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A Multi-Attribute Decision-Making Approach for the Analysis of Vendor Management Using Novel Complex Picture Fuzzy Hamy Mean Operators

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Abstract: Vendor management systems (VMSs) are web-based software packages that can be used to manage businesses. The performance of the VMSs can be assessed using multi-attribute decision-making (MADM) techniques under uncertain situations. This article aims to analyze and assess the performance of VMSs using MADM techniques, especially when the uncertainty is of complex nature. To achieve the goals, we aim to explore Hany mean (HM) operators in the environment of complex picture fuzzy (CPF) sets (CPFSs). We introduce CPF Hamy mean (CPFHM) and CPF weighted HM (CPFWHM) operators. Moreover, the reliability of the newly proposed HM operators is examined by taking into account the properties of idempotency, monotonicity, and boundedness. A case study of VMS is briefly discussed, and a comprehensive numerical example is carried out to assess VMSs using the MADM technique based on CPFHM operators. The sensitivity analysis and comprehensive comparative analysis of the proposed work are discussed to point out the significance of the newly established results.

Keywords: ambiguous and vague information; complex picture fuzzy numbers; Hamy mean operators; multi-attribute decision-making technique; vendor management systems

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1. Introduction

The majority of fields, including engineering, economics, and management, involve some type of decision-making difficulties. All of the information about using the alternatives has been traditionally thought to be taken in the form of unambiguous numbers. The processing of the data fuzziness and uncertainty is essential because they change regularly in real-life scenarios. A VMS is a tool that allows businesses to manage every step of the vendor management process, from the initial point of interaction through the final steps of concluding a sale or establishing a new business relationship. They typically feature particular modules or apps that deal with procedures such as on-boarding new vendors or processing vendor payments because they have a modular approach. Many aspects can move in vendor relationship management. There are purchase orders, purchase requisitions, order confirmations, performance monitoring, vendor screening procedures, and so on.

The decision-making technique plays a vital role in the process of aggregating information and creates a lot of interactions for several research scholars. Every field is full of uncertain, imprecise, and hazy information. To deal with human opinion in the form of uncertain and vague information, Zadeh [1] gave the concept of a fuzzy set (FS) with a degree of truth index (TI). El-Bably and Abo-Tabl [2,3] presented an innovative concept of FSs in the frame work of a rough set under some topological reduction. El Sayed et al. [4] explored the theory of topological techniques to handle the current situations of COVID-19 by using the model of nano-topology. Atanassov [5] generalized the theory of FS in the framework of an intuitionistic fuzzy set (IFS) having TI and falsity index (FI), where the sum of TI $\breve{\mho}(\varrho)$ and FI $\tilde{\Upsilon}(\varrho)$ restricted is less and equal to 0 and 1, i.e., $0 \leq \breve{\mho}(\varrho) + \tilde{\Upsilon}(\varrho) \leq 1$. In some scenarios IFS has failed; if the TI is 0.65 and FI is 0.55, then the sum of TI and FI is $0.65 + 0.55 = 1.2 \notin [0,\ 1]$. To cope with this situation, Yager [6] presented the concepts of a Pythagorean fuzzy set (PyFS); according to PyFS, the sum of the square of TI and FI are less than or equal to 0 and 1, and from the above example, $0 \leq \breve{\mho}^2(\varrho) + \tilde{\Upsilon}^2(\varrho) \leq 1$, so $(0.65)^2 + (0.55)^2 = 0.73 \in [0,\ 1]$. Yager [7] also developed the concept of a q-rung orthopair fuzzy set (q-ROFS) by generalizing the idea of PyFS. Cuong [8,9] introduced a new concept of picture fuzzy (PF) set (PFS), which contains four types of characteristic functions, TI, abstinence index (AI), FI, and refusal index (RI). The structure of PFS has the sum of three terms, and TI, AI, and FI are restricted in $[0,\ 1]$. Lu et al. [10] generalized the concepts of PFSs in the framework of PF rough sets to solve real-life problems under the system of MADM techniques. Several research scholars worked in different fields of research to find the limitations of the above-discussed phenomenon seen in [11–14].

Aggregation operators are convenient mathematical models to investigate fuzzy information, for the study of above discussed existing AOs, we analyzed research works to recognize how to deal with ambiguity and uncertainty in complex information. Several research scholars presented their research methodologies to solve MADM techniques. For instance, Xu [15] presented some AOs of IFS to investigate fuzziness data. Xu and Xia [16] generalized IFSs and developed a list of AOs to solve the MADM process. Biswas and Deb [17] introduced a list of new AOs by utilizing the Schweizer and Sklar power operations under the system of PyFSs. Garg [18] presented some AOs of PyFSs by using the operations of Einstein T-norm (TNM) and T-conorm (TCNM). Mahmood and Ali [19] explained a new technique of AOs by using the VIKOR method in the environment of complex q-rung orthopair sets. Riaz and Hashmi [20] elaborated on AOs based on Linear Diophantine FSs and solve a MADM technique to investigate a suitable candidate for a multinational company. Liu [21] extended algebraic AOs and Einstein AOs to develop some new AOs by using the operations of Hamacher TNM and TCNM under the system interval-valued IFSs (IVIFSs). Hussain et al. [22] presented some AOs by utilizing the basic operations of Aczel Alsina TNM and TCNM to select a suitable candidate for a multinational company. Liu et al. [23] generalized similarity measures based on interval-valued PFS (IVPFS) and studied a MADM technique to solve real-life problems. Mahmood et al. [24] established a series of new AOs based on the bipolar valued fuzzy hesitant system and their special cases. Garg [25] explained some new AOs based on PFSs and also studied a MADM technique to solve a numerical example related to our daily life. Wei [26] presented some AOs of arithmetic and geometric operators by utilizing the basic operations of Hamacher TNM and TCNM. We also studied the theory of generalized FS in different fuzzy environments seen in references [27–30].

The preceding aggregation operators and their associated methodologies are frequently utilized by researchers, but it has been determined from these studies that these works consider the data under the FS, IFSs, or their modifications, which are only to handle the uncertainty and vagueness that exist in the data. The partial ignorance of the data and their variations at a specific point in the time during implementation, however, is something that none of the existing models is capable of recognizing. Additionally, in daily life, change in the phase (periodicity) of the data corresponds with uncertainty and ambiguity that is present in the data. There is information loss during the process as a result of the present theories' inability to adequately account for this information. To overcome this situation, Ramot et al. [31] introduced the complex fuzzy set (CFS), in which the range of the TI is expanding from real numbers to complex numbers with the unit circle. Traditionally, fuzzy logic was generalized to complex fuzzy logic by Ramot et al. [32] in which the sets employed in the reasoning process are CFSs, characterized by complex-valued TI functions. In a later study, Greenfield et al. expanded on the CFS idea by considering the TI as an interval number rather than a single integer. A systematic review of CFSs and logic was done by Yazdanbakhsh and Dick [33], and they explained their finding. Alkouri and Salleh [34] extended the concepts of CFS in the framework of complex IFS (CIFS) by adding the new term of FI in CFS. They extended the range of both TI and FI to a unit circle in a complex system. Furthermore, they defined fundamental operations of CIFS such as union, intersection, and complement of CIFSs. Garg and Rani [35] utilized the MADM technique to solve real life problems by using the AOs of complex IFSs. Ullah et al. [36] generalized the concepts of CFS and CIFS in the framework of complex PyFS to find distance measures by using the technique of pattern recognition. Liu et al. [37] presented a new concepts of complex q-ROFS (Cq-ROFS) by the generalization of CPyFSs with sum of qth power of TI and FI. Rong et al. [38] developed a new list of AOs of MacLaurin symmetric mean operators under the system of Cq-ROFS. Akram et al. [39] proposed a new theory of complex PFS (CPFS), as an extension of CFSs, CIFSs, CPyFSs, and Cq-ROFSs by utilizing the basic operations of Hamacher AOs.

The HM tools are used to aggregate uncertain and vague information in a different framework of fuzzy environment. Firstly, the theory of HM operators was discovered by Hara et al. [40] in 1998. He obtained different inequalities by classifying the arithmetic and geometric inequalities. Recently a lot of research done on this topic. Qin [41] explored the concept of HM operator to cope with vagueness and imprecision under the system of interval type 2-fuzzy and he also discussed their application based on MADM techniques. Wu et al. [42] expanded the ideas of HM operators in the framework of interval-valued intuitionistic fuzzy Dombi HM operators to find suitable tourism destinations. Li et al. [43] utilized the theory of HM operator to select a suitable supplier for a motor vehicle under the system of IFSs. Wu et al. [44] also explored the concepts of HM operators in a new research area to evaluate construction engineering schemes based on the 2-tuple linguistic neutrosophic system. Li et al. [45] provided some new AOs by using the operational laws of HM operators based on PyFSs and also established an application to find the best supplier system based on the MADM technique. Liu et al. [46] also introduced some new AOs of IF uncertain linguistic HM operators with an application of a healthcare waste administration authority. Wu et al. [47] elaborated the concept of HM and dual HM (DHM) operators to develop a series of new AOs based on IVIFSs and also discussed an application to find the best tourism place. Wang et al. [48] developed some AOs by using the idea of HM and DHM operators under the system of q-rung orthopair fuzzy sets and gave an application for the selection of enterprise resource management authority. Xing et al. [49] developed some AOs to handle uncertain and vague information by using new operational laws of interactive HM and DHM operators. Sinani et al. [50] introduced a series of AOs by using the operation operator based on rough numbers. Wei et al. [51] developed some AOs to fuse uncertain information under the system of dual hesitant PyFSs with the help of the MADM approaches. Liu et al. [52] presented some convenient AOs by generalizing the concept of HM and DHM tools in the framework of interval neutrosophic power sets. Garg et al. [53] illustrated a list of AOs by using the operations of HM operators in the framework of a q-rung orthopair fuzzy set (q-ROFS). Ali et al. [54] presented a series of AOs by utilizing the theory of HM operators under the system of complex interval-valued q-ROFS (CIVq-ROFSs).

Keeping in mind the significance of CPFSs, we developed some new AOs by using the concept of the HM tool in the framework of CPFS. A CPFS has two aspects of information in the form of amplitude terms and phase terms of TI, AI, and FI. In this article, a list of AOs discusses CPFHM, CPFWHM, CPFDHM and CPFWDHM operators with some basic properties such as idempotency, monotonicity, and boundedness. We also study some numerical examples to support our proposed methodologies. We established an application based on VMS to find the flexibility and reliability of our proposed techniques. With the help of a practical numerical example, we evaluate suitable software for VMS. To check validity and compatibility, we study a comprehensive comparative study to contrast the results of existing AOs with the results of the discussed technique.

The structure of this article is organized as follows: In Section 1, we review the history of our research work for the improvement of this article. In Section 2, we study all the notions related to PFSs, CPFSs, and their basic operations. In Section 3, we recall existing concepts of HM and GHM operators and also discuss their basic properties. In Section 4, we utilize the basic operations of HM operators to introduce some new AOs such as CPFHM and CPFWHM operators with their characteristics. In Section 5, we also present some new AOs of CPFGHM and CPFWGHM operators. We also present some numerical examples to find the feasibility of our proposed approaches. In Section 6, we establish a strategy for the MADM process under the system of CPFSs. We also provide an application in the framework of VMS. To check the competitiveness and flexibility of our proposed AOs, we illustrate a numerical example based on CPF information. In Section 7, to find the validity and rationality of our proposed work, we make comparison results of our proposed approaches with some existing AOs. In Section 8, we summarize the whole article in a paragraph.

2. Preliminaries

This section aims to recall notions of PFSs, CPFSs, and their basic operational laws. We applied these operational laws to develop our proposed methodology. First, we want to define the meaning of some symbols and letters in Table 1, as follows.

Table 1. Symbols and their meanings.

Symbols	Meanings	Symbols	Meanings
$\ddot{\bar{\mho}}$	Universal set	ϕ_v	Falsity Index of phase term
ϱ	Element belonging to Universal set	$\hat{S}$	Score function
$\breve{\mho}_\mu$	Truth Index/(TI) of amplitude term	$\underset{\cdot}{A}$	Accuracy function
$\ddot{\varepsilon}_A$	Abstinence Index /(AI) of amplitude term	$\mathfrak{N}_{i_j}$	Weight vector
$\tilde{\Upsilon}_\nu$	Falsity Index/(FI) of amplitude term	$C_n^{ш}$	Binomial Coefficient
$\mathfrak{E}$	CPFS	$\sqrt{-1}$	Unit circle
ψ_μ	Truth Index of phase term	$\acute{r}_{\mathfrak{E}}$	Hesitancy Index
φ_A	Abstinence Index of phase term	$\bar{I}$	Complement of CPFV

The concepts of PFSs were developed by Cuong [8] and is given as follows:

Definition 1. *[8] Consider $\ddot{\bar{\mho}}$ to be an empty set. A PFS $\mathcal{Y}$ is defined as:*

$$\mathcal{Y} = \left\{ \left(\varrho,\ \breve{\mho}_\mu(\varrho),\ \ddot{\varepsilon}_A(\varrho), \tilde{\Upsilon}_\nu(\varrho) \right) \middle| \varrho \right\}$$

where $\breve{\mho}_\mu(\varrho),\ \tilde{\Upsilon}_A(\varrho), \tilde{\Upsilon}_\nu(\varrho) \in [0,1]$. Truth index is denoted (TI) by the $\breve{\mho}_\mu(\varrho)$, abstinence index (AI) is denoted by the $\ddot{\varepsilon}_A(\varrho)$, and falsity index (FI) is denoted by the $\tilde{\Upsilon}_\nu(\varrho)$, such that:

$$0 < \breve{\mho}_\mu(\varrho) + \ddot{\varepsilon}_A(\varrho) + \tilde{\Upsilon}_\nu(\varrho) < 1$$

A picture fuzzy value (PFV) represented by $\mathcal{T} = \left(\breve{\mho}_\mu(\varrho),\ \ddot{\varepsilon}_A(\varrho), \tilde{\Upsilon}_\nu(\varrho) \right)$.

The theory of the following Definition was proposed by Akram et al. [39].

Definition 2. *[39] A CPFS is formed as:*

$$\mathfrak{E} = \left\{ \left(\varrho,\ \breve{\mho}_\mu(\varrho) e^{2i\pi\psi_\mu(\varrho)},\ \ddot{\varepsilon}_A(\varrho) e^{2i\pi\varphi_A(\varrho)}, \tilde{\Upsilon}_\nu(\varrho) e^{2i\pi\phi_\nu(\varrho)} \right) \middle| \varrho \in \ddot{\bar{\mho}} \right\},\ i = \sqrt{-1}$$

where $\breve{\mho}_\mu(\varrho),\ \ddot{\varepsilon}_A(\varrho)$ and $\tilde{\Upsilon}_\nu(\varrho) \in [0,1]$ be amplitude terms and $\psi_\mu(\varrho),\ \varphi_A(\varrho)$, and $\phi_\nu(\varrho) \in [0,1]$ be the phase terms. TI, AI, and FI for amplitude terms are represented by the $\breve{\mho}_\mu(\varrho),\ \ddot{\varepsilon}_A(\varrho)$ and

$\check{\Upsilon}_\nu(\varrho)$, *respectively. Similarly, TI, AI and FI for phase terms are represented by the* $\psi_\mu(\varrho)$, $\varphi_A(\varrho)$, *and* $\phi_\nu(\varrho)$, *respectively. A CPFS must satisfy the following condition:*

$$0 \le \breve{\upsilon}_\mu(\varrho) + \hat{\varepsilon}_A(\varrho) + \check{\Upsilon}_\nu(\varrho) \le 1, \text{ and } 0 \le \psi_\mu(\varrho) + \varphi_A(\varrho) + \phi_\nu(\varrho) \le 1,\ \forall \varrho \in \bar{\text{Ü}},$$

A hesitancy index of a CPFS $\acute{\text{r}}_{\mathfrak{E}}$ *is denoted by* $\acute{\text{r}}_{\mathfrak{E}} = 1 - \left(\breve{\upsilon}_\mu(\varrho) + \hat{\varepsilon}_A(\varrho) + \check{\Upsilon}_\nu(\varrho)\right) e^{2\pi i(1-(\psi_\mu(\varrho)+\varphi_A(\varrho)+\phi_\nu(\varrho)))}$. *Let a complex PFV (CPFV) be denoted by* $I = (\breve{\upsilon}_\mu(\varrho)e^{2\pi i\psi_\mu(\varrho)}, \hat{\varepsilon}_A(\varrho)e^{2\pi i\varphi_A(\varrho)}, \check{\Upsilon}_\nu(\varrho)e^{2\pi i\phi_\nu(\varrho)})$.

