

# The Gravity of Light-Waves

Jan Willem van Holten 

Nikhef, Science Park 105, 1098 XG Amsterdam, The Netherlands; v.holten@nikhef.nl; Tel.: +31-20-592-5131

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**Abstract:** Light waves carry along their own gravitational field; for simple plain electromagnetic waves, the gravitational field takes the form of a  $pp$ -wave. I present the corresponding exact solution of the Einstein–Maxwell equations and discuss the dynamics of classical particles and quantum fields in this gravitational and electromagnetic background.

**Keywords:** general relativity; light waves; quantum fields

## 1. Setting the Stage

The gravitational properties of light waves have been studied extensively in the literature [1–12]. In this lecture, I first describe the exact solutions of Einstein–Maxwell equations presented e.g., in [3,6] and then proceed to discuss some applications building on [7].

The topic of this exposition concerns plain electromagnetic waves propagating in a fixed direction chosen to be the  $z$ -axis of the co-ordinate system. As they propagate at the universal speed  $c$ , taken to be unity:  $c = 1$  in natural units, it is useful to introduce light-cone co-ordinates  $u = t - z$ ,  $v = t + z$ . Then, the electromagnetic waves to be discussed are described by a transverse vector potential

$$A_i(u) = \int \frac{dk}{2\pi} (a_i(k) \sin ku + b_i(k) \cos ku), \quad i = (x, y). \quad (1)$$

This expression explicitly makes use of the superposition principle for electromagnetic fields, guaranteed in Minkowski space by the linearity of Maxwell’s equations and well-established experimentally. The corresponding Minkowskian energy-momentum tensor is

$$T_{\mu\nu} = F_{\mu\lambda} F_{\nu}{}^{\lambda} - \frac{1}{4} \eta_{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda}, \quad (2)$$

the only non-vanishing component of which in light-cone co-ordinates is

$$T_{uu} = \frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2). \quad (3)$$

Here, the components of the transverse electric and magnetic fields are expressed in terms of the vector potential (1) by

$$E_i(u) = -\varepsilon_{ij} B_j(u) = \partial_u A_i(u). \quad (4)$$

The same expression for light-waves also holds in general relativity, the corresponding solution of the Einstein equations being described by the special Brinkmann metric [13,14]

$$ds^2 = -dudv - \Phi(u, x, y) du^2 + dx^2 + dy^2. \quad (5)$$

For this class of metrics, the only non-vanishing components of the connection are

$$\Gamma_{uu}{}^v = \partial_u \Phi, \quad \Gamma_{iu}{}^v = 2\Gamma_{uu}{}^i = \partial_i \Phi, \quad (6)$$

and the complete Riemann tensor is given by the components

$$R_{uiuj} = -\frac{1}{2} \partial_i \partial_j \Phi. \tag{7}$$

As a result, the Ricci tensor is fully specified by

$$R_{uu} = -\frac{1}{2} \left( \partial_x^2 + \partial_y^2 \right) \Phi, \tag{8}$$

which matches the form of the energy-momentum tensor (3) and thus allows solutions of the Einstein equations specified by

$$\Phi = 2\pi G(x^2 + y^2) \left( \mathbf{E}^2 + \mathbf{B}^2 \right) (u) + \Phi_0(u, x^i), \tag{9}$$

with  $\Phi_0$  representing a free gravitational wave of plane-fronted or *pp*-type.

## 2. Geodesics

The motion of electrically neutral test particles in a light-wave (1) is described by the geodesics  $X^\mu(\tau)$  of the *pp*-wave space-time (5). They are found by solving the geodesic equation

$$\ddot{X}^\mu + \Gamma_{\lambda\nu}^\mu \dot{X}^\lambda \dot{X}^\nu = 0, \tag{10}$$

the overdot denoting a derivative w.r.t. proper time  $\tau$ . In a different context, using different co-ordinates this equation was considered in [15]; here, we follow the discussion of [6,7]. The equation for the geodesic light-cone co-ordinate  $U(\tau)$  is especially simple, as its momentum (representing a Killing vector) is conserved:

$$\dot{U} = \gamma = \text{constant}. \tag{11}$$

Another conservation law is found from the Hamiltonian constraint obtained by substitution of the proper time in the line element:

$$-1 = -\dot{U}\dot{V} - \Phi(U, X^i)\dot{U}^2 + \dot{X}^i{}^2 \Leftrightarrow \frac{1}{\gamma^2} = \frac{1 - \mathbf{v}^2}{(1 - v_z)^2} + \Phi, \tag{12}$$

where  $\mathbf{v} = d\mathbf{X}/dT$  is the velocity in the observer frame. Finally, using equation (11) to substitute  $U$  for  $\tau$ , the equations for the transverse co-ordinates become

$$\frac{d^2 X^i}{dU^2} + \frac{1}{2} \frac{\partial \Phi}{\partial X^i} = 0. \tag{13}$$

