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# String Sigma Models on Curved Supermanifolds

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Received: 15 January 2018; Accepted: 20 April 2018; Published: 24 April 2018

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**Abstract:** We use the techniques of integral forms to analyze the easiest example of two-dimensional sigma models on a supermanifold. We write the action as an integral of a top integral form over a $D = 2$ supermanifold, and we show how to interpolate between different superspace actions. Then, we consider curved supermanifolds, and we show that the definitions used for flat supermanifolds can also be used for curved supermanifolds. We prove it by first considering the case of a curved rigid supermanifold and then the case of a generic curved supermanifold described by a single superfield $E$.

**Keywords:** supermanifolds; sigma models; string models

## 1. Introduction

During the last few years, some new structures in the geometry of supermanifolds have been uncovered [1–3]. The conventional[1] exterior bundle of a supermanifold $\Lambda^*[\mathcal{M}]$ is not the complete bundle needed to construct a geometric theory of integration, to define the Hodge dual operation and to study the cohomology. One has to take into account also the complexes of pseudo and integral forms [4–6]. We call this bundle the integral superspace.

In the present notes, we consider some new developments, and we study also the case of the curved integral superspace.

One of the key ingredients of string theory (in the Ramond–Neveu–Schwarz formulation) is the worldsheet supersymmetry needed to remove the unphysical tachyon from the spectrum, to describe fermionic vertex operators and to construct a supersymmetric spectrum. All of these properties are deeply related to the worldsheet supersymmetry, and they are clearly displayed by using the superspace approach in two dimensions.

Perturbative string theory is described by a non-linear sigma model for maps:

$$\phi^m(z,\bar{z}), \lambda^m(z,\bar{z}), \bar{\lambda}^m(z,\bar{z}) \tag{1}$$

from the worldsheet Riemann surface $\Sigma^{(1)}$ (with one complex dimension) to a 10-dimensional target space $\mathcal{M}^{(10)}$, $(m = 0, \ldots, 9)$ with an action given by (for a flat surface $\Sigma = \mathbb{C}$):

$$S[\phi,\lambda,\bar{\lambda}] = \int_\Sigma d^2 z \mathcal{L}(\phi,\lambda,\bar{\lambda})\,. \tag{2}$$

1 This is usually called the bundle of superforms, generated by the direct sum of exterior products of differential forms on the supermanifold.

where $\phi, \lambda$ denote respectively the bosonic and the fermionic fields.

To generalize it to any surface, one has to couple the action to $D = 2$ gravity in the usual way, namely by promoting the derivatives to covariant derivatives and adding the couplings with the $D = 2$ curvature. That can be easily done by considering the action as a two-form to be integrated on $\Sigma$ using the intrinsic definition of differential forms, Hodge duals and the differential $d$. To avoid using the Hodge dual, one can pass to first order formalism by introducing some auxiliary fields. Then, we have:

$$S[\phi, \lambda, \bar{\lambda}] = \int_{\Sigma} \mathcal{L}^{(2)}(\phi, d\phi, \lambda, d\lambda, \bar{\lambda}, d\bar{\lambda}, V^{\pm\pm}, \omega)\,. \tag{3}$$

where the two-form Lagrangian[2] $\mathcal{L}^{(2)}(\phi, d\phi, \lambda, d\lambda, \bar{\lambda}, d\bar{\lambda}, V^{\pm\pm}, \omega)$ depends on the fields $(\phi^m, \lambda^m, \bar{\lambda}^m)$, their differentials $(d\phi^m, d\lambda^m, d\bar{\lambda}^m)$, the $2d$ vielbeins $V^{\pm\pm}$ and the $SO(1,1)$ spin connection $\omega$. To make the supersymmetry manifest, one can rewrite the action (1) in the superspace formalism by condensing all fields into a superfield:

$$\Phi^m = \phi^m + \lambda^m \theta^+ + \bar{\lambda}^m \theta^- + f\theta^+\theta^- \tag{4}$$

(we introduce the two anticommuting coordinates $\theta^\pm$, their corresponding derivatives $D_\pm$ and the auxiliary field $f$) as follows:

$$S[\Phi] = \int [d^2 z d^2 \theta] \mathcal{L}(\Phi)\,. \tag{5}$$

The integration is extended to the superspace using the Berezin integration rules. In the same way as above, in order to generalize it to any super Riemann surface $\mathcal{S}\Sigma$ or, more generally, to any complex $D = 1$ supermanifold, we need to rewrite the action (5) as an integral of an integral form on $\mathcal{S}\Sigma$. As will be explained in the forthcoming section and as is discussed in the literature [4–6], the Lagrangian $\mathcal{L}(\Phi)$ is replaced by a $(2|0)$-superform $\mathcal{L}^{(2|0)}(\Phi, d\Phi, V^{\pm\pm}, \psi^\pm, \omega)$. It is well known that a superform cannot be integrated on the supermanifold $\mathcal{S}\Sigma$, and it must be converted into an integral form by multiplying it by a Picture Changing Operator (PCO) $\mathbb{Y}^{(0|2)}(V^{\pm\pm}, \psi^\pm, \omega)$. The latter is the Poincaré dual of the immersion of the bosonic submanifold into the supermanifold $\mathcal{S}\Sigma$. The action is written as:

$$S[\Phi] = \int_{\mathcal{S}\Sigma} \mathcal{L}^{(2|0)}(\Phi, d\Phi, V^{\pm\pm}, \psi^\pm, \omega) \wedge \mathbb{Y}^{(0|2)}(V^{\pm\pm}, \psi^\pm, \omega) \tag{6}$$

The PCO $\mathbb{Y}^{(0|2)}$ is a $(0|2)$-integral form, is $d$-closed and not exact. If we shift it by an exact term $\mathbb{Y} + d\Lambda$, the action is left invariant if $d\mathcal{L}^{(2|0)}(\Phi, d\Phi, V^{\pm\pm}, \psi^\pm, \omega) = 0$. That can be obtained in the presence of auxiliary fields and can be verified using the Bianchi identities for the torsion $T^{\pm\pm}$, the gravitinos field strengths $\rho^\pm$ and the curvature $R$. The choice of the PCO allows one to interpolate between different superspace frameworks with different manifest supersymmetries.

The action (6) is invariant under superdiffeomorphisms by construction since it is an integral of a top integral form. Therefore, it can be written for any solution of the Bianchi identity for any supermanifold compatible with them. As will be shown in the following, we can write the most general solution of the Bianchi identities in terms of an unconstrained superfield $E$.