Definition 3. *[55] Consider* $I = (\breve{\upsilon}_\mu(\varrho)e^{2i\pi\psi_\mu(\varrho)}, \hat{\varepsilon}_A(\varrho)e^{2i\pi\varphi_A(\varrho)}, \check{\Upsilon}_\nu(\varrho)e^{2i\pi\phi_\nu(\varrho)})$, $I_1 = (\breve{\upsilon}_{\mu_1}(\varrho)e^{2i\pi\psi_{\mu_1}(\varrho)}, \hat{\varepsilon}_{A_1}(\varrho)e^{2i\pi\varphi_{A_1}(\varrho)}, \check{\Upsilon}_{\nu_1}(\varrho)e^{2i\pi\phi_{\nu_1}(\varrho)})$ *and* $I_2 = (\breve{\upsilon}_{\mu_2}(\varrho)e^{2i\pi\psi_{\mu_2}(\varrho)}, \hat{\varepsilon}_{A_2}(\varrho) e^{2i\pi\varphi_{A_2}(\varrho)}, \check{\Upsilon}_{\nu_2}(\varrho)e^{2i\pi\phi_{\nu_2}(\varrho)})$ *be any three CPFSs. Then some basic operational laws are defined as:*

1. $I_1 \subseteq I_2$ *, if and only if* $\breve{\upsilon}_{\mu_1}(\varrho) \le \breve{\upsilon}_{\mu_2}(\varrho)$, $\hat{\varepsilon}_{A_1}(\varrho) \le \hat{\varepsilon}_{A_2}(\varrho)$ *and* $\check{\Upsilon}_{\nu_1}(\varrho) \ge \check{\Upsilon}_{\nu_2}(\varrho)$ *for amplitude terms and* $\psi_{\mu_1}(\varrho) \le \psi_{\mu_2}(\varrho)$, $\varphi_{A_1}(\varrho) \le \varphi_{A_2}(\varrho)$ *, and* $\phi_{\nu_1}(\varrho) \ge \phi_{\nu_2}(\varrho)$ *for phase terms. For all* $\varrho \in \bar{\text{Ü}}$
2. $\bar{I} = \{(\varrho, \breve{\upsilon}_{\mu_I}(\varrho)e^{2i\pi\psi_{\mu_I}(\varrho)}, \hat{\varepsilon}_{AI}(\varrho)e^{2i\pi\varphi_{A_I}(\varrho)}, \check{\Upsilon}_{\nu_I}(\varrho)e^{2i\pi\phi_{\nu_I}(\varrho)})|\varrho \in \bar{\text{Ü}}\}$
3. $I_1 \cap I_2 = \{(\varrho, (\breve{\upsilon}_{\mu_1}(\varrho) \wedge \breve{\upsilon}_{\mu_2}(\varrho))\ e^{2i\pi(\psi_{\mu_1}(\varrho)\wedge\psi_{\mu_2}(\varrho))}, (\hat{\varepsilon}_{A_1}(\varrho) \vee \hat{\varepsilon}_{A_2}(\varrho))e^{2i\pi(\varphi_{A_1}(\varrho)\vee\varphi_{A_2}(\varrho))}, (\check{\Upsilon}_{\nu_1}(\varrho) \vee \check{\Upsilon}_{\nu_2}(\varrho))e^{2i\pi(\phi_{\nu_1}(\varrho)\vee\phi_{\nu_2}(\varrho))})|\varrho \in \bar{\text{Ü}}\}$
4. $I_1 \uplus I_2 = \{(\varrho, (\breve{\upsilon}_{\mu_1}(\varrho) \vee \breve{\upsilon}_{\mu_2}(\varrho))\ e^{2i\pi(\psi_{\mu_1}(\varrho)\vee\psi_{\mu_2}(\varrho))}, (\hat{\varepsilon}_{A_1}(\varrho) \wedge \hat{\varepsilon}_{A_2}(\varrho))e^{2i\pi(\varphi_{A_1}(\varrho)\wedge\varphi_{A_2}(\varrho))}, (\check{\Upsilon}_{\nu_1}(\varrho) \wedge \check{\Upsilon}_{\nu_2}(\varrho))e^{2i\pi(\phi_{\nu_1}(\varrho)\wedge\phi_{\nu_2}(\varrho))})|\varrho \in \bar{\text{Ü}}\}$

   where symbol $\wedge$ and $\vee$ *represent the minimum and maximum respectively.*

Definition 4. *Consider* $I = \left(\breve{\upsilon}_\mu(\varrho)e^{2\pi i\psi_\mu(\varrho)}, \hat{\varepsilon}_A(\varrho)e^{2\pi i\varphi_A(\varrho)}, \check{\Upsilon}_\nu(\varrho)e^{2\pi i\phi_\nu(\varrho)}\right)$ *is a CPFV. Then score functions are defined as:*

$$\hat{\text{S}}(I) = \frac{\left(3 + \left(\breve{\upsilon}_\mu(\varrho) - \hat{\varepsilon}_A(\varrho) - \check{\Upsilon}_\nu(\varrho)\right) + \left(\psi_\mu(\varrho) - \varphi_A(\varrho) - \phi_\nu(\varrho)\right)\right)}{6}$$

where $\hat{\text{S}}(I) \in [-1, 1]$.

Definition 5. *Consider* $I = \left(\breve{\upsilon}_\mu(\varrho)e^{2\pi i\psi_\mu(\varrho)}, \hat{\varepsilon}_A(\varrho)e^{2\pi i\varphi_A(\varrho)}, \check{\Upsilon}_\nu(\varrho)e^{2\pi i\phi_\nu(\varrho)}\right)$ *is a CPFV. Then accuracy functions are defined as:*

$$\text{Ą}(I) = \frac{\left(\breve{\upsilon}_\mu(\varrho) + \hat{\varepsilon}_A(\varrho) + \check{\Upsilon}_\nu(\varrho)\right) + \left(\psi_\mu(\varrho) + \varphi_A(\varrho) + \phi_\nu(\varrho)\right)}{3}$$

where $\text{Ą}(I) \in [0, 2]$.

Example 1. *Let* $I_1 = \left(0.30e^{2\pi i(0.09)}, 0.17e^{2\pi i(0.12)}, 0.42e^{2\pi i(0.32)}\right)$, $I_2 = \left(0.68e^{2\pi i(0.29)}, 0.07e^{2\pi i(0.52)}, 0.16e^{2\pi i(0.06)}\right)$ *and* $I_3 = \left(0.37e^{2\pi i(0.22)}, 0.25e^{2\pi i(0.32)}, 0.17e^{2\pi i(0.09)}\right)$ *be three CPFVs. The score function and accuracy function is defined as follows:*

$$\hat{\text{S}}(I_1) = \frac{(3 + (0.30 - 0.17 - 0.42) + (0.09 - 0.12 - 0.32))}{6} = 0.3050 \in [0, 1]$$

$$\hat{\text{S}}(I_2) = \frac{(3 + (0.68 - 0.07 - 0.16) + (0.29 - 0.52 - 0.06))}{6} = 0.3550 \in [0, 1]$$

$$\hat{\text{S}}(I_3) = \frac{(3 + (0.37 - 0.25 - 0.17) + (0.22 - 0.32 - 0.09))}{6} = 0.5433 \in [0, 1]$$

and

$$\text{Ą}(I_1) = \frac{(0.30 + 0.17 + 0.42) + (0.09 + 0.12 + 0.32)}{3} = 0.6500 \in [0, 1]$$

$$Ą(I_2) = \frac{(0.68 + 0.07 + 0.16) + (0.29 + 0.52 + 0.06)}{3} = 0.4833 \in [0,1]$$
$$Ą(I_3) = \frac{(0.37 + 0.25 + 0.17) + (0.22 + 0.32 + 0.09)}{3} = 0.4067 \in [0,1]$$

Remark 1. *Consider* $I_1 = \left(\breve{\upsilon}_{\mu_1}(\varrho)e^{2\pi i\psi_{\mu_1}(\varrho)}, \ddot{\varepsilon}_{A_1}(\varrho)e^{2\pi i\varphi_{A_1}(\varrho)}, \tilde{\Upsilon}_{\nu_1}(\varrho)e^{2\pi i\phi_{\nu_1}(\varrho)}\right)$ *and* $I_2 = \left(\breve{\upsilon}_{\mu_2}(\varrho)e^{2\pi i\psi_{\mu_2}(\varrho)}, \ddot{\varepsilon}_{A_2}(\varrho)e^{2\pi i\varphi_{A_2}(\varrho)}, \tilde{\Upsilon}_{\nu_2}(\varrho)e^{2\pi i\phi_{\nu_2}(\varrho)}\right)$ *are two CPFVs. Then some rules of score function and accuracy function such as if* $I_1 < I_2$ *, then* $\hat{S}(I_1) < \hat{S}(I_2)$*, if* $I_1 > I_2$*, then* $\hat{S}(I_1) > \hat{S}(I_2)$*. Similarly, if* $\hat{S}(I_1) = \hat{S}(I_2)$*, then following conditions must be satisfied:*

I. If $Ą(I_1) < Ą(I_2)$, then $I_1 < I_2$.
II. If $Ą(I_1) = Ą(I_2)$, then $I_1 = I_2$.

Definition 6. *Consider* $I = \left(\breve{\upsilon}_{\mu}(\varrho)e^{2\pi i\psi_{\mu}(\varrho)}, \ddot{\varepsilon}_{A}(\varrho)e^{2\pi i\varphi_{A}(\varrho)}, \tilde{\Upsilon}_{\nu}(\varrho)e^{2\pi i\phi_{\nu}(\varrho)}\right)$, $I_1 = \left(\breve{\upsilon}_{\mu_1}(\varrho)e^{2\pi i\psi_{\mu_1}(\varrho)}, \ddot{\varepsilon}_{A_1}(\varrho)e^{2\pi i\varphi_{A_1}(\varrho)}, \tilde{\Upsilon}_{\nu_1}(\varrho)e^{2\pi i\phi_{\nu_1}(\varrho)}\right)$ *and* $I_2 = \left(\breve{\upsilon}_{\mu_2}(\varrho)e^{2\pi i\psi_{\mu_2}(\varrho)}, \ddot{\varepsilon}_{A_2}(\varrho) e^{2\pi i\varphi_{A_2}(\varrho)}, \tilde{\Upsilon}_{\nu_2}(\varrho)e^{2\pi i\phi_{\nu_2}(\varrho)}\right)$ *are three CPFVs. The fundamental operations of CPFSs are defined as:*

I. $$I_1 \oplus I_2 = \begin{pmatrix} \left(\breve{\upsilon}_{\mu_1}(\varrho) + \breve{\upsilon}_{\mu_2}(\varrho) - \breve{\upsilon}_{\mu_1}(\varrho).\breve{\upsilon}_{\mu_2}(\varrho)\right)e^{2\pi i(\psi_{\mu_1}(\varrho)+\psi_{\mu_2}(\varrho)-\psi_{\mu_1}(\varrho).\psi_{\mu_2}(\varrho))}, \\ \left(\ddot{\varepsilon}_{A_1}(\varrho).\ddot{\varepsilon}_{A_2}(\varrho)\right)e^{2\pi i(\varphi_{A_1}(\varrho).\varphi_{A_2}(\varrho))}, \\ \left(\tilde{\Upsilon}_{\nu_1}(\varrho).\ \tilde{\Upsilon}_{\nu_2}(\varrho)\right)e^{2\pi i(\phi_{\nu_1}(\varrho).\phi_{\nu_2}(\varrho))} \end{pmatrix}$$

II. $$I_1 \otimes I_2 = \begin{pmatrix} \left(\breve{\upsilon}_{\mu_1}(\varrho).\ \breve{\upsilon}_{\mu_2}(\varrho)\right)e^{2\pi i(\psi_{\mu_1}(\varrho).\psi_{\mu_2}(\varrho))}, \\ \left(\ddot{\varepsilon}_{A_1}(\varrho) + \ddot{\varepsilon}_{A_2}(\varrho) - \ddot{\varepsilon}_{A_1}(\varrho).\ddot{\varepsilon}_{AI_2}(\varrho)\right)e^{2\pi i(\varphi_{A_1}(\varrho)+\varphi_{A_2}(\varrho)-\varphi_{A_1}(\varrho).\varphi_{A_2}(\varrho))}, \\ \left(\tilde{\Upsilon}_{\nu_1}(\varrho) + \tilde{\Upsilon}_{\nu_2}(\varrho) - \tilde{\Upsilon}_{\nu_1}(\varrho).\tilde{\Upsilon}_{\nu_2}(\varrho)\right)e^{2\pi i(\phi_{\nu_1}(\varrho)+\phi_{\nu_2}(\varrho)-\phi_{\nu_1}(\varrho).\phi_{\nu_2}(\varrho))} \end{pmatrix}$$

III. $$"\Omega.I = \begin{pmatrix} \left(1 - \left(1 - \breve{\upsilon}_{\mu_I}(\varrho)\right)^{"\Omega}\right)e^{2\pi i(1-(1-\psi_{\mu_I}(\varrho))^{"\Omega})}, \\ \left(\ddot{\varepsilon}_{A_I}(\varrho)\right)^{"\Omega}e^{2\pi i(\varphi_{A_I}(\varrho))^{"\Omega}}, \\ \left(\tilde{\Upsilon}_{\nu_I}(\varrho)\right)^{"\Omega}e^{2\pi i(\phi_{\nu_I}(\varrho))^{"\Omega}} \end{pmatrix}, "\Omega > 0$$

IV. $$I^{"\Omega} = \begin{pmatrix} \left(\breve{\upsilon}_{\mu I}(\varrho)\right)^{"\Omega}e^{2\pi i(\psi_{\mu_I}(\varrho))^{"\Omega}}, \\ \left(1 - \left(1 - \ddot{\varepsilon}_{A_I}(\varrho)\right)^{"\Omega}\right)e^{2\pi i(1-(1-\varphi_{A_I}(\varrho))^{"\Omega})}, \\ \left(1 - \left(1 - \tilde{\Upsilon}_{\nu_I}(\varrho)\right)^{"\Omega}\right)e^{2\pi i(1-(1-\phi_{\nu_I}(\varrho))^{"\Omega})} \end{pmatrix}, "\Omega > 0$$

3. Previous Study

This section aims to recall the concepts of the HM operator since the HM operator is a very useful tool to aggregate real numbers. Moreover, we use the concepts of HM operator for further development of this article.

Definition 7. *[40] The HM operator is defined as:*

$$HM^{(ꚉ)}(I_1, I_2, \ldots, I_n) = \frac{\sum_{1\le i_1<,\ \ldots,<i_{ꚉ}\le n}\left(\prod_{i=1}^{ꚉ} I_{i_j}\right)^{\frac{1}{ꚉ}}}{C_n^{ꚉ}} \tag{1}$$

where $C_k^{ꚉ}$ *denotes the binomial coefficient, i.e.,* $C_n^{ꚉ} = \frac{n!}{ꚉ!(n-ꚉ)!}$*, and* ꚉ *is such that* $1 \le ꚉ \le n$*.*

The HM operator must satisfy the following axioms.

1. $HM^{(ꚉ)}(I_1, I_2, \ldots, I_k) = I$ if $I_i = I,\ (i = 1, 2, 3, \ldots, k)$.
2. $HM^{(ꚉ)}(I_1, I_2, \ldots, I_k) \le HM^{(ꚉ)}(\acute{\omega}_1, \acute{\omega}_2, \ldots, \acute{\omega}_k)$ if $I_i \le \acute{\omega}_i,\ (i = 1, 2, 3, \ldots, k)$.

3. $min(I_i) \leq HM^{(\text{щ})}(I_1, I_2, \ldots, I_k) \leq max I_i$.
4. For arithmetic mean operator $HM^{(\text{щ})}(I_1, I_2, \ldots, I_k) = \frac{1}{k}\sum_{i=1}^{k} I_i$
5. For geometric mean operator $HM^{(\text{щ})}(I_1, I_2, \ldots, I_k) = (\prod_{i=1}^{k} I_i)^{\frac{1}{\text{щ}}}$

Now we study the notion of DHM operators given by the [56].

