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Article

Super Virasoro Algebras From Chiral Supergravity

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Abstract: In this note, we construct Noether charges for the chiral supergravity, which contains the Lorentz Chern–Simons term, by applying Wald's prescription to the vielbein formalism. We investigate the AdS_3/CFT_2 correspondence by using the vielbein formalism. The asymptotic symmetry group is carefully examined by taking into account the local Lorentz transformation, and we construct super Virasoro algebras with central extensions from the chiral supergravity.

Keywords: AdS/CFT; supergravity; chiral gravity

1. Introduction

The three dimensional gravity with negative cosmological constant has been one of the interesting testing grounds to uncover quantum natures of gravity. Especially the gauge/gravity correspondence has been investigated from various aspects for decades.

The vacuum solution of the three dimensional gravity with negative cosmological constant is described by global AdS_3 geometry [1]. In 1986, Brown and Henneaux showed that the asymptotic symmetry group of the AdS_3 geometry consists of left and right Virasoro algebras, and they succeeded to evaluate the same central charges for both algebras [2]. This is a prototype of the gauge/gravity correspondence, which was conjectured sophisticatedly in the context of superstring theory [3]. The three dimensional theory also contains Banados–Teitelboim–Zanelli (BTZ) black hole solution [4,5]. Moreover, the entropy of the BTZ black hole is statistically explained by using the Cardy formula for the boundary conformal field theory (CFT) [6]. It is well known that the three dimensional gravity theory can also be described by the gauge Chern–Simons theory [7,8]. The Virasoro algebras can be derived by using this alternative formulation [9], and the black hole entropy is statistically explained in Reference [10].

There are many important works on the three dimensional gravity, but we just focus on three kinds of generalizations on the Virasoro algebras at the boundary. First one is to deal with the supergravity [11]. As expected, the asymptotic symmetry group enhances to super Virasoro algebras and the central charges can be evaluated including fermionic sector [12,13]. Second one is to add chiral terms to the theory. The three dimensional gravity with the gravitational or Lorentz Chern–Simons term is called topologically massive gravity (TMG) [14,15]. In this theory it has been studied that the central charges for left and right modes are asymmetric [16–24]. Third one is to consider higher derivative corrections, such as R^2 terms. In this case, central charges are modified by some conformal factors [24–26].

The purpose of this note is to consider the supergravity with negative cosmological constant which contains the Lorentz Chern–Simons term. The supergravity with the Lorentz Chern–Simons term, which is called the topologically massive supergravity (TMSG), is constructed by Deser and Kay [27], and the cosmological constant is added to the TMSG by Deser (CTMSG) [28]. There are two parameters in CTMSG : the cosmological constant $-\frac{2}{\ell^2}$ and the coefficient of the Lorentz Chern–Simons term β. It is known that fluctuation around the AdS_3 geometry contains negative energy mode for generic ℓ and β [22]. The exception occurs at the critical point $|\beta/\ell| = 1$, and the theory is called chiral supergravity [22,29]. Since we need stable AdS_3 background to explore the gauge/gravity correspondence, the chiral supergravity is investigated in this note. We employ Wald's prescription to construct the Noether charge for the chiral theory [30–32]. Especially we formulate the chiral supergravity in the vielbein formalism [33]. The charges are covariant under the general coordinate transformation, and it is possible to evaluate the asymmetric central charges for left and right modes explicitly. As a result, super Virasoro algebras at the boundary are explicitly constructed, which are expected from the viewpoint of AdS/CFT correspondence [29]. The vielbein formalism is applicable to all supergravity theories [33,34], and this work will be useful to test the gauge/gravity correspondence in superstring theory and M-theory at quantum level [35,36].

In Section 2, we explain some basic properties of the CTMSG. In Section 3, we construct the current for the general coordinate transformation and that for the local supersymmetry. We review the asymptotic symmetry group of the AdS_3 in Section 4. The super Virasoro algebras for the chiral supergravity are constructed and the central charges for left and right movers are derived in Section 5. Section 6 is devoted to the conclusion and discussion.

2. Cosmologically Topologically Massive Supergravity

The topologically massive supergravity (TMSG) is the three dimensional supergravity with Lorentz Chern–Simons term which was constructed by Deser and Kay [27]. Deser also generalized the theory by adding the cosmological constant (CTMSG) [28]. In this section we review the equations of motion for the CTMSG. Fields of the CTMSG consist of a vielbein $e^a{}_\mu$ and a Majorana gravitino ψ_μ. Here μ, ν are used for space-time indices and $a, b = 0, 1, 2$ are for local Lorentz ones. In this note we consider $\mathcal{N} = (1, 0)$ CTMSG (If the sign of ℓ is flipped, we obtain $\mathcal{N} = (0, 1)$ CTMSG. Although the bulk gravity has three dimensions, by taking into account the AdS/CFT correspondence, we use the notation $\mathcal{N} = (1, 0)$ in the boundary CFT.), and the Lagrangian is given by

$$\mathcal{L} = \frac{e}{16\pi G_{\rm N}}\Big\{R + \frac{2}{\ell^2} - \frac{1}{2}\overline{\psi_\rho}\gamma^{\mu\nu\rho}\psi_{\mu\nu} + \frac{\beta}{2}\epsilon^{\mu\nu\rho}\Big(\omega_\mu{}^a{}_b\partial_\nu\omega_\rho{}^b{}_a + \frac{2}{3}\omega_\mu{}^a{}_b\omega_\nu{}^b{}_c\omega_\rho{}^c{}_a\Big) - \frac{\beta}{2}\overline{D_\rho\psi_\sigma}\gamma^{\mu\nu}\gamma^{\rho\sigma}D_\mu\psi_\nu\Big\} \tag{1}$$

Here $G_{\rm N}$ and $-2/\ell^2$ are the gravitational constant and the negative cosmological one, respectively. β is a coefficient for the nonchiral part. Since we evaluate physical quantities in the background of AdS_3 with $\psi_\mu = 0$ in later sections, below we consider the Lagrangian up to $\mathcal{O}(\psi^3)$.

In Equation (1), two kinds of covariant derivatives are defined,

$$D_\mu\psi_\nu = \partial_\mu\psi_\nu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\psi_\nu, \qquad \mathcal{D}_\mu\psi_\nu = D_\mu\psi_\nu + \frac{1}{2\ell}\gamma_\mu\psi_\nu \tag{2}$$

and the field strength of the Majorana gravitino is given by $\psi_{\mu\nu} \equiv \mathcal{D}_\mu\psi_\nu - \mathcal{D}_\nu\psi_\mu$. The gamma matrix in three dimensions satisfy the Clifford algebra $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$, and $\eta^{ab} = \mathrm{diag}(-1, 1, 1)$. The gamma matrix with spacetime index is defined as $\gamma^\mu = e^\mu{}_a\gamma^a$, and a completely antisymmetric tensor $\gamma^{\mu_1\cdots\mu_n}$ is defined so that a coefficient of each term becomes $1/n!$. $\gamma^{\mu\nu\rho} = \epsilon^{\mu\nu\rho}\mathbf{1}$ is a completely antisymmetric tensor in three dimensions.