For quadratic *pp*-waves  $\Phi(U, X^i) = \kappa_{ij}(U) X^i X^j$ , this takes the form of a parametric oscillator equation

$$\frac{d^2 X^i}{dU^2} + \kappa_{ij}(U) X^j = 0. \tag{14}$$

For light-like geodesics, the equations are essentially the same, except that the Hamiltonian constraint is replaced by

$$\frac{1 - \mathbf{v}^2}{(1 - v_z)^2} + \Phi = 0. \tag{15}$$

Note that, in Minkowski space, where  $\Phi = 0$ , this reduces to  $\mathbf{v}^2 = c^2 = 1$ . These equations take an especially simple form for circularly polarized light waves sharply peaked around a central frequency

$$A_x(u) = \int \frac{dk}{2\pi} a(k) \cos ku, \quad A_y(u) = \int \frac{dk}{2\pi} a(k) \sin ku, \tag{16}$$

where the domain of  $a(k)$  is centered around the value  $k_0$  with width  $\Delta k$  and central amplitude  $a_0 = a(k_0)$ . Then,

$$2\pi G (\mathbf{E}^2 + \mathbf{B}^2) = \mu^2 \sim G \Delta k k_0^2 a_0^2, \tag{17}$$

is a constant; as a result,

$$\Phi = \mu^2 (x^2 + y^2) \tag{18}$$

with a constant coefficient  $\mu^2$ . Then, Equation (14) reduces to a simple harmonic oscillator equation with angular frequency  $\mu$  in the  $U$ -domain.

### 3. Field Theory

In the previous section, we studied the equation of motion of test particles, supposed to have negligible back reaction on the gravitational field described by the metric (5). Similarly, one can study the dynamics of fields in this background space-time in the limit in which the fields are weak enough that their gravitational back reaction can be neglected. First, we consider a scalar field  $\Psi(x)$  described by the Klein–Gordon equation

$$(-\square_{pp} + m^2) \Psi = 0, \quad \square_{pp} = -4\partial_u \partial_v + 4\Phi(u, x^i) \partial_v^2 + \partial_x^2 + \partial_y^2. \tag{19}$$

It is convenient to consider the Fourier expansion w.r.t. the light-cone variables  $(u, v)$ :

$$\Psi(u, v, x^i) = \frac{1}{2\pi} \int dsdq \psi(s, q, x^i) e^{-i(su+qv)}. \tag{20}$$

Note that

$$su + qv = Et - pz, \quad E = s + q, \quad p = s - q. \tag{21}$$

Then, the amplitudes  $\psi$  satisfy the equation

$$[\partial_x^2 + \partial_y^2 + 4sq - 4q^2\Phi(-i\partial_s, x^i) - m^2] \psi = 0. \tag{22}$$

This equation can be solved explicitly for the circularly polarized wave packets which lead to the simple quadratic amplitude (18). Then,

$$(4sq - m^2) \psi = (-\partial_x^2 - \partial_y^2 + 4\mu^2 q^2 (x^2 + y^2)) \psi. \tag{23}$$

The right-hand side describes a couple of quantum oscillators with frequency  $\omega = 2\mu|q|$  possessing an eigenvalue spectrum

$$2\mu|q| (n_x + n_y + 1) \equiv 4\sigma|q|, \quad n_i = 0, 1, 2, \dots \tag{24}$$

Thus, Equation (23) reduces to

$$4sq - 4\sigma|q| = m^2 \text{ or } \begin{cases} (E - \sigma)^2 = (p - \sigma)^2 + m^2, & q > 0, \\ (E + \sigma)^2 = (p + \sigma)^2 + m^2, & q < 0. \end{cases} \tag{25}$$

The final result for the scalar field then becomes

$$\Psi(u, v, x^i) = \frac{1}{2\pi} \int_0^\infty \frac{dq}{\sqrt{q}} \sum_{n_i=0}^\infty \left( a_{n_i}(q) e^{-iqv - i\left(\frac{m^2}{4q} + \sigma\right)u} + a_{n_i}^*(q) e^{iqv + i\left(\frac{m^2}{4q} + \sigma\right)u} \right) \times \sqrt{\frac{2\mu}{\pi}} \prod_{j=x,y} \left[ \frac{H_{n_j}(\xi_j)}{\sqrt{2^{n_j} n_j!}} e^{-\xi_j^2} \right], \quad \xi_j = \sqrt{2\mu q} x_j. \tag{26}$$

Obviously, in a second-quantized context for this theory, the amplitudes  $(a_n, a_n^*)$  are to be interpreted as annihilation- and creation-operators [7].