The paper has the following structure: In Section 2, we summarize the geometry of integral forms. In Section 3, we discuss the PCOs and their properties. In Section 4, we discuss the action (2) in components and the Bianchi identities for the field strengths for the superfield $\Phi$. In Section 4.1, we derive the action (6), and we show the relation between the component action and the superfield

[2] Here and in the following, we define an action as an integral of a top form on the worldvolume of the theory in order to have manifest diffeomorphism invariance. As a consequence, in the case of supermanifolds, we consider integral top forms in place of usual top forms.

action. In Section 5, we consider the preliminary case of a rigid curved supermanifold based on the supercoset space $Osp(1|2)/SO(1,1)$. We show the relation between the volume form and the curvature. In Section 6, we study the general case of 2 $D$ supergravity $N = 1$. In particular, it is shown that the PCOs in the curved space are closed because of the torsion constraints.

## 2. Integral Forms and Integration

The integral forms are the crucial ingredients to define a geometric integration theory for supermanifolds inheriting all the good properties of differential forms integration theory in conventional (purely bosonic) geometry. In the following section, we briefly describe the notations and the most relevant definitions (see [1,3–5,7]).

We consider a supermanifold with $n$ bosonic and $m$ fermionic dimensions, denoted here and in the following by $\mathcal{M}^{(n|m)}$, locally isomorphic to the superspace $\mathbb{R}^{(n|m)}$. The local coordinates in an open set are denoted by $(x^a, \theta^\alpha)$. When necessary, in Sections 4–6, we introduce supermanifolds locally isomorphic to the complex superspace $\mathbb{C}^{(n|m)}$. In this case, the coordinates will be denoted by $(z^a, \bar{z}^a, \theta^\alpha, \bar{\theta}^\alpha)$; the formalism, mutatis mutandis, is the same.

A $(p|q)$ pseudoform $\omega^{(p|q)}$ has the following structure:

$$\omega^{(p|q)} = \omega(x,\theta) dx^{a_1} \dots dx^{a_r} d\theta^{\alpha_1} \dots d\theta^{\alpha_s} \delta^{(b_1)}(d\theta^{\beta_1}) \dots \delta^{(b_q)}(d\theta^{\beta_q}) \tag{7}$$

where the graded wedge product between $dx, d\theta$ and $\delta$'s is understood and $p$ and $q$ correspond respectively to the form number and the picture number, with $0 \le q \le m$ and $p = r + s - \sum_{i=1}^{i=q} b_i$ and $0 \le r \le n$. In a given monomial, the $d\theta^a$ appearing in the product are different from those appearing in the deltas $\delta(d\theta)$, and $\omega(x,\theta)$ is a set of superfields with index structure $\omega_{[a_1 \dots a_r](\alpha_1 \dots \alpha_s)[\beta_1 \dots \beta_q]}(x,\theta)$.

The index $b$ on the delta $\delta^{(b)}(d\theta^\alpha)$ denotes the degree of the derivative of the delta function with respect to its argument. The total picture of $\omega^{(p|q)}$ corresponds to the total number of delta functions and its derivatives. We call $\omega^{(p|q)}$ a superform if $q = 0$ and an integral form if $q = m$; otherwise, it is called a pseudoform. The total form degree is given by $p = r + s - \sum_{i=1}^{i=q} b_i$ since the derivatives act effectively as negative forms and the delta functions carry zero form degree. We recall the following properties:

$$d\theta^\alpha \delta(d\theta^\alpha) = 0, \; d\delta^{(b)}(d\theta^\alpha) = 0 \,, \; d\theta^\alpha \delta^{(b)}(d\theta^\alpha) = -b\delta^{(b-1)}(d\theta^\alpha) \,, \;\; b > 0 \,. \tag{8}$$

The index $\alpha$ is not summed. The indices $a_1 \dots a_r$ and $\beta_1 \dots \beta_q$ are anti-symmetrized; the indices $\alpha_1 \dots \alpha_s$ are symmetrized because of the rules of the graded wedge product:

$$dx^a dx^b = -dx^b dx^a \,, \quad dx^a d\theta^\alpha = d\theta^\alpha dx^a \,, \quad d\theta^\alpha d\theta^\beta = d\theta^\beta d\theta^\alpha \,, \tag{9}$$

$$\delta(d\theta^\alpha)\delta(d\theta^\beta) = -\delta(d\theta^\beta)\delta(d\theta^\alpha) \,, \tag{10}$$

$$dx^a \delta(d\theta^\alpha) = -\delta(d\theta^\alpha) dx^a \,, \quad d\theta^\alpha \delta(d\theta^\beta) = \delta(d\theta^\beta) d\theta^\alpha \,. \tag{11}$$

As usual, the module of $(p|q)$ pseudoforms is denoted by $\Omega^{(p|q)}$; if $q = 0$ or $q = m$, it is finitely generated.

It is possible to define the integral over the superspace $\mathbb{R}^{(n|m)}$ of an integral top form $\omega^{(n|m)}$ that can be written locally as:

$$\omega^{(n|m)} = f(x,\theta) dx^1 \dots dx^n \delta(d\theta^1) \dots \delta(d\theta^m) \tag{12}$$

where $f(x,\theta)$ is a superfield. By changing the one-forms $dx^a, d\theta^\alpha$ as $dx^a \to E^a = E^a_m dx^m + E^a_\mu d\theta^\mu$ and $d\theta^\alpha \to E^\alpha = E^\alpha_m dx^m + E^\alpha_\mu d\theta^\mu$, we get:

$$\omega \to \mathrm{sdet}(E)\, f(x,\theta) dx^1 \dots dx^n \delta(d\theta^1) \dots \delta(d\theta^m) \tag{13}$$

where $\mathrm{sdet}(E)$ is the superdeterminant of the supervielbein $(E^a, E^\alpha)$.

The integral form $\omega^{(n|m)}$ can be also viewed as a superfunction $\omega(x, \theta, dx, d\theta)$ on the odd dual[3] $T^*(\mathbb{R}^{(n|m)})$ acting superlinearly on the parity reversed tangent bundle $\Pi T(\mathbb{R}^{(n|m)})$, and its integral is defined as follows:

$$I[\omega] \equiv \int_{\mathbb{R}^{(n|m)}} \omega^{(n|m)} \equiv \int_{T^*(\mathbb{R}^{(n|m)})=\mathbb{R}^{(n+m|m+n)}} \omega(x, \theta, dx, d\theta)[dxd\theta\, d(dx)d(d\theta)] \tag{14}$$

where the order of the integration variables is kept fixed. The symbol $[dxd\theta\, d(dx)d(d\theta)]$ denotes the Berezin integration "measure", and it is invariant under any coordinate transformation on $\mathbb{R}^{(n|m)}$. It is a section of the Berezinian bundle of $T^*(\mathbb{R}^{(n|m)})$ (a super line bundle that generalizes the determinant bundle of a purely bosonic manifold). The sections of the determinant bundle transform with the determinant of the Jacobian and the sections of the Berezinian with the superdeterminant of the super-Jacobian. The Berezinian bundle of $T^*\mathcal{M}^{(n|m)}$ is always trivial, but the Berezinian bundle of $\mathcal{M}^{(n|m)}$ in general is non-trivial. The integrations over the fermionic variables $\theta$ and $dx$ are Berezin integrals, and those over the bosonic variables $x$ and $d\theta$ are Lebesgue integrals (we assume that $\omega(x, \theta, dx, d\theta)$ has compact support in the variables $x$, and it is a product of Dirac's delta distributions in the $d\theta$ variables). A similar approach for a superform would not be possible because the polynomial dependence on the $d\theta$ leads to a divergent integral.