The spin connection is expressed in terms of the vielbein and the Majorana gravitino by requiring $D_\mu\big(2ee^\mu{}_a e^\nu{}_b\big) = \frac{1}{4}e\overline{\psi_\rho}\gamma^{\rho\mu\nu}\gamma_{ab}\psi_\mu$. After standard calculations, the explicit forms of the spin connection and its variation can be obtained as

$$\omega_{\rho ab} = e^\mu{}_{[a}e^\nu{}_{b]}\Big(- e_{\rho c}\partial_\mu e^c{}_\nu + e_{\mu c}\partial_\nu e^c{}_\rho - e_{\mu c}\partial_\rho e^c{}_\nu + \frac{1}{4}\overline{\psi_\mu}\gamma_\rho\psi_\nu - \frac{1}{4}\overline{\psi_\nu}\gamma_\mu\psi_\rho + \frac{1}{4}\overline{\psi_\rho}\gamma_\mu\psi_\nu\Big), \tag{3}$$

$$\delta\omega_{\rho ab} = e^\mu{}_{[a}e^\nu{}_{b]}\Big(- e_{\rho c}D_\mu\delta e^c{}_\nu + e_{\mu c}D_\nu\delta e^c{}_\rho - e_{\mu c}D_\rho\delta e^c{}_\nu + \frac{1}{2}\overline{\delta\psi_\mu}\gamma_\rho\psi_\nu - \frac{1}{2}\overline{\delta\psi_\nu}\gamma_\mu\psi_\rho + \frac{1}{2}\overline{\delta\psi_\rho}\gamma_\mu\psi_\nu\Big) \tag{4}$$

Then , up to $\mathcal{O}(\psi^2)$, the variation of the Lagrangian (1) becomes

$$\begin{aligned} 16\pi G_{\rm N}\delta\mathcal{L} = {} & 2e\Big\{R^a{}_\mu - \frac{1}{2}e^a{}_\mu\Big(R + \frac{2}{\ell^2}\Big)\Big\}\delta e^\mu{}_a - e\overline{\delta\psi_\rho}\gamma^{\rho\mu\nu}\psi_{\mu\nu} \\ & + \frac{\beta}{2}\big(- e\epsilon^{\rho\mu\nu}R^{ab}{}_{\mu\nu}\delta\omega_{\rho ab} + 2e\overline{\delta\psi_\sigma}\gamma^{ab}\gamma^{\rho\sigma}D_\rho D_a\psi_b\big) \\ & + \partial_\mu\Big(2ee^\mu{}_a e^\nu{}_b\delta\omega_\nu{}^{ab} + e\overline{\psi_\nu}\gamma^{\mu\nu\rho}\delta\psi_\rho + \frac{\beta}{2}e\epsilon^{\mu\nu\rho}\omega_{\nu ab}\delta\omega_\rho{}^{ab} - \beta e\overline{\delta\psi_\nu}\gamma^{\rho\sigma}\gamma^{\mu\nu}D_\rho\psi_\sigma\Big) \end{aligned} \tag{5}$$

In the above calculation, we used $D_\mu\gamma^a = 0$ and $D_\gamma(e\gamma^{\mu\nu\rho}) = 0$. Note also that $D_\rho(e\gamma^{\rho\mu_1\cdots\mu_n}) = \mathcal{O}(\psi^2)$ from Equation (3), and we employed this relation to derive Equation (5).

Let us evaluate the first term in the second line in Equation (5). The Riemann tensor in three dimensions is written in terms of the Ricci tensor and the scalar curvature as

$$\begin{aligned} & R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho} - \frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R, \\ & \epsilon^{\rho\mu\nu}R^{ab}{}_{\mu\nu} = 2\epsilon^{\rho a\sigma}R^b{}_\sigma - 2\epsilon^{\rho b\sigma}R^a{}_\sigma - \epsilon^{\rho ab}R \end{aligned} \tag{6}$$

By using Equations (4) and (6), the first term in the second line in Equation (5) is evaluated as

$$-\frac{\beta}{2}e\epsilon^{\rho\mu\nu}R^{ab}{}_{\mu\nu}\delta\omega_{\rho ab} = 2\beta e\epsilon^{\mu\nu\rho}C_{a\rho}\Big(- D_\mu\delta e^a{}_\nu + \frac{1}{2}\overline{\delta\psi_\mu}\gamma^a\psi_\nu\Big) \tag{7}$$

In this calculation we used

$$C_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R, \qquad \epsilon^{\rho\mu\nu} R^{ab}{}_{\mu\nu} = 2\epsilon^{\rho a\sigma} C^{b}{}_{\sigma} - 2\epsilon^{\rho b\sigma} C^{a}{}_{\sigma} \tag{8}$$

and we neglected terms of $\mathcal{O}(\psi^2)$.

Finally the variation of the Lagrangian Equation (5) is expressed as

$$\delta\mathcal{L} = \frac{e}{16\pi G_{\mathrm{N}}} \left(2G^{a}{}_{\mu} \delta e^{\mu}{}_{a} + \overline{\delta\psi_{\rho}} \Psi^{\rho} \right) + \frac{1}{16\pi G_{\mathrm{N}}} \partial_{\mu} \left(e\Theta^{\mu}(\delta) \right) \tag{9}$$

In the above we defined

$$\begin{aligned} \Theta^{\mu}(\delta) = {} & 2e^{\mu}{}_{a} e^{\nu}{}_{b} \delta\omega_{\nu}{}^{ab} + \bar{\psi}_{\nu} \gamma^{\mu\nu\rho} \delta\psi_{\rho} \\ & + \frac{\beta}{2} \epsilon^{\mu\nu\rho} \omega_{\nu ab} \delta\omega_{\rho}{}^{ab} - 2\beta\epsilon^{\mu\nu\rho} C_{a\rho} \delta e^{a}{}_{\nu} - \beta \overline{\delta\psi_{\nu}} \gamma^{\rho\sigma} \gamma^{\mu\nu} D_{\rho} \psi_{\sigma} \end{aligned} \tag{10}$$

and

$$\begin{aligned} G^{a}{}_{\mu} &\equiv R^{a}{}_{\mu} - \frac{1}{2} e^{a}{}_{\mu} \left(R + \frac{2}{\ell^2} \right) + \beta\epsilon^{a\nu\rho} e^{b}{}_{\mu} D_{\nu} C_{b\rho}, \\ \Psi^{\rho} &\equiv -\gamma^{\mu\nu\rho} \psi_{\mu\nu} + \beta\epsilon^{\mu\nu\rho} C_{a\nu} \gamma^{a} \psi_{\mu} - \beta\gamma^{ab} \gamma^{\rho\sigma} D_{\sigma} D_{a} \psi_{b} \end{aligned} \tag{11}$$

The equations of motion for the CTMSG are given by $G^{a}{}_{\mu} = 0$ and $\Psi^{\rho} = 0$.

