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Proposal for an Electromagnetic Mass Formula for the X17 Particle

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Abstract: Recent observations of anomalous angular correlations of electron–positron pairs in several nuclear reactions have indicated the existence of a hypothetical neutral boson of rest mass ~17 MeV/c^2, called the X17 particle. Similarly, one has interpreted an independent set of experiments on photon pair spectra around the invariant mass ~38 MeV/c^2, by assuming the existence of the so-called E38 particle. In the present paper, we derive analytical mass formulas for the X17 particle and the E38 particle, on the basis of quantum electrodynamics. We shall use the exact solutions of the Dirac equation of the joint system of a charged particle and plane waves of the quantized electromagnetic radiation. When these solutions are applied to a proton, they lead to dressed radiation quanta with a rest mass of 17.0087 MeV/c^2, which may be identified with the X17 vector bosons. A similar consideration, applied to the udd quarks of the neutron, yields dressed quanta, whose mass equals 37.9938 MeV/c^2, corresponding to the E38 particle. These formulas, besides the Sommerfeld fine structure constant and the masses of the nucleons, do not contain any adjustable parameters. The present analysis also delivers the value 0.846299 fm for the proton radius.

Keywords: hypothetical X17 particle; hypothetical E38 particle; quantized Volkov states; plasmons; Sommerfeld fine structure constant; Compton wavelength of the proton; proton size

1. Introduction

The recent experimental results on anomalous internal pair creation in the nuclear reactions ^{7}Li(p,e$^-$e$^+$)^{8}Be and ^{3}H(p,e$^-$e$^+$)^{4}He have given firm indications of the existence of a light, neutral boson [1–5]. According to these experiments and their subsequent refinements, performed by ATOMKI scientists, this type of anomalous angular correlation of electron–positron pairs has been called an ATOMKI nuclear anomaly. The assumption that a new hypothetical particle, called the X17 particle, mediates the energy and momentum from the excited nuclei to the internally created pairs can well explain all the characteristics of such anomalous angular correlations [1–5]. This particle seems to be a vector boson with the invariant mass 17.01(16) MeV/c^2 [2].

Among the by now existing theoretical interpretations of the ATOMKI anomaly, we mention the fifth-force interpretation [6,7] and an extension of the standard model, which explains a potential presence of spin-1 gauge bosons with a mass of about 17 MeV [8]. In contrast to these, in the meantime, it has been argued [9] that the existence of a hypothetical X17 particle is not the only possible explanation for the observed anomalous angular correlations, if one takes into account higher-order processes along the usual calculations. On the other hand, it is an open question whether an angular peak calculated this way [9] represents merely a special qualitative coincidence with experience [1,2], or the validity of this reasoning goes beyond the discussed examples. In another independent set of experiments [10], an anomalous angular correlation of photon pairs has been observed at the invariant mass of about 38 MeV/c^2; this phenomenon can also be explained by the existence of a new hypothetical particle, called the E38 particle [10]. A nice and coherent theoretical frame for explaining the possibility of the existence of both the X17 particle and

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the E38 particle has been worked out by Wong [11–13]. According to this scheme, a light quark and antiquark are electromagnetically bound to form a 1 + 1 dimensional quantum electrodynamical (QED) meson, which is "extrapolated" to a physical QED meson. Wong has shown that if the flux-tube radius is properly chosen in 3 + 1 dimensions, then the calculated masses of the obtained isoscalar and isovector QED mesons are quite close to those of the hypothetical X17 and E38 particles [13]. In the present paper, we shall not rely on such a sophisticated background, but *a priori* use the original QED formalism in 3 + 1 dimensions, and consider the interaction of free charged particles, minimally coupled to the quantized electromagnetic radiation [14].

In the following stages, we shall use the exact solutions of the Dirac equation of the joint system of a charged particle and plane waves of the quantized electromagnetic radiation. These type of solutions, which may be called quantized Volkov states, have long been known [15–18], and occasionally used for the non-perturbative description of high-order multiphoton processes, like high-harmonic generation in a strong laser field in a nonlinear Compton scattering on electrons [18]. Concerning the latter subject, we just mention that high-harmonic production on noble gas atoms or solids, etc., induced by strong laser fields, has recently received much attention because the generation of attosecond radiation pulses is based on this process [19]. The theoretical description of these phenomena relies on various non-perturbative treatments, though, mostly in the external field approximation, and in the non-relativistic regime, which also incorporate the interaction with the electromagnetic radiation "up to infinite order". The interested reader may find the basic references concerning the classical Volkov states [20] in [21].

In Sections 2 and 3, it will be shown that when the quantized Volkov solutions are applied to a proton, they lead to a rest mass of value 17.0087 MeV$/c^2$ for the dressed radiation quanta. These transverse quanta may perhaps be identified with the excitations of the X17 vector bosons. We note that the analysis to be presented at the end of Section 3 naturally interrelates the proton size to its Compton wavelengths, and delivers a proton radius of 0.846299 fm, which is quite close to recent experimental values (see considerations at the end of Section 3). In Section 4, we give a simple plasmon interpretation of the derived dispersion relation (effective mass). This gives us a short-cut, in addition to showing that a similar consideration, also based on the quantized Volkov states, but now applied to the udd quarks of the neutron, yields dressed radiation quanta with a non-vanishing rest mass. The derived mass equals 37.9938 MeV$/c^2$, which may correspond to the hypotetical E38 particle. We emphasize that in both cases we express the mass by simple analytical formulas. We also point out that, besides the Sommerfeld fine structure constant and the proton or neutron mass, our derived mass formulas do not contain any adjustable parameters, but merely some numerical statistical factors. In Section 5, a brief summary closes the paper.

