begin{aligned} T_{\mu\nu} = & -2\tilde{\xi}_{\mu\nu}\tilde{\square}\tilde{\xi} - \tilde{\xi}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{\xi} + 2\tilde{\xi}_{\mu\rho}\tilde{\square}\tilde{\xi}^\rho_\nu + \tilde{\xi}^{\rho\sigma}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{\xi}_{\rho\sigma} \\ & + 2\tilde{\xi}^\rho_\mu\tilde{\nabla}_\nu\tilde{\nabla}_\rho\tilde{\xi} + 2\tilde{\xi}^\rho_\mu\tilde{\nabla}_\rho\tilde{\nabla}_\nu\tilde{\xi} + 2\tilde{\xi}^{\rho\sigma}\tilde{\nabla}_\rho\tilde{\nabla}_\sigma\tilde{\xi}_{\mu\nu} \\ & - 2\tilde{\xi}^\rho_\mu\tilde{\nabla}_\sigma\tilde{\nabla}_\rho\tilde{\xi}^\sigma_\nu - 2\tilde{\xi}^\rho_\mu\tilde{\nabla}_\sigma\tilde{\nabla}_\nu\tilde{\xi}^\sigma_\rho - 2\tilde{\xi}^{\rho\sigma}\tilde{\nabla}_\mu\tilde{\nabla}_\sigma\tilde{\xi}_{\nu\rho} \\ & + \tilde{g}_{\mu\nu} \left[\frac{1}{2}\tilde{\xi}\tilde{\square}\tilde{\xi} - \frac{1}{2}\tilde{\xi}^{\rho\sigma}\tilde{\square}\tilde{\xi}_{\rho\sigma} - \tilde{\xi}^{\rho\sigma}\tilde{\nabla}_\rho\tilde{\nabla}_\sigma\tilde{\xi} + \tilde{\xi}^{\rho\sigma}\tilde{\nabla}_\lambda\tilde{\nabla}_\sigma\tilde{\xi}^\lambda_\rho \right] \\ & - 2m_2^2 \left[\tilde{\xi}^\rho_\mu\tilde{\xi}_{\nu\rho} - \tilde{\xi}_{\mu\nu}\tilde{\xi} \right] + \frac{1}{2}m_2^2\tilde{g}_{\mu\nu} \left[\tilde{\xi}^{\rho\sigma}\tilde{\xi}_{\rho\sigma} - \tilde{\xi}\tilde{\xi} \right]. \end{aligned} \tag{57}$$

In order to show that the field can violate the null energy condition, we construct a counterexample. We consider the special case in which the tensor field is aligned with the metric:

$$\tilde{\xi}_{\mu\nu} = \frac{1}{4}g_{\mu\nu}\tilde{\xi}. \tag{58}$$

This results in an energy momentum tensor given by

$$T_{\mu\nu} = -\frac{1}{16} (g_{\mu\nu}\tilde{\xi}\tilde{\square}\tilde{\xi} + \tilde{\xi}\nabla_\mu\nabla_\nu\tilde{\xi} + \tilde{\xi}\nabla_\nu\nabla_\mu\tilde{\xi}). \tag{59}$$

Hence,

$$\begin{aligned} T_{\mu\nu}v^\mu v^\nu &= -\frac{1}{8}\zeta\nabla_\mu\nabla_\nu\zeta v^\mu v^\nu \\ &= -\frac{1}{8}(k_\mu v^\mu \zeta)^2 \\ &\leq 0, \end{aligned} \tag{60}$$

where v is an arbitrary null-like vector and where we assumed the field ζ to be an eigenvector of $\nabla_\mu\nabla_\nu$ with eigenvector $k_\mu k_\nu$, as is the case if the field exhibits sinusoidal behaviour with wave vector k .

Since the spin-2 field can violate the null energy condition, it can violate the strong energy condition as well. We conclude that the massive spin-2 field can resolve both kinds of singularities, since it does not satisfy any of the required energy conditions.

The fact that the ghost field can resolve singularities is less of a surprise, if one takes into account that the ghost field leads to a repulsive contribution to Newton's potential [42,43], and could thus result in an effective repulsive force at small distances.

4. Conclusions and Outlook

It is well known that the classical singularity theorems by Penrose and Hawking [1,2] only hold if general relativity is assumed. Quantum gravity, however, leads to deviations from general relativity, as can easily be shown using effective field theory methods. Furthermore, one of the main objectives of quantum gravity theories is to resolve singularities. In this work, we have discussed the validity of the classical singularity theorems in the context of the unique effective field theory for quantum gravity at second order in curvature.