Definition 8. *[56] The DHM operator is particularized as:*

$$DHM^{(\text{щ})}(I_1, I_2, \ldots, I_n) = \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(\frac{\sum_{j=1}^{\text{щ}} I_j}{\text{щ}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}} \tag{2}$$

Definition 9. *[45] Consider* $I_j = \left(\breve{\upsilon}_{\mu_j}(\varrho), \tilde{\Upsilon}_{\nu_j}(\varrho)\right), j = 1, 2, \ldots, k$ *be the family of PyFVs. Then PyF Hamy mean (PyFHM) operator is particularized as:*

$$PyFHM^{(\text{щ})}(I_1, I_2, \ldots, I_n) = \frac{\underset{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n}{\oplus} \left(\overset{\text{щ}}{\underset{i=1}{\otimes}} I_{i_j}\right)^{\frac{1}{\text{щ}}}}{C_n^{\text{щ}}}$$

$$= \left(\begin{array}{c} \sqrt{1 - \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\left(\prod_{j=1}^{\text{щ}} \breve{\upsilon}_{\mu_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)^2\right)\right)^{\frac{1}{C_n^{\text{щ}}}}}, \\ \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \sqrt{\left(1 - \left(\prod_{j=1}^{\text{щ}} \left(1 - \left(\tilde{\Upsilon}_{\nu_j}(\varrho)\right)^2\right)\right)^{\frac{1}{\text{щ}}}\right)}\right)^{\frac{1}{C_n^{\text{щ}}}} \end{array}\right)$$

Definition 10. *[47] Consider* $I_j = \left(\left[\breve{\upsilon}_{\mu j}(\varrho), \tilde{\Upsilon}_{\nu_j}(\varrho)\right], \left[t_j(\varrho),\ u_j(\varrho)\right]\right), j = 1, 2, \ldots, k,$ *to be any collection of interval-valued IFNs (IVIFNs). Then IVIF Hamy mean (IVIFHM) operator is particularized as:*

$$IVIFHM^{(\text{щ})}(I_1, I_2, \ldots, I_n) = \frac{\underset{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n}{\oplus} \left(\overset{\text{щ}}{\underset{i=1}{\otimes}} I_{i_j}\right)^{\frac{1}{\text{щ}}}}{C_n^{\text{щ}}}$$

$$IVIFHM^{(\text{щ})}(I_1, I_2, \ldots, I_n) = \left(\begin{array}{c} \left[\begin{array}{c} 1 - \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \breve{\upsilon}_{\mu_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}}, \\ 1 - \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \tilde{\Upsilon}_{\nu_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}} \end{array}\right], \\ \left[\begin{array}{c} \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \left(1 - t_{\mu_j}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}}, \\ \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \left(1 - u_{\nu_j}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}} \end{array}\right] \end{array}\right)$$

By utilizing theory of HM tool, we generalized concepts of CPFSs having two aspects of TI, AI, and FI in amplitude and phase terms. We also introduced some new AOs such as CPFHM and CPFWHM operators with their basic properties.

4. Complex Picture Fuzzy Hamy Mean Operators

Now we utilize the concept of HM operator to discover some new AOs under the system of CPF information. We establish AOs of CPFHM and CPFWHM operators with their basic properties of idempotency, monotonicity, and boundedness.

Definition 11. *Consider* $I_j = \left(\text{ʊ̌}_{\mu_j}(\varrho)e^{2i\pi\psi_{\mu j}(\varrho)}, \text{ɛ̏}_{A_j}(\varrho)e^{2i\pi\varphi_{Aj}(\varrho)}, \tilde{\Upsilon}_{\nu_j}(\varrho)e^{2i\pi\phi_{\nu_j}(\varrho)}\right), j = 1, 2, \ldots, k$, *to be the any family of CPFVs. Then, the CPFHM operator is given as:*

$$CPFHM^{(\text{ɰ})}(I_1, I_2, \ldots, I_n) = \frac{\bigoplus_{1\leq i_1<, \ldots,<i_{\text{ɰ}}\leq n}\left(\bigotimes_{i=1}^{\text{ɰ}} I_j\right)^{\frac{1}{\text{ɰ}}}}{C_n^{\text{ɰ}}} \tag{3}$$

Theorem 1. *Consider* $I_j = \left(\text{ʊ̌}_{\mu_j}(\varrho)e^{2i\pi\psi_{\mu j}(\varrho)}, \text{ɛ̏}_{A_j}(\varrho)e^{2i\pi\varphi_{Aj}(\varrho)}, \tilde{\Upsilon}_{\nu_j}(\varrho)e^{2i\pi\phi_{\nu_j}(\varrho)}\right), j = 1, 2, \ldots, k$ *to be any family of CPFVs. Then, accumulated value is also a CPFV.*

$$CPFHM^{(\text{ɰ})}(I_1, I_2, \ldots, I_n) = \begin{pmatrix} 1 - \left(\prod_{1\leq i_1<, \ldots,<i_{\text{ɰ}}\leq n}\left(1 - \left(\prod_{j=1}^{\text{ɰ}} \text{ʊ̌}_{\mu_j}(\varrho)\right)^{\frac{1}{\text{ɰ}}}\right)\right)^{\frac{1}{C_n^{\text{ɰ}}}} . \\ e^{2\pi i\left(1-\left(\prod_{1\leq i_1<, \ldots,<i_{\text{ɰ}}\leq n}\left(1-\left(\prod_{j=1}^{\text{ɰ}} \psi_{\mu_j}(\varrho)\right)^{\frac{1}{\text{ɰ}}}\right)\right)^{\frac{1}{C_n^{\text{ɰ}}}}\right)}, \\ \left(\prod_{1\leq i_1<, \ldots,<i_{\text{ɰ}}\leq n}\left(1 - \left(\prod_{j=1}^{\text{ɰ}}\left(1 - \text{ɛ̏}_{A_j}(\varrho)\right)\right)^{\frac{1}{\text{ɰ}}}\right)\right)^{\frac{1}{C_n^{\text{ɰ}}}} . \\ e^{2\pi i\left(\left(\prod_{1\leq i_1<, \ldots,<i_{\text{ɰ}}\leq n}\left(1-\left(\prod_{j=1}^{\text{ɰ}}\left(1-\varphi_{Aj}(\varrho)\right)\right)^{\frac{1}{\text{ɰ}}}\right)\right)^{\frac{1}{C_n^{\text{ɰ}}}}\right)} \\ \left(\prod_{1\leq i_1<, \ldots,<i_{\text{ɰ}}\leq n}\left(1 - \left(\prod_{j=1}^{\text{ɰ}}\left(1 - \tilde{\Upsilon}_{\nu_j}(\varrho)\right)\right)^{\frac{1}{\text{ɰ}}}\right)\right)^{\frac{1}{C_n^{\text{ɰ}}}} . \\ e^{2\pi i\left(\left(\prod_{1\leq i_1<, \ldots,<i_{\text{ɰ}}\leq n}\left(1-\left(\prod_{j=1}^{\text{ɰ}}\left(1-\phi_{\nu_j}(\varrho)\right)\right)^{\frac{1}{\text{ɰ}}}\right)\right)^{\frac{1}{C_n^{\text{ɰ}}}}\right)} \end{pmatrix} \tag{4}$$

Proof of this theorem given in Appendix A.

Further, we have to prove the basic properties of CPFHM operators such as idempotency, monotonicity, and boundedness under the basic operations of CPFHM.

Theorem 2. *(Idempotency Property) Consider* $I_j = \left(\text{ʊ̌}_{\mu_j}(\varrho)e^{2i\pi\psi_{\mu j}(\varrho)}, \text{ɛ̏}_{A_j}(\varrho)e^{2i\pi\varphi_{Aj}(\varrho)}, \tilde{\Upsilon}_{\nu_j}(\varrho)e^{2i\pi\phi_{\nu_j}(\varrho)}\right), j = 1, 2, \ldots, k$ *to be the family of all same CPFVs. Then, CPFHM is given as:*

$$CPFHM^{\text{ɰ}}(I_1, I_2, \ldots, I_n) = I$$

We studied the proof of this theorem in Appendix A.

Theorem 3. *(Monotonicity Property), Consider* $I_j = \left(\breve{\upsilon}_{\mu_j}(\varrho)e^{2i\pi\psi_{\mu j}(\varrho)},\ \hat{\varepsilon}_{A_j}(\varrho)e^{2i\pi\varphi_{Aj}(\varrho)},\ \tilde{\Upsilon}_{\nu_j}(\varrho)e^{2i\pi\phi_{\nu_j}(\varrho)}\right)$*, and* $R_j(\varrho) = \left(g_{\mu_j}(\varrho)e^{2i\pi\alpha_{\mu j}(\varrho)},\ t_{A_j}(\varrho)e^{2i\pi\gamma_{Aj}(\varrho)}, h_{\nu j}(\varrho)e^{2i\pi\beta_{\nu j}(\varrho)}\right), j = 1, 2, \ldots, k$ *are any two CPFSs. If* $I_j(\varrho) \leq R_j(\varrho)$. $\breve{\upsilon}_{\mu_j}(\varrho) \leq g_{\mu_j}(\varrho)$, $\psi_{\mu j}(\varrho) \leq \alpha_{\mu_j}(\varrho)$, $\hat{\varepsilon}_{A_j}(\varrho) \leq t_{A_j}(\varrho)$, $\varphi_{Aj}(\varrho) \leq \gamma_{Aj}(\varrho)$ *and* $\tilde{\Upsilon}_{\nu j}(\varrho) \leq h_{\nu j}(\varrho)$, $\phi_{\nu j}(\varrho) \leq \beta_{\nu j}(\varrho)$ *then:*

$$CPFHM^{\varrho}(I_1, I_2, \ldots, I_n) \leq CPFHM^{\varrho}(R_1, R_2, \ldots, R_n)$$

We discussed the proof of the Theorem 3 in Appendix A.

Theorem 4. *(Boundedness Property), Consider* $I_j = \left(\breve{\upsilon}_{\mu_j}(\varrho)e^{2i\pi\psi_{\mu j}(\varrho)},\ \hat{\varepsilon}_{A_j}(\varrho)e^{2i\pi\varphi_{Aj}(\varrho)},\ \tilde{\Upsilon}_{\nu_j}(\varrho)e^{2i\pi\phi_{\nu_j}(\varrho)}\right), j = 1, 2, \ldots, k,$ *to be the family of CPFVs, if* $I_j^- = min(I_1,\ I_2,\ I_3,\ \ldots,\ I_n)$ *and* $I_j^+ = max(I_1,\ I_2,\ I_3,\ \ldots,\ I_n)$ *Then:*

$$I^- \leq CPFHM^{щ}(I_1, I_2, \ldots, I_n) \leq I^+$$

Proof: From the Theorem 2:

$$CPFHM^{щ}(I_1, I_2, \ldots, I_n) = I^-$$

$$CPFHM^{щ}(I_1, I_2, \ldots, I_n) = I^+$$

□

From The Theorem 3,

$$I^- \leq CPFHM^{щ}(I_1, I_2, \ldots, I_n) \leq I^+$$

Now we discuss the CPFWHM operator by utilizing the basic operations of the HM operator. To solve the MADM techniques, the decision maker uses a weight vector of all attributes given by the experts.

Definition 12. *Consider* $I_j = \left(\breve{\upsilon}_{\mu_j}(\varrho)e^{2i\pi\psi_{\mu j}(\varrho)},\ \hat{\varepsilon}_{A_j}(\varrho)e^{2i\pi\varphi_{Aj}(\varrho)}, \tilde{\Upsilon}_{\nu_j}(\varrho)e^{2i\pi\phi_{\nu_j}(\varrho)}\right), j = 1, 2, \ldots, k,$ *to be the family of CPFVs and corresponding weight vectors* $\mathfrak{N}_i = (\mathfrak{N}_1, \mathfrak{N}_2,\ \ldots,\ \mathfrak{N}_n)^T$, $\mathfrak{N}_i \in [0, 1]$ *and* $\sum_{i=1}^{n}\ \mathfrak{N}_i = 1$. *Then:*

$$CPFWHM^{(щ)}(I_1, I_2, \ldots, I_n) = \frac{\underset{1 \leq i_1 <, \ldots, < i_щ \leq n}{\oplus}\left(1 - \prod_{j=1}^{щ} \mathfrak{N}_{i_j}\right)\left(\bigotimes_{j=1}^{щ}\left(I_{i_j}\right)\right)^{\frac{1}{щ}}}{C_n^{щ}} \tag{5}$$

Theorem 5. *Consider* $I_j = \left(\breve{\upsilon}_{\mu_j}(\varrho)e^{2i\pi\psi_{\mu j}(\varrho)},\ \hat{\varepsilon}_{A_j}(\varrho)e^{2i\pi\varphi_{Aj}(\varrho)}, \tilde{\Upsilon}_{\nu_j}(\varrho)e^{2i\pi\phi_{\nu_j}(\varrho)}\right), j = 1, 2, \ldots, k,$ *to be the family of CPFVs, Then the accumulated index of the CPFWHM operator is also a CPFV:*

$$CPFWHM^{(\text{ɰ})}(I_1, I_2, \ldots, I_n) = \begin{pmatrix} \begin{array}{c} 1 - \left(\prod_{1 \le i_1 <, \ldots, < i_{\text{ɰ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{ɰ}} \breve{\upsilon}_{\mu_{i_j}}(\varrho) \right)^{\frac{1}{\text{ɰ}}} \right)^{(1 - \prod_{j=1}^{\text{ɰ}} \mathfrak{N}_{i_j})} \right)^{\frac{1}{C_n^{\text{ɰ}}}} \\ e^{2\pi i \left(1 - \left(\prod_{1 \le i_1 <, \ldots, < i_{\text{ɰ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{ɰ}} \psi_{\mu_{i_j}}(\varrho) \right)^{\frac{1}{\text{ɰ}}} \right)^{(1 - \prod_{j=1}^{\text{ɰ}} \mathfrak{N}_{i_j})} \right)^{\frac{1}{C_n^{\text{ɰ}}}} \right)}, \\ \left(\prod_{1 \le i_1 <, \ldots, < i_{\text{ɰ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{ɰ}} \left(1 - \left(\ddot{\varepsilon}_{A_j}(\varrho) \right) \right) \right)^{\frac{1}{\text{ɰ}}} \right)^{(1 - \prod_{j=1}^{\text{ɰ}} \mathfrak{N}_{i_j})} \right)^{\frac{1}{C_n^{\text{ɰ}}}} \\ e^{2\pi i \left(\left(\prod_{1 \le i_1 <, \ldots, < i_{\text{ɰ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{ɰ}} \left(1 - (\varphi_{A_j}(\varrho)) \right) \right)^{\frac{1}{\text{ɰ}}} \right)^{(1 - \prod_{j=1}^{\text{ɰ}} \mathfrak{N}_{i_j})} \right)^{\frac{1}{C_n^{\text{ɰ}}}} \right)}, \\ \left(\prod_{1 \le i_1 <, \ldots, < i_{\text{ɰ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{ɰ}} \left(1 - \left(\tilde{\Upsilon}_{\nu_{i_j}}(\varrho) \right) \right) \right)^{\frac{1}{\text{ɰ}}} \right)^{(1 - \prod_{j=1}^{\text{ɰ}} \mathfrak{N}_{i_j})} \right)^{\frac{1}{C_n^{\text{ɰ}}}} \\ e^{2\pi i \left(\left(\prod_{1 \le i_1 <, \ldots, < i_{\text{ɰ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{ɰ}} \left(1 - (\phi_{\nu_{i_j}}(\varrho)) \right) \right)^{\frac{1}{\text{ɰ}}} \right)^{(1 - \prod_{j=1}^{\text{ɰ}} \mathfrak{N}_{i_j})} \right)^{\frac{1}{C_n^{\text{ɰ}}}} \right)} \end{array} \end{pmatrix} \quad (6)$$

Proof of this theorem is given in Appendix A.

We established a numerical example to support the CPFWHM operator by using the methodology of the Definition 12.