3. Currents for the CTMSG

The action of the CTMSG is invariant under the general coordinate transformation and the local supersymmetry. In this section we will construct currents for these transformations via Wald's procedure [30,31].

3.1. Current for the General Coordinate Invariance

Let us consider the general coordinate transformation $x'^{\mu} = x^{\mu} - \xi^{\mu}$. The vielbein and the spin connection transform as vector fields, and these behave like

$$\begin{aligned} \delta_{\xi} e^{a}{}_{\mu} &= \xi^{\nu} \partial_{\nu} e^{a}{}_{\mu} + \partial_{\mu} \xi^{\nu} e^{a}{}_{\nu} = D_{\mu} \xi^{a} - \xi^{\nu} \omega_{\nu}{}^{a}{}_{\mu}, \\ \delta_{\xi} \omega_{\nu}{}^{ab} &= \xi^{\rho} \partial_{\rho} \omega_{\nu}{}^{ab} + \partial_{\nu} \xi^{\rho} \omega_{\rho}{}^{ab} = \xi^{\rho} R^{ab}{}_{\rho\nu} + D_{\nu} (\xi^{\rho} \omega_{\rho}{}^{ab}) \end{aligned} \tag{12}$$

Below we apply Wald's procedure to construct the current for the general coordinate transformation [30,31] (Noether's procedure is generalized to the gravitational Chern–Simons term in Reference [32]).

First, by imposing the equations of motion $G^{a}{}_{\mu} = 0$ and $\Psi^{\rho} = 0$, the variation of the Lagrangian Equation (9) becomes

$$\delta_{\xi}\mathcal{L} = \frac{1}{16\pi G_{\mathrm{N}}} \partial_{\mu} (e\Theta^{\mu}(\xi)) \tag{13}$$

And the explicit form of $e\Theta^{\mu}(\xi)$ up to $\mathcal{O}(\psi)$ is evaluated as (Although Equation (10) is expressed up to $\mathcal{O}(\psi^3)$ for the general coordinate transformation, we also need to know the correct equations of motion

Equation (11) up to $\mathcal{O}(\psi^3)$ to obtain fermionic bilinear terms of $Q_{\mu\nu}(\xi)$. Thus, we evaluate $\Theta^\mu(\xi)$ up to $\mathcal{O}(\psi)$, which is enough to obtain the super Virasoro algebras in Section 5.)

$$\begin{aligned}
e\Theta^\mu(\xi) &= 2ee^\mu{}_a e^\nu{}_b \delta_\xi \omega_\nu{}^{ab} + \frac{\beta}{2} e\epsilon^{\mu\nu\rho}\omega_{\nu ab}\delta_\xi \omega_\rho{}^{ab} - 2\beta e\epsilon^{\mu\nu\rho} C_{a\rho}\delta_\xi e^a{}_\nu \\
&= 2eR^\mu{}_\nu \xi^\nu + 2ee^\mu{}_a e^\nu{}_b D_\nu(\xi^\rho \omega_\rho{}^{ab}) - \frac{\beta}{2} e\epsilon^{\mu\nu\rho}\omega_{\nu ab}\xi^\sigma R^{ab}{}_{\rho\sigma} + \frac{\beta}{2} e\epsilon^{\mu\nu\rho}\omega_{\nu ab} D_\rho(\xi^\sigma \omega_\sigma{}^{ab}) \\
&\quad - 2\beta e\epsilon^{\mu\nu\rho} C_{a\rho} D_\nu \xi^a + 2\beta e\epsilon^{\mu b\rho} C^a{}_\rho \xi^\sigma \omega_{\sigma ab} \\
&= 2eG^\mu{}_\nu \xi^\nu + e\Big(R + \frac{2}{\ell^2}\Big)\xi^\mu + \partial_\nu\Big(2ee^\mu{}_a e^\nu{}_b \xi^\rho \omega_\rho{}^{ab} - 2\beta e\epsilon^{\mu\nu\rho} C_{\rho\sigma}\xi^\sigma - \frac{\beta}{2} e\epsilon^{\mu\nu\rho}\omega_{\rho ab}\omega_\sigma{}^{ab}\xi^\sigma\Big) \\
&\quad - \frac{\beta}{2} e\epsilon^{\mu\nu\rho}\omega_{\nu ab} R^{ab}{}_{\rho\sigma}\xi^\sigma - \frac{\beta}{4} e\epsilon^{\mu\nu\rho}\omega_{\sigma ab} R^{ab}{}_{\nu\rho}\xi^\sigma - \frac{\beta}{2} e\epsilon^{\mu\nu\rho}\omega_\nu{}^a{}_b \omega_\rho{}^b{}_c \omega_\sigma{}^c{}_a \xi^\sigma \\
&= 2eG^\mu{}_\nu \xi^\nu + \xi^\mu \mathcal{L} + \partial_\nu\big(eQ^{\mu\nu}(\xi)\big)
\end{aligned} \tag{14}$$

Here, we defined the antisymmetric tensor,

$$eQ^{\mu\nu}(\xi) = 2ee^\mu{}_a e^\nu{}_b \xi^\rho \omega_\rho{}^{ab} - \frac{\beta}{2} e\epsilon^{\mu\nu\rho}\big(4C_{\rho\sigma} + \omega_{\rho ab}\omega_\sigma{}^{ab}\big)\xi^\sigma \tag{15}$$

In order to obtain the last line in Equation (14), we used the relation $\epsilon^{\mu\nu\rho}A_{\nu\rho\sigma}\xi^\sigma = \frac{1}{3}\epsilon^{\nu\rho\sigma}A_{\nu\rho\sigma}\xi^\mu$ for a completely antisymmetric tensor $A_{\mu\nu\rho}$.

Second, since the Lagrangian of the CTMSG is covariant under the general coordinate transformation, its variation behaves as a scalar field like

$$\delta_\xi \mathcal{L} = \partial_\mu\big(\xi^\mu \mathcal{L}\big) \tag{16}$$

Note that the Lorentz Chern–Simons term is invariant under the general coordinate transformation.

