

Fault-Tolerant Consensus Control of Positive Networked Systems

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Abstract: In this paper, we explore the consensus of positive networked systems with actuator faults. Firstly, the undirected and strongly connected topology is established with graph theory. The positive system theory is used to analyze the positive consensus of the closed-loop networked systems. State feedback gains are derived utilizing Algebraic Riccati Inequalities. Bounded multiplicative faults are regarded as uncertainties in the system matrix, while treating additive faults as external disturbances. Further, this transformation refocuses the analysis on the consensus problem with an L_2 -gain. Subsequently, the Genetic Algorithm is employed to optimize the L_2 performance criteria. Finally, the effectiveness of the proposed theory is validated through simulations involving both single-input electric circuit systems and multi-input networked systems.

Keywords: positive networked system; actuator fault; graph theory; L_2 -gain; genetic algorithm

1. Introduction

With the development of information technology, the theory of networked systems has made significant progress. Networked systems treat each node as an individual agent, utilizing concepts from graph theory to describe the exchange of information among nodes. Networked systems are extensively applied in various fields [1–4], including biology, traffic, electrical engineering and robots. Research on the coordination of networked systems primarily focuses on topics such as consensus control [5–7], formation control [8–10], and containment control [11–13].

In networked systems, individuals work successfully to accomplish complex tasks through effective allocation. The consensus of agents is the key to achieving coordinated control in networked systems. This consensus is described using relevant knowledge in graph theory to illustrate the connectivity relationships among nodes. The communication topology is primarily categorized into directed and undirected networks [14]. Gong et al. [15] design the distributed observer for multi-agent systems, improve the consensus performance with prescribed-time control. The propose prescribed-time consensus control ensures that the consensus error and estimate error are prescribed-time stable. In [16], the event-based controller is designed to ensure the consensus of networked system. Based on the general dynamic model of networked Euler-Lagrange systems, an event-based communication and an event-based controller can achieve the consensus of networked systems. He et al. [17] propose sliding mode control for achieving consensus of networked control systems. An integral sliding mode protocol is developed for a nominal networked system to ensure the consensus of networked systems within a finite time.

Notably, the states, data, and information in networked systems exhibit positive characteristics. Positive system means that, for non-negative inputs and initial conditions, their state variables and outputs are always greater than zero in [18]. Currently, there is extensive research on positive single systems [19–22], but there are few achievements in designing positive networked systems. By improving the current theory of positive single systems, it can be extended to the consensus analysis of positive networked systems. Finite-time control with L_1 -gain is proposed for positive Markov jump systems in [23].



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The conditions of positive system and finite-time L_1 -gain are guaranteed by the design's even-triggered controller. Cui et al. [24] investigate the positive problem of homogeneous systems with input delay. They establish positivity and stability for coupled equations with different degrees of homogeneity. The positive edge consensus of networked systems under actuator saturation is addressed in [25]. The study establishes an edge network based on the nodal network and provides ranges for the algebraic connectivity of the edge network. It conducts a comprehensive analysis of global consensus convergence.

Additionally, the transmission of information in networked systems is susceptible to high-frequency signal interference, and variations in signals within communication protocols could lead to sensor and actuator failures. In engineering practice, if faults are not detected and resolved promptly, there is a high likelihood of mission failure and substantial losses. Therefore, enhancing the safety, reliability, and fault tolerance of networked systems is of paramount importance. The consensus of a multi-agent system with actuator faults is investigated in [26]. The approach uses some first-order filters to estimate the desired signal, while smooth functions are applied to compensate for bounded actuator faults. This ensures the asymptotical stability of a closed-loop system. Chen et al. [27] investigate the tracking problem for nonlinear networked systems under the influence of actuator faults and data dropouts. Based on the linearized dynamic model, an iterative fault-tolerant control is introduced, relying on randomly received input/output data. In [28], the fault-tolerant consensus of multi-agent systems with input saturation is addressed. A improved low-and-high gain feedback controller is introduced to handle actuator faults and saturation. By combining adaptive techniques, the consensus error can converge to the region near the origin within a finite time.

This paper investigates the positive consensus of a networked system (PCNS) subject to actuator faults. Both single-input and multi-input networked systems are considered. Our contributions can be summarized as follows:

1. Utilizing the theories of networked systems and positive systems, control protocols are designed based on state feedback. Subsequently, this paper discusses the positive consensus problem with an L_2 -gain under actuator faults.
2. Actuator faults are categorized into multiplicative and additive faults. Bounded multiplicative faults are treated as uncertainties in the system matrices, while additive faults are considered external disturbances. Subsequently, the actuator fault problem is transformed into addressing external disturbances with an L_2 -gain, with a specific focus on analyzing disturbance rejection performance.
3. The proposed approach focuses on the inherent performance of networked systems rather than relying on techniques like observers for fault detection. Positive consensus constraints are derived using Algebraic Riccati Inequalities, and the Genetic Algorithm is employed to solve the nonlinear optimization problem.