Example 2. *Let* $I_1 = \left(0.28e^{2\pi i(0.42)}, 0.36e^{2\pi i(0.18)}, 0.33e^{2\pi i(0.19)}\right)$, $I_2 = \left(0.15e^{2\pi i(0.07)}, 0.52e^{2\pi i(0.09)}, 0.15e^{2\pi i(0.66)}\right)$, $I_3 = \left(0.64e^{2\pi i(0.15)}, 0.09e^{2\pi i(0.42)}, 0.16e^{2\pi i(0.15)}\right)$ *are three CPFVs with corresponding weight vectors* $\mathfrak{N} = (0.45, 0.35, 20)$, *suppose that* $\text{ɰ} = 2$. *Then,*

$$CPFWHM^{(\text{ɰ})}(I_1, I_2, \ldots, I_n) = \begin{pmatrix} \begin{array}{c} \left(1 - \left(\prod_{1 \le i_1 <, \ldots, < i_{\text{ɰ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{ɰ}} \breve{\upsilon}_{\mu_{i_j}}(\varrho) \right)^{\frac{1}{\text{ɰ}}} \right)^{(1 - \prod_{j=1}^{\text{ɰ}} \mathfrak{N}_{i_j})} \right)^{\frac{1}{C_n^{\text{ɰ}}}} \right) \\ e^{2\pi i \left(\left(1 - \left(\prod_{1 \le i_1 <, \ldots, < i_{\text{ɰ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{ɰ}} \psi_{\mu_{i_j}}(\varrho) \right)^{\frac{1}{\text{ɰ}}} \right)^{(1 - \prod_{j=1}^{\text{ɰ}} \mathfrak{N}_{i_j})} \right)^{\frac{1}{C_n^{\text{ɰ}}}} \right) \right)}, \\ \left(\prod_{1 \le i_1 <, \ldots, < i_{\text{ɰ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{ɰ}} \left(1 - \ddot{\varepsilon}_{A_j}(\varrho) \right) \right)^{\frac{1}{\text{ɰ}}} \right)^{(1 - \prod_{j=1}^{\text{ɰ}} \mathfrak{N}_{i_j})} \right)^{\frac{1}{C_n^{\text{ɰ}}}} \\ e^{2\pi i \left(\left(\prod_{1 \le i_1 <, \ldots, < i_{\text{ɰ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{ɰ}} (1 - \varphi_{A_j}(\varrho)) \right)^{\frac{1}{\text{ɰ}}} \right)^{(1 - \prod_{j=1}^{\text{ɰ}} \mathfrak{N}_{i_j})} \right)^{\frac{1}{C_n^{\text{ɰ}}}} \right)}, \\ \left(\prod_{1 \le i_1 <, \ldots, < i_{\text{ɰ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{ɰ}} \left(1 - \tilde{\Upsilon}_{\nu_{i_j}}(\varrho) \right) \right)^{\frac{1}{\text{ɰ}}} \right)^{(1 - \prod_{j=1}^{\text{ɰ}} \mathfrak{N}_{i_j})} \right)^{\frac{1}{C_n^{\text{ɰ}}}} \\ e^{2\pi i \left(\left(\left(\prod_{1 \le i_1 <, \ldots, < i_{\text{ɰ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{ɰ}} (1 - \phi_{\nu_{i_j}}(\varrho)) \right)^{\frac{1}{\text{ɰ}}} \right)^{(1 - \prod_{j=1}^{\text{ɰ}} \mathfrak{N}_{i_j})} \right) \right)^{\frac{1}{C_n^{\text{ɰ}}}} \right)} \end{array} \end{pmatrix}$$

$$
= \left(\begin{array}{c}
1-\left(\begin{array}{c}\left(1-(0.28\times 0.15)^{0.5}\right)^{(1-(0.45\times 0.35))}.\\ \left(1-(0.28\times 0.64)^{0.5}\right)^{(1-(0.45\times 0.20))}\\ \left(1-(0.15\times 0.64)^{0.5}\right)^{(1-(0.35\times 0.20))}\end{array}\right)^{\frac{1}{6}} \\
e^{2\pi i\left(1-\left(\begin{array}{c}(1-(0.42\times 0.07)^{0.5})^{(1-(1-0.20\times 0.15))}.\\ (1-(0.42\times 0.15)^{0.5})^{(1-(0.45\times 0.20))}\\ (1-(0.07\times 0.15)^{0.5})^{(1-(0.35\times 0.20))}\end{array}\right)^{\frac{1}{6}}\right)}, \\
\left(\begin{array}{c}\left(1-((1-0.36)\times(1-0.52))^{0.5}\right)^{(1-(0.45\times 0.35))}.\\ \left(1-((1-0.36)\times(1-0.09))^{0.5}\right)^{(1-(0.45\times 0.20))}\\ \left(1-((1-0.52)\times(1-0.09))^{0.5}\right)^{(1-(0.35\times 0.20))}\end{array}\right)^{\frac{1}{6}} \\
e^{2\pi i\left(\left(\begin{array}{c}(1-((1-0.18)\times(1-0.09))^{0.5})^{(1-(0.45\times 0.35))}.\\ (1-((1-0.18)\times(1-0.42))^{0.5})^{(1-(0.45\times 0.20))}\\ (1-((1-0.09)\times(1-0.42))^{0.5})^{(1-(0.35\times 0.20))}\end{array}\right)^{\frac{1}{6}}\right)}, \\
\left(\begin{array}{c}\left(1-((1-0.33)\times(1-0.15))^{0.5}\right)^{(1-(0.45\times 0.35))}.\\ \left(1-((1-0.33)\times(1-0.16))^{0.5}\right)^{(1-(0.45\times 0.20))}\\ \left(1-((1-0.15)\times(1-0.16))^{0.5}\right)^{(1-(0.35\times 0.20))}\end{array}\right)^{\frac{1}{6}} \\
e^{2\pi i\left(\left(\begin{array}{c}(1-((1-0.19)\times(1-0.66))^{0.5})^{(1-(0.45\times 0.35))}.\\ (1-((1-0.19)\times(1-0.15))^{0.5})^{(1-(0.45\times 0.20))}\\ (1-((1-0.66)\times(1-0.15))^{0.5})^{(1-(0.35\times 0.20))}\end{array}\right)^{\frac{1}{6}}\right)}
\end{array}\right)
$$

$$
= \left(0.0981e^{2\pi i(0.0309)}, 0.5793e^{2\pi i(0.4433)},\ 0.4166e^{2\pi i(0.5770)}\right)
$$

Theorem 6. *(Idempotency Property), Consider* $I_j = \left(\breve{\upsilon}_{\mu_j}(\varrho)e^{2\pi i\psi_{\mu j}(\varrho)},\ \ddot{\varepsilon}_{A_j}(\varrho)e^{2\pi i\varphi_{Aj}(\varrho)},\ \tilde{\Upsilon}_{\nu_j}(\varrho)e^{2\pi i\phi_{\nu_j}(\varrho)}\right)$*, $j = 1, 2, \ldots, k$, to be the family of all identical CPFVs. Then:*

$$
CPFWHM^{\text{ɯ}}(I_1, I_2, \ldots, I_n) = I
$$

Proof: Proof is analogously. □

Theorem 7. *(Monotonicity Property), Consider* $I_j = \left(\breve{\upsilon}_{\mu_j}(\varrho)e^{2\pi i\psi_{\mu j}(\varrho)},\ \ddot{\varepsilon}_{A_j}(\varrho)e^{2\pi i\varphi_{Aj}(\varrho)},\ \tilde{\Upsilon}_{\nu_j}(\varrho)e^{2\pi i\phi_{\nu_j}(\varrho)}\right)$*, and* $R_j(\varrho) = \left(g_{\mu_j}(\varrho)e^{2\pi i\alpha_{\mu j}(\varrho)},\ t_{A_j}(\varrho)e^{2\pi i\gamma_{Aj}(\varrho)}, h_{\nu j}(\varrho)e^{2\pi i\beta_{\nu j}(\varrho)}\right)$*, $j = 1, 2, \ldots, k$ are any two CPFSs. If $I_j(\varrho) \le R_j(\varrho)$. $\breve{\upsilon}_{\mu_j}(\varrho) \le g_{\mu_j}(\varrho)$, $\psi_{\mu j}(\varrho) \le \alpha_{\mu_j}(\varrho)$, $\ddot{\varepsilon}_{A_j}(\varrho) \le t_{A_j}(\varrho)$, $\varphi_{Aj}(\varrho) \le \gamma_{Aj}(\varrho)$ and $\tilde{\Upsilon}_{\nu j}(\varrho) \le h_{\nu j}(\varrho)$, $\phi_{\nu j}(\varrho) \le \beta_{\nu j}(\varrho)$. Then:*

Proof: Straightforward. □

Theorem 8. *(Boundedness Property),* *Consider* $I_j = \left(\ddot{\upsilon}_{\mu_j}(\varrho)e^{2\pi i\psi_{\mu j}(\varrho)},\ \hat{\varepsilon}_{A_j}(\varrho)e^{2\pi i\varphi_{Aj}(\varrho)}, \tilde{\Upsilon}_{\nu_j}(\varrho)e^{2\pi i\phi_{\nu_j}(\varrho)}\right), j = 1, 2, \ldots, k,$ *to be the family of CPFVs, if:*

$$I_j^- = min(I_1,\ I_2,\ I_3,\ \ldots,\ I_n)$$

and

$$I_j^+ = max(I_1,\ I_2,\ I_3,\ \ldots,\ I_n)$$

then

$$I^- \leq CPFWHM^{ɰ}(I_1, I_2, \ldots, I_n) \leq I^+$$

From boundedness property:

$$CPFWHM^{ɰ}(I_1, I_2, \ldots, I_n) = I^-$$
$$CPFWHM^{ɰ}(I_1, I_2, \ldots, I_n) = I^+$$

From monotonicity property

$$I^- \leq CPFWHM^{ɰ}(I_1, I_2, \ldots, I_n) \leq I^+$$

We explored the proof of the Theorem 8 in Appendix A.

5. Complex Picture Fuzzy Dual Hamy Mean Operators

We establish AOs of CPFDHM and CPFWDHM operators by using the basic idea of DHM operator under the system of CPF information. To find the validity of our discussion strategy, we gave a numerical example.

Definition 13. *Consider* $I_j = \left(\ddot{\upsilon}_{\mu_j}(\varrho)e^{2\pi i\psi_{\mu j}(\varrho)},\ \hat{\varepsilon}_{A_j}(\varrho)e^{2\pi i\varphi_{Aj}(\varrho)}, \tilde{\Upsilon}_{\nu_j}(\varrho)e^{2\pi i\phi_{\nu_j}(\varrho)}\right), j = 1, 2, \ldots, k,$ *to be the family of CPFVs. Then CPFDHM operator is given as:*

$$CPFDHM^{(ɰ)}(I_1, I_2, \ldots, I_n) = \left(\prod_{1 \leq i_1 <, \ldots, < i_{ɰ} \leq n} \left(\frac{\sum_{j=1}^{ɰ} I_{i_j}}{ɰ}\right)\right)^{\frac{1}{C_n^{ɰ}}} \quad (7)$$

Theorem 9. *Consider* $I_j = \left(\ddot{\upsilon}_{\mu_j}(\varrho)e^{2\pi i\psi_{\mu j}(\varrho)},\ \hat{\varepsilon}_{A_j}(\varrho)e^{2\pi i\varphi_{Aj}(\varrho)}, \tilde{\Upsilon}_{\nu_j}(\varrho)e^{2\pi i\phi_{\nu_j}(\varrho)}\right), j = 1, 2, \ldots, k,$ *to be the family of CPFVs. Then CPFDHM operator is given as:*

$$CPFDHM^{(ɰ)}(I_1, I_2, \ldots, I_n) = \left(\prod_{1 \leq i_1 <, \ldots, < i_{ɰ} \leq n} \left(\frac{\sum_{j=1}^{ɰ} I_{i_j}}{ɰ}\right)\right)^{\frac{1}{C_n^{ɰ}}} \quad (8)$$

Proof: The proof is analogous to the proof of Theorem 1. □

Remark 2. *All the properties of CPFWHM operator such as idempotency, monotonicity, and boundedness are prove similar to Theorems 2, 3 and 4.*

We elaborated the concept of DHM tool to establish a new AOs of CPFDHM operator under the system of CPFSs.

Definition 14. *Consider* $I_j = \left(\breve{\upsilon}_{\mu_j}(\varrho)e^{2\pi i\psi_{\mu j}(\varrho)},\ \hat{\varepsilon}_{A_j}(\varrho)e^{2\pi i\varphi_{Aj}(\varrho)}, \tilde{\Upsilon}_{\nu_j}(\varrho)e^{2\pi i\phi_{\nu_j}(\varrho)}\right), j = 1, 2, \ldots, k$, *to be the family of CPFVs, with corresponding weight vectors* $\mathfrak{N}_i = (\mathfrak{N}_1, \mathfrak{N}_2, \ldots, \mathfrak{N}_n)^T$, $\mathfrak{N}_i \in [0,1]$ *and* $\sum_{i=1}^{n} \mathfrak{N}_i = 1$. *Then:*

$$CPFWDHM^{(ɰ)}(I_1, I_2, \ldots, I_n) = \frac{\underset{1\le i_1<,\ \ldots,<i_ɰ\le n}{\otimes}\left(1-\prod_{j=1}^{ɰ}\mathfrak{N}_{i_j}\right)\left(\overset{ɰ}{\underset{j=1}{\oplus}}\left(I_{i_j}\right)\right)^{\frac{1}{ɰ}}}{C_n^{ɰ}} \tag{9}$$

Theorem 10. *Consider* $I_j = \left(\breve{\upsilon}_{\mu_j}(\varrho)e^{2i\pi\psi_{\mu j}(\varrho)},\ \hat{\varepsilon}_{A_j}(\varrho)e^{2i\pi\varphi_{Aj}(\varrho)}, \tilde{\Upsilon}_{\nu_j}(\varrho)e^{2i\pi\phi_{\nu_j}(\varrho)}\right), j = 1, 2, \ldots, k$, *to be the family of CPFV, then:*

$$CPFWDHM^{(ɰ)}(I_1, I_2, \ldots, I_n)$$
$$= \begin{pmatrix} \left(\prod\limits_{1\le i_1<,\ \ldots,<i_ɰ\le n}\left(1-\left(\prod\limits_{j=1}^{ɰ}\left(1-\left(\breve{\upsilon}_{\mu_{i_j}}(\varrho)\right)\right)\right)^{\frac{1}{ɰ}}\right)^{\left(1-\prod_{j=1}^{ɰ}\mathfrak{N}_{i_j}\right)}\right)^{\frac{1}{C_n^{ɰ}}} \\ e^{2\pi i\left(\left(\prod_{1\le i_1<,\ \ldots,<i_ɰ\le n}\left(1-\left(\prod_{j=1}^{ɰ}\left(1-\left(\psi_{\mu_{i_j}}(\varrho)\right)\right)\right)^{\frac{1}{ɰ}}\right)^{\left(1-\prod_{j=1}^{ɰ}\mathfrak{N}_{i_j}\right)}\right)^{\frac{1}{C_n^{ɰ}}}\right)}, \\ 1-\left(\prod\limits_{1\le i_1<,\ \ldots,<i_ɰ\le n}\left(1-\left(\prod\limits_{j=1}^{ɰ}\hat{\varepsilon}_{A_j}(\varrho)\right)^{\frac{1}{ɰ}}\right)^{\left(1-\prod_{j=1}^{ɰ}\mathfrak{N}_{i_j}\right)}\right)^{\frac{1}{C_n^{ɰ}}} \\ e^{2\pi i\left(1-\left(\prod_{1\le i_1<,\ \ldots,<i_ɰ\le n}\left(1-\left(\prod_{j=1}^{ɰ}\varphi_{Aj}(\varrho)\right)^{\frac{1}{ɰ}}\right)^{\left(1-\prod_{j=1}^{ɰ}\mathfrak{N}_{i_j}\right)}\right)^{\frac{1}{C_n^{ɰ}}}\right)}, \\ 1-\left(\prod\limits_{1\le i_1<,\ \ldots,<i_ɰ\le n}\left(1-\left(\prod\limits_{j=1}^{ɰ}\tilde{\Upsilon}_{\nu_{i_j}}(\varrho)\right)^{\frac{1}{ɰ}}\right)^{\left(1-\prod_{j=1}^{ɰ}\mathfrak{N}_{i_j}\right)}\right)^{\frac{1}{C_n^{ɰ}}} \\ e^{2\pi i\left(1-\left(\prod_{1\le i_1<,\ \ldots,<i_ɰ\le n}\left(1-\left(\prod_{j=1}^{ɰ}\phi_{\nu_{i_j}}(\varrho)\right)^{\frac{1}{ɰ}}\right)^{\left(1-\prod_{j=1}^{ɰ}\mathfrak{N}_{i_j}\right)}\right)^{\frac{1}{C_n^{ɰ}}}\right)} \end{pmatrix} \tag{10}$$

Proof: The proof is similar to the proof of Theorem 5. □

To support Definition 14, we establish the following practice Example 3 by utilizing the idea of CPFWDHM operator.