We have considered singularity theorems by making an explicit mapping to the Einstein frame. The local terms in this theory give rise to an additional scalar and tensor field at second order in curvature. Moreover, the inclusion of the non-local terms at this order only gives rise to a shift in the mass of these fields.

We have shown that the massive spin-2 ghost field can violate the null energy condition and thus the strong energy condition as well. This is independent of its unknown mass. Although this is expected from a ghost field, it shows that the ghost field can be useful for resolving singularities in quantum gravity. We emphasise that the ghost field in effective theories for quantum gravity is not problematic, since it results from integrating out the low energy quantum degrees of freedom. In this framework, it must thus be treated as a classical field, and not be quantised again [43].

Furthermore, we have shown that the scalar field satisfies the null energy condition, but may violate the strong energy condition. The latter is a necessary assumption of Hawking's original theorem that proves the necessity of a big bang singularity. Violation of the strong energy condition depends on the unknown mass of the scalar field. However, using a singularity theorem with weakened energy conditions [6], we have shown that the scalar field can lead to avoidance of the big bang singularity for the entire mass range that is allowed by experiment.

We note that explicit examples of singularity avoidance in our framework have already been found in a big bounce scenario [20] and a collapse to a black hole [19]. Moreover, other examples of singularity resolution in various theories such as higher derivative gravity [44,45], string theory [46] and polynomial gravity models [47] have been found. The topic has also extensively been discussed within many ultraviolet complete approaches to quantum gravity.

It is important to notice that the breakdown of the assumptions of Hawking's and Penrose's singularity theorem does not imply the non-existence of singularities. However, it does imply that singularities can potentially be avoided. If the assumptions for the singularity theorems hold, the singular solutions of general relativity are the necessary endpoint of a collapsing star or universe. When considering perturbative corrections in the effective field theory approach, it is expected that these singular solutions such as the Kerr black hole will remain to be viable solutions. However, it is

possible that new solutions such as the ones discussed in [48–50] are present when the higher order curvature corrections are taken into account. If the singularity theorems are no longer applicable, such non-singular solutions can become the physically relevant solutions.

Finally, we should notice that the results discussed in this paper only hold up to second order in curvature. Inclusion of higher orders might force us back into a regime where the singularity theorems hold or might draw us further away from this regime. The effects of these higher order terms are sub-leading but not negligible, as singularities form in highly curved regions of space-time. It is interesting, however, that singularities can potentially already be resolved at the second order in curvature without making assumptions about the correct UV-complete theory of quantum gravity. This fact may help guide the way to singularity resolution in ultraviolet complete theories of quantum gravity.

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Appendix A. Classical Singularity Theorems

Appendix A.1. Hawking’s Cosmological Singularity Theorem

In this appendix, we state and prove Hawking’s singularity theorem [2].

Theorem A1. *Let \mathcal{M} be a globally hyperbolic n -dimensional space-time with $n \geq 2$ and a Cauchy surface S . Assume that $\exists C > 0$ such that $\theta_x < -C \forall x \in S$, where $\theta = \frac{1}{2}g^{\mu\nu}\partial_\tau g_{\nu\mu}$ is the expansion parameter. Furthermore, assume that matter within this space-time satisfies the strong energy condition*

$$\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)t^\mu t^\nu \geq 0 \tag{A1}$$

for every normalised time-like vector t^μ everywhere in the future of the Cauchy surface S . Then, the space-time \mathcal{M} is geodesically incomplete towards the future of S . Moreover, if $\theta_x > C \forall x \in S$ and the strong energy condition is satisfied everywhere in the past of S , then \mathcal{M} is geodesically incomplete towards the past of S .

Proof. Consider an n -dimensional globally hyperbolic space-time \mathcal{M} with Cauchy surface S . Then, we can find an open neighborhood $\hat{S} \supset S$ and a coordinate system on \hat{S} such that the metric is given by

$$ds^2 = -dt^2 + g_{ij}(t, \vec{x})dx^i dx^j. \tag{A2}$$

In order to prove Hawking’s singularity theorem [2], we can write down the Raychaudhuri equation [51]:

$$\frac{d\theta}{d\tau} = -\frac{\theta^2}{n-1} - \sigma_{\mu\nu}\sigma^{\nu\mu} - R_{\mu\nu}t^\mu t^\nu, \tag{A3}$$

where the expansion θ and shear $\sigma_{\mu\nu}$ are given by

$$\theta = \frac{1}{2}g^{\mu\nu}\partial_\tau g_{\nu\mu} = \frac{\dot{V}}{V}, \tag{A4}$$