2. The Quantized Volkov States of the System of a Charged Particle and Electromagnetic Plane Waves

The Dirac equation of the joint system of a charged particle and plane waves of the quantized electromagnetic radiation reads $H\Psi = i\hbar\partial\Psi/\partial t$, where H is the Hamiltonian,

$$H = c\boldsymbol{\alpha}\cdot(\hat{\boldsymbol{p}} - e\hat{\boldsymbol{A}}/c) + \beta mc^2 + H_{rad}, H_e = c\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} + \beta mc^2, H_{\rm int} = -e\boldsymbol{\alpha}\cdot\boldsymbol{A}, \tag{1}$$

where e and m are the charge and rest mass of the particle, respectively, c is the velocity of light in a vacuum. $\hat{\boldsymbol{p}} = -i\hbar\nabla = -i\hbar\partial/\partial\boldsymbol{r}$ is the momentum operator of the particle in coordinate representation, and $\hbar = h/2\pi$ is Planck's constant divided by 2π. The vector potential $\hat{\boldsymbol{A}}(\boldsymbol{r})$ of the radiation field represents two co-propagating linearly polarized plane waves,

$$\hat{\boldsymbol{A}}(\boldsymbol{r}) = c\sqrt{\frac{2\pi\hbar}{\omega V}}[\boldsymbol{\varepsilon}_1(a_1 e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + a_1^+ e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}) + \boldsymbol{\varepsilon}_2(a_2 e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + a_2^+ e^{-i\boldsymbol{k}\cdot\boldsymbol{r}})], \omega = ck_0 = c|\boldsymbol{k}|, \tag{2}$$

where the two polarization vectors and the propagation vector, $(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{k})$, form a right orthogonal system, and V is the quantization volume. The meaning of other notations and conventions [22] used by us in the present paper are summarized in the footnote [1] below. We note that the propagation direction will be taken as the z-direction, when we occasionally write out the Cartesian components of vectors. The quantized amplitudes a_1, a_2 and a_1^+, a_2^+ are the photon absorption and emission operators, respectively, which satisfy the usual commutation rules $[a_1, a_1^+] = 1$, $[a_2, a_2^+] = 1$. All the other commutators of a_1, a_2 and a_1^+, a_2^+ are zero. H_{rad} denotes the sum of the Hamiltonians of the two quantized oscillators with the same frequency

$$H_{rad} = \hbar\omega(a_1^+ a_1 + \frac{1}{2}) + \hbar\omega(a_2^+ a_2 + \frac{1}{2}). \tag{3}$$

Since the Hamiltonian does not depend on time, the system has stationary states, $\Psi = \exp(-iEt/\hbar)\psi$, and the corresponding energy eigenvalue equation is

$$[c\boldsymbol{\alpha} \cdot (\hat{\boldsymbol{p}} - e\boldsymbol{A}(\boldsymbol{r})/c) + \beta mc^2 + H_{rad}]\psi_{E,\boldsymbol{P}} = E\psi_{E,\boldsymbol{P}}. \tag{4}$$

Moreover, because $[H, \hat{\boldsymbol{p}} + \hat{\boldsymbol{p}}_{rad}] = 0$, we can choose $\psi_{E,\boldsymbol{P}}$ as a simultaneous eigenstate of the total energy $H = H_e + H_{\text{int}} + H_{rad}$ and the sum of the momentum $\hat{\boldsymbol{p}}$ of the particle and that of the field components $\hat{\boldsymbol{p}}_{rad} = \hbar\boldsymbol{k}(a_1^+ a_1 + a_2^+ a_2 + 1)$,

$$[-i\hbar\nabla + \hbar\boldsymbol{k}(a_1^+ a_1 + a_2^+ a_2 + 1)]\psi_{E,\boldsymbol{P}} = \boldsymbol{P}\psi_{E,\boldsymbol{P}}. \tag{5}$$

According to Equation (5), the particle's momentum operator $\hat{\boldsymbol{p}}$ in Equation (4) can be replaced by $\boldsymbol{P} - \hat{\boldsymbol{p}}_{rad}$. Furthermore, from Equation (4) we see that $\psi_{E,\boldsymbol{P}}$ can be written as

$$\psi_{E,\boldsymbol{P}}(\boldsymbol{r}) = \exp[(i/\hbar)\boldsymbol{P} \cdot \boldsymbol{r} - i\boldsymbol{k} \cdot \boldsymbol{r}(a_1^+ a_1 + a_2^+ a_2 + 1)]\Phi_{E,\boldsymbol{P}}, \tag{6}$$

where $\Phi_{E,\boldsymbol{P}}$ does not depend on the position of the charged particle (but it still depends on the spinor variables and on the field variables). The transformation generated by $\exp[-i\boldsymbol{k} \cdot \boldsymbol{r}(a_1^+ a_1 + a_2^+ a_2 + 1)]$ eliminates the $\boldsymbol{r}$—dependence of the vector potential,

$$\exp[+i\boldsymbol{k} \cdot \boldsymbol{r}(a_1^+ a_1 + a_2^+ a_2 + 1)]\hat{\boldsymbol{A}}(\boldsymbol{r}) \exp[-i\boldsymbol{k} \cdot \boldsymbol{r}(a_1^+ a_1 + a_2^+ a_2 + 1)] = \hat{\boldsymbol{A}}(0). \tag{7}$$