#### 4. Electromagnetic Fluctuations in a Light-Wave Background

On top of an electromagnetic wave described by Equation (1), there can be fluctuations of the electromagnetic field. The general form of the Maxwell field then is of the form

$$A_\mu(u, v, x^i) = \delta_\mu^i A_i^{wave}(u) + a_\mu(u, v, x^i). \tag{27}$$

Because of the linearity of Maxwell’s equations, the field equations for the wave background and the fluctuations separate. The fluctuating field equations in the gravitational  $pp$ -wave background are derived from the action

$$S = \int dudvdxdy \left[ (\partial_u a_v - \partial_v a_u)^2 + (\partial_u a_i - \partial_i a_u) (\partial_v a_i - \partial_i a_v) - \Phi (\partial_v a_i - \partial_i a_v)^2 - \frac{1}{8} (\partial_i a_j - \partial_j a_i)^2 \right], \tag{28}$$

and read

$$\begin{aligned} \frac{\delta S}{\delta a_u} &= 4\partial_v \partial_u a_v - \Delta_\perp a_v - 2\partial_v \left( \partial_v a_u + \partial_u a_v - \frac{1}{2} \partial_i a_i \right) = 0, \\ \frac{\delta S}{\delta a_v} &= 4\partial_v \partial_u a_u - \Delta_\perp a_u - 2\partial_u \left( \partial_v a_u + \partial_u a_v - \frac{1}{2} \partial_i a_i \right) + 2\partial_i [\Phi (\partial_i a_v - \partial_v a_i)] = 0, \\ \frac{\delta S}{\delta a_i} &= -2\partial_v \partial_u a_i + \frac{1}{2} \Delta_\perp a_i + \partial_i \left( \partial_v a_u + \partial_u a_v - \frac{1}{2} \partial_j a_j \right) - 2\partial_v [\Phi (\partial_i a_v - \partial_v a_i)] = 0, \end{aligned} \tag{29}$$

where  $\Delta_\perp = \partial_x^2 + \partial_y^2$ . As the fluctuating field equations possess their own gauge invariance they can be restricted without loss of generality by the constraint

$$\partial_v a_u + \partial_u a_v - \frac{1}{2} \partial_i a_i = 0. \tag{30}$$

However, this does not yet exhaust the freedom to make gauge transformations, as the condition (30) is repected by special gauge transformations

$$a'_\mu = a_\mu + \partial_\mu \alpha, \quad \text{with} \quad (4\partial_u \partial_v - \Delta_\perp) \alpha = 0. \tag{31}$$

As can be seen from the first Equation (29), these transformations can be used to eliminate the component  $a_v$  by taking

$$\partial_v \alpha = -a_v \quad \Rightarrow \quad a'_v = 0. \tag{32}$$

We are then left with a fluctuating field component  $a_u$  restricted by (30):

$$\partial_v a_u = \frac{1}{2} \partial_i a_i, \quad (33)$$

implying  $a_u$  to satisfy the Gauss law constraint

$$\partial_i [\partial_i a_u - 2(\partial_u - \Phi \partial_v) a_i] = 0. \quad (34)$$

The only remaining dynamical degrees of freedom are now the transverse components  $a_i$  which are solutions of the Klein–Gordon type of equations

$$\left( -2\partial_u \partial_v + 2\Phi \partial_v^2 + \frac{1}{2} \Delta_{\perp} \right) a_i = 0. \quad (35)$$

For  $pp$ -backgrounds of the special form (18), these solutions take the form (26) with  $m^2 = 0$ .

In the full theory, the gravitational field must also fluctuate in a corresponding fashion. In the limit where the fluctuations are due to irreducible quantum noise, a corresponding quantum effect must be present in the space-time curvature. In view of the result (9) for the photon fluctuations in the light-beam itself, these are expected to take the form of associated spin-0 graviton excitations.

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