This article is organized as follows: Section 2 provides a review of essential preliminary knowledge. In Section 3, we present the main results of the networked system, including positive consensus and the conditions for L_2 -gain design and analysis. Section 4 provides numerical examples that include both single-input electric circuit systems and multi-input networked systems. Finally, the conclusion is summarized in Section 5.

Notation: \mathbb{R}^n denotes the n -dimensional real vector, A^T is the transpose of matrix A , $A \succ 0$ ($A \prec 0$) denotes that all elements of A are positive (negative). $A \otimes B$ denotes the Kronecker product of matrices A and B . $L_2[0, +\infty)$ means the space of square integrable vector functions over $[0, +\infty)$.

2. Preliminaries and Problem Formulation

2.1. Preliminaries

The networked systems with communication topology that can be characterized by a undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, $\mathcal{V} = \{1, 2, \dots, N\}$ is the vertex set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set. Two agents i and j can establish communication if and only if $(i, j) \in \mathcal{E}$. The adjacency

matrix $\Gamma \in \mathbb{R}^{N \times N}$ of graph \mathcal{G} is defined as $[\Gamma]_{ij} = [\Gamma]_{ji} = 1$ if $(j, i) \in \mathcal{E}$ and $[\Gamma]_{ij} = 0$ otherwise. The Laplacian matrix $L \in \mathbb{R}^{N \times N}$ of graph \mathcal{G} is defined as

$$L = \begin{bmatrix} \sum_{j=1}^N [\Gamma]_{1j} & -[\Gamma]_{12} & \cdots & -[\Gamma]_{1N} \\ -[\Gamma]_{21} & \sum_{j=1}^N [\Gamma]_{2j} & \cdots & -[\Gamma]_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -[\Gamma]_{N1} & -[\Gamma]_{N2} & \cdots & \sum_{j=1}^N [\Gamma]_{Nj} \end{bmatrix} \tag{1}$$

Because the Laplacian matrix L is symmetric, its eigenvalues are all real numbers, denoted as $\lambda_i, i \in N$ and ordered as $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$.

Consider the following linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2}$$

where $x(t)$ and $u(t)$ represent the system state and input, respectively. Some essential definitions and lemmas are presented below.

Definition 1 ([29]). Matrix $A \in \mathbb{R}^{n \times n}$ is called Metzler, if all its off-diagonal elements are non-negative.

Definition 2 ([30]). Square matrix A is called Hurwitz matrix, if every eigenvalue of A has strictly negative real part.

Definition 3 ([31]). The system (2) is positive if the system state $x(t) \succeq 0, t \geq 0$ with initial condition $x_0 \succeq 0$ and $u(t) \succeq 0$.

Lemma 1 ([31]). The system (2) is positive if and only if matrix A is Metzler and matrix B is non-negative.

Lemma 2 ([32]). The matrix A is Hurwitz if and only if there is a positive definite symmetric matrix P with

$$A^T P + PA < 0 \tag{3}$$

2.2. Problem Formulation

Consider a networked system with N agents on an undirected graph \mathcal{G} ; the dynamic of agent i can be described by

$$\dot{x}_i(t) = Ax_i(t) + Bv_i(t) \tag{4}$$

where $x_i(t) = [x_{i1}, x_{i2}, \dots, x_{in}]^T \in \mathbb{R}^n$ and $v_i(t) \in \mathbb{R}^n$ are the state and input of agent i .

The following actuator fault is considered

$$v_i(t) = \rho_i(t)u_i(t) + w_i(t) \tag{5}$$

where $\rho_i(t)$ with $0.2 \leq \rho_i(t) \leq 1.0$ denotes the unknown time-varying actuator multiplicative fault, and $w_i(t)$ is the unknown time-varying actuator additive fault.

Remark 1. In this paper, only consider actuator efficiency factor $\rho_i(t)$ with $0.2 \leq \rho_i(t) \leq 1.0$; $\rho_i(t) < 0.2$ means that there is a major failure in the actuator, which may reduce system control performance and even lead to system instability. To improve the performance and sensitivity of the control system, we need to repair or replace the actuator in time.

Assumption 1. The additive fault $w_i(t)$ is a positive unknown time-varying function and bounded with $\int_0^{+\infty} w_i^T(\tau)w_i(\tau)d\tau \leq a$, where a is a positive scalar, i.e., $w_i(t) \in L_2[0, +\infty)$.

Definition 4 ([33]). *The networked system (2) is said to have L_2 -gain less than or equal to γ if there exists $\gamma > 0$ such that:*

(1) *The networked system (4) is asymptotically stable.*

(2) *When the external disturbance $w(t) \in L_2[0, +\infty)$, under zero initial conditions $x(0) = 0$, there holds*

$$\int_0^{+\infty} \|x(t)\|^2 dt \leq \gamma^2 \int_0^{+\infty} \|w(t)\|^2 dt \tag{6}$$

where $x(t)$ is the state of the networked system, and $w(t)$ is the external disturbance.