Thus, we have from (4), (6) and (7)

$$\left\{c\boldsymbol{\alpha} \cdot [\boldsymbol{P} - \hbar\boldsymbol{k}(a_1^+ a_1 + a_2^+ a_2 + 1) - e\hat{\boldsymbol{A}}(0)/c] + \beta mc^2 + \hbar\omega(a_1^+ a_1 + a_2^+ a_2 + 1)\right\}\Phi_{E,\boldsymbol{P}} = E\Phi_{E,\boldsymbol{P}} \tag{8}$$

In order to have a manifestly covariant form of Equation (8), we multiply from the left with the $\beta = \gamma^0$ matrix (and use the conventions summarized in footnote 1), yielding

$$\{\not{Q} - \not{k}(a_1^+ a_1 + a_2^+ a_2 + 1) - g[\not{\varepsilon}_1(a_1 + a_1^+) + \not{\varepsilon}_2(a_2 + a_2^+)] - \kappa\}\Phi_Q = 0, \tag{9}$$

where we have introduced the notations

$$Q_0 = E/\hbar c, \boldsymbol{Q} = \boldsymbol{P}/\hbar, Q^\mu = (Q_0, \boldsymbol{Q}), k^\mu = (k_0, \boldsymbol{k}), \varepsilon_{1,2}^\mu = (0, \boldsymbol{\varepsilon}_{1,2}), \tag{10}$$

$$\kappa = \frac{mc}{\hbar}, g := \frac{e}{\hbar}\sqrt{\frac{2\pi\hbar}{\omega V}} = \frac{e}{\sqrt{\hbar c}}\sqrt{\frac{\lambda}{V}}, \lambda = \frac{2\pi c}{\omega}. \tag{11}$$

The normalized four momentum $Q^\mu = (Q_0, \boldsymbol{Q})$, introduced in (10) is the total four-momentum of the system "charged particle plus the monochromatic electromagnetic plane wave components", the latter part represented by the vector potential (2).

In [16,18], we proved that the solution to (9) has to have the following structure

$$\Phi_Q = \left\{1 + \frac{g}{2(k \cdot Q)}\not{k}[\not{\varepsilon}_1(a_1 + a_1^+) + \not{\varepsilon}_2(a_2 + a_2^+)]\right\}X_Q. \tag{12}$$

By putting this into Equation (9), and multiplying the equation from the left with the inverse

$$\left\{1-\frac{g}{2(k\cdot Q)}\not{k}[\not{\epsilon}_1(a_1+a_1^+)+\not{\epsilon}_2(a_2+a_2^+)]\right\}, \tag{13}$$

we receive the equation for the new state X_Q,

$$\left\{\not{Q}-\not{k}\begin{bmatrix} (1+b)(a_1^+a_1+a_2^+a_2+1)- \\ -(g/k\cdot Q)[Q_x(a_1+a_1^+)+Q_y(a_2+a_2^+)]+ \\ +\frac{1}{2}b(a_1^2+a_1^{+2}+a_2^2+a_2^{+2}) \end{bmatrix}-\kappa\right\}X_Q=0, \tag{14}$$

where

$$b:=\frac{g^2}{k\cdot Q},\ g=\frac{e}{\sqrt{\hbar c}}\sqrt{\frac{2\pi}{k_0V}}. \tag{15}$$

In Equation (15), we introduced the parameter b, which is proportional with the squared coupling strength g, defined in Equation (11). Equation (14) shows that, with the help of the Ansatz (12), we managed to ensure that all the boson operators have a common matrix coefficient, and now the diagonalization of the boson part can be carried out.

As a first step towards the diagonalization of the expression in the bracket, on the left-hand side of Equation (14), we introduce the unitary operators

$$C_1(\theta):=\exp[-\frac{1}{2}\theta(a_1^{+2}-a_1^2)],\ C_2(\theta):=\exp[-\frac{1}{2}\theta(a_2^{+2}-a_2^2)], \tag{16}$$

with the help of which the quadratic off-diagonal terms can be eliminated. The effect of the Bogolyubov transformation, generated by the operator $C_1(\theta)$, is expressed by the formulas

$$a_1\to C_1^{-1}(\theta)a_1C_1(\theta)=a_1\cosh\theta-a_1^+\mathrm{sinh}\theta,$$

$$a_1^+\to C_1^{-1}(\theta)a_1^+C_1(\theta)=a_1^+\cosh\theta-a_1\mathrm{sinh}\theta. \tag{17}$$

Analogous formulas are valid for the transformation of the quantized amplitudes a_2 and a_2^+. If we choose the by now unknown common parameter θ to satisfy

$$b\cosh 2\theta=(1+b)\mathrm{sinh}2\theta,\ \mathrm{tanh}2\theta=\frac{b}{1+b}, \tag{18}$$

then the sum of the coefficients of the resulting off-diagonal quadratic terms in the bracket in (14) is zero,

$$\left\{\sqrt{1+2b}(a_1^+a_1+a_2^+a_2+1)-(g/k\cdot Q)[Q_x(a_1+a_1^+)+Q_y(a_2+a_2^+)]\mathrm{e}^{-\theta}\right\}C_2^{-1}C_1^{-1}X_Q. \tag{19}$$

The remaining interaction terms in (19), which are linear in the boson amplitudes, $\propto g(a_1+a_1)$ and $\propto g(a_2+a_2^+)$, can be eliminated with the help of the displacement operators

$$D_1(\tau_1):=\exp[\tau_1(a_1^+-a_1)],\ D_2(\tau_2):=\exp[\tau_2(a_2^+-a_2)]. \tag{20}$$