Example 3. *Let* $I_1 = \left(0.42e^{2\pi i(0.18)},\ 0.04e^{2\pi i(0.36)}, 0.16e^{2\pi i(0.23)}\right)$, $I_2 = \left(0.08e^{2\pi i(0.16)},\ 0.62e^{2\pi i(0.27)},\ 0.26e^{2\pi i(0.19)}\right)$, $I_3 = \left(0.53e^{2\pi i(0.22)}, 0.12e^{2\pi i(0.32)},\ 0.33e^{2\pi i(0.22)}\right)$ *are three CPFVs with corresponding weight vectors* $\mathfrak{N} = (0.45,\ 0.35,\ 0.20)$, *and suppose that* $ɰ = 2$. *Then*

$$CPFWDHM^{(щ)}(I_1, I_2, \ldots, I_n) = \begin{pmatrix} \left(\prod_{1\le i_1<,\ldots,<i_щ\le n}\left(1-\left(\prod_{j=1}^{щ}\left(1-\left(\breve{\upsilon}_{\mu_{i_j}}(\varrho)\right)\right)\right)^{\frac{1}{щ}}\right)^{(1-\prod_{j=1}^{щ}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{щ}}} e^{2\pi i\left(\left(\prod_{1\le i_1<,\ldots,<i_щ\le n}\left(1-\left(\prod_{j=1}^{щ}\left(1-\left(\psi_{\mu_{i_j}}(\varrho)\right)\right)\right)^{\frac{1}{щ}}\right)^{(1-\prod_{j=1}^{щ}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{щ}}}\right)}, \\ 1-\left(\prod_{1\le i_1<,\ldots,<i_щ\le n}\left(1-\left(\prod_{j=1}^{щ}\hat{\varepsilon}_{A_j}(\varrho)\right)^{\frac{1}{щ}}\right)^{(1-\prod_{j=1}^{щ}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{щ}}} e^{2\pi i\left(1-\left(\prod_{1\le i_1<,\ldots,<i_щ\le n}\left(1-\left(\prod_{j=1}^{щ}\varphi_{Aj}(\varrho)\right)^{\frac{1}{щ}}\right)^{(1-\prod_{j=1}^{щ}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{щ}}}\right)}, \\ 1-\left(\prod_{1\le i_1<,\ldots,<i_щ\le n}\left(1-\left(\prod_{j=1}^{щ}\tilde{\Upsilon}_{\nu_{i_j}}(\varrho)\right)^{\frac{1}{щ}}\right)^{(1-\prod_{j=1}^{щ}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{щ}}} e^{2\pi i\left(1-\left(\prod_{1\le i_1<,\ldots,<i_щ\le n}\left(1-\left(\prod_{j=1}^{щ}\phi_{\nu_{i_j}}(\varrho)\right)^{\frac{1}{щ}}\right)^{(1-\prod_{j=1}^{щ}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{щ}}}\right)} \end{pmatrix}$$
$$= \left(0.6149e^{2\pi i(0.3800)}, 0.0320e^{2\pi i(0.0896)}, \ 0.0546e^{2\pi i(0.0407)}\right)$$

Remark 3. *All the properties of CPFWDHM operator like idempotency, monotonicity, and boundedness are proved similar to Theorems 2, 3 and 4.*

6. MADM Techniques and Its Algorithm

In this section, we study a method to solve the procedure of the MADM technique under the system of PFSs. We also apply our discussed approaches like CPFWHM and CPFWDHM operators. Consider $ӥ = (ӥ_1, ӥ_2, \ldots, ӥ_n)$ be a discrete set of alternatives, which can be evaluated by using characteristics (set of attributes) $Æ = (Æ_1, Æ_2, \ldots, Æ_n)$ with corresponding weight vectors $Æ = (\mathfrak{N}_1, \mathfrak{N}_2, \ldots, \mathfrak{N}_n)^T, \mathfrak{N}, \in [0,1]$ and $\sum_{i=1}^{n} \mathfrak{N}_i = 1$. Each alternative has information on the environment of CPFSs. After accumulation of the information results in the state of CPFVs,

$I_{i_j} = \left(\breve{\upsilon}_{\mu_j}(\varrho)e^{2\pi i\psi_{\mu j}(\varrho)}, \ \hat{\varepsilon}_{A_j}(\varrho)e^{2\pi i\varphi_{Aj}(\varrho)}, \tilde{\Upsilon}_{\nu_j}(\varrho)e^{2\pi i\phi_{\nu_j}(\varrho)}\right), j = 1, 2, \ldots, k$, these results must satisfy such conditions:

$0 \le \breve{\upsilon}_{\mu_j}(\varrho) + \hat{\varepsilon}_{A_j}(\varrho) + \tilde{\Upsilon}_{\nu_j}(\varrho) \le 1$ and $0 \le \psi_{\mu j}(\varrho) + \varphi_{Aj}(\varrho) + \phi_{\nu_j}(\varrho) \le 1$.

A decision matrix $Đ = (å_{ij})_{m\times n}$ is depicted in the following form:

$$Đ = \begin{bmatrix} I_{11} & I_{12} & \cdots & I_{1n} \\ I_{21} & I_{22} & \cdots & I_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ I_{m1} & I_{m2} & \cdots & I_{mn} \end{bmatrix}$$

To solve a MADM technique, we follow the steps of the following algorithm.

Steps 1: A decision maker constructs a decision matrix having information in form of alternative $щ = (щ_1, щ_2, \ldots, щ_n)$ and attributes $Æ = (Æ_1, Æ_2, \ldots, Æ_n)$ with corresponding weight vectors $\mathfrak{N}_i = (\mathfrak{N}_1, \mathfrak{N}_2, \ldots, \mathfrak{N}_n)^T, \ i = 1, 2, 3, \ldots, n$. All above-discussed information is packed in a decision matrix $Đ = (å_{ij})_{m\times n}$.

Step 2: Transformation of a decision matrix into a normalization matrix. The attributes can be divided into two types of criteria, cost type, and benefit type. If the cost factor involves then we have to transform the decision matrix into a normalizing matrix otherwise there is no need to transform the decision matrix. We can normalize the decision matrix by using the following technique.

$$Đ = \left(\mathring{a}_{ij}\right) = \left\{\begin{matrix} \left(\tilde{\Upsilon}_{\nu_j}(\varrho)e^{2\pi i\phi_{\nu_j}(\varrho)},\ \ddot{\varepsilon}_{A_j}(\varrho)e^{2\pi i\varphi_{Aj}(\varrho)}, \breve{\upsilon}_{\mu_j}(\varrho)e^{2\pi i\psi_{\mu j}(\varrho)}\right) & if\ j\ is\ cost\ factor \\ \left(\breve{\upsilon}_{\mu_j}(\varrho)e^{2\pi i\psi_{\mu j}(\varrho)},\ \ddot{\varepsilon}_{A_j}(\varrho)e^{2\pi i\varphi_{Aj}(\varrho)}, \tilde{\Upsilon}_{\nu_j}(\varrho)e^{2\pi i\phi_{\nu_j}(\varrho)}\right) & if\ j\ is\ benefit\ factor \end{matrix}\right\}$$

Steps 3: Accumulate CPF information depicted in the decision matrix by using our discussed approaches of CPFWHM and CPFWDHM operators.

Step 4: Investigate score values of the consequences of CPFWHM and CPFWDHM operators by using Definition 4.

Step 5: To find a suitable alternative, we have to make ranking and ordering of the score values.

A compressive flowchart explaining all the steps of algorithm is given below in Figure 1.

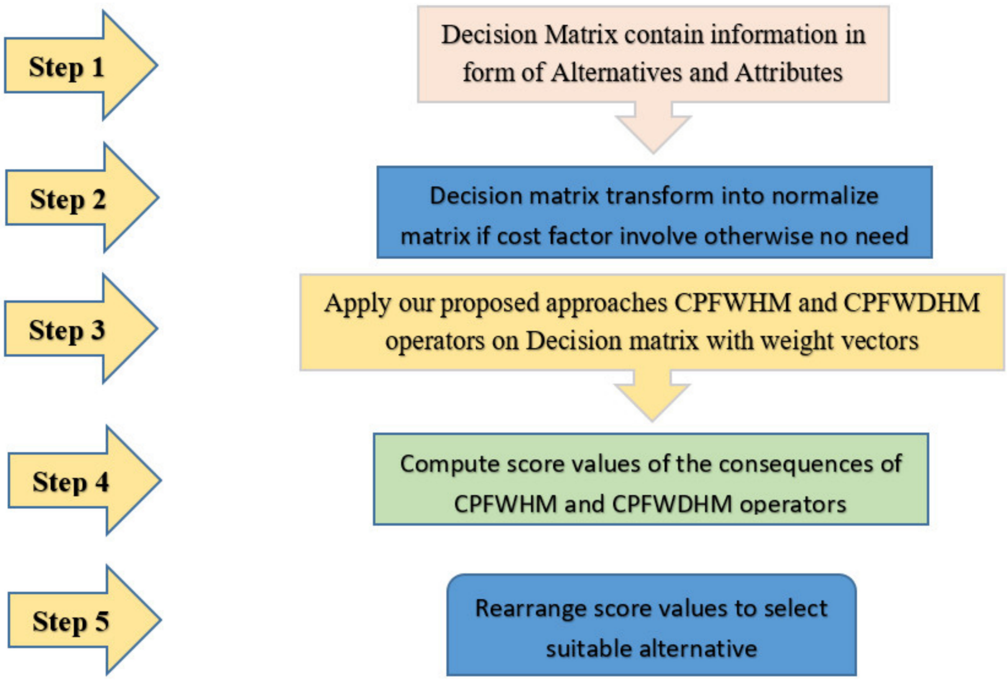

Figure 1. Flowchart of algorithm.

6.1. Application

A VMS is a program or piece of software that automates all of an organization's vendor-related tasks. An organization's communication and collaboration with vendors can be an important mechanism for these systems. On a VMS, a business can also effectively approve and monitor a vendor's portfolio and performance. A VMS enables your business to collect purchase orders from managers, optimize flexible worker onboarding, automate transactions, save and collect data from every stage of your contingent worker hiring process, and compile key performance indicators such as spending tracking, candidate information, payroll and invoice data. A vendor management system is often adopted by a business directly to manage its independent talent pool or by an MSP on its behalf. By improving the supply chain system and reducing the risk of operational disruptions, vendor management also enables firms to better controls and management of vendors. Additionally, it helps businesses ensure quality and timely delivery of various goods and services, which improves customer satisfaction levels. As a last advantage, the vendor management process enables companies in developing long-lasting and reputable relationships with their vendors, which leads to better rates being secured. A lot of research scholars worked on the theory of VMS to try to improve the mechanism of the VMS. Savaşaneril and Erkip [57] analyzed the purpose and advantages of vendor management software. Solyal and Süral [58] proposed the solution for inventory control under the system of VMS.

6.2. Numerical Example

In this numerical example, we evaluate the suitable software for VMS by observing the various qualities of different software presented by different multinational companies. The reliability and lifespan of a software for VMS depend on manufacturing and the degree of testing qualities. Consider we have to choose a suitable software for VMS from four different types of software $ß_{þ}$, $(þ = 1,\ 2,\ 3,\ 4)$ according to observing a few qualities (attributes) $ɳ_{þ}$, $(þ = 1,\ 2,\ 3)$ by assigning the experts. We select the best software for VMS based on the following characteristics: $ɳ_1$ represents ease of navigation and setup; $ɳ_2$ represents a large capacity to manage, order, invoices, deliveries, and payments; and $ɳ_3$ represents product performance and warranty.

The experts assign different weight vectors $\mathfrak{N} = (0.35,\ 0.40,\ 0.25)$ to the attributes according to their characteristics. By using our proposed methodology, we select a suitable object from the given information by the decision maker. To investigate the best software for a VMS, we follow the above-discussed algorithm and its steps.

Step 1: The decision maker collects information under the system of CPFNs (this information is present in Table 2 which contains alternative and attributes).

Step 2: There is no need to transform the decision matrix because the cost factor does not involve the types of attributes.

Step 3: Accumulate the given information of CPFNs which is displayed in Table 2 by using CPFWHM and CPFWDHM operators. These AOs are used to deduce results of alternatives in form of CPFNs depicted in Table 3. The results of CPFNs representing in Table 3 for the parametric value of $ɰ = 2$.

Table 2. The decision matrix in the form of CPFVs.

	$ɳ_1$		$ɳ_2$
$ß_1$	$\left(0.36e^{2\pi i(0.09)}, 0.15e^{2\pi i(0.36)},\ 0.09e^{2\pi i(0.19)}\right)$	$ß_1$	$\left(0.56e^{2\pi i(0.09)}, 0.12e^{2\pi i(0.44)},\ 0.17e^{2\pi i(0.23)}\right)$
$ß_2$	$\left(0.17e^{2\pi i(0.46)}, 0.35e^{2\pi i(0.09)},\ 0.45e^{2\pi i(0.32)}\right)$	$ß_2$	$\left(0.24e^{2\pi i(0.42)}, 0.17e^{2\pi i(0.38)},\ 0.42e^{2\pi i(0.16)}\right)$
$ß_3$	$\left(0.15e^{2\pi i(0.08)}, 0.45e^{2\pi i(0.36)},\ 0.18e^{2\pi i(0.43)}\right)$	$ß_3$	$\left(0.03e^{2\pi i(0.39)}, 0.07e^{2\pi i(0.15)},\ 0.35e^{2\pi i(0.41)}\right)$
$ß_4$	$\left(0.48e^{2\pi i(0.47)}, 0.07e^{2\pi i(0.15)},\ 0.25e^{2\pi i(0.28)}\right)$	$ß_4$	$\left(0.23e^{2\pi i(0.37)}, 0.17e^{2\pi i(0.34)},\ 0.07e^{2\pi i(0.26)}\right)$
	$ɳ_3$		
$ß_1$	$\left(0.43e^{2\pi i(0.42)}, 0.15e^{2\pi i(0.27)},\ 0.06e^{2\pi i(0.09)}\right)$		
$ß_2$	$\left(0.09e^{2\pi i(0.12)}, 0.09e^{2\pi i(0.06)},\ 0.42e^{2\pi i(0.24)}\right)$		
$ß_3$	$\left(0.33e^{2\pi i(0.17)}, 0.28e^{2\pi i(0.33)},\ 0.07e^{2\pi i(0.38)}\right)$		
$ß_4$	$\left(0.38e^{2\pi i(0.62)}, 0.37e^{2\pi i(0.26)},\ 0.05e^{2\pi i(0.07)}\right)$		

Table 3. Aggregated values by the CPFWHM and CPFWDHM.

CPFWHM	**CPFWDHM**
$\left(0.4087e^{2\pi i(0.1459)}, 0.1736e^{2\pi i(0.3995)},\ 0.1334e^{2\pi i(0.2047)}\right)$	$\left(0.4931e^{2\pi i(0.2291)}, 0.1254e^{2\pi i(0.3202)},\ 0.0888e^{2\pi i(0.1448)}\right)$
$\left(0.1415e^{2\pi i(0.2765)}, 0.2379e^{2\pi i(0.1982)},\ 0.4715e^{2\pi i(0.2804)}\right)$	$\left(0.2007e^{2\pi i(0.3803)}, 0.1641e^{2\pi i(0.1227)},\ 0.3938e^{2\pi i(0.2115)}\right)$
$\left(0.1201e^{2\pi i(0.1670)}, 0.3093e^{2\pi i(0.3235)},\ 0.2335e^{2\pi i(0.4486)}\right)$	$\left(0.1201e^{2\pi i(0.1670)}, 0.3093e^{2\pi i(0.3235)},\ 0.2335e^{2\pi i(0.4486)}\right)$
$\left(0.3232e^{2\pi i(0.4436)}, 0.2391e^{2\pi i(0.2907)},\ 0.1457e^{2\pi i(0.2406)}\right)$	$\left(0.4088e^{2\pi i(0.5326)}, 0.1590e^{2\pi i(0.2182)},\ 0.0907e^{2\pi i(0.1643)}\right)$

Step 4: Evaluate score values by using the results of CPFWHM and CPFWDHM operators depicted in Table 3. Computed score values are presented in Table 4.

Table 4. Score values of different software applications for a VMS.

