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# Stability Analysis of Linear Systems under Time-Varying Samplings by a Non-Standard Discretization Method

Xiefu Jiang \*, Zongming Yin and Jinjing Wu

School of Automation, Hangzhou Dianzi University, Hangzhou 310018, Zhejiang, China; yinzm@hdu.edu.cn (Z.Y.); 13575760842@163.com (J.W.)

\* Correspondence: jiangxf@hdu.edu.cn

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**Abstract:** This paper is concerned with the stability of linear systems under time-varying sampling. First, the closed-loop sampled-data system under study is represented by a discrete-time system using a non-standard discretization method. Second, by introducing a new sampled-date-based integral inequality, the sufficient condition on stability is formulated by using a simple Lyapunov function. The stability criterion has lower computational complexity, while having less conservatism compared with those obtained by a classical input delay approach. Third, when the system is subject to parameter uncertainties, a robust stability criterion is derived for uncertain systems under time-varying sampling. Finally, three examples are given to show the effectiveness of the proposed method.

**Keywords:** stability; time-varying sampling; discretization; linear matrix inequality (LMI)

## 1. Introduction

Recently, sampled-data systems have been widely applied in digital control systems and networked control systems [1–5]. More and more attention has been focused on the stability analysis and synthesis of sampled-data systems [6–14]. The sampled-data systems are often those including continuous-time state and discrete-time control, simultaneously [15]. Consider the following linear system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ x(0) = x_0 \end{cases} \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  and  $u(t) \in \mathbb{R}^m$  are the state vector and input vector, respectively;  $x_0$  is the initial condition of the system (1);  $A$  and  $B$  are known parameter matrices of appropriate dimensions. The following sampled-data control law is assumed by a zero-order holder for the system (1):

$$u(t) = Kx(t_k), \quad t \in [t_k, t_{k+1}) \quad (2)$$

where  $K$  is the given controller gain of (2) and  $t_k$  are the sampling instants satisfying  $0 = t_0 < t_1 < \dots < t_k < \dots$ . Let  $t_{k+1} - t_k = h(k)$ ,  $k = 1, 2, \dots$ .  $x(t_k)$  are the state vectors at the instants  $t_k$ .

Substituting (2) into (1) yields:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1x(t_k), \\ x(0) = x_0 \end{cases} \quad (3)$$

for  $t \in [t_k, t_{k+1})$ ,  $k = 1, 2, \dots$ , where  $A_1 = BK$ .

Recalling some existing results reported in the literature, there are mainly three approaches to dealing with stability analysis and synthesis of the sampled-data system (3) based on linear matrix inequality (LMI) techniques.

- The first approach is called an input delay approach [7]. The input delay approach is very popular in the analysis of sampled-data systems. This approach has been applied by constructing time-independent Lyapunov–Krasovskii functionals or Razumikhin-type functions to derive stability criteria for linear sampled-data systems under constant/time-varying sampling. The idea of the input delay approach is to represent  $x(t_k)$  as  $x(t_k) = x(t - (t - t_k))$ . By introducing an artificial delay  $\tau(t) = t - t_k$  for  $t \in [t_k, t_{k+1})$ ,  $k = 1, 2, \dots$ , the system (3) is thus modeled as the following time-delay system:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1x(t - \tau(t)), \\ x(0) = x_0 \end{cases} \quad (4)$$

Clearly,  $\tau(t)$  is a piecewise function with its time-derivative being one, i.e.,  $\dot{\tau}(t) = 1$  for  $t \neq t_k$ . Notice that  $\tau(t) = t - t_k \leq t_{k+1} - t_k = h(k)$ . Therefore, a stability criterion of (4) can be obtained in terms of LMIs using the Razumikhin method or the Lyapunov–Krasovskii functional method [16–18]. The input delay approach is developed by introducing time-dependent Lyapunov–Krasovskii functionals [5].

- The second approach is the so-called impulsive model approach [5,19]. The impulsive model approach is to model a sampled-data system as an impulsive system. By choosing a piecewise time-dependent Lyapunov–Krasovskii functional or a discontinuous Lyapunov–Krasovskii functional, less stability criteria can be derived [20].