We use the well-known displacement properties [23]

$$D_1^{-1}(\tau_1)a_1D_1(\tau_1)=a_1+\tau_1,\ D_2^{-1}(\tau_2)a_2D_2(\tau_2)=a_2+\tau_2, \tag{21}$$

and choose the values of the by now unknown parameters τ_1 and τ_2 as

$$Q_x(g/k\cdot Q)\mathrm{e}^{-\theta}=\tau_1\sqrt{1+2b},\ Q_y(g/k\cdot Q)\mathrm{e}^{-\theta}=\tau_2\sqrt{1+2b}, \tag{22}$$

then the linear off-diagonal terms can also be eliminated. Therefore, from (14), with the help of the unitary transformations (16), (18), (20) and (22), we receive the following diagonal equation for the transformed state

$$[\not{Q} - \not{k}\sqrt{1+2b}(a_1^+ a_1 + a_2^+ a_2 + 1 - \tau_1^2 - \tau_2^2) - \kappa] D_2^{-1} D_1^{-1} C_2^{-1} C_1^{-1} X_Q = 0. \tag{23}$$

Concerning the boson part, the solutions to (23) can immediately be taken as photon number eigenstates,

$$D_2^{-1} D_1^{-1} C_2^{-1} C_1^{-1} X_Q = |n_1, n_2\rangle u_Q, \quad X_Q = C_1 C_2 D_1 D_2 |n_1, n_2\rangle u_Q \quad (n_1, n_2 = 0, 1, 2, \ldots), \tag{24}$$

where $|n_1, n_2\rangle \equiv |n_1\rangle_1 |n_2\rangle_2$ are the number eigenstates in the product Hilbert space $\mathcal{H}_\gamma$ of the two orthogonal modes. At the same time, the bispinor u_Q satisfies the equation

$$[\not{Q} - \not{k}\sqrt{1+2b}(n_1 + n_2 + 1 - \tau_1^2 - \tau_2^2) - \kappa] u_Q = 0, \tag{25}$$

where the parameter b was defined in (15), and τ_1, τ_2 are given by (22). We write the bispinor Equation (25) in a short-hand form,

$$[\not{Q} - \not{k} g_Q(n) - \kappa] u_Q = 0, \; g_Q(n) := \sqrt{1+2b}(n + 1 - \tau_1^2 - \tau_2^2), \; n := n_1 + n_2. \tag{26}$$

The first equation in Equation (26) is formally a usual bispinor equation for a free particle of four momentum $p = Q - k g_Q(n)$. Moreover, since $k^2 = 0$, and $g_Q(n)$ depends on Q only from the combination $k \cdot Q$ (see the definition of the parameter $b = g^2/k \cdot Q$ in Equation (15)), we have $k \cdot Q = k \cdot p$, and hence $g_Q(n) = g_p(n)$. Accordingly, Q can be expressed in terms of p as $Q = p + k g_p(n)$, where p is on the free mass-shell, i.e., $p^2 = \kappa^2$. The latter statement is a consequence of the well-known bispinor equation $(\not{p} - \kappa) u_p = 0$. Collecting the results in (6), (12), and (24), the exact solution of the eigenvalue Equation (4) reads

$$\begin{aligned} \psi_{E,\boldsymbol{p}} = \exp[-i\boldsymbol{k} \cdot \boldsymbol{r}(a_1^+ a_1 + a_2^+ a_2 + 1)] \Big\{ 1 + \tfrac{g}{2(k \cdot p)} \not{k} [\not{\epsilon}_1 (a_1 + a_1^+) + \not{\epsilon}_2 (a_2 + a_2^+)] \Big\} \cdot \\ \times C_1(\theta) C_2(\theta) D_1(\tau_1) D_2(\tau_2) |n_1, n_2\rangle u_p \exp\{ i[\boldsymbol{p} + g_p(n)\boldsymbol{k}] \cdot \boldsymbol{r} \} \end{aligned} \tag{27}$$

As has been shown in the considerations following Equation (26), the total four-momentum Q^μ is the sum of the parameter p^μ and $k^\mu g_p(n)$, where $k^2 = 0$ and p^μ are on the free mass-shell ($p^2 = \kappa^2$). Of course, if the coupling constant goes to zero ($g \to 0$), the joint state of the system becomes a simple product state $|p\rangle u_p |n_1, n_2\rangle$ in the Hilbert space $\mathcal{H}_e \otimes \mathcal{H}_\gamma$ of the charge and the radiation. Since we are dealing with an interacting system, of course, the total energy and momentum have a more complex structure than that of the unperturbed system. According to (26), the total energy Q_0 and momentum $\boldsymbol{Q}$ can be written as

$$Q_0 = \overline{p}_0 + \overline{k}_0(n_1 + n_2 + 1), \quad \overline{p}_0 = p_0 - k_0 \frac{g^2}{(k \cdot p)^2}(p_x^2 + p_y^2), \quad \overline{k}_0 = k_0\sqrt{1+2b}, \tag{28}$$

$$\boldsymbol{Q} = \bar{\boldsymbol{p}} + \boldsymbol{k}(n_1 + n_2 + 1), \; \bar{\boldsymbol{p}} = \boldsymbol{p} - \boldsymbol{k} \frac{g^2}{(k \cdot p)} \left[\frac{p_x^2 + p_y^2}{(k \cdot p)} - (n_1 + n_2 + 1) \right]. \tag{29}$$

Equations (28) and (29) can be understood so that the dressing of the electromagnetic radiation, due to the interaction with a real charged particle, results in a modified dispersion relation, which is shown by the square root in the last equation in (28)

$$\overline{k}_0^2 - \left|\boldsymbol{k}\right|^2 = 2bk_0^2, \quad b = \frac{g^2}{k \cdot p}, \quad 2bk_0^2 = \frac{e^2}{\hbar c} \frac{4\pi}{V} \frac{1}{p_0 - p_z}, \quad p_0 = \sqrt{\boldsymbol{p}^2 + \kappa^2}, \tag{30}$$

where we have taken into account (15). In Equation (30), the quantity $2bk_0^2$ is an effective mass term for the dressed radiation at a given momentum parameter. Notice that $2bk_0^2$ does not depend on the frequency; thus, the final dispersion relation, which we are to calculate in Session 3, will be valid for any frequencies. In the next session, we will sum up this expression with respect to the Lorentz invariant density of states, so that we receive the final analytic formula for the dispersion of the dressed electromagnetic radiation.