Operators	$\hat{S}(ß_1)$	$\hat{S}(ß_2)$	$\hat{S}(ß_3)$	$\hat{S}(ß_4)$	Ranking and Ordering
CPFWHM	0.4406	0.3717	0.3287	0.4751	$ß_4 > ß_1 > ß_2 > ß_3$
CPFWDHM	0.5072	0.4482	0.4112	0.5515	$ß_4 > ß_1 > ß_2 > ß_3$

Step 5: Rearrange the results of score values to determine a suitable alternative by ordering and ranking the score values.

The following graphical representation explores the results of score values of CPFWHM and CPFWDHM operators in Figure 2.

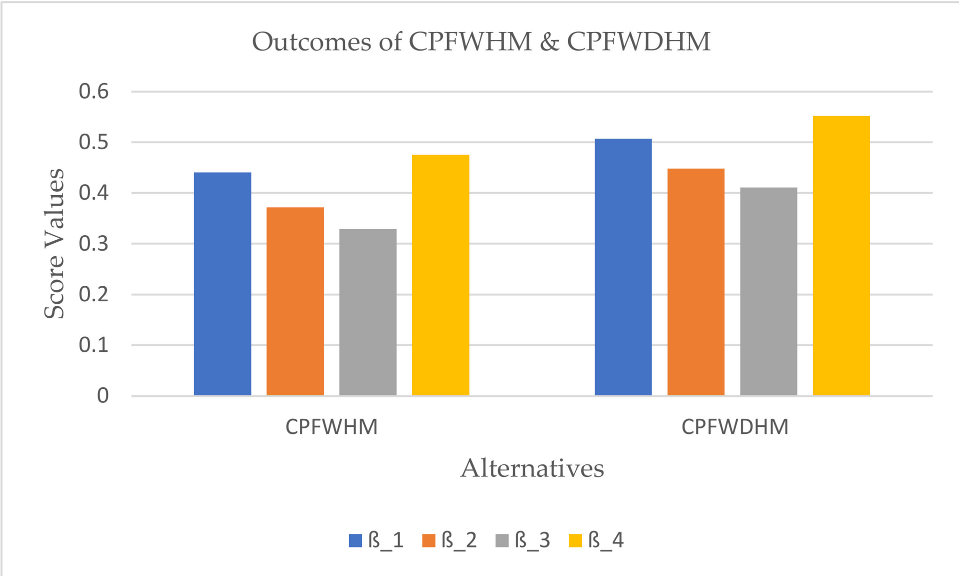

Figure 2. Score values of tourist destinations.

6.3. Influence Study

To find flexibility and reliability of our proposed methodologies, we use a different value of ɰ in binomial coefficient $C_n^{ɰ} = \frac{n!}{ɰ!(n-ɰ)!}$. We observe if the parametric value of ɰ increases, then the score values are obtained by the CPFWHM and CPFWDHM operators. We also observed if we increase the magnitude of the parametric value of ɰ, then there is no change in the ordering and ranking of the score values. All the score values which are obtained by the CPFWHM and CPFWDHM operators are shown in the following Table 5. After evaluating the score values, we see $ß_4$ is a suitable alternative for both AOs. Moreover, we represent score values geometrically in Figures 3 and 4 obtained by the CPFWHM and CPFWDHM operators, respectively.

Table 5. Ranking and ordering of the consequences of CPFWHM and CPFWDHM operators.

Operators	Parameters	$\hat{S}(ß_1)$	$\hat{S}(ß_2)$	$\hat{S}(ß_3)$	$\hat{S}(ß_4)$	Ranking and Ordering
CPFWHM	$ɰ = 1$	0.3734	0.3093	0.2788	0.4087	$ß_4 > ß_1 > ß_2 > ß_3$
	$ɰ = 2$	0.4406	0.3717	0.3287	0.4751	$ß_4 > ß_1 > ß_2 > ß_3$
	$ɰ = 3$	0.4589	0.3838	0.3407	0.4919	$ß_4 > ß_1 > ß_2 > ß_3$
CPFWDHM	$ɰ = 1$	0.5565	0.4916	0.4533	0.5935	$ß_4 > ß_1 > ß_2 > ß_3$
	$ɰ = 2$	0.5072	0.4482	0.4112	0.5515	$ß_4 > ß_1 > ß_2 > ß_3$
	$ɰ = 3$	0.4817	0.4569	0.3974	0.5181	$ß_4 > ß_1 > ß_2 > ß_3$

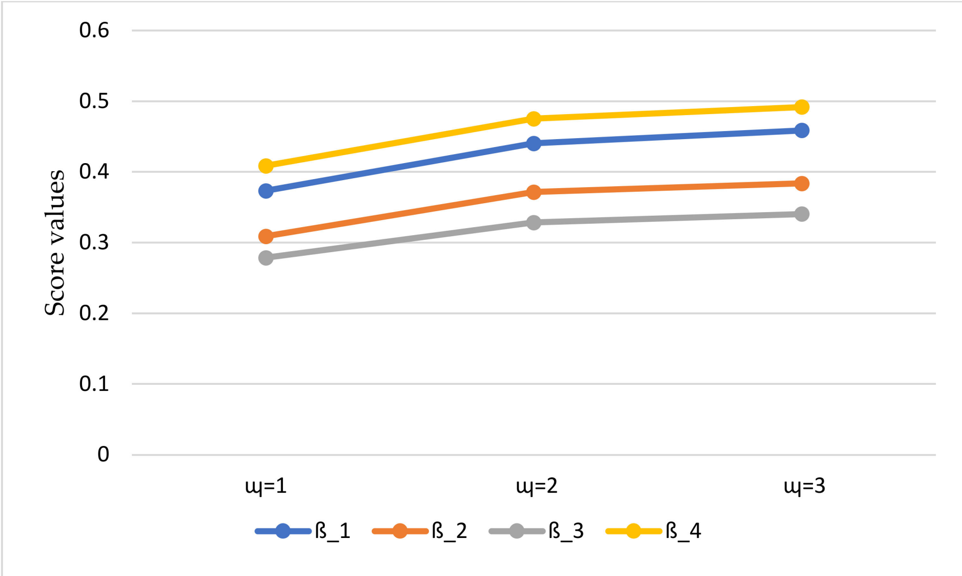

Figure 3. Results of the CPFWHM operator for different values of щ.

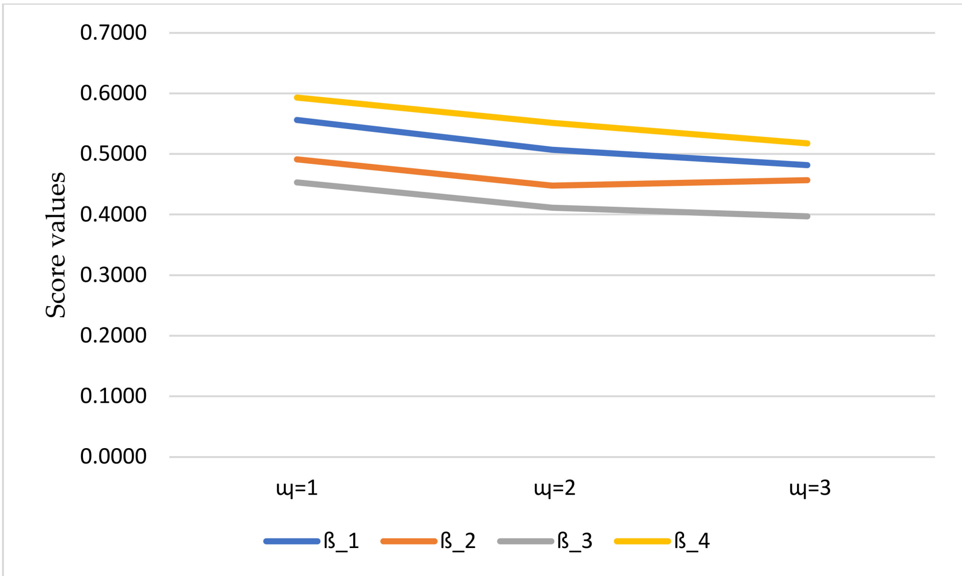

Figure 4. Results of the CPFWDHM operator for different values of щ.

7. Comparative Analysis

In this section, we contrast the results of existing AOs with the results of our proposed methodology. We applied existing AOs to the decision matrix developed by Garg and Rani [59], Akram et al. [39,60], Zhang et al. [61] and Ullah et al. [36]. We observed some existing AOs are unable to deal with the decision matrix shown in Table 2. The existing AOs [59–61] and [36] failed with the information given by the decision maker. We also study the consequences of AOs [39] shown in the following Table 6, which is obtained by the decision matrix shown in Table 2.

Table 6. Results of the comparative study.

Operator	Environment	Results
CIFWHM operator (current work)	CPFSs	$ß_3 > ß_3 > ß_3 > ß_3$
CIFWDHM operator (current work)	CPFSs	$ß_3 > ß_3 > ß_3 > ß_3$
CPFHWA Akram et al. [39]	CPFSs	$ß_4 > ß_1 > ß_2 > ß_3$
CPFHWG Akram et al. [39]	CPFSs	$ß_4 > ß_1 > ß_2 > ß_3$
Akram et al. [60]	CIFSs	Failed
Akram et al. [60]	CIFSs	Failed
Ullah et al. [36]	CPyFSs	Failed
Garg and Rani [59]	CIVIFSs	Failed
Zhang et al. [61]	PFSs	Failed

The following graphical interpretation shows results of our proposed AOs and CPF Hamacher weighted (CPFHW) averaging (CPFHWA) and CPFHW geometric (CPFHWG) operators in the Figure 5.

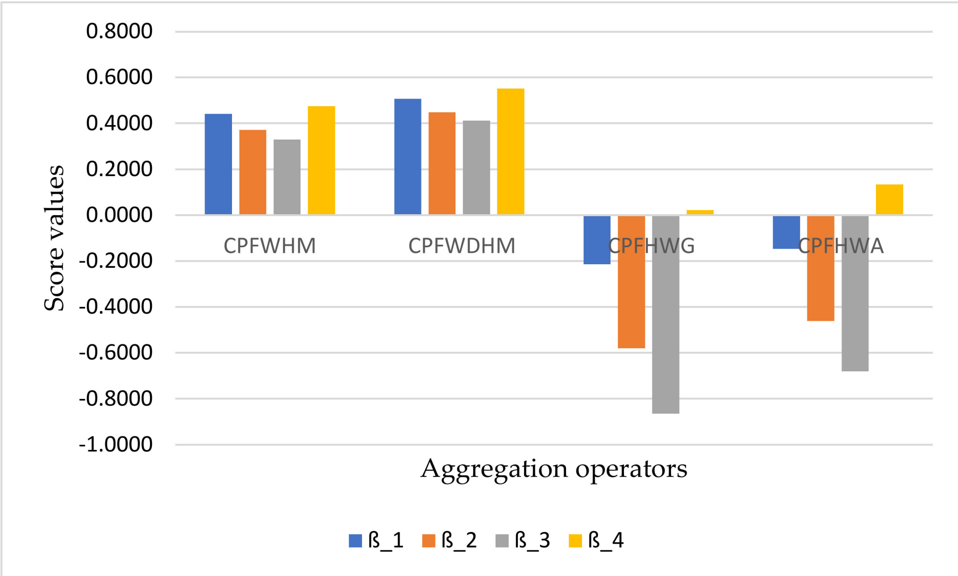

Figure 5. Comparison of existing AOs with our proposed methodologies.

8. Conclusions

To cope with uncertainty and vagueness, we established a series of new AOs under the system of CPFSs. A CPFS contains two aspects of MV, AV, and NMV in the form of amplitude and phase terms. A CPFS is superior and flexible because CPFSs are the extension of IFSs, PyFSs, q-ROFSs, CIFSs, CPyFSs, and PFSs. We deduced some new AOs of CPFHM and CPFWDHM operators by using the operational laws of the HM tool under the environment of CPFS with some basic characteristics such as idempotency, monotonicity, and boundedness. We also generalized concepts of HM operators in the framework of CPFDHM and CPFWDHM operators. To support our proposed methodology, we interpreted some examples. We established an application based on VMS under the system of CPFSs. A VMS is a software application that is utilized to handle vendors, ordering, invoices, and delivery procedures in several shopping malls, restaurants, and other numerous companies. To find the reliability and validity of our proposed AOs, we evaluated a numerical example to show usefulness and compatibility by using the technique of the MADM process under VMS. We also demonstrated a comprehensive comparative study to compare the results of our proposed methodology with existing AOs.

In future, we will elaborate our proposed work in the framework of picture fuzzy Maclaurin symmetric operators [62] and a further extension in the environment of a bipolar soft set [63]. Further, we will also extend our invented approaches in the framework of rough sets under the system of topological techniques [64].

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Appendix A

Proof of Theorem 1. This theorem has two parts. First, we derive the formula given in Equation (6) as follows:

$$\bigotimes_{i=1}^{\text{щ}} I_j = \begin{pmatrix} \prod_{j=1}^{\text{щ}} \breve{\upsilon}_{\mu_j}(\varrho) e^{2\pi i\left(\prod_{j=1}^{\text{щ}}\psi_{\mu_j}(\varrho)\right)}, \\ \left(1-\prod_{j=1}^{\text{щ}}\left(1-\ddot{\varepsilon}_{A_j}(\varrho)\right)\right).e^{2\pi i\left(1-\prod_{j=1}^{\text{щ}}\left(1-\varphi_{Aj}(\varrho)\right)\right)}, \\ \left(1-\prod_{j=1}^{\text{щ}}\left(1-\tilde{\Upsilon}_{\nu_j}(\varrho)\right)\right).e^{2\pi i\left(1-\prod_{j=1}^{\text{щ}}\left(1-\phi_{\nu_j}(\varrho)\right)\right)} \end{pmatrix}$$

$$\left(\bigotimes_{i=1}^{\text{щ}} I_j\right)^{\frac{1}{\text{щ}}} = \begin{pmatrix} \left(\left(\prod_{j=1}^{X} \breve{\upsilon}_{\mu_j}(\varrho)\right)^{\frac{1}{\text{щ}}} e^{2\pi i\left(\prod_{j=1}^{\text{щ}}\psi_{\mu j}(\varrho)\right)^{\frac{1}{\text{щ}}}}\right), \\ \left(1-\prod_{j=1}^{\text{щ}}\left(1-\ddot{\varepsilon}_{A_j}(\varrho)\right)^{\frac{1}{\text{щ}}} e^{2\pi i\left(1-\prod_{j=1}^{\text{щ}}\left(1-\varphi_{Aj}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}}\right) \\ \left(1-\prod_{j=1}^{X}\left(1-\tilde{\Upsilon}_{\nu_j}(\varrho)\right)^{\frac{1}{\text{щ}}} e^{2\pi i\left(1-\prod_{j=1}^{\text{щ}}\left(1-\phi_{\nu_j}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}}\right) \end{pmatrix}$$

$$\bigoplus_{1\le i_t<,\ldots,<i_t}\left(\bigotimes_{i=1}^{n} I_j\right)^{\frac{1}{\text{щ}}} = \begin{pmatrix} \left(1-\prod_{1\le i_t<,\ldots,<i_t}\left(1-\left(\prod_{j=1}^{\text{щ}}\breve{\upsilon}_{\mu_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)e^{2\pi i\left(1-\prod_{1\le i_t<,\ldots,<i_t}\left(1-\left(\prod_{j=1}^{\text{щ}}\psi_{\mu j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)\right)}\right), \\ \left(\prod_{1\le i_t<,\ldots,<i_t}\left(1-\prod_{j=1}^{X}\left(\ddot{\varepsilon}_{A_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)e^{2\pi i\left(\prod_{1\le i_t<,\ldots,<i_t}\left(1-\left(\prod_{j=1}^{\text{щ}}\varphi_{Aj}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)\right)}\right) \\ \left(\prod_{1\le i_t<,\ldots,<i_t}\left(1-\prod_{j=1}^{X}\left(\tilde{\Upsilon}_{\nu_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)e^{2\pi i\left(\prod_{1\le i_t<,\ldots,<i_t}\left(1-\left(\prod_{j=1}^{\text{щ}}\phi_{\nu_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)\right)}\right) \end{pmatrix}$$

$$CPFHM^{\text{щ}}(I_1, I_2, \ldots, I_n) = \begin{pmatrix} 1 - \left(\prod_{1 \le i_1 <, \ldots, < i_{\text{щ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \breve{\upsilon}_{\mu_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}} \\ e^{2\pi i\left(1 - \left(\prod_{1 \le i_1 <,\ldots,< i_{\text{щ}} \le n}\left(1 - \left(\prod_{j=1}^{\text{щ}} \psi_{\mu j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}}\right)}, \\ \left(\prod_{1 \le i_1 <, \ldots, < i_{\text{щ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \left(1 - \hat{\hat{\varepsilon}}_{A_j}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}} \\ e^{2\pi i\left(\prod_{1 \le i_1 <, \ldots, < i_{\text{щ}} \le n}\left(1 - \left(\prod_{j=1}^{\text{щ}}\left(1 - \varphi_{A j}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}}} \\ \left(\prod_{1 \le i_1 <, \ldots, < i_{\text{щ}} \le n} \left(1 - \left(\prod_{j=1}^{n} \left(1 - \tilde{\Upsilon}_{\nu_j}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}} \\ e^{2\pi i\left(\prod_{1 \le i_1 <, \ldots, < i_{\text{щ}} \le n}\left(1 - \left(\prod_{j=1}^{\text{щ}}\left(1 - \phi_{\nu_j}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}}} \end{pmatrix}$$