As usual, this definition can be extended to supermanifolds $\mathcal{M}^{(n|m)}$ by using bosonic partitions of unity.

See again Witten [3] for a more detailed discussion on the symbol $[dxd\theta d(dx)d(d\theta)]$ and many other important aspects of the integration theory of integral forms.

According to the previous discussion, if a superform $\omega^{(n|0)}$ with form degree $n$ (equal to the bosonic dimension of the reduced bosonic submanifold $\mathcal{M}^{(n)} \hookrightarrow \mathcal{M}^{(n|m)}$) and picture number zero is multiplied by a $(0|m)$ integral form $\gamma^{(0|m)}$, we can define the integral on the supermanifold of the product:

$$\int_{\mathcal{M}^{(n|m)}} \omega^{(n|0)} \wedge \gamma^{(0|m)}. \tag{15}$$

This type of integral can be given a geometrical interpretation in terms of the reduced bosonic submanifold $\mathcal{M}^{(n)}$ of the supermanifold and the corresponding Poincaré dual (see [5]).

## 3. PCOs and Their Properties

In this section, we recall a few definitions and useful computations about the PCOs in our notations. For more details, see [2,6].

We start with the picture lowering operators that map cohomology classes in picture $q$ to cohomology classes in picture $r < q$.

Given an integral form, we can obtain a superform by acting on it with operators decreasing the picture number. Consider the following integral operator:

$$\delta(\iota_D) = \int_{-\infty}^{\infty} \exp\left(it\iota_D\right) dt \tag{16}$$

where $D$ is an odd vector field with $\{D, D\} \neq 0$[4] and $\iota_D$ is the contraction along the vector $D$. The contraction $\iota_D$ is an even operator.

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3 In order to make contact with the standard physics literature, we adopt the conventions that $d$ is an odd operator and $dx$ (an odd form) is dual to the even vector $\frac{\partial}{\partial x}$. The same holds for the even form $d\theta$ dual to the odd vector $\frac{\partial}{\partial \theta}$. As clearly explained for example in the Appendix of the paper [8], if one introduces also the natural concept of even differential (in order to make contact with the standard definition of the cotangent bundle of a manifold), our cotangent bundle (that we consider as the bundle of one-forms) should, more appropriately, be denoted by $\Pi T^*$.

4 Here and in the following, $\{,\}$ is the anticommutator (i.e., the graded commutator).

For example, if we decompose $D$ on a basis $D = D^\alpha \partial_{\theta^\alpha}$, where the $D^\alpha$ are even coefficients and $\{\partial_{\theta^\alpha}\}$ is a basis of the odd vector fields and take $\omega = \omega_\beta d\theta^\beta \in \Omega^{(1|0)}$, we have:

$$\iota_D \omega = D^\alpha \omega_\alpha = D^\alpha \frac{\partial \omega}{\partial d\theta^\alpha} \in \Omega^{(0|0)} . \tag{17}$$

In addition, due to $\{D, D\} \neq 0$, we have also that $\iota_D^2 \neq 0$. The differential operator $\delta(\iota_\alpha) \equiv \delta(\iota_D)$ (with $D = \partial_{\theta^\alpha}$) acts on the space of integral forms as follows (we neglect the possible introduction of derivatives of delta forms, but that generalization can be easily done):

$$\begin{aligned} \delta(\iota_\alpha) \textstyle\prod_{\beta=1}^m \delta(d\theta^\beta) &= \pm \textstyle\int_{-\infty}^{\infty} \exp\left(it\iota_\alpha\right) \delta(d\theta^\alpha) \prod_{\beta=1\neq\alpha}^m \delta(d\theta^\beta) dt \\ &= \pm \textstyle\int_{-\infty}^{\infty} \delta(d\theta^\alpha + it) \prod_{\beta=1\neq\alpha}^m \delta(d\theta^\beta) dt = \mp i \prod_{\beta=1\neq\alpha}^m \delta(d\theta^\beta) \end{aligned} \tag{18}$$

where the sign $\pm$ is due to the anticommutativity of the delta forms and it depends on the index $\alpha$. We have used also the fact that $\exp\left(it\iota_\alpha\right)$ represents a finite translation of $d\theta^\alpha$. The result contains $m-1$ delta forms, and therefore, it has picture $m-1$. It follows that $\delta(\iota_\alpha)$ is an odd operator.

We can define also the Heaviside step operator $\Theta(\iota_D)$:

$$\Theta(\iota_D) = \lim_{\epsilon \to 0^+} -i \int_{-\infty}^{\infty} \frac{1}{t - i\epsilon} \exp\left(it\iota_D\right) dt \tag{19}$$

The operators $\delta(\iota_D)$ and $\Theta(\iota_D)$ have the usual formal distributional properties: $\iota_D \delta(\iota_D) = 0$ , $\iota_D \delta'(\iota_D) = -\delta(\iota_D)$ and $\iota_D \Theta(\iota_D) = \delta(\iota_D)$.

In order to map cohomology classes into cohomology classes decreasing the picture number, we introduce the operator (see [2]):

$$Z_D = [d, \Theta(\iota_D)] \tag{20}$$

In the simplest case $D = \partial_{\theta^\alpha}$, we have:

$$Z_{\partial_{\theta^\alpha}} = i\delta(\iota_\alpha)\partial_{\theta^\alpha} \equiv Z_\alpha \tag{21}$$

The operator $Z_\alpha$ is the composition of two operators acting on different quantities: $\partial_{\theta^\alpha}$ acts only on functions, and $\delta(\iota_\alpha)$ acts only on delta forms.

In order to further reduce the picture, we simply iterate operators of type $Z$. An alternative description of $Z$ in terms of the Voronov integral transform can be found in [6].