At the end of the present section, we note that the exact solutions (27) form an orthogonal and complete set (together with the associated negaton states [24]) in the Hilbert space $\mathcal{H}_e \otimes \mathcal{H}_\gamma$ of the system under discussion. In the following discussions, these properties will not be used; nevertheless, with the detailed derivation of the exact solution (27) shown above, we wish to demonstrate that we have not applied any approximation in obtaining the mass term, displayed in the third equation of (30). In the next session, we proceed to analyse further this term $2bk_0^2$ in (30), from which the total effective mass will be derived.

3. Derivation of an Analytic Formula for the Rest Mass of the Hypothetical X17 Particle

In the present section, we shall calculate the contribution of the mass term $2bk_0^2$ of the dressed radiation (see the first equation in (30)), by summing up over the particle parameters within the Fermi sphere at zero temperature. This summation over the parameters $\boldsymbol{p} = (2\pi/L)(n_x, n_y, n_z)$, where $L^3 = V$ is the quantization volume, and $n_{x,y,z} = 0, \pm 1, \pm 2, \ldots$, can be done in the usual way, by going over to the (Lorentz invariant) integral,

$$2\sum \Delta n_x \Delta n_y \Delta n_z \frac{\kappa}{p_0} k_0^2 2\beta = \frac{e^2}{\hbar c} 4\pi \frac{\kappa^3}{(2\pi)^3} \frac{1}{\kappa} 2 \times \int \frac{d^3 y}{y_0 (y_0 - y_z)}, \quad \boldsymbol{y} := \boldsymbol{p}/\kappa. \tag{31}$$

In Equation (31), the factor 2 comes from the two spin states, and we have introduced the dimensionless momentum parameter $\boldsymbol{y}$, in terms of which $y_0 = \sqrt{1+\boldsymbol{y}^2}$ (henceforth, the modulus $|\boldsymbol{y}|$ will simply be denoted by $y := |\boldsymbol{y}|$). Having performed the angular integrals in momentum space, we have

$$\int \frac{d^3 y}{y_0(y_0 - y_z)} = 2\pi \int_0^{y_1} dy \frac{y}{\sqrt{1+y^2}} \log \frac{\sqrt{1+y^2}+y}{\sqrt{1+y^2}-y},$$

where y_1 denotes the dimensionless Fermi momentum. By introducing the new integration variable t, by the relation $y = \sinh t$, the radial integral can be easily performed, so we have

$$\int \frac{d^3 y}{y_0(y_0 - y_z)} = 2\pi \times 2 \times \{ \sqrt{1+y_1^2} \log(\sqrt{1+y_1^2} + y_1) - y_1 \}. \tag{32}$$

By taking the explicit result for the integral in (32), the sum (31) over the particle parameters, then becomes

$$\hbar^2 c^2 \times 2\sum \Delta n_x \Delta n_y \Delta n_z \frac{\kappa}{p_0} k_0^2 2\beta = \frac{1}{\pi} \frac{e^2}{\hbar c} (mc^2)^2 \times 4 f(y_1), \tag{33}$$

$$f(y) := \sqrt{1+y^2} \log(\sqrt{1+y^2} + y) - y. \tag{34}$$

We note that in Equation (33) we reintroduced the factor $\hbar^2 c^2$, since we used normalized energy–momentum variables (of the wave number dimension), as is shown in Equation (10). In the above formula, the dimensionless quantity $\alpha = e^2/\hbar c$ is just the Sommerfeld fine structure constant, whose numerical value is $\alpha = 0.00729735 = 1/137.035999084$ If we put the electron mass $m_e = 0.510998950\ \mathrm{MeV}/c^2$ to the mass (squared) term in Equation (33), we would receive for the first factor on the right-hand-side of Equation (33) the following equivalent energy

$$\sqrt{\frac{1}{\pi} \frac{e^2}{\hbar c}} (m_e c^2) = 0.0246279\ \mathrm{MeV}. \tag{35}$$

Concerning this analytic Formula (35), we mention that such a formula for the photon self-energy was derived by Wentzel [25] in 1948 from the Tomonaga–Schwinger equation [26,27], on the basis of perturbation theory up to order e^2. It is essential, however, that Wentzel considered the propagation in an electron–positron *vacuum*, in contrast to our calculations, where the dressing of the radiation is caused by *real particles*. If we put the proton mass $m_p = 938.272088\ \mathrm{MeV}/c^2$ to the mass (squared) term in Equation (33), we receive for the first factor on the right-hand-side of Equation (33)

$$\sqrt{\frac{1}{\pi}\frac{e^2}{\hbar c}}(m_p c^2) = 45.2206\ \mathrm{MeV}. \tag{36}$$