$$\sigma_\nu^\mu = \frac{1}{2}\left(g^{\mu\rho}\partial_\tau g_{\rho\nu} - \frac{1}{n-1}\delta_\nu^\mu g^{\rho\sigma}\partial_\tau g_{\sigma\rho}\right), \tag{A5}$$

where we defined

$$V = \sqrt{\det(g)} \tag{A6}$$

and the time-derivative by $\dot{V} = \partial_\tau V$. Furthermore, θ and $\sigma_{\mu\nu}$ are taken along a time-like path γ parametrised by τ with normalised tangent vectors t^μ , and $\gamma(0) \in S$.

If we use the Einstein field equation, we can rewrite the Raychaudhuri equation to

$$\frac{d\theta}{d\tau} = -\frac{\theta^2}{n-1} - \sigma_{\mu\nu}\sigma^{\nu\mu} - \kappa^2 \left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \right) t^\mu t^\nu. \tag{A7}$$

Assuming the strong energy condition

$$\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \right) t^\mu t^\nu \geq 0, \tag{A8}$$

we find

$$\frac{d\theta}{d\tau} \leq -\frac{\theta^2}{n-1}. \tag{A9}$$

Hence,

$$\frac{d}{d\tau} \theta^{-1} \geq \frac{1}{n-1}. \tag{A10}$$

Assume $\exists C > 0$ such that $\theta_x(0) < -C \forall x \in S$; then, we can integrate (A10) and obtain

$$\frac{1}{\theta(\tau)} \geq \frac{\tau}{n-1} - \frac{1}{C}. \tag{A11}$$

Hence for $\tau \in \left(-\infty, \frac{n-1}{C}\right)$

$$\theta(\tau) \leq -\left(\frac{1}{C} - \frac{\tau}{n-1}\right)^{-1}. \tag{A12}$$

We can rewrite in terms of V and integrate to find

$$0 \leq V(\tau) \leq V(0) \left(1 - \frac{C\tau}{n-1}\right)^{n-1}. \tag{A13}$$

Therefore,

$$\lim_{\tau \rightarrow \frac{n-1}{C}} V(\tau) = 0. \tag{A14}$$

We thus conclude that any geodesic emanating from the Cauchy surface will develop a focal point for $0 < \tau \leq \frac{n-1}{C}$. Furthermore, since S is a Cauchy surface and \mathcal{M} is globally hyperbolic, any point $y \in \mathcal{M}$ is connected to a point $x \in S$ through a causal path of maximal proper time. We thus conclude that no geodesic $\gamma(\tau)$ can be extended to $\tau \geq \frac{n-1}{C}$. Therefore, the space-time is geodesically incomplete towards the future. This proves the future version of the theorem. The past version immediately follows by inverting the time direction in the proof. \square

We conclude this subsection by mentioning an immediate result of the theorem: if there exists a Cauchy surface S such that the Hubble parameter $H \geq H_0 > 0$ on the entire surface S , and the strong energy condition is expected to hold anywhere in the past of this surface, then the space-time is geodesically incomplete towards the past. More precisely, no geodesic can be extended beyond $\tau = H_0^{-1}$ towards the past. To see this, we recall that the Hubble constant given by

$$H = \frac{\dot{a}}{a} = (n-1) \frac{\dot{V}}{V} \tag{A15}$$

for the FLRW metric

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2. \tag{A16}$$

Appendix A.2. Penrose’s Black Hole Singularity Theorem

In this appendix, we state and prove Penrose’s singularity theorem [1]. Here, we closely follow the proof provided in [8].

Theorem A2. *Let \mathcal{M} be a globally hyperbolic n -dimensional space-time with $n \geq 3$ and a non-compact Cauchy surface S . Assume that \mathcal{M} contains a compact trapped surface⁷ U . Furthermore, assume that matter within this space-time satisfies the null energy condition*

$$T_{\mu\nu}v^\mu v^\nu \geq 0 \tag{A17}$$

for every null-like vector v^μ everywhere in the future of the trapped surface U . Then, the space-time \mathcal{M} is null-geodesically incomplete towards the future of U .