The $Z$ operator is in general not invertible, but it is possible to find a non-unique operator $Y$ such that $Z \circ Y$ is an isomorphism in the cohomology. These operators are the called picture raising operators. The operators of type $Y$ are non-trivial elements of the de Rham cohomology.

We apply a PCO of type $Y$ on a given form by taking the graded wedge product; given $\omega$ in $\Omega^{(p|q)}$, we have:

$$\omega \xrightarrow{Y} \omega \wedge Y \in \Omega^{(p|q+1)} , \tag{22}$$

Notice that if $q = m$, then $\omega \wedge Y = 0$. In addition, if $d\omega = 0$, then $d(\omega \wedge Y) = 0$ (by applying the Leibniz rule), and if $\omega \neq dK$, then it follows that also $\omega \wedge Y \neq dU$, where $U$ is a form in $\Omega^{(p-1|q+1)}$. Therefore, given an element of the cohomology $H^{(p|q)}$, the new form $\omega \wedge Y$ is an element of $H^{(p|q+1)}$.

For a simple example in $\mathbb{R}^{(1|1)}$, we can consider the PCO $Y = \theta\delta(d\theta)$, corresponding to the vector $\partial_\theta$; we have $Z \circ Y = Y \circ Z = 1$

More general forms for $Z$ and $Y$ can be constructed, for example starting with the vector $Q = \partial_\theta + \theta\partial_x$.

For example, if $\varphi = g(x)\theta dx\delta(d\theta)$ is a generic top integral form in $\Omega^{(1|1)}\left(\mathbb{R}^{(1|1)}\right)$, the explicit computation using the formula $Z = [d, \Theta(\iota_Q)]$ is:

$$\begin{aligned} Z_Q[\varphi] &= d[\Theta(\iota_Q)\varphi] = d\Big[\Theta(\iota_Q)g(x)\theta dx\delta(d\theta)\Big] \\ &= d\Big[\lim_{\epsilon\to 0^+} -i\int_{-\infty}^{\infty}\frac{1}{t-i\epsilon}g(x)\theta dx\delta(d\theta+it)dt\Big] = \\ &= d\left[-\frac{g(x)\theta dx}{d\theta}\right] = -g(x)dx\,. \end{aligned} \tag{23}$$

The last expression is clearly closed. Note that in the above computations, we have introduced formally the inverse of the (commuting) superform $d\theta$. Using terminology borrowed from superstring theory, we can say that, even though in a computation we need an object that lives in the large Hilbert space, the result is still in the small Hilbert space.

Note that the negative powers of the superform $d\theta$ are well defined only in the complexes of superforms (i.e., in picture 0). In this case, the inverse of the $d\theta$ and its powers are closed and exact and behave with respect to the graded wedge product as negative degree superforms of picture 0. In picture $\neq 0$, negative powers are not defined because of the distributional relation $d\theta\delta\,(d\theta) = 0$.

A PCO of type $Y$ invariant under the rigid supersymmetry transformations (generated by the vector $Q$) $\delta_\epsilon x = \epsilon\theta$ and $\delta_\epsilon\theta = \epsilon$ is, for example, given by:

$$Y_Q = (dx + \theta d\theta)\delta'(d\theta) \tag{24}$$

We have:

$$Y_Q Z_Q[\varphi] = -g(x)dx \wedge (dx + \theta d\theta)\delta'(d\theta) = g(x)\theta dx\delta(d\theta) = \varphi\,. \tag{25}$$

## 4. Rheonomic Sigma Model

We consider a flat complex superspace with bosonic coordinates $(z = z^{++}, \bar{z} = z^{--})$ and Grassmannian coordinates $(\theta = \theta^+, \bar{\theta} = \theta^-)$. The charges $\pm$ are assigned according to the transformation properties of the coordinates $z, \theta$ under the Lorentz group $SO(1,1)$. The latter being unidimensional, the irreducible representations are parametrized by their charges:

$$x^{\pm\pm} \to e^{\pm i\alpha}x^{\pm\pm}\,, \quad \theta^\pm \to e^{\pm\frac{i\alpha}{2}}\theta^\pm\,. \tag{26}$$

where $\alpha$ is a parameter of the Lorentz transformation.

We introduce the differentials $(dx^{\pm\pm}, d\theta^\pm)$ and the flat supervielbeins:

$$V^{\pm\pm} = dz^{\pm\pm} + \theta^\pm d\theta^\pm\,, \qquad \psi^\pm = d\theta^\pm\,, \tag{27}$$

invariant under the rigid supersymmetry $\delta\theta^\pm = \epsilon^\pm$ and $\delta x^{\pm\pm} = \epsilon^\pm\theta^\pm$. They satisfy the Maurer–Cartan (MC) algebra:

$$dV^{\pm\pm} = \psi^\pm \wedge \psi^\pm\,, \qquad d\psi^\pm = 0\,. \tag{28}$$

We first consider the non-chiral multiplet. This is described by a superfield $\Phi$ with the decomposition:

$$\begin{aligned} \Phi &= \phi + \lambda\theta^+ + \bar{\lambda}\theta^- + f\theta^+\theta^- \\ W &= D_+\Phi\,, \\ \bar{W} &= D_-\Phi\,, \\ F &= D_-D_+\Phi\,. \end{aligned} \tag{29}$$

where $D_+ = \partial_{\theta^+} - \frac{1}{2}\theta^+\partial_{++}$ and $D_- = \partial_{\theta^-} - \frac{1}{2}\theta^-\partial_{--}$ (with $\partial_{++} = \partial_{z^{++}}$ and $\partial_{--} = \partial_{z^{--}}$). They satisfy the algebra $D_+^2 = -\partial_{++}$ and $D_-^2 = -\partial_{--}$ and anticommute $D_-D_+ + D_+D_- = 0$. The component fields $\phi, \lambda, \bar{\lambda}$ and $f$ are spacetime fields, and they depend only on $z^{\pm\pm}$. On the other hand, $(\Phi, W, \bar{W}, F)$ are the superfields whose first components are the component fields. $W$ and $\bar{W}$ are anticommuting superfields.