Subtracting Equation (16) from Equation (13), we obtain the conservation law of the current. The current for the general coordinate invariance is expressed as

$$\begin{aligned}
eJ^\mu(\xi) &= \frac{1}{16\pi G_{\mathrm{N}}}\big\{e\Theta^\mu(\xi) - 16\pi G_{\mathrm{N}}\xi^\mu\mathcal{L} + \partial_\nu\big(e\tilde{Q}^{\mu\nu}(\xi)\big)\big\} \\
&= \frac{1}{16\pi G_{\mathrm{N}}}\partial_\nu\big(eQ^{\mu\nu}(\xi) + e\tilde{Q}^{\mu\nu}(\xi)\big)
\end{aligned} \tag{17}$$

Here the equation of motion $G^\mu{}_\nu = 0$ is used, and $\tilde{Q}^{\mu\nu}(\xi)$ is an antisymmetric tensor. According to the Wald's procedure, in order to make the Hamiltonian well defined, the variation of $\tilde{Q}^{\mu\nu}(\xi)$ should become

$$\delta\big(e\tilde{Q}^{\mu\nu}(\xi)\big) = e\big(\xi^\mu\Theta^\nu(\delta) - \xi^\nu\Theta^\mu(\delta)\big) \tag{18}$$

Then the variation of the current is evaluated as

$$\delta\big(eJ^\mu(\xi)\big) = \frac{1}{16\pi G_{\mathrm{N}}}\partial_\nu\big\{\delta\big(eQ^{\mu\nu}(\xi)\big) + e\big(\xi^\mu\Theta^\nu(\delta) - \xi^\nu\Theta^\mu(\delta)\big)\big\} \tag{19}$$

We will use this expression to derive the Virasoro algebras from the chiral supergravity in Section 5.

3.2. Supercurrent

Let us construct the supercurrent for the CTMSG. Under the local supersymmetry transformation, the vielbein and the Majorana gravitino transform as

$$\delta_\epsilon e^a{}_\mu = \bar{\epsilon}\gamma^a\psi_\mu, \qquad \delta_\epsilon\psi_\mu = 2\mathcal{D}_\mu\epsilon \tag{20}$$

Here $\epsilon(x)$ represents a spacetime dependent parameter which belongs to the Majorana representation. From these, we see that the variation of the spin connection and that of the field strength of the Majorana gravitino become

$$\begin{aligned}\delta_\epsilon\omega_{\rho ab} &= e^\mu{}_{[a}e^\nu{}_{b]}\Big(-\bar{\epsilon}\gamma_\rho D_\mu\psi_\nu + \bar{\epsilon}\gamma_\mu D_\nu\psi_\rho - \bar{\epsilon}\gamma_\mu D_\rho\psi_\nu \\ &\qquad - \frac{1}{2\ell}\bar{\epsilon}\gamma_\mu\gamma_\rho\psi_\nu + \frac{1}{2\ell}\bar{\epsilon}\gamma_\nu\gamma_\mu\psi_\rho - \frac{1}{2\ell}\bar{\epsilon}\gamma_\rho\gamma_\mu\psi_\nu\Big), \\ \delta_\epsilon\psi_{\mu\nu} &= \frac{1}{2}R_{ab\mu\nu}\gamma^{ab}\epsilon + \frac{1}{\ell^2}\gamma_{\mu\nu}\epsilon\end{aligned} \tag{21}$$

up to $\mathcal{O}(\psi^2)$.

First, by imposing the equations of motion $G^a{}_\mu = 0$ and $\Psi^\rho = 0$, the variation of the Lagrangian Equation (9) becomes

$$\delta_\epsilon\mathcal{L} = \frac{1}{16\pi G_{\rm N}}\partial_\mu(e\Theta^\mu(\epsilon)) \tag{22}$$

and the explicit form of $\Theta^\mu(\epsilon)$ up to $\mathcal{O}(\psi^2)$ is evaluated as

$$\begin{aligned}\Theta^\mu(\epsilon) &= 2e^\mu{}_a e^\nu{}_b\delta_\epsilon\omega_\nu{}^{ab} + \bar{\psi}_\nu\gamma^{\mu\nu\rho}\delta_\epsilon\psi_\rho \\ &\quad + \frac{\beta}{2}\epsilon^{\mu\nu\rho}\omega_{\nu ab}\delta_\epsilon\omega_\rho{}^{ab} - 2\beta\epsilon^{\mu\nu\rho}C_{a\rho}\delta_\epsilon e^a{}_\nu - \beta\overline{\delta_\epsilon\psi_\nu}\gamma^{\rho\sigma}\gamma^{\mu\nu}D_\rho\psi_\sigma\end{aligned} \tag{23}$$

Next, by consulting the calculations in Section 2, the variation of the Lagrangian under the local supersymmetry is evaluated as

$$\begin{aligned}16\pi G_{\rm N}\delta_\epsilon\mathcal{L} &= 2e\Big\{R^a{}_\mu - \frac{1}{2}e^a{}_\mu\Big(R + \frac{2}{\ell^2}\Big)\Big\}\delta_\epsilon e^\mu{}_a - \frac{1}{2}e\overline{\delta_\epsilon\psi_\rho}\gamma^{\rho\mu\nu}\psi_{\mu\nu} - \frac{1}{2}e\overline{\psi_\rho}\gamma^{\rho\mu\nu}\delta_\epsilon\psi_{\mu\nu} \\ &\quad + 2\beta e\epsilon^{\mu\nu\rho}C_{a\rho}\Big(- D_\mu\delta_\epsilon e^a{}_\nu + \frac{1}{2}\overline{\delta_\epsilon\psi_\mu}\gamma^a\psi_\nu\Big) - e\beta\overline{D_\rho\delta_\epsilon\psi_\sigma}\gamma^{\mu\nu}\gamma^{\rho\sigma}D_\mu\psi_\nu \\ &\quad + \partial_\mu\Big(2ee^\mu{}_a e^\nu{}_b\delta_\epsilon\omega_\nu{}^{ab} + \frac{\beta}{2}e\epsilon^{\mu\nu\rho}\omega_{\nu ab}\delta_\epsilon\omega_\rho{}^{ab}\Big)\end{aligned} \tag{24}$$

In the above, terms of $\mathcal{O}(\psi^2)$ are neglected. The second and third terms in the first line of Equation (24) are deformed as

$$\begin{aligned}&-\partial_\rho\big(e\bar{\epsilon}\gamma^{\rho\mu\nu}\psi_{\mu\nu}\big) + e\bar{\epsilon}\gamma^{\rho\mu\nu}\mathcal{D}_\rho\psi_{\mu\nu} + \frac{1}{4}eR_{ab\mu\nu}\bar{\epsilon}\gamma^{ab}\gamma^{\rho\mu\nu}\psi_\rho + \frac{1}{2\ell^2}e\bar{\epsilon}\gamma_{\mu\nu}\gamma^{\rho\mu\nu}\psi_\rho \\ &\qquad = -\partial_\mu\big(e\bar{\epsilon}\gamma^{\mu\nu\rho}\psi_{\nu\rho}\big) + 2e\Big\{R^a{}_\mu - \frac{1}{2}e^a{}_\mu\Big(R + \frac{2}{\ell^2}\Big)\Big\}\bar{\epsilon}\gamma^\mu\psi_a\end{aligned} \tag{25}$$