It should be mentioned that, although some less conservative stability criteria can be derived using the above two approaches, the chosen Lyapunov–Krasovskii functionals are commonly complicated. Since the obtained LMIs require more scalar decision variables, the total numerical complexity of the stability criteria is definitely much higher.

- The third approach is a discrete-time approach [3,21–25], by which a sampled-data system is equivalently transformed into a finite-dimensional discrete-time system, where inter-sampling information of the systems can be maintained. The discrete-time approach assumes the sampling period to be a constant, i.e.,  $h(k) = h$ , where  $h$  is a positive constant. Under such an assumption, the system (3) is often represented as the following form by using a standard discretization technique [3,24–26]:

$$x(t_{k+1}) = G(h)x(t_k), \quad (5)$$

where  $G(h) = e^{Ah} + \int_0^h e^{Ar}A_1dr$ . In this situation, one can draw a conclusion that the system (3) is asymptotically stable if and only if there exists a real matrix  $P > 0$  such that:

$$G^T(h)PG(h) - P < 0. \quad (6)$$

Thus, a maximum allowable constant sampling  $h_{max}$  of  $h$  can be obtained such that (6) holds. However, if the above assumption is not satisfied, that is the sampling is not uniform, the standard discretization approach can hardly be utilized to represent the system (3) as (5) [25,26]. As a result, the discrete-time approach based on a standard discretization technique may not be applicable in this case.

This paper focuses on the stability of sampled-data systems under time-varying sampling. Different from the three aforementioned approaches, a non-standard discretization method is introduced to model the system (3) as a discrete-time system. By establishing a new sampled-data-based integral inequality, the sufficient conditions on the stability and robust stability

of sampled-data systems under time-varying sampling are proposed, which are of much lower computational complexity. Finally, the effectiveness of the proposed method is demonstrated by three numerical examples.

## 2. Main Results

In this section, we introduce a non-standard discretization method to model the system (3) under time-varying sampling as a discrete-time system. Notice that the asymptotical stability of (3) is equivalent to that of the following discrete-time system:

$$\begin{cases} x(t_{k+1}) = x(t_k) + \int_{t_k}^{t_{k+1}} [Ax(s) + A_1x(t_k)]ds, \\ x(0) = x_0 \end{cases} \tag{7}$$

which can be rewritten as:

$$\begin{cases} x(t_{k+1}) = (I + h(k)A_1)x(t_k) + \int_{t_k}^{t_{k+1}} Ax(s)ds, \\ x(0) = x_0 \end{cases} \tag{8}$$

Clearly, inter-sampling information of the system (3) is maintained by (8).

**Remark 1.** It is clear that the discrete-time system (8) is different from the one in (5). In fact, the system (5) is obtained by a standard discretization technique, while the system (8) is not. Thus, we refer to the discretization technique used in (8) as a non-standard discretization method. It should be pointed out that the discrete-time system (8) allows the sampling to be non-uniform, while only uniform sampling applies in the system (5).

Replacing  $t_{k+1}$  and  $t_k$  with  $t$  in (7), respectively, one has:

$$x(t) = x(t_k) + \int_{t_k}^t X(s)ds = x(t_{k+1}) - \int_t^{t_{k+1}} X(s)ds$$

for  $t \in [t_k, t_{k+1}), k = 1, 2, \dots$ , where  $X(s) = Ax(s) + A_1x(t_k)$ .

### 2.1. A New Sampled-Data-Based Integral Inequality

In the following, we first introduce a new sampled-data-based integral inequality, which is useful in the derivative of our main results.