Proof. Consider a globally hyperbolic n -dimensional space-time with non-compact Cauchy surface S , and a compact trapped surface U . Then, we can find an open neighborhood $\hat{U} \supset U$ and a coordinate system on \hat{U} such that the metric is given by (cf. [8,52])

$$ds^2 = -2e^q dv du + g_{AB}(dx^A + c^A dv)(dx^B + c^B dv), \tag{A18}$$

where x^A is an arbitrary but fixed local coordinate system on the $(n - 2)$ -dimensional surface U . Furthermore, q and c are respectively a scalar and vector function of the coordinates. In this metric, we can evaluate the Ricci tensor and find

$$R_{uu} = -\frac{1}{2}\partial_u \left(g^{AB}\partial_u g_{AB} \right) - \frac{1}{4} \left(g^{AC}\partial_u g_{BC} \right) \left(g^{BD}\partial_u g_{DA} \right). \tag{A19}$$

We can define the area of a bundle of orthogonal null geodesics locally by

$$A = \sqrt{\det(g_{AB})}, \tag{A20}$$

which allows us to define the null expansion as

$$\theta = \frac{\dot{A}}{A} = \frac{1}{2}g^{AB}\partial_u g_{BA}, \tag{A21}$$

where the dot represents a derivative with respect to u . Furthermore, we can define the null shear by

$$\sigma_B^A = \frac{1}{2} \left(g^{AC}\partial_u g_{CB} - \frac{1}{n-2} \delta_B^A g^{CD}\partial_u g_{DC} \right). \tag{A22}$$

We then find the null Raychaudhuri equation given by

$$\frac{d\theta}{du} = -\frac{\theta^2}{n-2} - \sigma_{AB}\sigma^{BA} - R_{uu}. \tag{A23}$$

Furthermore, we can use the Einstein equation and the fact $g_{uu} = 0$ to write

$$R_{uu} = \kappa^2 T_{uu}. \tag{A24}$$

⁷ A codimension 2 spacelike and achronal submanifold such that the null expansion parameter is negative everywhere on U for each family of orthogonal future going null geodesics.

Imposing the null energy condition results in

$$\frac{d}{du} \theta^{-1} \geq \frac{1}{n-2}. \tag{A25}$$

Using that U is a trapped surface $\exists C > 0$ such that $\theta_x < -C \forall x \in U$, one can integrate this equation in a similar way as was done in the proof of Theorem A1. One obtains

$$\lim_{u \rightarrow \frac{n-2}{C}} A(u) = 0. \tag{A26}$$

Therefore, all future going null-like geodesics develop a focal point for an affine distance $0 < u \leq \frac{n-2}{C}$.

Let us now assume that all null geodesics can be extended beyond this focal point, and let us pick such a geodesic l arbitrarily. Then, at least a small segment of this geodesic is prompt, and lies in the lightcone $\partial J^+(U)$. Furthermore, the part of l that lies in $\partial J^+(U)$ is connected, and the part beyond its first focal point cannot be in $\partial J^+(U)$, since it is not prompt. Therefore, $l \cap \partial J^+(U)$ is a finite non-empty interval which has to be closed, since $\partial J^+(U)$ is closed in \mathcal{M} .

If we take an arbitrary point $p \in \partial J^+(U)$, then this point can be reached by a null geodesic originating from U . This point is thus determined by the point $q \in U$, where the geodesic emanates, the value of the affine parameter u measured along the geodesic and the direction (i.e., ingoing or outgoing) of the geodesic. Since U is compact and since the affine parameters measured along the geodesics range over a compact interval, we find that $\partial J^+(U)$ is compact.

However, by construction, $\partial J^+(U)$ is an achronal codimension 1 submanifold of \mathcal{M} . Furthermore, by assumption, \mathcal{M} is a globally hyperbolic manifold with noncompact Cauchy hypersurface S , and thus does not allow for an achronal codimension 1 submanifold (see, e.g., [8]). Hence, we arrive at a contradiction and conclude that at least one of the future going null geodesics orthogonal to U cannot be extended beyond an affine distance $(n-2)/C$, which proves the theorem. \square

Appendix B. Singularity Theorems for Weakened Energy Conditions

In this section, we state a theorem and its proof from [6]. The theorem is similar to Hakwing’s cosmological singularity theorem, but uses relaxed conditions on the energy momentum tensor.