and the second line of Equation (24) is calculated like

$$\begin{aligned}&-2\beta e\epsilon^{\mu\nu\rho}C_{a\rho}\Big(\bar{\epsilon}\gamma^a D_\mu\psi_\nu + \frac{1}{2\ell}\bar{\epsilon}\gamma_\mu\gamma^a\psi_\nu\Big) + \frac{\beta}{4}eR_{ab\rho\sigma}\bar{\epsilon}\gamma^{ab}\gamma^{\mu\nu}\gamma^{\rho\sigma}D_\mu\psi_\nu + \frac{\beta}{\ell}e\overline{D_\rho\epsilon}\gamma_\sigma\gamma^{\mu\nu}\gamma^{\rho\sigma}D_\mu\psi_\nu \\ &\qquad = \frac{\beta}{\ell}e\epsilon^{\mu\nu\rho}C_{a\rho}\bar{\epsilon}\gamma_\nu\gamma^a\psi_\mu + \frac{2\beta}{\ell}e\overline{D_\rho\epsilon}\gamma^{\mu\nu\rho}D_\mu\psi_\nu \\ &\qquad = \partial_\mu\Big(\frac{2\beta}{\ell}e\bar{\epsilon}\gamma^{\mu\nu\rho}D_\nu\psi_\rho\Big)\end{aligned} \tag{26}$$

In order to derive the above expressions, we noted $\gamma^{\mu\nu\rho} = \epsilon^{\mu\nu\rho}\mathbf{1}$, $\gamma^{\mu\nu} = \epsilon^{\mu\nu\rho}\gamma_\rho$, $\gamma^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho}\gamma_{\nu\rho}$, and used relations below.

$$\begin{aligned}
\gamma^{\mu\nu}\gamma^{\rho\sigma} &= -2g^{\rho[\mu}\gamma^{\nu]\sigma} + 2g^{\sigma[\mu}\gamma^{\nu]\rho} - 2g^{\rho[\mu}g^{\nu]\sigma}, \\
\gamma_\sigma\gamma^{\mu\nu}\gamma^{\rho\sigma} &= 2g^{\rho[\mu}\gamma^{\nu]} + 2\gamma^{[\mu}\gamma^{\nu]\rho} = 2\gamma^{\mu\nu\rho}, \\
-\epsilon^{\mu\nu\rho}R_{a\rho}\bar{\epsilon}\gamma^a D_\mu\psi_\nu &= -R_{a\rho}\bar{\epsilon}\gamma^{\mu\nu\rho}\gamma^a D_\mu\psi_\nu = -2\bar{\epsilon}R^{[\mu}{}_\rho\gamma^{\nu]\rho}D_\mu\psi_\nu - R\bar{\epsilon}\gamma^{\mu\nu}D_\mu\psi_\nu, \\
R_{ab\rho\sigma}\gamma^{ab}\gamma^{\mu\nu}\gamma^{\rho\sigma} &= 16R^{[\mu}{}_\rho\gamma^{\nu]\rho} + 6R\gamma^{\mu\nu}
\end{aligned} \tag{27}$$

Eventually the variation of the Lagrangian for the CTMSG (24) becomes

$$16\pi G_{\rm N}\delta_\epsilon\mathcal{L} = \partial_\mu\Big(2ee^\mu{}_a e^\nu{}_b\delta_\epsilon\omega_\nu{}^{ab} - e\bar{\epsilon}\gamma^{\mu\nu\rho}\psi_{\nu\rho} + \frac{\beta}{2}e\epsilon^{\mu\nu\rho}\omega_{\nu ab}\delta_\epsilon\omega_\rho{}^{ab} + \frac{2\beta}{\ell}e\bar{\epsilon}\gamma^{\mu\nu\rho}D_\nu\psi_\rho\Big) \tag{28}$$

Thus, the CTMSG is invariant under the local supersymmetry.

By subtracting Equation (28) from Equation (22), it is possible to obtain the current conservation for the local supersymmetry. The supercurrent for the CTMSG is expressed as

$$\begin{aligned}
eS^\mu(\epsilon) &= \frac{e}{16\pi G_{\rm N}}\Big(\bar{\psi}_\nu\gamma^{\mu\nu\rho}\delta_\epsilon\psi_\rho + \bar{\epsilon}\gamma^{\mu\nu\rho}\psi_{\nu\rho} \\
&\quad - 2\beta\epsilon^{\mu\nu\rho}C_{a\rho}\delta_\epsilon e^a{}_\nu - \beta\overline{\delta_\epsilon\psi_\nu}\gamma^{ab}\gamma^{\mu\nu}D_a\psi_b - \frac{2\beta}{\ell}\bar{\epsilon}\gamma^{\mu\nu\rho}D_\nu\psi_\rho\Big) \\
&= \frac{1}{16\pi G_{\rm N}}\partial_\nu\big(eU^{\mu\nu}(\epsilon)\big)
\end{aligned} \tag{29}$$

Here the antisymmetric tensor $U^{\mu\nu}(\epsilon)$ is given by

$$U^{\mu\nu}(\epsilon) = -2\epsilon^{\mu\nu\rho}\bar{\epsilon}\psi_\rho - 2\beta\bar{\epsilon}\gamma^{ab}\gamma^{\mu\nu}D_a\psi_b \tag{30}$$

In order to derive Equation (29), we used the second line in Equation (27) and imposed the equation of motion $\Psi^\mu = 0$.

4. Asymptotic Symmetry Group for AdS$_3$ Geometry

In this section we briefly review the asymptotic behavior of AdS$_3$ geometry including supersymmetry. At the spatial infinity $r \to \infty$, the metric of AdS$_3$ geometry becomes

$$ds^2 = -N^2dt^2 + r^2d\phi^2 + N^{-2}dr^2, \qquad N = \frac{r}{\ell} \tag{31}$$

where t, ϕ and r are time, angular and radial directions, respectively. This background corresponds to the massless BTZ black hole. The Riemann tensor is simply given by $R_{\mu\nu\rho\sigma} = -\frac{1}{\ell^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma})$. In the background of the massless BTZ black hole, the vielbein and the spin connection become

$$e^0 = \frac{r}{\ell}dt, \qquad e^1 = rd\phi, \qquad e^2 = \frac{\ell}{r}dr, \tag{32}$$

$$\omega_t{}^0{}_2 = \frac{r}{\ell^2}, \qquad \omega_\phi{}^1{}_2 = \frac{r}{\ell} \tag{33}$$

$\mu, \nu = t, \phi, r$ are used for spacetime indices and $a, b = 0, 1, 2$ are done for local Lorentz ones.