Objective PCNS: Consider the networked system (4) with the initial states $x_i(0) \succeq 0$, $i = 1, 2, \dots, N$. Design a state-feedback gain K to achieve the consensus of the networked system, i.e., $\lim_{t \rightarrow +\infty} (x_i(t) - x_j(t)) = 0, \forall i, j = 1, 2, \dots, N$, while having an L_2 -gain $\gamma > 0$. Moreover, the state of each agent is non-negative, that is, $x_i(t) \succeq 0, i = 1, 2, \dots, N$ with $t \geq 0$.

Consider the state-feedback protocol [34]

$$u_i(t) = K \sum_{j=1}^N [\Gamma]_{ij} (x_j - x_i), i = 1, 2, \dots, N \tag{7}$$

where K is the feedback control matrix, and we can define the following state and input of the networked system

$$\begin{aligned} x(t) &= [x_1^T(t), x_2^T(t), \dots, x_N^T(t)] \in \mathbb{R}^{nN} \\ u(t) &= [u_1^T(t), u_2^T(t), \dots, u_N^T(t)] \in \mathbb{R}^{nN} \\ w(t) &= [w_1^T(t), w_2^T(t), \dots, w_N^T(t)] \in \mathbb{R}^{nN} \end{aligned}$$

Then the closed-loop of networked system is formulated as

$$\dot{x}(t) = \mathbb{A}x(t) + \mathbb{B}w(t) \tag{8}$$

where $\mathbb{A} = I_n \otimes A - L \otimes \rho BK$ and $\mathbb{B} = I_n \otimes B$.

By expanding \mathbb{A} , the following matrix can be obtained

$$\mathbb{A} = \begin{bmatrix} A - \sum_{j=1}^N [\Gamma]_{1j} \rho BK & [\Gamma]_{12} \rho BK & \cdots & [\Gamma]_{1N} \rho BK \\ [\Gamma]_{21} \rho BK & A - \sum_{j=1}^N [\Gamma]_{2j} \rho BK & \cdots & [\Gamma]_{2N} \rho BK \\ \vdots & \vdots & \ddots & \vdots \\ [\Gamma]_{N1} \rho BK & [\Gamma]_{N2} \rho BK & \cdots & A - \sum_{j=1}^N [\Gamma]_{Nj} \rho BK \end{bmatrix} \tag{9}$$

In the following, we define $l_i = \sum_{j=1}^N [\Gamma]_{1j}$ and $l_{\max} = \max(l_i), i = 1, 2, \dots, N$.

3. Main Results

In this section, some theoretical results for the positive consensus of a networked system with L_2 -gain are analyzed based on graph theory and positive systems theory.

Proposition 1 ([35]). *Consider a networked system with an undirected and complete graph, PCNS is solved by the designed feedback gain K , satisfying the following conditions:*

- (1) $BK \succeq 0$.
- (2) $A - l_{\max} BK$ is Metzler.
- (3) $A - \lambda_i BK, i \in N \setminus \{1\}$ are Hurwitz.

Based on Lemma 1, the positivity of networked system can be assured by the statement \mathbb{A} is Metzler. According to Lemma 1, the matrix \mathbb{A} is Metzler if and only if the off-diagonal block matrices $[\Gamma]_{ij} \rho BK$ are non-negative and the diagonal block matrices $A - \sum_{j=1}^N [\Gamma]_{ij} \rho BK$ are Metzler. Therefore, $BK \succeq 0$ and $A - l_{\max} \rho BK$ can ensure that the

networked system is positive. Furthermore, the consensus is guaranteed by the Hurwitz matrices $A - \lambda_i BK, i \in N \setminus \{1\}$. We utilize the Riccati method [34] to address the PCNS, enhancing flexibility in establishing the controller gain by the Algebraic Riccati Inequality (ARI).

Theorem 1. *The PCNS with L_2 -gain can be achieved with the control protocol $K = B^T P$ if and only if there are positive definite matrix P values and scalars $\zeta, \epsilon > 0$ such that*

$$B \succeq 0, BK \succeq 0 \tag{10}$$

$$A - l_{\max} \rho_{\max} BK + \zeta I_n \succeq 0 \tag{11}$$

$$A^T P + PA - 2\rho_{\min} \lambda_2 PBB^T P + \frac{1}{\gamma^2} \lambda_N PBB^T P \leq -\epsilon P \tag{12}$$

Proof. Proof can be carried out through three steps.

(1) Positivity analysis

According to Lemma 1, the positivity of the networked system (8) can be ensured if and only if $B \succeq 0, BK \succeq 0$ and the matrix $A - l_{\max} \rho_{\max} BK$ is Metzler.