According to Equation (33), the energy equivalent of the rest mass of the radiation, $m_X c^2$, dressed by a proton environment, will be equal to the product of (36) and the factor $\sqrt{4f(y_1)}$, where the function $f(y)$ has been defined in Equation (34),

$$m_X c^2 = \sqrt{\frac{1}{\pi}\frac{e^2}{\hbar c}}(m_p c^2)\sqrt{4f(y_1)}. \tag{37}$$

In order to find the value of the Fermi momentum y_1, we use the following normalization condition. The density of the charged particle in real space can be determined by calculating a similar integral to the one we have seen above,

$$n = \frac{N}{L^3} = \frac{\kappa^3}{(2\pi)^3} 2 \times \int \frac{d^3y}{y_0}, \quad n = \frac{\kappa^3}{\pi^2} g(y_1), \tag{38}$$

$$g(y) := \frac{1}{2}[y\sqrt{1+y^2} - \log(y + \sqrt{1+y^2})]. \tag{39}$$

By multiplying the density by an interaction volume $4\pi r_s^3/3$ of a sphere of radius r_s, the normalization for one particle reads,

$$\frac{4(\kappa r_s)^3}{3\pi} g(y_1) = 1. \tag{40}$$

First, let us make an estimate, on the basis of the power expansions of the functions $f(y)$ and $g(y)$, which have been defined in Equations (34) and (39), respectively,

$$f(y) = \frac{y^3}{3} - \frac{2y^5}{15} + \frac{8y^7}{105} + O(y^9), \tag{41}$$

$$g(y) = \frac{y^3}{3} - \frac{y^5}{10} + \frac{3y^7}{56} + O(y^9). \tag{42}$$

We see that for small values of y, say, in the non-relativistic region, where, for instance $y = p/\kappa = 1/4$, the higher terms are of the order of 1% smaller than the leading term $y^3/3$. Accordingly, from (40) we have

$$\frac{4}{3} y^3 \approx \frac{3\pi}{(r_s\kappa)^3} = \frac{4}{9\pi}\frac{27\pi^2}{4(r_s\kappa)^3}. \tag{43}$$

Now, let us take $r_s = 4/\kappa$ (which is the inverse of the non-relativistic wave number mentioned above) in the normalization Equation (40), with $g(y) \approx y^3/3$. Then we receive

$$\frac{4}{3} y^3 \approx \frac{4}{9\pi}\frac{27\pi^2}{256} = \frac{4}{9\pi} \times 1.04093. \tag{44}$$

Notice that the value of the numerical factor $27\pi^2/256$ in (44) happens to be very close to unity. By taking (41) into account, Equation (44) means, at the same time, that

$$4f(y_1) \approx \frac{4}{3}y_1^3 \approx \frac{4}{9\pi}. \tag{45}$$

On the basis of Equations (33), (36), (37) and (45), the proposed analytic formula reads

$$m_X c^2 = \frac{2}{3\pi}\sqrt{\frac{e^2}{\hbar c}}(m_p c^2) = 17.0087 \text{ MeV}. \tag{46}$$

The numerical value in Equation (46) for $m_X = 17.0087 \text{ MeV}/c^2$ is in very good agreement with the experimental invariant mass of the hypothetical X17 particle [2].

Now, the question arises whether we can reproduce such a good agreement, if we use the exact normalization condition (40) instead of the non-relativistic approximate formulas, relying on Equations (41) and (42). The answer is affirmative; however, we have to take a slightly larger radius, $r'_s = (4/\kappa) \times 1.00602$, for the interaction sphere. In reality, if we choose, then the normalization condition (see also Equations (38) and (39)) reads

$$\frac{4(\kappa r'_s)^3}{3\pi} g(y_1) = 1, \quad r'_s = (4/\kappa) \times 1.00602 = 0.846299 \text{ fm}. \tag{47}$$

The numerical solution of the above normalization condition in Equation (47) yields the value for the dimensionless Fermi momentum $y_1 = 0.487446$. By inserting this Fermi momentum into the original relativistic expression (37), we have

$$m_X c^2 = \sqrt{\frac{1}{\pi}\frac{e^2}{\hbar c}(m_p c^2)}\sqrt{4f(y_1)} = 17.0087 \text{ MeV}, \quad y_1 = p_{Fermi}/\kappa = 0.487446. \tag{48}$$

The numerical value of the rest energy in Equation (48) coincides with the one in (46), expressed by the proposed analytic formula. According to Equation (45), and the second equation of Equation (48), the reason for this perfect agreement is based on the numerical equality of $f(y_1 = 0.487446) = 0.0353679$ and $1/9\pi = 0.0353678$.

At the end of the present section, we note that the above derivation may be interrelated to the recent investigations on the proton radius. In order to derive the mass formula in a non-relativistic approximation, first we took the radius of the interaction sphere $r_s = 4/\kappa$ (see Equations (43) and (44)), i.e.,

$$r_s = 4/\kappa = 4\frac{\hbar}{m_p c}, \quad r_s = \frac{2}{\pi}\frac{h}{m_p c} = \frac{2}{\pi}\lambda_p, \tag{49}$$

where $\lambda_p = h/m_p c = 1.32141$ fm is the Compton wavelength of the proton. This radius in Equation (49), $r_s = 2\lambda_p/\pi = 0.8412356$ fm, is quite close to the recently measured (smaller) value for the proton charge radius,$r_p = 0.84184(67)$ fm [28]. We also point out that in a theoretical investigation on neutron decay [29], $r_p = 2\lambda_p/\pi$ has been defined as the proton radius. In the normalization condition (47) above, we had to take a slightly larger value for the radius of the interaction sphere, $r'_s = (2\lambda_p/\pi) \times 1.00602$, which is 0.846299 fm. In both cases, the derived invariant mass of the hypothetical X17 particle is in good agreement with the experimental value. Thus, one may also even say that the precise measurements, performed by now on this invariant mass of X17 [1–5], supports the identification of $r_s = 2\lambda_p/\pi$ with the proton charge radius (at zero temperature).