Since we are interested in the boundary behavior of the symmetry group, we explore general coordinate transformation $x'^{\mu} = x^{\mu} - \xi^{\mu}$ which does not change the geometry of AdS_3 only at the spatial infinity. The condition to be imposed for the variation of the metric is written as follows.

$$\delta_{\xi} g_{\mu\nu} = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(r^{-1}) \\ \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(r^{-1}) \\ \mathcal{O}(r^{-1}) & \mathcal{O}(r^{-1}) & \mathcal{O}(r^{-4}) \end{pmatrix} \tag{34}$$

The behaviors of the diagonal components of $\delta_{\xi} g_{\mu\nu}$ are determined so that these go to zero faster than the background Equation (31) as r goes to infinity. Then the behaviors of ξ^{μ} and off diagonal components of $\delta_{\xi} g_{\mu\nu}$ around the boundary are simultaneously fixed. After some calculations, the general coordinate transformation ξ^{μ} which satisfy the above condition is solved as

$$\begin{aligned} \xi^{t} &= \ell\big(T_{+}(x^{+}) + T_{-}(x^{-})\big), \\ \xi^{\phi} &= T_{+}(x^{+}) - T_{-}(x^{-}), \\ \xi^{r} &= -r\big(\partial_{+}T_{+}(x^{+}) + \partial_{-}T_{-}(x^{-})\big) \end{aligned} \tag{35}$$

where $x^{\pm} = \frac{t}{\ell} \pm \phi$ and $\partial_{\pm} = \frac{1}{2}(\ell\partial_t \pm \partial_{\phi})$. The isometry group only at the boundary is called asymptotic symmetry group. The the asymptotic symmetry group is parametrized by arbitrary functions $T_{+}(x^{+})$ and $T_{-}(x^{-})$, and we often expand these by

$$T_{\pm,n}(x^{\pm}) = \frac{1}{2} e^{inx^{\pm}} \tag{36}$$

Now let us calculate the transformation of the vielbein under Equation (35). As discussed in Reference [33], the transformation should be combined with local Lorentz transformation $\delta_{\Lambda} e^{a}{}_{\mu} = \Lambda^{a}{}_{b} e^{b}{}_{\mu}$, where

$$\Lambda^{a}{}_{b} = \begin{pmatrix} 0 & -\partial_{+}T_{+} + \partial_{-}T_{-} & \frac{\ell}{r}\big(\partial_{+}^{2}T_{+} + \partial_{-}^{2}T_{-}\big) \\ -\partial_{+}T_{+} + \partial_{-}T_{-} & 0 & -\frac{\ell}{r}\big(\partial_{+}^{2}T_{+} - \partial_{-}^{2}T_{-}\big) \\ \frac{\ell}{r}\big(\partial_{+}^{2}T_{+} + \partial_{-}^{2}T_{-}\big) & \frac{\ell}{r}\big(\partial_{+}^{2}T_{+} - \partial_{-}^{2}T_{-}\big) & 0 \end{pmatrix} \tag{37}$$

Then the variation $\delta_{\xi} e^{a}{}_{\mu} = \xi^{\rho}\partial_{\rho} e^{a}{}_{\mu} + \partial_{\mu}\xi^{\rho} e^{a}{}_{\rho} + \Lambda^{a}{}_{b} e^{b}{}_{\mu}$ is evaluated as

$$\delta_{\xi} e^{a}{}_{\mu} = \begin{pmatrix} 0 & 0 & \frac{\ell^2}{r^2}\big(\partial_{+}^{2}T_{+} + \partial_{-}^{2}T_{-}\big) \\ 0 & 0 & -\frac{\ell^2}{r^2}\big(\partial_{+}^{2}T_{+} - \partial_{-}^{2}T_{-}\big) \\ 0 & 0 & 0 \end{pmatrix} \tag{38}$$

This variation goes to zero faster than the background Equation (32). In a similar way, the transformation of the spin connection is given by $\delta_{\xi}\omega_{\mu}{}^{ab} = \xi^{\rho}\partial_{\rho}\omega_{\mu}{}^{ab} + \partial_{\mu}\xi^{\rho}\omega_{\rho}{}^{ab} - \partial_{\mu}\Lambda^{ab} + \Lambda^{a}{}_{c}\omega_{\mu}{}^{cb} + \Lambda^{b}{}_{c}\omega_{\mu}{}^{ac}$. After some calculations, the variation of the spin connection becomes

$$\delta_{\xi}\omega_{t}{}^{a}{}_{b} = \begin{pmatrix} 0 & 0 & -\frac{1}{r}\big(\partial_{+}^{3}T_{+} + \partial_{-}^{3}T_{-}\big) \\ 0 & 0 & \frac{1}{r}\big(\partial_{+}^{3}T_{+} - \partial_{-}^{3}T_{-}\big) \\ -\frac{1}{r}\big(\partial_{+}^{3}T_{+} + \partial_{-}^{3}T_{-}\big) & -\frac{1}{r}\big(\partial_{+}^{3}T_{+} - \partial_{-}^{3}T_{-}\big) & 0 \end{pmatrix},$$

$$
{}_{\xi}\omega_{\phi}{}^{a}{}_{b} = \begin{pmatrix} 0 & 0 & -\frac{\ell}{r}\left(\partial_{+}^{3}T_{+} - \partial_{-}^{3}T_{-}\right) \\ 0 & 0 & \frac{\ell}{r}\left(\partial_{+}^{3}T_{+} + \partial_{-}^{3}T_{-}\right) \\ -\frac{\ell}{r}\left(\partial_{+}^{3}T_{+} - \partial_{-}^{3}T_{-}\right) & -\frac{\ell}{r}\left(\partial_{+}^{3}T_{+} + \partial_{-}^{3}T_{-}\right) & 0 \end{pmatrix}, \tag{39}
$$

$$
\delta_{\xi}\omega_{r}{}^{a}{}_{b} = \begin{pmatrix} 0 & 0 & \frac{\ell}{r^2}\left(\partial_{+}^{2}T_{+} + \partial_{-}^{2}T_{-}\right) \\ 0 & 0 & -\frac{\ell}{r^2}\left(\partial_{+}^{2}T_{+} - \partial_{-}^{2}T_{-}\right) \\ \frac{\ell}{r^2}\left(\partial_{+}^{2}T_{+} + \partial_{-}^{2}T_{-}\right) & \frac{\ell}{r^2}\left(\partial_{+}^{2}T_{+} - \partial_{-}^{2}T_{-}\right) & 0 \end{pmatrix}
$$

The variation of the spin connection also goes to zero faster that the background Equation (33). These results will be employed to calculate central charges in the next section.