(2) Consensus analysis

Construct a Lyapunov function candidate

$$V = x^T (L \otimes P)x \tag{13}$$

Differentiating V , we have

$$\begin{aligned} \dot{V} &= x^T \left(L \otimes (A^T P + PA) - 2\rho L^2 \otimes PBB^T P \right) x + 2x^T (L \otimes PB)w \\ &= x^T \left(L \otimes (A^T P + PA) - 2\rho L^2 \otimes PBB^T P \right) x + 2x^T (L \otimes PB)w + \gamma^2 w^T w - \gamma^2 w^T w \\ &= x^T \left(L \otimes (A^T P + PA) - 2\rho L^2 \otimes PBB^T P \right) x + \gamma^2 w^T w + \frac{1}{\gamma^2} x^T (L \otimes PB)(L \otimes PB)^T x \\ &\quad - \gamma^2 \left(w - \frac{1}{\gamma^2} (L \otimes B^T P)x \right)^T \left(w - \frac{1}{\gamma^2} (L \otimes B^T P)x \right) \\ &\leq x^T \left(L \otimes (A^T P + PA) - 2\rho L^2 \otimes PBB^T P \right) x + \gamma^2 w^T w + \frac{1}{\gamma^2} x^T \left(L^2 \otimes PBB^T P \right) x \end{aligned} \tag{14}$$

Because L is a real symmetric matrix, one can conclude that there is an unitary matrix U such that

$$L = U^T \Lambda U$$

where $\Lambda = \text{diag}\{0, \lambda_2, \dots, \lambda_N\}$.

When the additive fault $w = 0$, let $x = (U^T \otimes I_n)\tilde{x}$; in the light of (12), we have

$$\begin{aligned} \dot{V} &\leq x^T \left(L \otimes (A^T P + PA) - 2\rho L^2 \otimes PBB^T P + \frac{1}{\gamma^2} (L^2 \otimes PBB^T P) \right) x \\ &= \tilde{x}^T \left(\Lambda \otimes (A^T P + PA) - 2\rho \Lambda^2 \otimes PBB^T P + \frac{1}{\gamma^2} \Lambda^2 \otimes PBB^T P \right) \tilde{x} \\ &= \sum_{j=1}^N \lambda_i \tilde{x}_i^T \left(A^T P + PA - 2\rho \lambda_i PBB^T P + \frac{1}{\gamma^2} \lambda_i PBB^T P \right) \tilde{x}_i \\ &\leq \sum_{j=1}^N \lambda_i \tilde{x}_i^T \left(A^T P + PA - 2\rho \lambda_2 PBB^T P + \frac{1}{\gamma^2} \lambda_N PBB^T P \right) \tilde{x}_i \\ &\leq -\epsilon \sum_{j=1}^N \lambda_i \tilde{x}_i^T P \tilde{x}_i \\ &\leq -\epsilon x^T (L \otimes P) x \end{aligned} \tag{15}$$

According to the Lyapunov stability theory, we have

$$\lim_{t \rightarrow +\infty} V = \lim_{t \rightarrow +\infty} x^T (L \otimes P) x = \frac{1}{2} \lim_{t \rightarrow +\infty} \sum_{j=1}^N \sum_{j=1}^N \Gamma_{ij} (x_i(t) - x_j(t))^T P (x_i(t) - x_j(t)) = 0$$

We can obtain

$$\lim_{t \rightarrow +\infty} \sum_{j=1}^N \sum_{j=1}^N \Gamma_{ij} \|x_i(t) - x_j(t)\|^2 = 0$$

When $(i, j) \in \mathcal{E}$, let $\underline{\Gamma} = \min_{(i,j) \in \mathcal{E}} \Gamma_{ij}$; we have

$$\lim_{t \rightarrow +\infty} \|x_i(t) - x_j(t)\| = 0 \tag{16}$$

Consider the undirected graph is connected; for all $(i, j) \notin \mathcal{E}$, there is a simple path $i, k_1, k_2, \dots, k_m, j$ (from i to j) such that $(i, k_1) \in \mathcal{E}, (k_1, k_2) \in \mathcal{E}, \dots, (k_m, j) \in \mathcal{E}$, then the following inequality holds

$$\begin{aligned} \|x_i(t) - x_j(t)\| &\leq \|x_i(t) - x_{k_1}(t)\| + \|x_{k_1}(t) - x_{k_2}(t)\| + \dots + \|x_{k_m}(t) - x_j(t)\| \\ &\leq \sum_{p < q} \|x_p(t) - x_q(t)\| \end{aligned}$$

Moreover, we have

$$\|x_i(t) - x_j(t)\| \leq \sum_{p < q} \|x_p(t) - x_q(t)\| \leq N \sqrt{\frac{\sum_{p < q} \|x_p(t) - x_q(t)\|^2}{N}}$$

Thus, we can obtain

$$\lim_{t \rightarrow +\infty} \|x_i(t) - x_j(t)\| = 0 \tag{17}$$

Consequently, it can be found that $\lim_{t \rightarrow +\infty} \|x_i(t) - x_j(t)\| = 0$, the positive consensus of the networked system (2) is achieved by (10)–(12).