Computing the differential of each superfield, we have the following relations:

$$\begin{aligned} d\Phi &= V^{++}\partial_{++}\Phi + V^{--}\partial_{--}\Phi + \psi^{+}W + \psi^{-}\bar{W}\,, \\ dW &= V^{++}\partial_{++}W + V^{--}\partial_{--}W - \psi^{+}\partial_{++}\Phi + \psi^{-}F\,, \\ d\bar{W} &= V^{++}\partial_{++}\bar{W} + V^{--}\partial_{--}\bar{W} - \psi^{-}\partial_{--}\Phi - \psi^{+}F\,, \\ dF &= V^{++}\partial_{++}F + V^{--}\bar{\partial}_{--}F + \psi^{+}\partial_{++}\bar{W} - \psi^{-}\partial_{--}W\,, \end{aligned} \tag{30}$$

The last field $F$ is the auxiliary field, and therefore, it vanishes when the theory is on-shell. Before writing the rheonomic Lagrangian for the multiplet, we first write the equations of motion. If we set $F = 0$, then we see from the last equation that:

$$\partial_{++}\bar{W} = 0\,, \quad \partial_{--}W = 0\,. \tag{31}$$

They imply that the superfield $W$ is holomorphic $W = W(z)$ and the superfield $\bar{W}$ is anti-holomorphic. Then, we can write Equation (30) with these constraints:

$$\begin{aligned} d\Phi &= V^{++}\partial_{++}\Phi + V^{--}\partial_{--}\Phi + \psi^{+}W + \psi^{-}\bar{W}\,, \\ dW &= V^{++}\partial_{++}W - \psi^{+}\partial_{++}\Phi\,, \\ d\bar{W} &= V^{--}\partial_{--}\bar{W} - \psi^{-}\partial_{--}\Phi\,, \end{aligned} \tag{32}$$

The consistency of the last two equations ($d^2 = 0$) implies that $\partial_{++}\partial_{--}\Phi = 0$. Then, we get that the rheonomic Equation (30) is compatible with the set of the equations of motion:

$$\partial_{++}\partial_{--}\Phi = 0\,, \qquad \partial_{++}\bar{W} = 0\,, \qquad \partial_{--}W = 0\,, \qquad F = 0\,. \tag{33}$$

which are the free equations of the $D = 2$ multiplet. The Klein–Gordon equation in $D = 2$ implies that the solution $\Phi = \Phi_h(z) + \Phi_{\bar{h}}(\bar{z})$ is split into holomorphic and anti-holomorphic parts, and therefore, we get the on-shell matching of the degrees of freedom. In particular, we can write on-shell holomorphic and anti-holomorphic superfields:

$$\Phi_h(z) = \phi(z) + \lambda(z)\theta^{+}\,, \quad \Phi_{\bar{h}}(\bar{z}) = \phi(\bar{z}) + \bar{\lambda}(\bar{z})\theta^{-}\,, \tag{34}$$

factorizing into left- and right-movers.

Let us now write the Lagrangian. We introduce two additional superfields $\xi$ and $\bar{\xi}$. Then, we have [9–11]:

$$\begin{aligned} \mathcal{L}^{(2|0)} &= (\xi V^{++} + \bar{\xi}V^{--}) \wedge (d\Phi - \psi^{+}W - \psi^{-}\bar{W}) + \left(\xi\bar{\xi} + \tfrac{F^2}{2}\right) V^{++} \wedge V^{--} \\ &+ W dW \wedge V^{++} - \bar{W} d\bar{W} \wedge V^{--} - d\Phi \wedge (W\psi^{+} - \bar{W}\psi^{-}) - W\bar{W}\,\psi^{+} \wedge \psi^{-} \end{aligned} \tag{35}$$

The equations of motion are given by:

$$\begin{aligned} & V^{++} \wedge (d\Phi - \psi^{+}W - \psi^{-}\bar{W}) + \bar{\xi}V^{++} \wedge V^{--} = 0\,, \\ & V^{--} \wedge (d\Phi - \psi^{+}W - \psi^{-}\bar{W}) + \xi V^{++} \wedge V^{--} = 0\,, \\ & (\xi V^{++} + \bar{\xi}V^{--})\psi^{+} + 2dW \wedge V^{++} - W\psi^{+} \wedge \psi^{+} + d\Phi \wedge \psi^{+} - \bar{W}\psi^{+} \wedge \psi^{-} = 0\,, \\ & (\xi V^{++} + \bar{\xi}V^{--})\psi^{-} - 2d\bar{W} \wedge V^{--} + \bar{W}\psi^{-} \wedge \psi^{-} - d\Phi \wedge \psi^{-} + W\psi^{+} \wedge \psi^{-} = 0\,, \\ & d(\xi V^{++} + \bar{\xi}V^{--}) + dW\psi^{+} - d\bar{W}\psi^{-} = 0\,, \\ & F = 0\,. \end{aligned} \tag{36}$$

They imply the on-shell differentials (32), the equations of motion (33) and the relations:

$$\xi = \partial_{++}\Phi\,, \quad \bar{\xi} = -\partial_{--}\Phi\,. \tag{37}$$

expressing the additional auxiliary fields $\xi$ and $\bar{\xi}$ in terms of $\Phi$. It is easy to check that they are consistent: acting with $d$ in the third and in the fourth equations, and using the fifth equation, one gets a trivial consistency check; and in the same way for all the others.

The action is a $(2|0)$ superform, and it can be verified that it is closed by using only the algebraic equations of motion for $\xi$ and $\bar{\xi}$, which are solved in (37), and using the curvature parametrization $d\Phi, dW, d\bar{W}$ and $dF$ given in (30). Note that those equations are off-shell parametrizations of the curvatures, and therefore, they do not need the equations of motion of the Lagrangian (35).

### 4.1. Sigma Model on Supermanifolds

To check whether this action leads to the correct component action, we use the PCO $\mathbb{Y}^{(0|2)} = \theta^+\theta^-\delta(\psi^+)\delta(\psi^-)$. Then, we have[5]

$$\begin{aligned} S &= \int_{S\Sigma} \mathcal{L}^{(2|0)} \wedge \mathbb{Y}^{(0|2)} \\ &= \int d^2z \Big[ (\xi_0 dz^{++} + \bar{\xi}_0 dz^{--}) \wedge d\phi + \Big( \xi_0\bar{\xi}_0 + \frac{f^2}{2} dz^{++} \wedge dz^{--} \Big) \\ &\quad + \lambda d\lambda \wedge dz^{++} + \bar{\lambda} d\bar{\lambda} \wedge dz^{--} \Big] \end{aligned} \tag{38}$$

where $\xi_0$ and $\bar{\xi}_0$ are the first components of the superfields $\xi$ and $\bar{\xi}$. Eliminating $\xi_0$ and $\bar{\xi}_0$, one finds the usual equations of motion for the $D = 2$ free sigma model.