Theorem A3. *Let \mathcal{M} be a globally hyperbolic n -dimensional space-time ($n \geq 2$) with a compact Cauchy surface S . Assuming that $\exists C \geq 0$ such that along every future directed geodesic γ issuing orthogonally from S , we have*

$$\liminf_{T \rightarrow \infty} \int_0^T e^{-\frac{2C\tau}{n-1}} R_{\mu\nu}(\tau) \hat{\gamma}^\mu(\tau) \hat{\gamma}^\nu(\tau) d\tau > \theta(x_0) + \frac{C}{2}, \tag{A27}$$

where $x_0 = \gamma(0) \in S$, $\theta(x_0)$ is the expansion at x_0 , and $\hat{\gamma}(\tau)$ is a normalised time-like tangent vector of $\gamma(\tau)$. Then, \mathcal{M} is geodesically incomplete towards the future of S . Moreover, if

$$\liminf_{T \rightarrow \infty} \int_0^T e^{-\frac{2C\tau}{n-1}} R_{\mu\nu}(\tau) \hat{\gamma}^\mu(\tau) \hat{\gamma}^\nu(\tau) d\tau > -\theta(x_0) + \frac{C}{2} \tag{A28}$$

with γ a past directed geodesic, then \mathcal{M} is geodesically incomplete towards the past of S .

For the proof, we will use the following lemma which is proved in [6].

Lemma A1. *Consider the initial value problem*

$$\begin{cases} \dot{x}(t) = \frac{x(t)^2}{q(t)} + p(t), \\ x(0) = x_0, \end{cases} \tag{A29}$$

where $q(t)$ and $p(t)$ are continuous on $[0, \infty)$ and $q(t) > 0 \forall t \in [0, \infty)$. If

$$\int_0^\infty q(t)^{-1} dt = \infty, \tag{A30}$$

$$\liminf_{T \rightarrow \infty} \int_0^T p(t) dt > -x_0, \tag{A31}$$

Equation (A29) has no solution on $[0, \infty)$. Moreover, it implies that $\lim_{t \rightarrow t_c} x(t) \rightarrow \infty$ for $t_c \in (0, \infty)$.

Proof of Theorem A3. We follow the same argument as in the proof of Theorem A1 and find the Raychaudhuri equation

$$\frac{d\theta}{d\tau} = -\frac{\theta^2}{n-1} - \sigma_{\mu\nu}\sigma^{\mu\nu} - R_{\mu\nu}t^\mu t^\nu, \tag{A32}$$

which can be rewritten to

$$\frac{dx(\tau)}{d\tau} = \frac{x(\tau)^2}{q(\tau)} + p(\tau) \tag{A33}$$

with

$$x(\tau) = -(\theta + C)e^{-\frac{2C\tau}{n-1}}, \tag{A34}$$

$$p(\tau) = \left(\frac{C^2}{n-1} + \sigma_{\mu\nu}\sigma^{\mu\nu} + R_{\mu\nu}t^\mu t^\nu \right) e^{-\frac{2C\tau}{n-1}}, \tag{A35}$$

$$q(\tau) = (n-1)e^{-\frac{2C\tau}{n-1}}. \tag{A36}$$

Then, $q(\tau)$ satisfies condition (A30), while $p(\tau)$ satisfies condition (A31), if

$$\liminf_{T \rightarrow \infty} \int_0^T \left(\frac{C^2}{n-1} + \sigma_{\mu\nu}\sigma^{\mu\nu} + R_{\mu\nu}t^\mu t^\nu \right) e^{-\frac{2C\tau}{n-1}} d\tau > \theta(0) + C, \tag{A37}$$

which is satisfied, if

$$\liminf_{T \rightarrow \infty} \int_0^T e^{-\frac{2C\tau}{n-1}} R_{\mu\nu}t^\mu t^\nu d\tau > \theta(0) + \frac{C}{2}. \tag{A38}$$

By assumption (A27), this holds for all geodesics emanating from the Cauchy surface S . Thus, $\lim_{\tau \rightarrow \tau_\gamma} x(\tau) \rightarrow \infty$ for some $\tau_\gamma \in (0, \infty)$, which immediately implies that $\lim_{\tau \rightarrow \tau_\gamma} \theta(\tau) \rightarrow -\infty$. Hence,

$$\forall \gamma : [0, \infty) \rightarrow \mathcal{M} \quad \text{with} \quad \gamma(0) \in S \quad \exists \tau_\gamma \in (0, \infty) \quad \text{s.t.} \quad \lim_{\tau \rightarrow \tau_\gamma} V(\tau) \rightarrow 0. \tag{A39}$$

By compactness of S , $\sup\{\tau_\gamma | \gamma : [0, \infty) \rightarrow \mathcal{M}, \gamma(0) \in S\} < \infty$. Furthermore, since \mathcal{M} is globally hyperbolic, every point $y \in J^+(S)$ can be connected through a geodesic γ with maximal proper time. The past version can be obtained with a similar proof by inverting the direction of time. \square

Let us finally note that one can derive a similar theorem for the black hole case [6].

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