(3) L_2 -gain analysis

The following is the L_2 -gain analysis for system (5) with the additive fault $w \neq 0$ under zero initial state. Combining (14) and (15), the following results can be obtained:

$$\dot{V} \leq -\epsilon x^T (L \otimes P) x + \gamma^2 w^T w \tag{18}$$

Furthermore, we have

$$\begin{aligned} \dot{V} &\leq -\epsilon x^T(L \otimes P)x + \gamma^2 w^T w \\ &= -\frac{\epsilon}{2} \sum_{j=1}^N \sum_{i=1}^N \Gamma_{ij} (x_i(t) - x_j(t))^T P (x_i(t) - x_j(t)) + \gamma^2 w^T w \\ &\leq -\frac{\epsilon}{2} \lambda_{\min}(P) \sum_{j=1}^N \sum_{i=1}^N \Gamma_{ij} \|x_i(t) - x_j(t)\|^2 + \gamma^2 w^T w \end{aligned} \tag{19}$$

Integrating (19) yields

$$V(\infty) - V(0) \leq -\frac{\epsilon}{2} \lambda_{\min}(P) \int_0^{+\infty} \sum_{j=1}^N \sum_{i=1}^N \Gamma_{ij} \|x_i(t) - x_j(t)\|^2 dt + \gamma^2 \int_0^{+\infty} \|w\|^2 dt \tag{20}$$

Consider $V(0) = 0$ and $V(t) > 0$, then (20) can be written as

$$\frac{\epsilon}{2} \lambda_{\min}(P) \int_0^{+\infty} \sum_{j=1}^N \sum_{i=1}^N \Gamma_{ij} \|x_i(t) - x_j(t)\|^2 dt \leq \gamma^2 \int_0^{+\infty} \|w\|^2 dt \tag{21}$$

Similarly, when $(i, j) \in \mathcal{E}$, let $\underline{\Gamma} = \min_{(i,j) \in \mathcal{E}} \Gamma_{ij}$, so we have

$$\int_0^{+\infty} \|x_i(t) - x_j(t)\|^2 dt \leq \frac{2\gamma^2}{\underline{\Gamma} \epsilon \lambda_{\min}(P)} \int_0^{+\infty} \|w\|^2 dt \tag{22}$$

Otherwise, for all $(i, j) \notin \mathcal{E}$, there is a simple path $i, k_1, k_2, \dots, k_m, j$ (from i to j) such that $(i, k_1) \in \mathcal{E}, (k_1, k_2) \in \mathcal{E}, \dots, (k_m, j) \in \mathcal{E}$, then we have

$$\begin{aligned} \|x_i(t) - x_j(t)\| &\leq \|x_i(t) - x_{k_1}(t)\| + \|x_{k_1}(t) - x_{k_2}(t)\| + \dots + \|x_{k_m}(t) - x_j(t)\| \\ &\leq \sum_{p < q} \|x_p(t) - x_q(t)\| \end{aligned}$$

Moreover, we have

$$\|x_i(t) - x_j(t)\| \leq \sum_{p < q} \|x_p(t) - x_q(t)\| \leq N \sqrt{\frac{\sum_{p < q} \|x_p(t) - x_q(t)\|^2}{N}}$$

Thus, the following inequality is satisfied

$$\int_0^{+\infty} \|x_i(t) - x_j(t)\|^2 dt \leq \frac{2N^2\gamma^2}{\underline{\Gamma} \epsilon \lambda_{\min}(P)} \int_0^{+\infty} \|w\|^2 dt \tag{23}$$

Considering both of the above cases, we can obtain

$$\int_0^{+\infty} \|x_i(t) - x_j(t)\|^2 dt \leq \tilde{\gamma}^2 \int_0^{+\infty} \|w\|^2 dt \tag{24}$$

where $\tilde{\gamma}^2 = \frac{2N^2\gamma^2}{\underline{\Gamma} \epsilon \lambda_{\min}(P)}$. That means the system (8) has the L_2 performance index $\tilde{\gamma}$. \square

Remark 2. In this paper, we consider networked systems with generalized L_2 -gain. Generalized L_2 performance primarily evaluates the effect of actuator faults on the consensus error of networked systems, while normal control system L_2 performance focuses on the influence of external disturbances on the control system output.

Remark 3. The consensus tracking error can be reduced by adjusting $\tilde{\gamma} = \sqrt{\frac{2N^2\gamma^2}{\underline{\Gamma} \epsilon \lambda_{\min}(P)}}$. Increasing the main diagonal elements of matrix P while decreasing the off-diagonal elements can increase $\lambda_{\min}(P)$. Similarly, decreasing γ contributes to reducing the consensus error. There is a non-linear relationship between matrix P and γ , a balance between γ and the matrix P is necessary. Furthermore, the maximum disturbance resilience of the control system is related to the system

itself and external disturbances. Therefore, when the state-feedback gain matrix $K = B^T P$ is fixed, the minimum L_2 -gain is also fixed. We can employ the Genetic Algorithm to solve the nonlinear optimization problem, seeking the optimal $\frac{\gamma^2}{\lambda_{\min}(P)}$ to reduce the system tracking error.