Choosing a different PCO of the form[6]:

$$\mathbb{Y}^{(0|2)} = V^{++}\delta'(\psi^+) \wedge V^{--}\delta'(\psi^-) , \tag{39}$$

which has again the correct picture number and is cohomologous to the previous one, leading to the superspace action (listing only the relevant terms):

$$\begin{aligned} S &= \int_{\mathcal{M}} [W\bar{W}\psi^+ \wedge \psi^- - d\Phi \wedge (W\psi^+ + \bar{W}\psi^-)] \wedge V^{++}\delta'(\psi^+) \wedge V^{--}\delta'(\psi^-) \\ &= \int_{\mathcal{M}} \Big( W\bar{W} - [(\iota_- d\phi)W + (\iota_+ d\phi)\bar{W}] \Big) V^{++} \wedge V^{--} \wedge \delta(\psi^+)\delta(\psi^-) . \end{aligned} \tag{40}$$

where $\iota_\pm$ are the derivatives with respect to $\psi^\pm$. The contractions give $\iota_+ d\Phi = D_+\Phi$ and $\iota_- d\Phi = \bar{D}_-\Phi$. Then, we get the superspace action:

$$S = \int [d^2z d^2\theta] \left( W\bar{W} - D_-\Phi W - D_+\Phi \bar{W} \right) . \tag{41}$$

The equations of motion are $W = D_+\Phi$ and $\bar{W} = D_-\Phi$. Hence, we obtain the usual $D = 2$ superspace free action in a flat background:

$$S = \int [d^2z d^2\theta] D_+\Phi D_-\Phi . \tag{42}$$

## 5. Geometry of $OSp(1|2)/SO(1,1)$

Let us consider the coset $OSp(1|2)/SO(1,1)$. The MC equations can be easily computed by using the notation $V^{\pm\pm}, \psi^\pm$ for the MC forms and $\nabla$ for the $SO(1,1)$ covariant derivate. The MC forms $V^{\pm\pm}$ have charge $\pm 2$, while $\psi^\pm$ have charge $\pm 1$. Then, we have:

$$\nabla V^{++} = \psi^+ \wedge \psi^+ , \quad \nabla V^{--} = \psi^- \wedge \psi^- , \quad \nabla \psi^+ = V^{++} \wedge \psi^- , \quad \nabla \psi^- = -V^{--} \wedge \psi^+ ,$$

5 We denote by $\int_{\mathcal{M}}$ the integral of an integral form on the supermanifold, by $\int [d^2z d^2\theta]$ the Berezin integral on the superspace and by $\int d^2z$ the usual integral on the reduced bosonic submanifold.

6 This form of the PCO recalls the string theory PCO $c\delta'(\gamma)$ where $c$ is the diffeomorphism ghost and $\gamma$ is the superghost.

Computing the Bianchi identities, we have:

$$\begin{aligned} \nabla^2 V^{\pm\pm} &= \pm V^{\pm\pm} \wedge R^{(2|0)}\,, \qquad \nabla^2 \psi^{\pm} = \pm \psi^{\pm} \wedge R^{(2|0)}\,, \\ R^{(2|0)} &= -V^{++} \wedge V^{--} + \psi^{+} \wedge \psi^{-}\,, \qquad \nabla R^{(2|0)} = 0\,, \end{aligned} \tag{43}$$

All the expressions have been constructed to respect the charge assignments. The superform $R^{(2|0)}$ is neutral and invariant.

The volume form is computed by observing that:

$$\mathrm{Vol}^{(2|2)} = V^{++} \wedge V^{--} \delta(\psi^{+})\delta(\psi^{-}) = \mathrm{Sdet}(E) d^2 z \delta^2(d\theta)\,. \tag{44}$$

where $\mathrm{Sdet}(E)$ is the Berezinian of the supervielbein $E$ of the super-coset manifold $OSp(1|2)/SO(1,1)$. It is SUSY invariant, and it is closed. This can be checked by observing that:

$$\begin{aligned} d\mathrm{Vol}^{(2|2)} = \nabla \mathrm{Vol}^{(2|2)} &= (\nabla V^{++} \wedge V^{--} - V^{++} \nabla V^{--})\, \delta(\psi^{+})\delta(\psi^{-}) \\ &+ V^{++} \wedge V^{--}\, (\nabla \delta(\psi^{+})\delta(\psi^{-}) - \delta(\psi^{+})\nabla \delta(\psi^{-})) \\ &+ (\psi^{+} \wedge \psi^{+}\ \wedge V^{--} - V^{++} \wedge \psi^{-} \wedge \psi^{-})\, \delta(\psi^{+})\delta(\psi^{-}) \\ &+ V^{++} \wedge V^{--} \Big( V^{++} \wedge \psi^{-} \delta'(\psi^{+})\delta(\psi^{-}) + \delta(\psi^{+}) V^{--} \wedge \psi^{+} \delta'(\psi^{-}) \Big) = 0 \end{aligned} \tag{45}$$

The first equality follows from the neutrality of the volume integral form $\mathrm{Vol}^{(2|2)}$, and therefore, we can use the covariant derivative instead of the differential $d$. The covariant differential $\nabla$ acts as derivative, and this leads to the last two lines. The third line cancels because of the Dirac delta functions multiplied by $\psi^{\pm}$, the fourth line vanishes since $V^{\pm\pm} \wedge V^{\pm\pm} = 0$.

The relevant set of pseudo-forms is contained in the rectangular diagram in Figure 1. The vertical arrows denote a PCO, which increases the picture number. There are additional sets outside the present rectangular set, but are unessential for the present discussion since they do not contain non-trivial cohomology classes (see [12]).

$$\begin{array}{ccccc} \Omega^{(0|0)} & \xrightarrow{\nabla} & \Omega^{(1|0)} & \xrightarrow{\nabla} & \Omega^{(2|0)} \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^{(0|1)} & \xrightarrow{\nabla} & \Omega^{(1|1)} & \xrightarrow{\nabla} & \Omega^{(2|1)} \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^{(0|2)} & \xrightarrow{\nabla} & \Omega^{(1|2)} & \xrightarrow{\nabla} & \Omega^{(2|2)} \end{array}$$

**Figure 1.** The pseudoform complexes.

Let us discuss the relevant cohomology spaces.

$$\begin{aligned} H^{(0|0)} &= \{1\}\,, \\ H^{(0|1)} &= \{V^{++} \wedge \delta'(\psi^{+}), V^{--}\delta'(\psi^{-})\}\,, \\ H^{(0|2)} &= \{V^{++} \wedge V^{--}\delta'(\psi^{+})\delta'(\psi^{-})\}\,, \\ H^{(2|0)} &= \{V^{++} \wedge V^{--} - \psi^{+} \wedge \psi^{-}\}\,, \\ H^{(2|1)} &= \{V^{++} \wedge \psi^{-}\delta(\psi^{+}), V^{--} \wedge \psi^{+}\delta(\psi^{-})\}\,, \\ H^{(2|2)} &= \{V^{++} \wedge V^{--}\delta(\psi^{+})\delta(\psi^{-})\}\,, \end{aligned} \tag{46}$$