Next let us explore local supersymmetric transformation $\epsilon(x)$ which satisfy the boundary condition at the spatial infinity. Notations are the same as in Reference [33]. Because $\psi_{\mu} = 0$ for AdS_3 solution, the condition for the supersymmetric variation is imposed as

$$
\delta_{\epsilon}\psi_{\mu} = \begin{pmatrix} \mathcal{O}(r^{-1/2}) & \mathcal{O}(r^{-1/2}) & \mathcal{O}(r^{-5/2}) \end{pmatrix} \tag{40}
$$

The solution of Equation (40) becomes

$$
\epsilon(x^{+}) = r^{1/2}\gamma^{0}\chi(x^{+}) + \ell r^{-1/2}\chi'(x^{+}) \tag{41}
$$

where $\chi(x^{+})$ is a Majorana fermion with $\gamma^{2}\chi = \chi$. The solution depends only on x^{+}, so the remaining local supersymmetry is chiral in this sense. We often expand $\chi(x^{+})$ and $\epsilon(x^{+})$ by Fourier modes,

$$
\chi_{s} = e^{isx^{+}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \epsilon_{s} = e^{isx^{+}}\begin{pmatrix} -r^{1/2} \\ i\ell s r^{-1/2} \end{pmatrix} \tag{42}
$$

which satisfy the following relation

$$
\chi_{s}^{T}\chi_{t} = 2T_{+,s+t}, \qquad \overline{\epsilon_{s}}\gamma^{\mu}\epsilon_{t} = -2i\xi^{\mu}_{+,s+t} \tag{43}
$$

From Equation (43), it is clear that $s+t$ should take some integer value. When $s, t \in \mathbb{Z} + \frac{1}{2}$, those modes are called in the Neveu–Schwarz sector. On the other hand, when $s, t \in \mathbb{Z}$, those modes are done in the Ramond sector.

5. Super Virasoro Algebras from Chiral Supergravity

So far we have constructed Noether currents for the CTMSG. Since the CTMSG has stable AdS_3 background for the critical point $|\beta/\ell| = 1$ [22,29], we consider the chiral supergravity below. Now we evaluate super Virasoro algebras at the boundary of the chiral supergravity. The Hamiltonian for the general coordinate transformation ξ^{μ} is given by

$$
H(\xi) = \int dr d\phi\, eJ^{t}(\xi) = \frac{1}{16\pi G_{\mathrm{N}}}\oint_{r=\infty} d\phi \left(eQ^{tr}(\xi) + e\tilde{Q}^{tr}(\xi)\right) \tag{44}
$$

The variation of the Hamiltonian is related to the Poisson bracket of the algebra as

$$
\delta_{\xi_2}H(\xi_1) = \{H(\xi_1), H(\xi_2)\} = H([\xi_1, \xi_2]) + K(\xi_1, \xi_2) \tag{45}
$$

The last term represents the central extension of the algebra. Let us evaluate the above quantity in the background of the massless BTZ black hole Equation (31) with $\psi_\mu = 0$. The energy of the massless black hole is zero, so $H(\xi) = 0$ in this background. Thus, $K(\xi_1, \xi_2) = \delta_{\xi_2} H(\xi_1)$ and it is evaluated like

$$\begin{aligned}
\delta_{\xi_2} H(\xi_1) &= \frac{1}{16\pi G_{\rm N}} \oint_{r=\infty} d\phi \left\{ \delta_{\xi_2} \left(e Q^{tr}(\xi_1) \right) + e\left(\xi_1^t \Theta^r(\xi_2) - \xi_1^r \Theta^t(\xi_2) \right) \right\} \\
&= \frac{1}{16\pi G_{\rm N}} \oint_{r=\infty} d\phi \left\{ \delta_{\xi_2} \left(2ee^t{}_a e^r{}_b \xi_1^\rho \omega_\rho{}^{ab} \right) + \frac{\beta}{2} \delta_{\xi_2} \left(4C_{\phi\sigma} + \omega_{\phi ab} \omega_\sigma{}^{ab} \right) \xi_1^\sigma \right. \\
&\qquad \left. + 4e \xi_1^{[t} e^{r]}{}_a e^\nu{}_b \delta_{\xi_2} \omega_\nu{}^{ab} + \beta e \xi_1^{[t} \epsilon^{r]\nu\rho} \left(4C_{a\nu} \delta_{\xi_2} e^a{}_\rho + \omega_{\nu ab} \delta_{\xi_2} \omega_\rho{}^{ab} \right) \right\} \\
&= -\frac{\ell}{4\pi G_{\rm N}} \oint_{r=\infty} d\phi \left\{ \left(1 - \frac{\beta}{\ell} \right) T_{1+} \partial_+^3 T_{2+} + \left(1 + \frac{\beta}{\ell} \right) T_{1-} \partial_-^3 T_{2-} \right\}
\end{aligned} \tag{46}$$

In order to derive the last expression, we used $e\epsilon^{t\phi r} = 1$, Equation (38), Equation (39) and following relations.

$$\begin{aligned}
\delta_{\xi_2} \left(4C_{\rho\sigma} + \omega_{\rho ab} \omega_\sigma{}^{ab} \right) &= \begin{pmatrix} \frac{4}{\ell^2}(\partial_+^3 T_{2+} + \partial_-^3 T_{2-}) & \frac{4}{\ell}(\partial_+^3 T_{2+} - \partial_-^3 T_{2-}) & 0 \\ \frac{4}{\ell}(\partial_+^3 T_{2+} - \partial_-^3 T_{2-}) & 4(\partial_+^3 T_{2+} + \partial_-^3 T_{2-}) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\delta_{\xi_2} \left(4C_{\rho\sigma} + \omega_{\rho ab} \omega_\sigma{}^{ab} \right) \xi_1^\sigma &= \begin{pmatrix} \frac{8}{\ell}(T_{1+} \partial_+^3 T_{2+} + T_{1-} \partial_-^3 T_{2-}) \\ 8(T_{1+} \partial_+^3 T_{2+} - T_{1-} \partial_-^3 T_{2-}) \\ 0 \end{pmatrix}, \\
4C_{a\nu} \delta_{\xi_2} e^a{}_\rho + \omega_{\nu ab} \delta_{\xi_2} \omega_\rho{}^{ab} &= \begin{pmatrix} \frac{2}{\ell^2}(\partial_+^3 T_{2+} + \partial_-^3 T_{2-}) & \frac{2}{\ell}(\partial_+^3 T_{2+} - \partial_-^3 T_{2-}) & 0 \\ \frac{2}{\ell}(\partial_+^3 T_{2+} - \partial_-^3 T_{2-}) & 2(\partial_+^3 T_{2+} + \partial_-^3 T_{2-}) & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{47}$$

Notice that left and right modes are separated in a nontrivial way in Equation (46).

Now we substitute the Fourier mode expansion of Equation (36). Then the variation of the Hamiltonian becomes

$$\delta_{\xi_{\pm,n}} H(\xi_{\pm,m}) = -i \frac{\ell}{8G_{\rm N}} \left(1 \mp \frac{\beta}{\ell} \right) m^3 \delta_{m+n,0} \tag{48}$$

This gives the central extensions of left and right Virasoro algebras. By expanding $H(\xi_{\pm,m}) = L_m^\pm e^{imx^\pm}$ and replacing the Poisson bracket with the commutator, we obtain Virasoro algebras for left and right modes.

$$\begin{aligned}
[L_m^+, L_n^+] &= (m-n) L_{m+n}^+ + \frac{c_+}{12} m^3 \delta_{m+n,0}, \\
[L_m^-, L_n^-] &= (m-n) L_{m+n}^- + \frac{c_-}{12} m^3 \delta_{m+n,0}
\end{aligned} \tag{49}$$

Here the central charges are given by

$$c_\pm = \frac{3\ell}{2G_{\rm N}} \left(1 \mp \frac{\beta}{\ell} \right) \tag{50}$$

Note that the sign is flipped compared with Reference [24] because of the definition $e\epsilon^{t\phi r} = 1$. At the critical point, one of the central charges vanishes.