Finally, the nonlinear constraint problem described in Theorem 1 can be transformed into an optimization L_2 -gain problem.

$$\begin{aligned} & \min_P \frac{\gamma^2}{\lambda_{\min}(P)} \\ \text{s.t. } & B \succeq 0, BK \succeq 0 \\ & A - l_{\max} \rho_{\max} BK + \zeta I_n \succeq 0 \\ & A^T P + PA - 2\rho_{\min} \lambda_2 P B B^T P + \frac{1}{\gamma^2} \lambda_N P B B^T P \leq -\epsilon P \end{aligned} \tag{25}$$

Remark 4. Theorem 1 provides conditions ensuring the existence of feasible solutions for positive consensus and L_2 -gain of the networked systems (2). Furthermore, combined with Theorem 1 and (24), we can formulate the optimization problem (25), which reveals the conditions for the optimal solution $\gamma^2 / \lambda_{\min}(P)$. By transforming the feasible solution into a nonlinear optimization problem, we can derive the control protocol K that minimizes the impact of actuator faults on consensus error, thereby enhancing the disturbance resilience of the networked system.

4. Numerical Simulation

To demonstrate the effectiveness of the proposed methods, some simulations are presented in this section.

4.1. Electric Circuit System

Consider a single-input electrical networked system with four positive electrical circuits with strongly connected and undirected graphs. The electrical circuit system is shown in Figure 1.

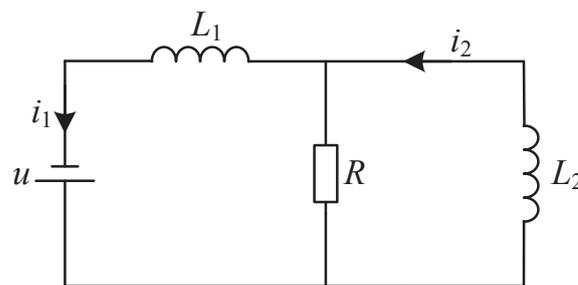


Figure 1. Electric circuit system.

According to Kirchoff’s voltage theorem, the system model is described as

$$\begin{cases} u(t) = L_1 \frac{di_1(t)}{dt} + R(i_1(t) - i_2(t)) \\ R(i_1(t) - i_2(t)) = L_2 \frac{di_2(t)}{dt} \end{cases} \tag{26}$$

Choosing $i_1(t)$ and $i_2(t)$ as the system states and u as the system input, the following system matrices can be obtained

$$A = \begin{bmatrix} -\frac{R_1}{L_1} & \frac{R_1}{L_1} \\ \frac{R_1}{L_2} & -\frac{R_1}{L_2} \end{bmatrix}, B = \begin{bmatrix} \frac{1}{L_1} \\ 0 \end{bmatrix}$$

where the Resistor is $R_1 = 1\Omega$ and the Inductors are $L_1 = L_2 = 1H$. Consider the actuator faults $\rho_i(t) = 0.4\sin(t) + 0.6$ and $w_i(t) = \exp(-t)$. The communication topology is represented by an undirected graph \mathcal{G} with the Laplacian matrix:

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

The initial conditions of the network are $x_1(0) = [1, 1]^T$, $x_2(0) = [1.4, 1.4]^T$, $x_3(0) = [1.8, 0.8]^T$, $x_4(0) = [0.5, 1.2]^T$. The eigenvalues of the Laplacian matrix are $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = 2$, and $\lambda_4 = 4$, respectively. The nonlinear optimization problem (25) can be solved using the Genetic Algorithm (GA). The termination condition is to end after 1000 iterations or when the target parameter $\frac{\gamma^2}{\lambda_{\min}(P)}$ changes by less than 1×10^{-4} in two consecutive iterations. Each element of the matrix P is kept within the range $[-10, 10]$. When $\epsilon = 1$, we can obtain

$$\left(\frac{\gamma^2}{\lambda_{\min}(P)}\right)^* = 0.798, P = \begin{bmatrix} 10 & 0.28 \\ 0.28 & 10 \end{bmatrix}$$

Then, we have

$$K = [10 \ 0.28], BK = \begin{bmatrix} 10 & 0.28 \\ 0 & 0 \end{bmatrix}, A - l_{\max}\rho_{\max}BK = \begin{bmatrix} -21 & 0.44 \\ 1 & -1 \end{bmatrix}$$

$$A - \rho_{\min}\lambda_2BK = \begin{bmatrix} -5 & 0.89 \\ 1 & -1 \end{bmatrix}, A - \rho_{\min}\lambda_4BK = \begin{bmatrix} -9 & 0.78 \\ 1 & -1 \end{bmatrix}$$

The eigenvalues of matrices $A - \rho_{\min}\lambda_2BK$ and $A - \rho_{\min}\lambda_4BK$ are $\{-5.21, -0.79\}$ and $\{-9.10, -0.90\}$, respectively. The results have confirmed that the conditions described in Proposition 1 hold.