**Lemma 1.** For a given matrix  $Q > 0$  of appropriate dimension, we have:

$$\begin{aligned} & \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} [x(s) - x(t_k)]^T Q [x(s) - x(t_k)] ds + \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} [x(t_{k+1}) - x(s)]^T Q [x(t_{k+1}) - x(s)] ds \\ & \leq \frac{3}{4} h(k) \int_{t_k}^{t_{k+1}} X^T(u) Q X(u) du \end{aligned}$$

**Proof.**

$$\begin{aligned} & \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} [x(s) - x(t_k)]^T Q [x(s) - x(t_k)] ds + \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} [x(t_{k+1}) - x(s)]^T Q [x(t_{k+1}) - x(s)] ds \\ & = \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} \left[ \int_{t_k}^s X(u) du \right]^T Q \left[ \int_{t_k}^s X(u) du \right] ds + \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} \left[ \int_s^{t_{k+1}} X(u) du \right]^T Q \left[ \int_s^{t_{k+1}} X(u) du \right] ds \\ & \leq \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} \left\{ (s - t_k) \int_{t_k}^s X^T(u) Q X(u) du \right\} ds + \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} \left\{ (t_{k+1} - s) \int_s^{t_{k+1}} X^T(u) Q X(u) du \right\} ds \end{aligned}$$

From the integral region of the above two integrals, which are given in Figure 1, we can obtain that:

$$\begin{aligned}
 & \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} [x(s) - x(t_k)]^T Q [x(s) - x(t_k)] ds + \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} [x(t_{k+1}) - x(s)]^T Q [x(t_{k+1}) - x(s)] ds \\
 \leq & \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} \left\{ (s - t_k) \int_{t_k}^s X^T(u) Q X(u) du \right\} ds + \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} \left\{ (t_{k+1} - s) \int_s^{t_{k+1}} X^T(u) Q X(u) du \right\} ds \\
 = & \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} \left\{ \int_u^{t_{k+1}} (s - t_k) X^T(u) Q X(u) ds \right\} du + \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} \left\{ \int_{t_k}^u (t_{k+1} - s) X^T(u) Q X(u) ds \right\} du \\
 = & \frac{1}{2h(k)} \int_{t_k}^{t_{k+1}} [h^2(k) - (u - t_k)^2] X^T(u) Q X(u) du + \frac{1}{2h(k)} \int_{t_k}^{t_{k+1}} [h^2(k) - (t_{k+1} - u)^2] X^T(u) Q X(u) du \\
 = & \frac{1}{2h(k)} \int_{t_k}^{t_{k+1}} [2h^2(k) - (u - t_k)^2 - (t_{k+1} - u)^2] X^T(u) Q X(u) du \\
 \leq & \frac{3}{4} h(k) \int_{t_k}^{t_{k+1}} X^T(u) Q X(u) du
 \end{aligned}$$

In the process of the above enlargements, we use the following fact:

$$-(u - t_k)^2 - (t_{k+1} - u)^2 = -2\left(u - \frac{t_k + t_{k+1}}{2}\right)^2 - \frac{1}{2}(t_{k+1} - t_k)^2 \leq -\frac{1}{2}h^2(k).$$

Thus, the proof is completed.  $\square$

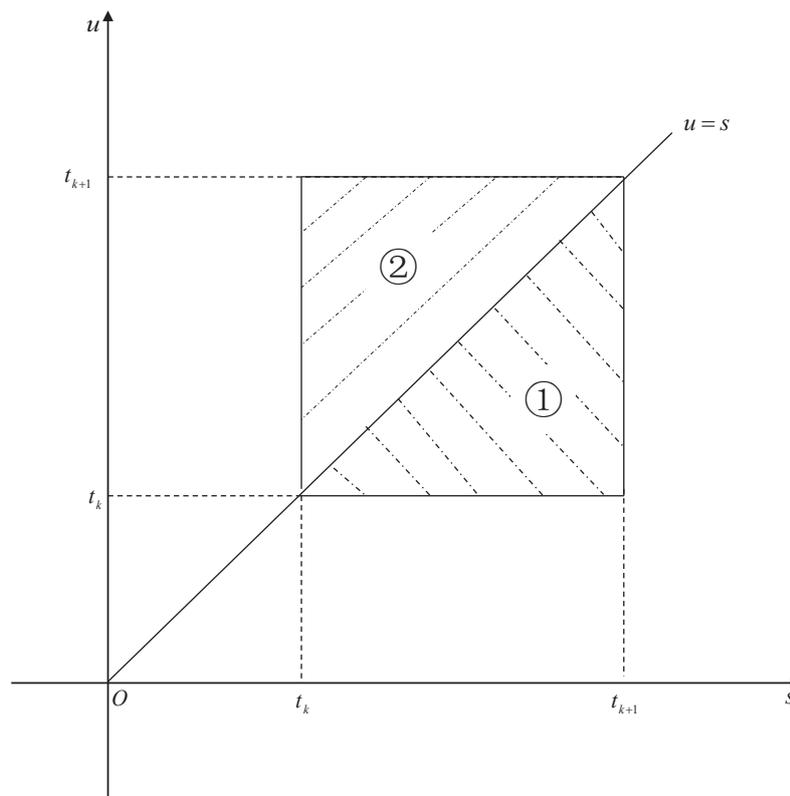


Figure 1. The integral regions.