It is easy to check the closure of all generators. In addition, all generators are neutral; for instance, in $V^{++} \wedge \delta'(\psi^{+})$, the charge $+2$ is compensated by the negative charge $-1$ of $\delta(\psi^{+})$ and the negative charge of the derivative of the delta form. It can be shown that:

$$\begin{aligned} H^{(2|0)} \wedge H^{(0|2)} &= (V^{++} \wedge V^{--} - \psi^{+} \wedge \psi^{-}) \wedge V^{++} \wedge V^{--}\delta'(\psi^{+})\delta'(\psi^{-}) \\ &\longrightarrow H^{(2|2)} = V^{++} \wedge V^{--}\delta(\psi^{+})\delta(\psi^{-}) \end{aligned} \tag{47}$$

This equation is rather suggestive. If we consider the cohomology class in $H^{(0|2)}$ as the total PCO $\mathbb{Y}^{(0|2)}$ and if we consider the cohomology class in $H^{(2|0)}$ as the Kähler form $K^{(2|0)}$ of our complex supermanifold, we find:

$$\mathbb{Y}^{(0|2)} \wedge R^{(2|0)} = \mathbb{Y}^{(0|2)} \wedge K^{(2|0)} = \mathrm{Vol}^{(2|2)}\,, \tag{48}$$

which is the super-Liouville form.

The PCO operator $\mathbb{Y}^{(0|2)}$ easily factorizes as $\mathbb{Y}^{(0|2)} = \mathbb{Y}^{(0|1)}_{+} \wedge \mathbb{Y}^{(0|1)}_{-}$ with $\mathbb{Y}^{(0|1)}_{\pm} = V^{\pm\pm} \wedge \delta'(\psi^{\pm})$. This factorization is very useful for the equations below.

Finally, we want to show that acting with the PCO $Z$, we can map the volume form $\mathrm{Vol}^{(2|2)}$ into the Kähler form $K^{(2|0)}$. For that, we define the PCOs:

$$Z^{(0|-1)}_{+} = [d, \Theta(\iota_{+})]\,, \qquad Z^{(0|-1)}_{-} = [d, \Theta(\iota_{-})]\,. \tag{49}$$

Acting with the first one on $\mathrm{Vol}^{(2|2)}$ (and using the fact that $d\mathrm{Vol}^{(2|2)} = 0$), we have:

$$\begin{aligned} Z^{(0|-1)}_{+}\Big(V^{++} \wedge V^{--}\delta(\psi^{+})\delta(\psi^{-})\Big) &= d\Big(\Theta(\iota_{+})V^{++} \wedge V^{--}\delta(\psi^{+})\delta(\psi^{-})\Big) \\ &= d\Big(V^{++} \wedge V^{--}\tfrac{1}{\psi^{+}}\delta(\psi^{-})\Big) \\ &= \psi^{+} \wedge V^{--}\delta(\psi^{-}) \in H^{(2|1)} \end{aligned} \tag{50}$$

It can be noticed that the final expression is chargeless; it is $d$-closed, and it is expressed in terms of supersymmetric invariant quantities. Furthermore, in the first step of the computation, we have used objects in the large Hilbert space, but the final result is again in the small Hilbert space (there is no inverse of the $\psi$'s)[7].

Let us act with the second PCO, $Z^{(0|-1)}_{-} = [d, \Theta(\iota_{-})]$. Again, we use the fact that the result of (50) is $d$-closed. We have:

$$\begin{aligned} Z^{(0|-1)}_{-}\Big(\psi^{+} \wedge V^{--}\delta(\psi^{-})\Big) &= d\Big(\Theta(\iota_{-})\psi^{+} \wedge V^{--}\delta(\psi^{-})\Big) \\ &= d\Big(\psi^{+} \wedge V^{--}\tfrac{1}{\psi^{-}}\Big) = V^{++} \wedge V^{--} - \psi^{+} \wedge \psi^{-} \in H^{(2|0)} \end{aligned} \tag{51}$$

Again, in the intermediate steps, we have expressions living in the large Hilbert space; however, the final result is in the small Hilbert space, and it is polynomial in the MC forms. This clearly shows how to act with the PCOs on the cohomology classes mapping from cohomology to cohomology. The same result is obtained by exchanging the two PCOs. The final result is:

$$Z^{(0|-1)}_{-} Z^{(0|-1)}_{+} \mathrm{Vol}^{(2|2)} = \frac{1}{4} R^{(2|0)} \tag{52}$$

mapping the volume form into the curvature of the manifold.

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7 Note the important point that expressions like $\frac{1}{\psi^{+}}\delta(\psi^{-})$ or $\frac{1}{\psi^{-}}\delta(\psi^{+})$ are well defined.

## 6. $D = 2$ Supergravity

As is well know, there are no dynamical gravitons and gravitinos in two dimensions; nonetheless, the geometric formulation of supergravity is interesting, and it is relevant in the present work. The definitions are:

$$\begin{aligned} & T^{\pm\pm} = \nabla V^{\pm\pm} \pm \frac{1}{2}\psi^{\pm} \wedge \psi^{\pm}\,, \\ & \rho^{\pm} = \nabla \psi^{\pm}\,, \\ & R = d\omega\,, \end{aligned} \tag{53}$$

where $\omega$ is the $SO(1,1)$ spin connection. These curvatures satisfy the following Bianchi identities:

$$\begin{aligned} & \nabla T^{\pm\pm} = \mp 2R \wedge V^{\pm\pm} \mp \rho^{\pm} \wedge \psi^{\pm} \\ & \nabla \rho^{\pm} = \mp R \wedge \psi^{\pm}\,, \\ & \nabla R = 0\,. \end{aligned} \tag{54}$$

The Bianchi identities can be solved by the following parametrization:

$$\begin{array}{rcl} T^{\pm\pm} & = & 0\,, \\ \rho^{\pm} & = & 4D_{\mp}E\, V^{++} \wedge V^{--} - 2E\, \psi^{\mp} \wedge V^{\pm\pm}\,, \\ R & = & -4(D_+ D_- E + E^2)\, V^{++} \wedge V^{--} \\ & & -2D_+ E\, V^{++} \wedge \psi^{-} + 2D_- E\, V^{--} \wedge \psi^{+} + E\, \psi^{+} \wedge \psi^{-}\,, \end{array} \tag{55}$$

where $E$ is a generic superfield $E(x,\theta) = E_0(x) + E_+(x)\theta^+ + E_-(x)\theta^- + E_1(x)\theta^+\theta^-$. There are no dynamical constraints on $E(x,\theta)$ since there are no equations of motion. Nonetheless, if we impose that:

$$D_+ E = D_- E = 0 \tag{56}$$

we immediately get $\partial_{++} E = \partial_{--} E = 0$, and therefore, $E = const$. If we set $E = \Lambda$, we get the well-known anti-de-Sitter solution:

$$\begin{array}{rcl} T^{\pm\pm} & = & 0\,, \\ \rho^{\pm} & = & -2\Lambda\, \psi^{\mp} \wedge V^{\pm\pm}\,, \\ R & = & -4\Lambda^2\, V^{++} \wedge V^{--} + \Lambda\, \psi^{+} \wedge \psi^{-}\,, \end{array} \tag{57}$$

describing the coset space $OSp(1|2)/SO(1,1)$.