Now, we prove that Equation (6) represents a CPFV, as follows:

(1) $\breve{\upsilon}_\mu(\varrho), \hat{\hat{\varepsilon}}_A(\varrho),\ \tilde{\Upsilon}_\nu\ (\varrho) \in [0,1], \psi_\mu(\varrho), \varphi_A(\varrho),\ \phi_\nu(\varrho) \in [0,1]$

(2) $0 \le \breve{\upsilon}_\mu\ (\varrho) + \hat{\hat{\varepsilon}}_A(\varrho) + \tilde{\Upsilon}_\nu(\varrho) \le 1$ and $0 \le \psi_\mu(\varrho) + \varphi_A(\varrho) + \phi_\nu(\varrho) \le 1$

$$\breve{\upsilon}_\mu\ (\varrho) = 1 - \left(\prod_{1 \le i_1 <, \ldots, < i_{\text{щ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \breve{\upsilon}_{\mu_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}}$$

$$\psi_\mu(\varrho) = 1 - \left(\prod_{1 \le i_1 <,\ldots,< i_{\text{щ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \psi_{\mu j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}}$$

$$\hat{\hat{\varepsilon}}_A(\varrho) = \left(\prod_{1 \le i_1 <, \ldots, < i_{\text{щ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \left(1 - \hat{\hat{\varepsilon}}_{A_j}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}}$$

$$\varphi_A(\varrho) = \left(\prod_{1 \le i_1 <, \ldots, < i_{\text{щ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{щ}} (1 - \varphi_{A j}(\varrho))\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}}$$

$$\tilde{\Upsilon}_\nu(\varrho) = \left(\prod_{1 \le i_1 <, \ldots, < i_{\text{щ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \left(1 - \tilde{\Upsilon}_{\nu_j}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}}$$

$$\phi_\nu(\varrho) = \left(\prod_{1 \le i_1 <, \ldots, < i_{\text{щ}} \le n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \left(1 - \phi_{\nu_j}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}}$$

Since $0 \le \breve{\upsilon}_\mu(\varrho) \le 1$ and $0 \le \psi_\mu(\varrho) \le 1$.

$$0 \le \breve{\upsilon}_\mu(\varrho) e^{2\pi i \psi_\mu(\varrho)} \le 1$$

$$0 \leq \prod_{j=1}^{\text{щ}} \ddot{\upsilon}_{\mu_j}(\varrho) e^{2\pi i \left(\prod_{j=1}^{\text{щ}} \psi_{\mu_j}(\varrho)\right)} \leq 1$$
$$0 \leq \prod_{j=1}^{\text{щ}} \ddot{\upsilon}_{\mu_j}(\varrho) e^{2\pi i \left(\prod_{j=1}^{\text{щ}} \psi_{\mu_j}(\varrho)\right)} \leq 1$$
$$0 \leq 1 - \left(\prod_{j=1}^{X} \ddot{\upsilon}_{\mu_j}(\varrho)\right)^{\frac{1}{\text{щ}}} e^{2\pi i \left(1 - \left(\prod_{j=1}^{\text{щ}} \psi_{\mu_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)} \leq 1$$
$$0 \leq \prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{X} \ddot{\upsilon}_{\mu_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right) . e^{2\pi i \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \psi_{\mu_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)\right)} \leq 1$$
$$0 \leq \left(1 - \prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \left(1 - \ddot{\upsilon}_{\mu_j}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}}$$
$$e^{2\pi i \left(\left(1 - \prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \left(1 - \psi_{\mu_j}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}}\right)} \leq 1$$

In a similar way,

$$0 \leq \ddot{\varepsilon}_A(\varrho) e^{2\pi i \varphi_A(\varrho)} \leq 1$$

and

$$0 \leq \tilde{\Upsilon}_\nu(\varrho) e^{2\pi i \phi_\nu(\varrho)} \leq 1$$

Since $0 \leq \ddot{\upsilon}_\mu(\varrho) e^{2\pi i \psi_\mu(\varrho)} \leq 1$, $0 \leq \ddot{\varepsilon}_A(\varrho) e^{2\pi i \varphi_A(\varrho)} \leq 1$ and $0 \leq \tilde{\Upsilon}_\nu(\varrho) e^{2\pi i \phi_\nu(\varrho)} \leq 1$, therefore,

$$0 \leq \ddot{\upsilon}_\mu(\varrho) e^{2\pi i \psi_\mu(\varrho)} + \ddot{\varepsilon}_A(\varrho) e^{2\pi i \varphi_A(\varrho)} + \tilde{\Upsilon}_\nu(\varrho) e^{2\pi i \phi_\nu(\varrho)} \leq 1$$

□

Proof of Theorem 2. Let $I_j = \left(\ddot{\upsilon}_{\mu_j}(\varrho) e^{2i\pi \psi_{\mu j}(\varrho)}, \ddot{\varepsilon}_{A_j}(\varrho) e^{2i\pi \varphi_{Aj}(\varrho)}, \tilde{\Upsilon}_{\nu_j}(\varrho) e^{2i\pi \phi_{\nu_j}(\varrho)}\right), j = 1, 2, \ldots, k$ be the family of all same CPFVs. Then CPFHM operator is as follows:

$$CPFHM^{(\text{щ})}(I_1, I_2, \ldots, I_n) = \begin{pmatrix} 1 - \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \ddot{\upsilon}_{\mu_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}} \\ e^{2\pi i \left(1 - \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \psi_{\mu j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}}\right)}, \\ \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \left(1 - \ddot{\varepsilon}_{A_j}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}} \\ e^{2\pi i \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \left(1 - \varphi_{Aj}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}}} \\ \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{n} \left(1 - \tilde{\Upsilon}_{\nu_j}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}} \\ e^{2\pi i \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \left(1 - \phi_{\nu_j}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}\right)\right)^{\frac{1}{C_n^{\text{щ}}}}} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\left(\left(1-\left(\prod_{j=1}^{щ}\breve{\upsilon}_{\mu}(\varrho)\right)^{\frac{1}{щ}}\right)\right)^{\frac{1}{C_n^{щ}}}.e^{2\pi i\left(1-\left(\left(1-\left(\prod_{j=1}^{щ}\psi_{\mu}(\varrho)\right)^{\frac{1}{щ}}\right)\right)^{\frac{1}{C_n^{щ}}}\right)}, \\ \left(\left(1-\left(\prod_{j=1}^{щ}(1-\hat{\varepsilon}_{A}(\varrho))\right)^{\frac{1}{щ}}\right)\right)^{\frac{1}{C_n^{щ}}}.e^{2\pi i\left(\left(\left(1-\left(\prod_{j=1}^{щ}(1-\varphi_{A}(\varrho))\right)^{\frac{1}{щ}}\right)\right)^{\frac{1}{C_n^{щ}}}\right)} \\ \left(\left(1-\left(\prod_{j=1}^{щ}(1-\tilde{\Upsilon}_{\nu}(\varrho))\right)^{\frac{1}{щ}}\right)\right)^{\frac{1}{C_n^{щ}}}.e^{2\pi i\left(\left(\left(1-\left(\prod_{j=1}^{щ}(1-\phi_{\nu}(\varrho))\right)^{\frac{1}{щ}}\right)\right)^{\frac{1}{C_n^{щ}}}\right)} \end{pmatrix}$$

$$= \left(\breve{\upsilon}_{\mu}(\varrho)e^{2i\pi\psi_{\mu}}, \hat{\varepsilon}_{A}(\varrho)e^{2i\pi\varphi_{A}}, \tilde{\Upsilon}_{\nu}(\varrho)e^{2i\pi\phi_{\nu}}\right) = I$$

□

Proof of Theorem 3. Since $I_j(\varrho) \le R_j(\varrho)$, $\breve{\upsilon}_{\mu_j}(\varrho) \le g_{\mu_j}(\varrho)$, $\psi_{\mu j}(\varrho) \le \alpha_{\mu_j}(\varrho)$, $\hat{\varepsilon}_{A_j}(\varrho) \le t_{A_j}(\varrho)$, $\varphi_{Aj}(\varrho) \le \gamma_{Aj}(\varrho)$ and $\tilde{\Upsilon}_{\nu j}(\varrho) \le h_{\nu j}(\varrho)$, $\phi_{\nu j}(\varrho) \le \beta_{\nu j}(\varrho)$, then:

$$\prod_{j=1}^{щ}\breve{\upsilon}_{\mu_j}(\varrho)e^{2\pi i(\prod_{j=1}^{щ}\psi_{\mu j}(\varrho))} \le \prod_{j=1}^{щ}g_{\mu_j}e^{2\pi i(\prod_{j=1}^{щ}\alpha_{\mu j}(\varrho))}$$

$$1-\left(\prod_{j=1}^{щ}\breve{\upsilon}_{\mu_j}(\varrho)\right)^{\frac{1}{щ}}e^{2\pi i\left(1-\left(\prod_{j=1}^{щ}\psi_{\mu j}(\varrho)\right)^{\frac{1}{щ}}\right)} \ge 1-\left(\prod_{j=1}^{щ}g_{\mu_j}(\varrho)\right)^{\frac{1}{щ}}e^{2\pi i\left(1-\left(\prod_{j=1}^{щ}\alpha_{\mu j}(\varrho)\right)^{\frac{1}{щ}}\right)}$$

$$\left(\prod_{1\le i_1<,\ldots,<i_{щ}\le i_n}\left(1-\left(\prod_{j=1}^{щ}\breve{\upsilon}_{\mu_j}(\varrho)\right)^{\frac{1}{щ}}\right)^{\frac{1}{C_n^{щ}}}\right)$$

$$e^{2\pi i\left(\prod_{1\le i_1<,\ldots,<i_{щ}\le i_n}\left(1-\left(\prod_{j=1}^{щ}\psi_{\mu j}(\varrho)\right)^{\frac{1}{щ}}\right)^{\frac{1}{C_n^{щ}}}\right)} \ge \left(\prod_{1\le i_1<,\ldots,<i_{щ}\le i_n}\left(1-\left(\prod_{j=1}^{щ}g_{\mu_j}(\varrho)\right)^{\frac{1}{щ}}\right)^{\frac{1}{C_n^{щ}}}\right)$$

$$e^{2\pi i\left(\prod_{1\le i_1<,\ldots,<i_{щ}\le i_n}\left(1-\left(\prod_{j=1}^{щ}\alpha_{\mu j}(\varrho)\right)^{\frac{1}{щ}}\right)^{\frac{1}{C_n^{щ}}}\right)}$$

$$1-\left(\prod_{1\le i_1<,\ldots,<i_{щ}\le i_n}\left(1-\left(\prod_{j=1}^{щ}\breve{\upsilon}_{\mu_j}(\varrho)\right)^{\frac{1}{щ}}\right)^{\frac{1}{C_n^{щ}}}\right)e^{2\pi i\left(1-\left(\prod_{1\le i_1<,\ldots,<i_{щ}\le i_n}\left(1-\left(\prod_{j=1}^{щ}\psi_{\mu j}(\varrho)\right)^{\frac{1}{щ}}\right)^{\frac{1}{C_n^{щ}}}\right)\right)} \le$$

$$1-\left(\prod_{1\le i_1<,\ldots,<i_{щ}\le i_n}\left(1-\left(\prod_{j=1}^{щ}g_{\mu_j}(\varrho)\right)^{\frac{1}{щ}}\right)^{\frac{1}{C_n^{щ}}}\right)e^{2\pi i\left(1-\left(\prod_{1\le i_1<,\ldots,<i_{щ}\le i_n}\left(1-\left(\prod_{j=1}^{щ}\alpha_{\mu j}(\varrho)\right)^{\frac{1}{щ}}\right)^{\frac{1}{C_n^{щ}}}\right)\right)}$$

Thus, the above equation can be written as $\breve{\upsilon}_{\mu_j}(\varrho)e^{2\pi i\psi_{\mu}(\varrho)} \le ge^{2\pi i\alpha_{\mu}(\varrho)}$. We also investigate the value of $\hat{\varepsilon}_{A_j}(\varrho)e^{2\pi i\varphi_{Aj}(\varrho)} \ge t_{A_j}(\varrho)e^{2\pi i\varphi_{Aj}(\varrho)}$ and $\tilde{\Upsilon}_{\nu_j}(\varrho)e^{2\pi i\phi_{\nu}(\varrho)} \ge h_{\nu}(\varrho)e^{2\pi i\beta_{\nu}(\varrho)}$, keeping in mind the step of the above equations.

1. If $\breve{\upsilon}_{\mu_j}(\varrho)e^{2\pi i\psi_{\mu}(\varrho)} < g_{\mu_j}(\varrho)e^{2\pi i\alpha_{\mu}(\varrho)}$, $\hat{\varepsilon}_{A_j}(\varrho)e^{2\pi i\varphi_{Aj}(\varrho)} \ge t_{A_j}(\varrho)e^{2\pi i\varphi_{Aj}(\varrho)}$ and $\tilde{\Upsilon}_{\nu}(\varrho)e^{2\pi i\phi_{\nu}(\varrho)} > h_{\nu}(\varrho)e^{2\pi i\beta_{\nu}(\varrho)}$, then:

$$CPFHM^{щ}(I_1, I_2, \ldots, I_n) < CPFHM^{щ}(R_1, R_2, \ldots, R_n)$$

2. If $\check{\upsilon}_{\mu_j}(\varrho)e^{2\pi i\psi_\mu(\varrho)} = g_{\mu_j}(\varrho)e^{2\pi i\alpha_\mu(\varrho)}$, $\hat{\varepsilon}_{A_j}(\varrho)e^{2\pi i\varphi_{Aj}(\varrho)} = t_{A_j}(\varrho)e^{2\pi i\varphi_{Aj}(\varrho)}$ and $\tilde{\Upsilon}_\nu(\varrho)e^{2\pi i\phi_\nu(\varrho)} > h_\nu(\varrho)e^{2\pi i\beta_\nu(\varrho)}$, then:

$$CPFHM^{ꟺ}(I_1, I_2, \ldots, I_n) = CPFHM^{ꟺ}(R_1, R_2, \ldots, R_n)$$

$\square$

Proof of Theorem 5. We have

$$\bigotimes_{j=1}^{ꟺ}\left(I_{i_j}\right) = \left(\begin{array}{c} \left(\prod_{j=1}^{ꟺ}\check{\upsilon}_{\mu_{i_j}}(\varrho)\right)e^{2\pi i\left(\prod_{j=1}^{ꟺ}\psi_{\mu_{i_j}}(\varrho)\right)}, \\ 1-\prod_{j=1}^{ꟺ}\left(1-\left(\hat{\varepsilon}_{A_j}(\varrho)\right)^n\right)e^{2\pi i\left(1-\prod_{j=1}^{ꟺ}\left(1-\left(\varphi_{Aj}(\varrho)\right)^n\right)\right)}, \\ 1-\prod_{j=1}^{ꟺ}\left(1-\left(\tilde{\Upsilon}_{v_{i_j}}(\varrho)\right)^n\right)e^{2\pi i\left(1-\prod_{j=1}^{ꟺ}\left(1-\left(\phi_{v_{i_j}}(\varrho)\right)^n\right)\right)} \end{array}\right)$$