Lemma 1 presents an integral inequality for the sum of two integral terms. Since these integral terms are related to sample-data information, Lemma 1 is called a sampled-data-based integral inequality, which plays an important role in the proof of the main results.

2.2. A Stability Criterion

Then, based on Lemma 1, we state the following result.

**Proposition 1.** For given scalars  $h_M \geq h_m > 0$ , the system (3) is asymptotically stable if there exist some matrices  $P > 0, Q > 0, M_1, M_2$  of appropriate dimensions such that the following LMI holds for  $h(k) = h_M$  and  $h(k) = h_m$ , simultaneously.

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & h(k)A_1^T Q \\ \Xi_{12}^T & \Xi_{22} & \Xi_{23} & 0 \\ \Xi_{13}^T & \Xi_{23}^T & -2Q & h(k)A^T Q \\ h(k)QA_1 & 0 & h(k)QA & -\frac{4}{3}Q \end{bmatrix} < 0 \tag{9}$$

where:

$$\begin{aligned} \Xi_{11} &= -P - Q - M_1^T(I + h(k)A_1) - (I + h(k)A_1)^T M_1 \\ \Xi_{12} &= M_1^T - (I + h(k)A_1)^T M_2 \\ \Xi_{13} &= Q - h(k)M_1^T A \\ \Xi_{22} &= P - Q + M_2^T + M_2 \\ \Xi_{23} &= Q - h(k)M_2^T A \end{aligned}$$

**Proof.** Choose the Lyapunov function candidate  $V(t_k)$  for (8) as  $V(t_k) = x^T(t_k)Px(t_k)$ , where  $P > 0$ . The forward difference of  $V(t_k)$  can be calculated as:

$$\begin{aligned} \Delta V(t_k) &= x^T(t_{k+1})Px(t_{k+1}) - x^T(t_k)Px(t_k) \\ &= x^T(t_{k+1})Px(t_{k+1}) - x^T(t_k)Px(t_k) \\ &\quad + \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} [x(s) - x(t_k)]^T Q [x(s) - x(t_k)] ds + \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} [x(t_{k+1}) - x(s)]^T Q [x(t_{k+1}) - x(s)] ds \\ &\quad - \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} [x(s) - x(t_k)]^T Q [x(s) - x(t_k)] ds - \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} [x(t_{k+1}) - x(s)]^T Q [x(t_{k+1}) - x(s)] ds \end{aligned}$$

where  $Q > 0$ . By Lemma 1, we have:

$$\begin{aligned} \Delta V(t_k) &\leq x^T(t_{k+1})Px(t_{k+1}) - x^T(t_k)Px(t_k) + \frac{3}{4}h(k) \int_{t_k}^{t_{k+1}} X^T(u)QX(u)du \\ &\quad - \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} [x(s) - x(t_k)]^T Q [x(s) - x(t_k)] ds - \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} [x(t_{k+1}) - x(s)]^T Q [x(t_{k+1}) - x(s)] ds \\ &= x^T(t_{k+1})Px(t_{k+1}) - x^T(t_k)Px(t_k) + \frac{3}{4}h(k) \int_{t_k}^{t_{k+1}} X^T(s)QX(s)ds \\ &\quad - \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} [x(s) - x(t_k)]^T Q [x(s) - x(t_k)] ds - \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} [x(t_{k+1}) - x(s)]^T Q [x(t_{k+1}) - x(s)] ds \\ &\quad + 2 [x^T(t_k)M_1^T + x^T(t_{k+1})M_2^T] \left[ x(t_{k+1}) - (I + h(k)A_1)x(t_k) - \int_{t_k}^{t_{k+1}} Ax(s)ds \right] \\ &= \frac{1}{h(k)} \int_{t_k}^{t_{k+1}} \begin{bmatrix} x^T(t_k) & x^T(t_{k+1}) & x^T(s) \end{bmatrix} \hat{\Xi} \begin{bmatrix} x(t_k) \\ x(t_{k+1}) \\ x(s) \end{bmatrix} ds \end{aligned} \tag{10}$$