Going back to a generic $E$, we consider the volume form:

$$\mathrm{Vol}^{(2|2)} = E\, V^{++} \wedge V^{--} \delta(\psi^+) \delta(\psi^-) \tag{58}$$

which is closed since it is a top integral form. Now, we act with the PCO $Z_+ = [d, \Theta(\iota_+)]$, and we get:

$$\begin{aligned} Z_+ \mathrm{Vol}^{(2|2)} & = d\Big(\Theta(\iota_+) \mathrm{Vol}^{(2|2)}\Big) = \Big(D_+ E\, V^{++} \wedge V^{--} - \tfrac{1}{2} E\, V^{--} \wedge \psi^+\Big) \delta(\psi^-) \\ & = \tfrac{1}{4} \rho^- \delta(\psi^-)\,. \end{aligned} \tag{59}$$

Notice that the Dirac deltas do not carry any charges, and the PCO $Z_+$ has a negative charge. Therefore, the result is consistent. In addition, the r.h.s. is closed, as can be easily verified by using the Bianchi identities:

$$\begin{aligned} \nabla\Big(\rho^- \delta(\psi^-)\Big) & = (\nabla \rho^-) \delta(\psi^-) - \rho^- \delta'(\psi^-) \wedge \nabla \psi^- \\ & = R \wedge \psi^- \delta(\psi^-) - \rho^- \delta'(\psi^-) \wedge \rho^- = 0 \end{aligned} \tag{60}$$

since $\rho^- \wedge \rho^- = 0$ and $\psi^- \delta(\psi^-) = 0$.

Let us now act with the second PCO $Z_-$. There are two ways to perform the computation: either using the complete expression given in the first line of (59) or using the Bianchi identities. With the second proposal, we observe:

$$\begin{aligned} Z_- Z_+ \mathrm{Vol}^{(2|2)} &= Z_-\left(\tfrac{1}{4}\rho^- \delta(\psi^-)\right) = \tfrac{1}{4}\nabla\left(\Theta(\iota_-)\rho^-\delta(\psi^-)\right) \\ &= \tfrac{1}{4}\nabla\left(\tfrac{\rho^-}{\psi^-}\right) = \tfrac{1}{4}R \end{aligned} \tag{61}$$

Notice that acting both with $Z_-$ and $Z_+$, the total charge is zero as for $R$. The result is closed, $dR = 0$, and it confirms the formula obtained for the curved rigid supermanifold (52). The result (61) is valid for any superfield $E$.

The PCOs $Y$ are defined as in the flat case:

$$Y^+ = V^{++}\delta'(\psi^+)\,, \quad Y^- = V^{--}\delta'(\psi^-)\,, \tag{62}$$

We can easily check their closure:

$$\begin{aligned} \nabla Y^+ &= \nabla V^{++}\delta'(\psi^+) + V^{++}\delta''(\psi^+)\nabla\psi^+ \\ &= \left(T^{++} + \frac{1}{2}\psi^+ \wedge \psi^+\right)\delta'(\psi^+) + V^{++}\delta''(\psi^+)\left(4D_\mp E\, V^{++}\wedge V^{--} - 2E\,\psi^- \wedge V^{++}\right) \\ &= T^{++}\delta'(\psi^+) \end{aligned} \tag{63}$$

Therefore, it is closed if $T^{++} = 0$. In the same way, we get for $Y^-$. Finally, we observe that:

$$Y^+ \wedge Y^- \wedge R = \mathrm{Vol}^{(2|2)} \tag{64}$$

This can also be obtained by observing that:

$$\begin{aligned} Z_+ Y^+ &= d\left(\Theta(\iota_+)V^{++}\delta'(\psi^+)\right) = d\left(\tfrac{V^{++}}{\psi^+\wedge\psi^+}\right) \\ &= \tfrac{1}{2} + 2\tfrac{V^{++}\wedge\rho^+}{\psi^+\wedge\psi^+\wedge\psi^+} = \tfrac{1}{2} \end{aligned} \tag{65}$$

since $V^{++} \wedge \rho^+ = 0$. In the same way, $Z_- Y^- = 1/2$. These equations are valid for any $E$. Using Equation (64), one can define an integral over the supermanifold:

$$\int_{\mathcal{S}\Sigma} R \wedge Y^+ \wedge Y^- = \int_{\mathcal{S}\Sigma} \mathrm{Vol}^{(2|2)} = \int_\Sigma D_+ D_- E\, V^{++} \wedge V^{--} \tag{66}$$

that might be interpreted as the Euler characteristic for supermanifolds.

To complete the program, one has to use the PCOs (62), for a generic background $E$ to rewrite the action (6) in that background. Choosing a different PCO gives an equivalent string sigma model with different manifest supersymmetry.

Finally, we would like to point out the relation between the PCO used in the action and the conventional PCO used for correlation computations in string theory [13]. The latter can be written as follows:

$$Y = c^{++}\delta'(\gamma^+)\,, \quad \bar{Y} = c^{--}\delta'(\gamma^-) \tag{67}$$

for the left- and right-moving sector, where $c^{\pm\pm}$ are Einstein's ghosts and $\gamma^\pm$ are the superghosts. They should be compared with (62). We further notice that BRSTtransformations of the $D = 2$ supervielbeins $V^{\pm\pm}, \psi^\pm$ are given by (see [14,15]):

$$QV^{\pm\pm} = dc^{\pm\pm} + \dots\,, \quad Q\psi^\pm = d\gamma^\pm + \dots \tag{68}$$

where the ellipsis denotes non-linear terms, and therefore, there should be a relation between the two types of PCO. We leave this to further investigations.

**Author Contributions:** All authors contributed equally to the work.

**Funding:** The paper is supported in part by: Fondi Ricerca Locale (ex 60%) Theoretical Physics.

**Acknowledgments:** The authors would like to acknowledge Leonardo Castellani and Pietro Fré for useful discussions and comments.

**Conflicts of Interest:** The authors declare no conflict of interest.

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