4. Plasmon Interpretation of the Dispersion Relation of the Dressed Radiation—Application to the Hypothetical E38 Particle

The plane wave quanta of the dressed radiation, which were derived in the previous section to model the X17 field, satisfy the energy–momentum relation

$$E(\boldsymbol{p}) = \sqrt{(c\boldsymbol{p})^2 + (m_X c^2)^2}, \quad \omega(\boldsymbol{k}) = \sqrt{(c\boldsymbol{k})^2 + \omega_X^2}, \tag{50}$$

where $m_X c^2$ is given by (37) or (46), and $\omega_X = m_X c^2/\hbar$. We introduced the frequency $\omega = E/\hbar$ and the wave vector $\boldsymbol{k} = \boldsymbol{p}/\hbar$ of an arbitrary component. The original zero-mass-shell relation $k_\mu k^\mu = 0$ (see Equation (2)) is replaced by $\bar{k}_\mu \bar{k}^\mu = \kappa_X^2$. Here, from the dispersion relation in Equation (50), the four-k-vector is $k_X = \left\{\bar{k}^\mu\right\} = (\omega(\boldsymbol{k})/c, \boldsymbol{k})$, and $\kappa_X = \omega_X/c$ is the Compton wave number of the X17 field. Assume now that a monochromatic component of frequency $\omega > \omega_X$ propagates, say, in the z-direction. Then its phase factor (in the vector potential and in the field strengths) is given by

$$\mathrm{e}^{-ik_X\cdot x} = \exp[-i\omega(t - n_m z/c)], \quad n_m(\omega) = \sqrt{1 - \frac{\omega_X^2}{\omega^2}}. \tag{51}$$

In Equation (51), the quantity $n_m(\omega)$ is analogous to the Drude index of refraction of a free electron plasma medium [30]. We are going to show that the role of ω_X is similar to that of the plasma frequency, ω_{plasma}, in a free electron gas. In the free electron plasma medium

$$\omega_{\text{plasma}}^2 = \frac{4\pi e^2 n_e}{m_e}, \quad n_{\text{plasma}}(\omega) = \sqrt{1 - \frac{\omega_{\text{plasma}}^2}{\omega^2}}, \tag{52}$$

where n_e is the free electron density, and m_e is the electron's mass [30]. We see that here the plasma frequency plays the role of an effective mass of the plasmon modes [31–33]. In order to make the correspondence, we use the definition $\omega_X = m_X c^2/\hbar$, and the mass formula in Equation (37), and the second equation of Equation (38) for the density, applied to protons

$$\omega_X^2 = \frac{4\pi e^2}{m_p}\frac{\kappa_p^3}{\pi^2} f(y_1), \; n_p = \frac{\kappa_p^3}{\pi^2} g(y_1), \; \omega_X^2 = \frac{4\pi e^2 n_p}{m_p} \times \frac{f(y_1)}{g(y_1)}. \tag{53}$$

The ratio of $f(y_1 = 0.487446)$ and $g(y_1 = 0.487446)$, appearing in Equation (53), is 0.978135, which practically, up to about 2%, equals to unity. Thus, to a good approximation, we are allowed to write the last equation of Equation (53) as

$$\omega_X^2 = \frac{4\pi e^2 n_p}{m_p}. \tag{54}$$

The expression (54) for $\omega_X = m_X c^2/\hbar$ is completely analogous to the formula for the classical plasma frequency ω_{plasma}, shown in Equation (52).

Concerning the hypothetical E38 particle of the invariant mass ~38 MeV [10], we consider the neutron's udd quarks, which, as charged particles, are interacting with the quantized electromagnetic radiation, and evolve according to three separate Dirac equations $H_j\Psi_j = i\hbar\partial\Psi_j/\partial t_j$, where H_j $(j = 1, 2, 3)$ are Hamiltonians of the type given in Equation (1). Furthermore, the terms $\beta_j m_n c^2$ in each Hamiltonian are considered as a common scalar potential, where m_n is the neutron mass. This procedure is formally in accord with the many-time formalism of quantum electrodynamics due to Dirac, Fock and Podolsky [34], being equivalent [35,36] to the Tomonaga–Schwinger theory [26,27], if one also takes symmetrization into account. The calculations can be performed in complete analogy with the ones in Sections 2 and 3. The exact stationary states of each equation can be derived with the help of that diagonalization procedure, which was detailed in Section 2. The Fermi integrals come out in the same form (see Equations (33), (34), (38) and (39)), but now the scaled wave numbers are defined as $\boldsymbol{y} := \boldsymbol{p}/\kappa_n$, where $\kappa_n = m_n c/\hbar$ is the neutron's Compton wave number. In comparison with the previous calculation, the first difference is represented by a different effective charge, $e\sqrt{2/3}$, because for the udd system we have

$$(e_d^2 + e_d^2 + e_u^2) = \left(\frac{4}{9} + \frac{1}{9} + \frac{1}{9}\right)e^2 = \frac{2}{3}e^2. \tag{55}$$