where  $M_1$  and  $M_2$  are two free-weighting  $n \times n$  matrices,

$$\hat{\Xi} = \begin{bmatrix} \hat{\Xi}_{11} & \hat{\Xi}_{12} & \hat{\Xi}_{13} \\ \hat{\Xi}_{12}^T & \hat{\Xi}_{22} & \hat{\Xi}_{23} \\ \hat{\Xi}_{13}^T & \hat{\Xi}_{23}^T & \hat{\Xi}_{33} \end{bmatrix}$$

with:

$$\begin{aligned}
 \hat{\Xi}_{11} &= -P - Q + \frac{3}{4}h^2(k)A_1^TQA_1 - M_1^T(I + h(k)A_1) - (I + h(k)A_1)^TM_1 \\
 \hat{\Xi}_{12} &= M_1^T - (I + h(k)A_1)^TM_2 \\
 \hat{\Xi}_{13} &= \frac{3}{4}h^2(k)A_1^TQA + Q - h(k)M_1^TA \\
 \hat{\Xi}_{22} &= P - Q + M_2^T + M_2 \\
 \hat{\Xi}_{23} &= Q - h(k)M_2^TA \\
 \hat{\Xi}_{33} &= \frac{3}{4}h^2(k)A^TQA - 2Q
 \end{aligned}$$

If the inequality (9) holds for  $h(k) = h_M$  and  $h(k) = h_m$ , simultaneously, then by the Schur complement, one has  $\hat{\Xi} < 0$ , which means that there exists a constant  $\lambda > 0$  such that  $\Delta V(t_k) \leq -\lambda x^T(t_k)x(t_k)$ . Therefore, the system (8) is asymptotically stable. This completes the proof.  $\square$

**Remark 2.** Proposition 1 is obtained based on a simple Lyapunov function rather than a complicated Lyapunov–Krasovskii functional involved if using the input delay approach and the impulsive model approach. Moreover, from (10), one can find that two free-weighting  $n \times n$  matrices  $M_1$  and  $M_2$  are introduced to build the relationship among  $x(t_k)$ ,  $x(t_{k+1})$  and  $x(s)$ .

**Remark 3.** For Proposition 1, the total number of scalar decision variables is  $\mathcal{M} = 2 \times \frac{n(n+1)}{2} + 2n^2 = 3n^2 + n$ , and the total row size of the LMIs is  $\mathcal{L} = 5n$ . The numerical complexity of Proposition 1 is proportional to  $\mathcal{L}\mathcal{M}^3$  [27]. From Table 1, one can find that the stability criterion provided by Proposition 1 requires less scalar decision variables and a lesser row size of the LMIs compared with some existing stability criteria.

**Table 1.** The total number of scalar decision variables and the total row size of LMIs.

Method	[19]	[5]	[12]	[9]	[28]	Proposition 1
$\mathcal{M}$	$4n^2 + n$	$8n^2 + n$	$5n^2 + 2n$	$5n^2 + 2n$	$10.5n^2 + 3.5n$	$3n^2 + n$
$\mathcal{L}$	$7n$	$10n$	$11n$	$7n$	$9n$	$5n$
$\mathcal{L}\mathcal{M}^3$	$7n(4n^2 + n)^3$	$10n(8n^2 + n)^3$	$11n(5n^2 + 2n)^3$	$7n(5n^2 + 2n)^3$	$9n(10.5n^2 + 3.5n)^3$	$5n(3n^2 + n)^3$