Figures 2 and 3 show the simulation results of the positive networked system. The networked system states are positive, and the agents achieve consensus. Control inputs under actuator fault are illustrated in Figure 4.

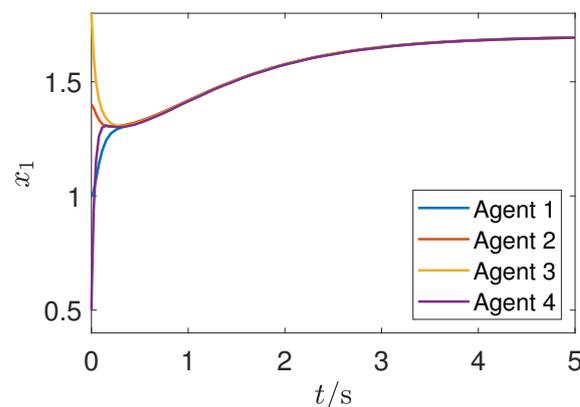


Figure 2. The state x_1 of networked system.

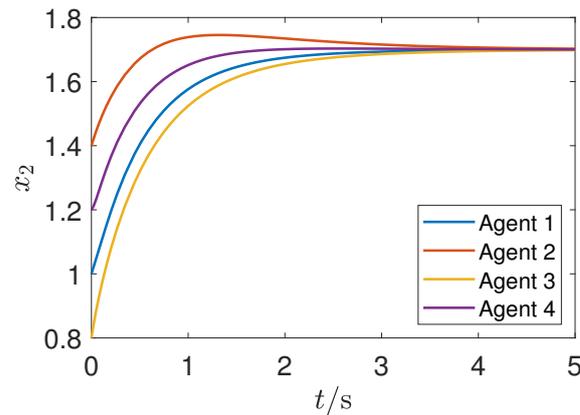


Figure 3. The state x_2 of networked system.

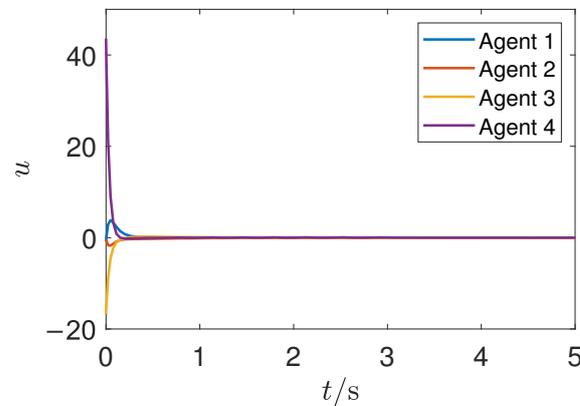


Figure 4. Control input u of networked system.

4.2. Multi-Input System

Further, we investigate a multi-input positive networked system, which can be described as follows:

$$A = \begin{bmatrix} -3 & 2 & 1 \\ 2 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The communication topology and actuator faults are the same as those in the electric circuit system. The initial conditions of the network are $x_1(0) = [1, 1, 1]^T$, $x_2(0) = [1.4, 1.4, 1.4]^T$, $x_3(0) = [1.8, 0.8, 0.8]^T$, $x_4(0) = [0.5, 1.2, 1.2]^T$. According to the Genetic Algorithm, we can obtain

$$\left(\frac{\gamma^2}{\lambda_{\min}(P)} \right)^* = 0.548, P = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

Then, we have

$$K = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \end{bmatrix}, BK = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A - l_{\max}\rho_{\max}BK = \begin{bmatrix} -23 & 2 & 0 \\ 2 & -22 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$A - \rho_{\min}\lambda_2BK = \begin{bmatrix} -7 & 2 & 1 \\ 2 & -6 & 0 \\ 1 & 0 & -2 \end{bmatrix}, A - \rho_{\min}\lambda_4BK = \begin{bmatrix} -11 & 2 & 1 \\ 2 & -10 & 0 \\ 1 & 0 & -10 \end{bmatrix}$$

The eigenvalues of matrices $A - \rho_{\min}\lambda_2BK$ and $A - \rho_{\min}\lambda_4BK$ are $\{-8.66, -4.58, -1.77\}$ and $\{-12.79, -10, -8.21\}$, respectively. The conditions in Proposition 1 are satisfied.

In the case of multi-input networked systems, the simulation results are presented in Figures 5–9. It can be found that as time t increases, the states of the networked system converge to zero while ensuring that all states remain greater than zero in Figures 5–7. Despite the presence of actuator faults, the system can still maintain its positivity while achieving consensus. The actuator outputs are illustrated in Figures 8 and 9.