We note that for a proton $(e_u^2 + e_u^2 + e_d^2) = e^2$, the effective charge is just the elementary charge. In terms of the plasma interpretation, the plasma frequency squared is the sum of the contributions, being proportional with the squared individual charges, if the medium consists of different charged particles. The other new element in our consideration is that now we take into account the singlet and triplet statistical weights separately, with respect to the d-quarks. As a result, the invariant mass consists of the following two terms

$$m'_E c^2 = \sqrt{\frac{1}{\pi}\frac{\frac{2}{3}e^2}{\hbar c}}(m_n c^2)\sqrt{4f(y_1)},\; m''_E c^2 = \sqrt{3}\sqrt{\frac{1}{\pi}\frac{\frac{2}{3}e^2}{\hbar c}}(m_n c^2)\sqrt{4f(y_1)}. \tag{56}$$

On the basis of Equation (56), by using a similar argumentation to that we had in the derivation of (46) for $m_X c^2$, we arrive at the final formula for the sum of the two expressions in (56),

$$m_E c^2 = (1+\sqrt{3})\frac{2}{3\pi}\sqrt{\frac{\frac{2}{3}e^2}{\hbar c}}(m_n c^2) = 37.9938 \text{ MeV}, \tag{57}$$

where we used $m_n c^2 = 939.565420$ MeV for the energy equivalent of the neutron mass. The calculated value of $m_E c^2$ in Equation (57) is in very good agreement with the measured value.

5. Summary

In the present paper, on the basis of 3 + 1 dimensional quantum electrodynamics, we derived an analytic mass formula for the X17 particle (see (46), (48)). We used the exact solutions of the Dirac equation of the joint system consisting of a charged particle and plane waves of the quantized electromagnetic radiation. These solutions, called quantized Volkov states, have long been known, and occasionally used for the non-perturbative description of high-order multiphoton processes. In Section 2, we gave a detailed derivation, and applied these solutions to the sytem of a proton and quantized plane wave radiation modes. The energy eigenvalues contain an additional mass term, from which we derived the formula $m_X = (2\alpha^{1/2}/3\pi)m_p$ for the invariant mass of the dressed radiation quanta. The calculated numerical value, $m_X = 17.0087\ \text{MeV}/c^2$, is very close to the nowadays accepted experimental value, stemming from measurements of anomalous angular correlations of electron–positron pairs created in nuclear processes [2]. Hence, the derived massive transverse quanta may be identified with the excitations of the hypothetical X17 vector bosons. The existence of these bosons was supported by the appearance of the anomalies encountered in several experiments by the ATOMKI scientists. At the end of Section 3, we saw that the present analysis naturally interrelates the proton size to its Compton wavelength, through the radius $r'_s = (2\lambda_p/\pi) \times 1.00602$ of the interaction sphere, which we had to suppose. Its numerical value is $r'_0 = 0.846299$ fm, which is quite close to recent experimental data. In Section 4, first we discussed the analogous roles played by the Compton frequency $\omega_X = m_X c^2/\hbar$ of the X17 particle and the plasma frequency in a free electron gas medium. A similar consideration, also based on the quantized Volkov states, but now applied to the udd quarks of the neutron, yields the mass formula $m_E = (1+3^{1/2})(2/3)^{1/2}(2\alpha^{1/2}/3\pi)m_n$ for the dressed electromagnetic radiation quanta, with the numerical value $m_E = 37.9938\ \text{MeV}/c^2$, which may correspond to the hypotetical E38 particle [10]. It is remarkable that, besides the Sommerfeld fine structure constant

$\alpha = e^2/\hbar c$, and the proton or neutron mass, our derived formulas do not contain any adjustable parameters, but merely some numerical statistical factors. The very good agreement of the calculated invariant masses of the hypothetical X17 and E38 particles with the experimental data may indicate that our theoretical treatment reflects the basic physical processes, which are responsible for the measured anomalies. However, various refinements are still needed for further progress of this theoretical model. For instance, the calculation of the decay rates as well as taking into account the finite temperature effects are left for further studies, which we plan to carry out in the near future.

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Note

1 In the standard representation, the Dirac matrices $\boldsymbol{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$ and β have the form $\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. In the first three equations, the "0" and "1" denote 2×2 zero and unit matrices, respectively. In the last three equations, $\sigma_{x,y,z}$ are the usual 2×2 Pauli matrices. The γ matrices are defined as $\gamma^{1,2,3} = \gamma_{x,y,z} = \beta\alpha_{x,y,z}$ and $\gamma^0 = \beta$, their commutation relations are $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$, and $\gamma^0\gamma^+_\mu\gamma^0 = \gamma_\mu$, where γ^+_μ denotes the adjoint (transposed conjugate) of γ_μ, see e.g., [22]. We shall also use manifestly covariant notations, so we summarize the conventions we follow. The Minkowski metric tensor $g_{\mu\nu} = g^{\mu\nu}$ has the components $g_{00} = 1 = -g_{ii}(i = 1,2,3)$ and $g_{\mu\nu} = 0$ if $\mu \neq \nu(\mu, \nu = 0,1,2,3)$. The scalar product of two four-vectors a and b is $a \cdot b = g_{\mu\nu}a^\mu b^\nu$, i.e., $a \cdot b = a_\nu b^\nu = a^0 b^0 - \boldsymbol{a} \cdot \boldsymbol{b}$, where $\boldsymbol{a} \cdot \boldsymbol{b}$ is the usual scalar product of three-vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Space–time coordinates are denoted by x^μ, where $x = \{x^\mu\} = (ct, \boldsymbol{r})$. The four-gradient is $\partial = \{\partial^\mu\} = (\partial/\partial ct, -\partial/\partial \boldsymbol{r})$, and $\partial_\mu = \partial/\partial x^\mu$. For products of the type $\gamma \cdot a \cdot = \gamma_\nu a^\nu = \not{a}$, we use the "slash" notation.

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