### 2.3. A Robust Stability Criterion

If there exist time-varying norm-bounded parameter uncertainties in (1), then (1) becomes:

$$\begin{cases} \dot{x}(t) = (A + \Delta A(t))x(t) + (B + \Delta B(t))u(t), \\ x(0) = x_0 \end{cases} \tag{11}$$

where  $\Delta A(t)$  and  $\Delta B(t)$  represent norm-bounded parameter uncertainties of the form:

$$[\Delta A(t) \ \Delta B(t)] = DF(t)[E \ E_b] \tag{12}$$

where  $D$ ,  $E$  and  $E_b$  are known real constant matrices with compatible dimensions; the unknown time-varying matrix  $F(t) \in \mathbb{R}^{l_1 \times l_2}$  satisfies:

$$F^T(t)F(t) \leq I \text{ for all } t \geq 0. \tag{13}$$

Similar to (3), the closed-loop system of (11) is given by:

$$\begin{cases} \dot{x}(t) = (A + \Delta A(t))x(t) + (A_1 + \Delta A_1(t))x(t_k), \\ x(0) = x_0 \end{cases} \tag{14}$$

where  $\Delta A_1(t) = \Delta B(t)K$ .

Extending Proposition 1 to the uncertain system (14) gives the following robust stability criterion.

**Proposition 2.** For given scalars  $h_M \geq h_m > 0$ , the system (14) is robustly stable if there exist some matrices  $P > 0, Q > 0, R > 0, M_1, M_2$  of appropriate dimensions such that the following LMI holds for  $h(k) = h_m$  and  $h(k) = h_M$ , simultaneously.

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & h(k)A_1^T Q & -h(k)M_1^T D \\ \Psi_{12}^T & \Psi_{22} & \Psi_{23} & 0 & \Psi_{25} \\ \Psi_{13}^T & \Psi_{23}^T & \Psi_{33} & h(k)A^T Q & 0 \\ h(k)QA_1 & 0 & h(k)QA & -\frac{4}{3}Q & h(k)QD \\ -h(k)D^T M_1 & \Psi_{25}^T & 0 & h(k)D^T Q & -h(k)R \end{bmatrix} < 0 \tag{15}$$

where:

$$\begin{aligned} \Psi_{11} &= -P - Q - M_1^T(I + h(k)A_1) - (I + h(k)A_1)^T M_1 + h(k)E_b^T R E_b \\ \Psi_{12} &= M_1^T - (I + h(k)A_1)^T M_2 \\ \Psi_{22} &= P - Q + M_2^T + M_2 \\ \Psi_{13} &= Q - h(k)M_1^T A + h(k)E_b^T R E \\ \Psi_{23} &= Q - h(k)M_2^T A \\ \Psi_{25} &= -h(k)M_2^T D \\ \Psi_{33} &= -2Q + h(k)E^T R E \end{aligned}$$

**Proof.** Replacing  $A$  and  $A_1$  in Proposition 1 with  $A + \Delta A(t)$  and  $A_1 + \Delta A_1(t)$ , respectively, one can see that the system (14) is robustly stable if the following inequality holds:

$$\Xi + h(k)Y_1^T F(t)Y_2 + h(k)Y_2^T F^T(t)Y_1 < 0$$

where:

$$Y_1 = [-D^T M_1 \quad -D^T M_2 \quad 0 \quad D^T Q^T], \quad Y_2 = [E_b \quad 0 \quad E \quad 0].$$

Then, the above matrix inequality is inferred by:

$$\Xi + h(k)Y_1^T R^{-1}Y_1 + h(k)Y_2^T R Y_2 < 0 \tag{16}$$

for any matrix  $R > 0$  of appropriate dimensions. By the Schur complement, (16) is equivalent to LMI (15). Thus, if the inequality (15) is satisfied, then so is (16), leading to robust stability of the system (14), which completes the proof. □

### 3. Numerical Examples

**Example 1.** Consider the following much-studied problem [29]:

$$\dot{x}(t) = -x(t_k), \quad t_k \leq t < t_{k+1}, \quad k = 0, 1, 2, \dots \tag{17}$$

The system (17) remains stable for all constant samplings less than two and becomes unstable for samplings greater than two [5]. By Proposition 1 in [5] and Proposition 1 in this paper,  $h_{max} = 2$  is found, but less scalar decision variables are needed by Proposition 1 in this paper from Table 1.