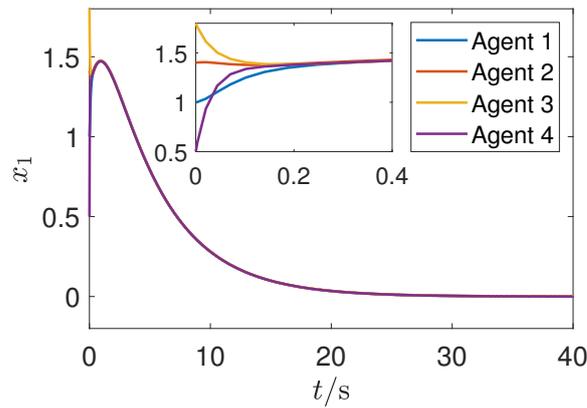


Figure 5. The state x_1 of multi-input system.

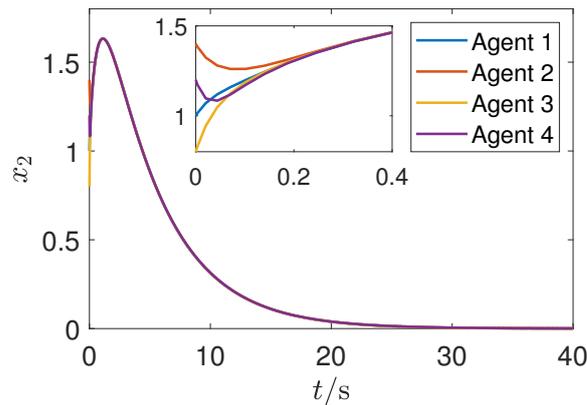


Figure 6. The state x_2 of multi-input system.

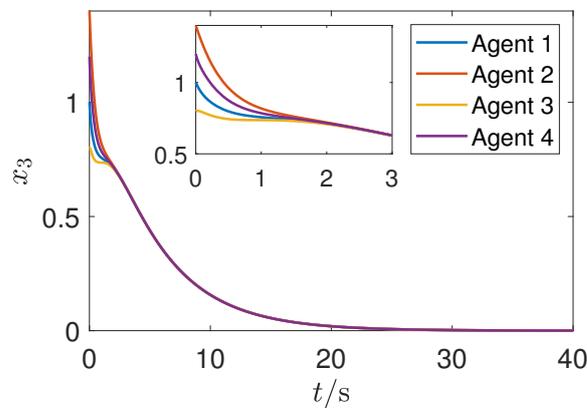


Figure 7. The state x_3 of multi-input system.

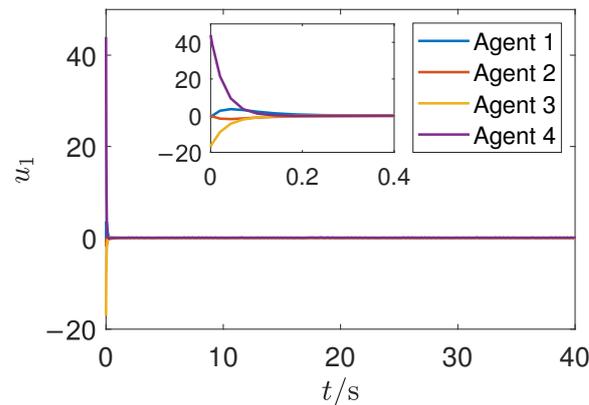


Figure 8. Control input u_1 of networked system.

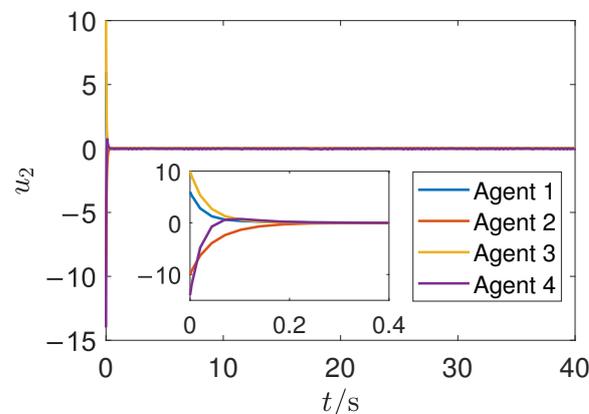


Figure 9. Control input u_2 of networked system.

5. Conclusions

This article has addressed the consensus of an undirected networked system under actuator faults. An approach has been employed, treating multiplicative faults as system matrix uncertainties and using the Algebraic Riccati Inequality to ensure the state consensus of the networked system. Additionally, additive faults have been regarded as external disturbances, thus transforming the actuator fault problem into involving L_2 -gain performance in the presence of external disturbances. This has ensured that the consensus errors of agents have met the L_2 performance index. Finally, the Genetic Algorithm has been employed to optimize L_2 performance criterion γ . In the future, we can explore the consensus of networked systems under input saturation and time delay, improve optimization algorithms for feedback gains, and enhance the control performance of the networked system.

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Abbreviations

The following abbreviations are used in this manuscript:

PCNS	Positive consensus of a networked system
ARI	Algebraic Riccati Inequality

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