Next let us evaluate the Poisson bracket of the supercharge. The supercharge for the local supersymmetry is written as

$$F(\epsilon) = \int dr d\phi\, eS^t(\epsilon) = \frac{1}{16\pi G_{\rm N}} \oint_{r=\infty} d\phi\, eU^{tr}(\epsilon) \tag{51}$$

It is obvious that the supercharge is zero in the background of $\psi_\mu = 0$. The variation of the supercurrent under the local supersymmetry is evaluated as

$$\delta_{\epsilon_2} F(\epsilon_1) = \{F(\epsilon_1), F(\epsilon_2)\} = H(\overline{\epsilon_1}\gamma\epsilon_2) + K(\epsilon_1, \epsilon_2) \tag{52}$$

where $K(\epsilon_1, \epsilon_2)$ is the central extension of the algebra. Let us evaluate the above quantity in the background of the massless BTZ black hole Equation (31) with $\psi_\mu = 0$. The energy of the massless black hole is zero, so $H(\xi) = 0$ in this background. Thus, $K(\epsilon_1, \epsilon_2) = \delta_{\epsilon_2} F(\epsilon_1)$ and its explicit form is calculated as

$$\begin{aligned}
\delta_{\epsilon_2} F(\epsilon_1) &= \frac{1}{16\pi G_{\rm N}} \oint_{r=\infty} d\phi \left(4\overline{\epsilon_1}\mathcal{D}_\phi \epsilon_2 - \frac{\beta}{2} eR_{\rho\sigma ab}\overline{\epsilon_1}\gamma^{ab}\gamma^{tr}\gamma^{\rho\sigma}\epsilon_2 - \frac{2\beta}{\ell} e\overline{\epsilon_1}\gamma^{ab}\gamma^{tr}\gamma_b D_a \epsilon_2 \right) \\
&= \frac{1}{4\pi G_{\rm N}} \left(1 - \frac{\beta}{\ell}\right) \oint_{r=\infty} d\phi\, \overline{\epsilon_1}\mathcal{D}_\phi \epsilon_2, \\
&= \frac{i\ell}{4\pi G_{\rm N}} \left(1 - \frac{\beta}{\ell}\right) \oint_{r=\infty} d\phi\, \chi_1^T \chi_2'
\end{aligned} \tag{53}$$

In the above we employed Equation (27). Let us substitute Fourier mode expansion of Equation (42). Then the variation of the supercharge is evaluated as

$$\delta_{\epsilon_t} F(\epsilon_s) = -i\frac{\ell}{2G_{\rm N}} \left(1 - \frac{\beta}{\ell}\right) s^2 \delta_{s+t,0} \tag{54}$$

This corresponds to the central extension of the super Virasoro algebra. Notice that $i\overline{\epsilon_s}\gamma^\mu\epsilon_t = 2\xi^\mu_{+,s+t}$. By expanding $F(\epsilon_s) = G_s e^{isx^+}$, the algebra is expressed as

$$\{G_s, G_t\} = 2L_{s+t} + \frac{c_+}{3} s^2 \delta_{s+t,0} \tag{55}$$

The Neveu–Schwarz sector corresponds to $s, t \in \mathbb{Z} + \frac{1}{2}$, and the Ramond sector does to $s, t \in \mathbb{Z}$.

Finally let us examine the variation of the supercharge under the general coordinate transformation [33]. When the transformation ξ^μ_+ depends only on x^+, we obtain

$$\delta_{\xi_+} F(\epsilon_1) = \{F(\epsilon_1), H(\xi_+)\} = -F(\delta_{\xi_+}\epsilon_1) \tag{56}$$

where $\delta_{\xi_+}\epsilon_1 = \xi^\rho_+ \partial_\rho \epsilon_1 + \frac{1}{4}\Lambda_{ab}\gamma^{ab}\epsilon_1$. Notice that the integral constant should be zero since $F(\epsilon) = 0$ for $\psi_\mu = 0$. By setting $\xi_+ = \xi_{+,m}$ and $\epsilon_1 = \epsilon_s$, we obtain

$$[L^+_m, G_s] = \left(\frac{m}{2} - s\right) G_{m+s} \tag{57}$$

In a similar way, it is possible to show $[L^-_m, G_s] = 0$. Therefore we conclude that there are left and right Virasoro algebras at the boundary with different central charges, and left mode is extended to the super Virasoro algebra.

6. Conclusions and Discussion

In this note, we investigated the chiral supergravity in three dimensions. The charges for the general coordinate transformation and local supersymmetry are explicitly constructed by applying Wald's prescription to the vielbein formalism. Commutation relations of the charges are explored in detail and super Virasoro algebras are constructed for AdS_3 background. Especially, the central extensions of the left and right super Virasoro algebras are evaluated by calculating the variations of the charges. The asymmetric central charges are obtained and those expressions are given by $c_\pm = \frac{3\ell}{2G_N}(1 \mp \frac{\beta}{\ell})$.

Note that the super Virasoro algebras Equations (49) and (55) are not in the canonical form. In order to make the expressions canonical, we just shift the zero point energy as

$$L_0^\pm \ \to \ L_0^\pm - \frac{c_\pm}{24} \tag{58}$$

Then the algebras become

$$\begin{aligned}
[L_m^\pm, L_n^\pm] &= (m-n)L_{m+n}^\pm + \frac{c_\pm}{12}(m^3 - m)\delta_{m+n,0}, \\
\{G_s, G_t\} &= 2L_{s+t} + \frac{c_+}{3}\left(s^2 - \frac{1}{4}\right)\delta_{s+t,0}
\end{aligned} \tag{59}$$

At the same time, the energy of the global AdS_3 geometry is shifted to zero. Thus, the effective central charge is the same as the central charge, and the entropy of the BTZ black hole can be correctly explained by the Cardy formula (As a review on the BTZ black hole entropy and Cardy formula, see Reference [37] for example.). Though this conclusion was obtained in the supersymmetric theory, it is also true for the bosonic case if we truncate the fermionic sector.

Since the vielbein formulation of the chiral supergravity is well established, it is interesting to apply these results to other geometries, such as warped AdS_3 [38,39], or Kerr/CFT correspondence [40]. For these cases, it is important to generalize the covariant formalism of refs. [41,42] to the chiral supergravity. It is also important to apply the vielbein formalism to the higher spin supergravity and derive the central charges [43–46].

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Conflicts of Interest

The author declares no conflict of interest.

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