**Example 2.** To illustrate the proposed stability criterion of linear systems with a constant/time-varying sampling, we consider a sampled-data system as follows: [30]

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} Kx(t_k) \quad (18)$$

where  $K = - \begin{bmatrix} 3.75 & 11.5 \end{bmatrix}$ . If we transfer the above system to the following system with an input delay:

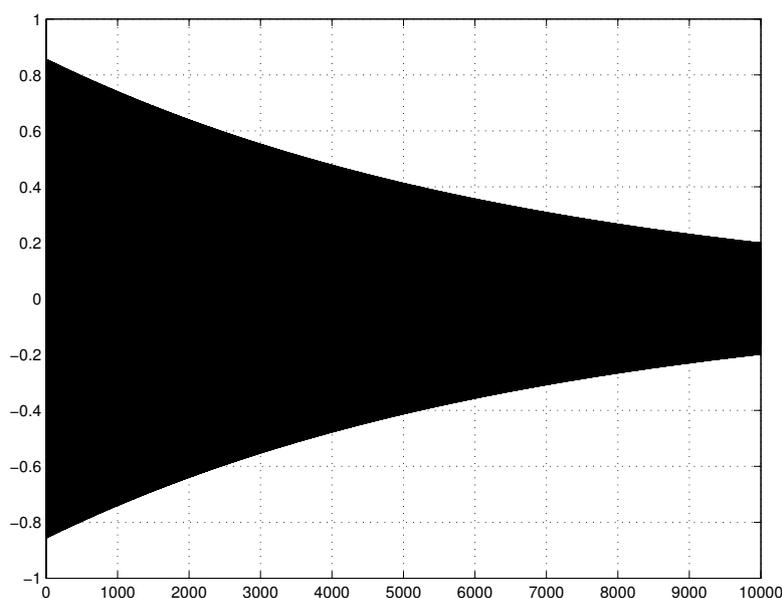
$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} Kx(t - \tau(t)) \quad (19)$$

where  $\tau(t) = t - t_k$ , then the system (19) becomes unstable for  $\tau(t) > 1.167144$ . This is because the analytical delay limit for the stability of (19) can be calculated as  $\tau^{\text{analytical}} = 1.167144$  for constant delay  $\tau(t) = \tau$ . Therefore, if we transfer (18) to a delay system (19), then we cannot guarantee the stability for samplings greater than 1.167144. Figure 2 shows the trajectories of states of (19) with constant delay  $\tau = 1.167144$ . Moreover, it was found by applying an eigenvalue-based analysis that the system (18) remains stable for all constant samplings less than 1.729 and becomes unstable for samplings greater than 1.729 [28].

To show the effectiveness of the proposed method, we now consider two cases of sampling.

Under Case I, the sampling is a constant. Applying Proposition 1, the maximum allowable sampling period can be obtained as  $h_{\max} = 1.6962$ , which means that the system (18) is asymptotically stable for the constant sampling  $h = 1.6962$ , which is demonstrated by Figure 3. To further show the effectiveness of the proposed method, we make a comparison with some existing results. The obtained maximum allowable sampling periods  $h$  by some existing methods in [5,12,19,31] are listed in Table 2. Moreover, the total numbers of scalar decision variables  $\mathcal{M}$  and the total row size  $\mathcal{L}$  of the corresponding LMIs are also given in this table. From this table, one can see that Proposition 1 outperforms [5,19,31] from both the maximum allowable period  $h$  and the computational complexity. Although the obtained maximum  $h$  by Proposition 1 is smaller than that by [12,28], the computational complexity of Proposition 1 is less than that in [12,28]. Therefore, as a tradeoff between the maximum sampling period  $h$  and the complexity, Proposition 1 is more effective than [12,28].

Under Case II, the sampling is time-varying. Then, by Proposition 1, we calculate the maximum allowable sampling interval that retains the stability of the system (18). For different  $h_m$ , the obtained results are listed in Table 3. However, these results cannot be derived using the standard discretization method [3,24–26].



**Figure 2.** The trajectories of states  $x_1$  and  $x_2$  of (19) with constant delay  $h = 1.167144$ .

Table 2.  $h$ ,  $\mathcal{M}$  and  $\mathcal{L}$  of LMIs.

Method	[19]	[5]	[12]	[31]	[28]	Proposition 1
$h$	1.3277	1.69	1.723	1.69	1.72	1.6962
$\mathcal{M}$	18	34	24	24	49	14
$\mathcal{L}$	14	20	22	14	18	10
$\mathcal{LM}^3$	81,648	786,080	304,128	193,536	2,117,682	27,440

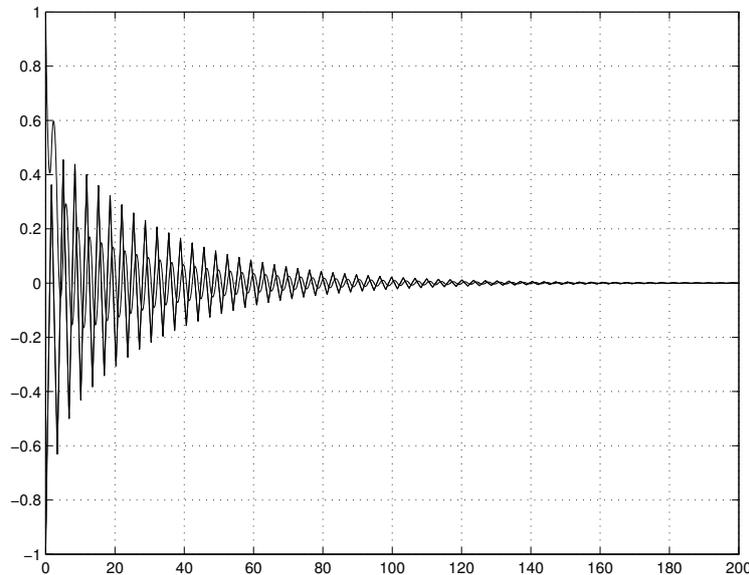


Figure 3. The trajectories of states  $x_1$  and  $x_2$  of (18) with constant sampling 1.6962.

Table 3. Maximum bounds of  $h_M$  for different  $h_m$  by Proposition 1.

$h_m$	0.001	0.01	0.1	0.5	1.0	1.5	1.69
$h_M$	1.6066	1.6180	1.6507	1.6868	1.6937	1.6957	1.6962

Example 3. Let us consider the uncertain sampled-data system (11) with matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -3.3235 & -0.0212 & 0.0184 & 0.0030 & -5.3449 & -0.8819 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0.0184 & 0.0030 & -118.1385 & -0.11188 & 5.3465 & 0.8822 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -0.0114 & -0.0019 & 0.0114 & 0.0019 & -3.3501 & -0.5454 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0.003445 & 0 & -0.00344628 & 0 & 0.00213 \end{bmatrix}^T,$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1 \end{bmatrix}, E = \begin{bmatrix} 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 \end{bmatrix},$$

$$E_b = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$K = \begin{bmatrix} -10,100 & 5034 & -3634 & 22,432 & 5426 & -7681 \end{bmatrix},$$

borrowed from the offshore steel jacket platforms [32]. We then apply Proposition 2 to obtain upper bounds  $h_M$  for different lower bounds  $h_m$  of samplings. The results are given in Table 4, which show the effectiveness of the given robust stability criterion for uncertain sampled-data systems.

**Table 4.** Maximum bounds of  $h_M$  for different  $h_m$  by Proposition 2.

$h_m$	0.1	0.11	0.12	0.13	0.14
$h_M$	0.1293	0.1408	0.1467	0.1493	0.1499

#### 4. Conclusions

The stability problem has been studied in this paper for linear systems under time-varying sampling. First, by using a non-standard discretization method, the sampled-data system has been represented by a linear discrete-time system. Then, by introducing a new sampled-data-based integral inequality, several sufficient conditions on stability and robust stability for sampled-data systems and the uncertain sampled-data systems, respectively, have been provided in terms of LMIs. These criteria are of lower computational complexity since less scalar decision variables and smaller row sizes of the LMIs are required. Finally, the effectiveness of the proposed criteria has been demonstrated by three numerical examples.

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