$$\left(\bigotimes_{j=1}^{ꟺ}\left(I_{i_j}\right)\right)^{\frac{1}{ꟺ}} = \left(\begin{array}{c} \left(\prod_{j=1}^{ꟺ}\check{\upsilon}_{\mu_{i_j}}(\varrho)\right)^{\frac{1}{ꟺ}}e^{2\pi i\left(\prod_{j=1}^{ꟺ}\psi_{\mu_{i_j}}(\varrho)\right)^{\frac{1}{ꟺ}}}, \\ \left(1-\left(\prod_{j=1}^{ꟺ}\left(1-\left(\hat{\varepsilon}_{A_j}(\varrho)\right)^n\right)\right)^{\frac{1}{ꟺ}}\right)e^{2\pi i\left(\left(1-\left(\prod_{j=1}^{ꟺ}\left(1-(\varphi_{Aj}(\varrho))^n\right)\right)^{\frac{1}{ꟺ}}\right)\right)}, \\ \left(1-\left(\prod_{j=1}^{ꟺ}\left(1-\left(\tilde{\Upsilon}_{\nu_{i_j}}(\varrho)\right)^n\right)\right)^{\frac{1}{ꟺ}}\right)e^{2\pi i\left(1-\left(\prod_{j=1}^{ꟺ}\left(1-\left(\phi_{\nu_{i_j}}(\varrho)\right)^n\right)\right)^{\frac{1}{ꟺ}}\right)} \end{array}\right)$$

$$\left(1-\prod_{j=1}^{ꟺ}\mathfrak{N}_{i_j}\right)\left(\bigotimes_{j=1}^{ꟺ}\left(I_{i_j}\right)\right)^{\frac{1}{ꟺ}} = \left(\begin{array}{c} \left(1-\left(\left(1-\left(\prod_{j=1}^{ꟺ}\check{\upsilon}_{\mu_{i_j}}(\varrho)\right)^{\frac{1}{ꟺ}}\right)^{\left(1-\prod_{j=1}^{ꟺ}\mathfrak{N}_{i_j}\right)}\right)\right) \\ e^{2\pi i\left(1-\left(\left(1-\left(\prod_{j=1}^{ꟺ}\psi_{\mu_{i_j}}(\varrho)\right)^{\frac{1}{ꟺ}}\right)^{\left(1-\prod_{j=1}^{ꟺ}\mathfrak{N}_{i_j}\right)}\right)\right)}, \\ \left(1-\left(\prod_{j=1}^{ꟺ}\left(1-\hat{\varepsilon}_{A_j}(\varrho)\right)\right)^{\frac{1}{ꟺ}}\right)^{\left(1-\prod_{j=1}^{ꟺ}\mathfrak{N}_{i_j}\right)} \\ e^{2\pi i\left(\left(1-\left(\prod_{j=1}^{ꟺ}\left(1-\varphi_{Aj}(\varrho)\right)\right)^{\frac{1}{ꟺ}}\right)^{\left(1-\prod_{j=1}^{ꟺ}\mathfrak{N}_{i_j}\right)}\right)}, \\ \left(1-\left(\prod_{j=1}^{ꟺ}\left(1-\tilde{\Upsilon}_{\nu_{i_j}}(\varrho)\right)\right)^{\frac{1}{ꟺ}}\right)^{\left(1-\prod_{j=1}^{ꟺ}\mathfrak{N}_{i_j}\right)} \\ e^{2\pi i\left(\left(1-\left(\prod_{j=1}^{ꟺ}\left(1-\phi_{\nu_{i_j}}(\varrho)\right)\right)^{\frac{1}{ꟺ}}\right)^{\left(1-\prod_{j=1}^{ꟺ}\mathfrak{N}_{i_j}\right)}\right)} \end{array}\right)$$

$$\bigoplus_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j}\right)\left(\bigotimes_{j=1}^{\text{ɰ}}\left(I_{i_j}\right)\right)^{\frac{1}{\text{ɰ}}}=\left(\begin{array}{c}\left(1-\left(\prod\limits_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod\limits_{j=1}^{\text{ɰ}}\breve{\mho}_{\mu_{i_j}}(\varrho)\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)\right) \\ e^{2\pi i\left(1-\left(\prod_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod_{j=1}^{\text{ɰ}}\psi_{\mu_{i_j}}(\varrho)\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)\right)}, \\ \left(\prod\limits_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod\limits_{j=1}^{\text{ɰ}}\left(1-\ddot{\varepsilon}_{A_j}(\varrho)\right)\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right), \\ e^{2\pi i\left(\left(\prod_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod_{j=1}^{\text{ɰ}}(1-\varphi_{Aj}(\varrho))\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)\right)} \\ \left(\prod\limits_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod\limits_{j=1}^{\text{ɰ}}\left(1-\tilde{\Upsilon}_{\nu_{i_j}}(\varrho)\right)\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right) \\ e^{2\pi i\left(\left(\prod_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod_{j=1}^{\text{ɰ}}(1-\phi_{\nu_{i_j}}(\varrho))\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)\right)}\end{array}\right)$$

$$\frac{\bigoplus_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j}\right)\left(\bigotimes_{j=1}^{\text{ɰ}}\left(I_{i_j}\right)\right)^{\frac{1}{\text{ɰ}}}}{C_n^{\text{ɰ}}}=\left(\begin{array}{c}\left(1-\left(\prod\limits_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod\limits_{j=1}^{\text{ɰ}}\breve{\mho}_{\mu_{i_j}}(\varrho)\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}}\right) \\ e^{2\pi i\left(\left(1-\left(\prod_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-(\prod_{j=1}^{\text{ɰ}}\psi_{\mu_{i_j}}(\varrho))^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}}\right)\right)}, \\ \left(\prod\limits_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod\limits_{j=1}^{\text{ɰ}}(1-\ddot{\varepsilon}_{A_j}(\varrho))\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}} \\ e^{2\pi i\left(\left(\prod_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod_{j=1}^{\text{ɰ}}(1-\varphi_{Aj}(\varrho))\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}}\right)}, \\ \left(\prod\limits_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod\limits_{j=1}^{\text{ɰ}}\left(1-\tilde{\Upsilon}_{\nu_{i_j}}(\varrho)\right)\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}} \\ e^{2\pi i\left(\left(\left(\prod_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod_{j=1}^{\text{ɰ}}(1-\phi_{\nu_{i_j}}(\varrho))\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)\right)^{\frac{1}{C_n^{\text{ɰ}}}}\right)}\end{array}\right)$$

$$CPFWHM^{(\text{ɰ})}(I_1,I_2,\dots,I_n)=\left(\begin{array}{c}\left(1-\left(\prod\limits_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod\limits_{j=1}^{\text{ɰ}}\breve{\mho}_{\mu_{i_j}}(\varrho)\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}}\right) \\ e^{2\pi i\left(\left(1-\left(\prod_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod_{j=1}^{\text{ɰ}}\psi_{\mu_{i_j}}(\varrho)\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}}\right)\right)}, \\ \left(\prod\limits_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod\limits_{j=1}^{\text{ɰ}}\left(1-\ddot{\varepsilon}_{A_j}(\varrho)\right)\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}} \\ e^{2\pi i\left(\left(\prod_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod_{j=1}^{\text{ɰ}}\left(1-\varphi_{Aj}(\varrho)\right)\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}}\right)}, \\ \left(\prod\limits_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod\limits_{j=1}^{\text{ɰ}}\left(1-\tilde{\Upsilon}_{\nu_{i_j}}(\varrho)\right)\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}} \\ e^{2\pi i\left(\left(\left(\prod_{1\le i_1<,\,\dots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod_{j=1}^{\text{ɰ}}\left(1-\phi_{\nu_{i_j}}(\varrho)\right)\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)\right)^{\frac{1}{C_n^{\text{ɰ}}}}\right)}\end{array}\right)$$

Now, we have to show that is a CPFV.

(1) $\breve{\upsilon}_\mu(\varrho),\ \ddot{\varepsilon}_A(\varrho),\ \tilde{\Upsilon}_\nu\ (\varrho) \in [0,1], \psi_\mu(\varrho), \varphi_A(\varrho),\ \phi_\nu(\varrho) \in [0,1]$

(2) $0 \leq \breve{\upsilon}_\mu\ (\varrho) + \ddot{\varepsilon}_A(\varrho) + \tilde{\Upsilon}_\nu(\varrho) \leq 1$ and $0 \leq \psi_\mu(\varrho) + \varphi_A(\varrho) + \phi_\nu(\varrho) \leq 1$

$$\breve{\upsilon}_\mu\ (\varrho) = \left(1 - \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \breve{\upsilon}_{\mu_{i_j}}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)^{(1-\prod_{j=1}^{\text{щ}} \mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{щ}}}}\right)$$

$$\psi_\mu(\varrho) = \left(\left(1 - \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \psi_{\mu_{i_j}}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)^{(1-\prod_{j=1}^{\text{щ}} \mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{щ}}}}\right)\right)$$

$$\ddot{\varepsilon}_A = \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \left(1 - \ddot{\varepsilon}_{A_j}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}\right)^{(1-\prod_{j=1}^{\text{щ}} \mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{щ}}}}$$

$$\varphi_A = \left(\left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{\text{щ}} (1 - \varphi_{A j}(\varrho))\right)^{\frac{1}{\text{щ}}}\right)^{(1-\prod_{j=1}^{\text{щ}} \mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{щ}}}}\right)$$

$$\tilde{\Upsilon}_\nu(\varrho) = \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \left(1 - \tilde{\Upsilon}_{\nu_{i_j}}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}\right)^{(1-\prod_{j=1}^{\text{щ}} \mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{щ}}}}$$

$$\phi_\nu(\varrho) = \left(\left(\left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(1 - \left(\prod_{j=1}^{\text{щ}} \left(1 - \phi_{\nu_{i_j}}(\varrho)\right)\right)^{\frac{1}{\text{щ}}}\right)^{(1-\prod_{j=1}^{\text{щ}} \mathfrak{N}_{i_j})}\right)\right)^{\frac{1}{C_n^{\text{щ}}}}\right)^{\frac{1}{C_n^{\text{щ}}}}$$

Since $0 \leq \breve{\upsilon}_\mu(\varrho) \leq 1$ and $0 \leq \psi_\mu(\varrho) \leq 1$, we have:

$$0 \leq \breve{\upsilon}_\mu(\varrho) e^{2\pi i \psi_\mu(\varrho)} \leq 1$$

$$0 \leq \prod_{j=1}^{\text{щ}} \breve{\upsilon}_{\mu_j}(\varrho) e^{2\pi i \left(\prod_{j=1}^{\text{щ}} \psi_{\mu_j}(\varrho)\right)} \leq 1$$

$$0 \leq \left(\prod_{j=1}^{\text{щ}} \breve{\upsilon}_{\mu_j}(\varrho)\right)^{\frac{1}{\text{щ}}} e^{2\pi i \left(\left(\prod_{j=1}^{\text{щ}} \psi_{\mu_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)} \leq 1$$

$$0 \leq \left(\left(\prod_{j=1}^{\text{щ}} \breve{\upsilon}_{\mu_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)^{1-\sum_{j=1}^{\text{щ}} \mathfrak{N}_{i_j}} e^{2\pi i \left(\left(\left(\prod_{j=1}^{\text{щ}} \psi_{\mu_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)^{(1-\prod_{j=1}^{\text{щ}} \mathfrak{N}_{i_j})}\right)} \leq 1$$

$$0 \leq \prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(\left(\prod_{j=1}^{\text{щ}} \breve{\upsilon}_{\mu_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)^{(1-\prod_{j=1}^{\text{щ}} \mathfrak{N}_{i_j})} e^{2\pi i \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(\left(\prod_{j=1}^{\text{щ}} \psi_{\mu_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)^{(1-\prod_{j=1}^{\text{щ}} \mathfrak{N}_{i_j})}\right)} \leq 1$$

$$0 \leq 1 - \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(\left(\prod_{j=1}^{\text{щ}} \breve{\upsilon}_{\mu_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)^{(1-\prod_{j=1}^{\text{щ}} \mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{щ}}}} e^{2\pi i \left(\left(1 - \left(\prod_{1 \leq i_1 <, \ldots, < i_{\text{щ}} \leq n} \left(\left(\left(\prod_{j=1}^{\text{щ}} \psi_{\mu_j}(\varrho)\right)^{\frac{1}{\text{щ}}}\right)^{n}\right)^{(1-\prod_{j=1}^{\text{щ}} \mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{щ}}}}\right)^{\frac{1}{n}}\right)} \leq 1$$

Similarly, we can prove the following equations.

$$0 \le \tilde{\Upsilon}_{\nu}(\varrho)e^{2\pi i\phi_{\nu}(\varrho)} \le 1, \text{ and } 0 \le \ddot{\varepsilon}_{A}(\varrho)e^{2\pi i\varphi_{A}(\varrho)} \le 1$$

Since $0 \le \ddot{\upsilon}_{\mu}(\varrho)e^{2\pi i\psi_{\mu}(\varrho)} \le 1$, $0 \le \ddot{\varepsilon}_{A}(\varrho)e^{2\pi i\varphi_{A}(\varrho)} \le 1$ and $0 \le \tilde{\Upsilon}_{\nu}(\varrho)e^{2\pi i\phi_{\nu}(\varrho)} \le 1$, therefore,

$$0 \le \ddot{\upsilon}_{\mu}(\varrho)e^{2\pi i\psi_{\mu}(\varrho)} + \ddot{\varepsilon}_{A}(\varrho)e^{2\pi i\varphi_{A}(\varrho)} + \tilde{\Upsilon}_{\nu}(\varrho)e^{2\pi i\phi_{\nu}(\varrho)} \le 1$$

□

Proof of Theorem 8. We prove this theorem by using previous Theorems 2 and 3.

From Theorem 5, we have:

$$CPFWHM^{(\text{ɰ})}\left(I_1^-, I_2^-, \ldots, I_n^-\right) = \begin{pmatrix} \left(1-\left(\prod\limits_{1\le i_1<,\ldots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod\limits_{j=1}^{\text{ɰ}}\min\left(\ddot{\upsilon}_{\mu_{i_j}}(\varrho)\right)\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}}\right) e^{2\pi i\left(\left(1-\left(\prod_{1\le i_1<,\ldots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod_{j=1}^{\text{ɰ}} min(\psi_{\mu_{i_j}}(\varrho))\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}}\right)\right)}, \\ \left(\prod\limits_{1\le i_1<,\ldots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod\limits_{j=1}^{\text{ɰ}}\left(1-max\left(\ddot{\varepsilon}_{A_j}(\varrho)\right)\right)\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}} e^{2\pi i\left(\left(\prod_{1\le i_1<,\ldots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod_{j=1}^{\text{ɰ}}(1-max(\varphi_{Aj}(\varrho)))\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}}\right)}, \\ \left(\prod\limits_{1\le i_1<,\ldots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod\limits_{j=1}^{\text{ɰ}}\left(1-max\left(\tilde{\Upsilon}_{\nu_{i_j}}(\varrho)\right)\right)\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}} e^{2\pi i\left(\left(\left(\prod_{1\le i_1<,\ldots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod_{j=1}^{\text{ɰ}}(1-max(\phi_{\nu_{i_j}}(\varrho)))\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)\right)^{\frac{1}{C_n^{\text{ɰ}}}}\right)} \end{pmatrix}$$

$$CPFWHM^{(\text{ɰ})}\left(I_1^+, I_2^+, \ldots, I_n^+\right) = \begin{pmatrix} \left(1-\left(\prod\limits_{1\le i_1<,\ldots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod\limits_{j=1}^{\text{ɰ}}max\left(\ddot{\upsilon}_{\mu_{i_j}}(\varrho)\right)\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}}\right) e^{2\pi i\left(\left(1-\left(\prod_{1\le i_1<,\ldots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod_{j=1}^{\text{ɰ}} max(\psi_{\mu_{i_j}}(\varrho))\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}}\right)\right)}, \\ \left(\prod\limits_{1\le i_1<,\ldots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod\limits_{j=1}^{\text{ɰ}}\left(1-min\left(\ddot{\varepsilon}_{A_j}(\varrho)\right)\right)\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}} e^{2\pi i\left(\left(\prod_{1\le i_1<,\ldots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod_{j=1}^{\text{ɰ}}(1-min(\varphi_{Aj}(\varrho)))\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}}\right)} \\ \left(\prod\limits_{1\le i_1<,\ldots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod\limits_{j=1}^{\text{ɰ}}\left(1-min\left(\tilde{\Upsilon}_{\nu_{i_j}}(\varrho)\right)\right)\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)^{\frac{1}{C_n^{\text{ɰ}}}} e^{2\pi i\left(\left(\left(\prod_{1\le i_1<,\ldots,<i_{\text{ɰ}}\le n}\left(1-\left(\prod_{j=1}^{\text{ɰ}}(1-min(\phi_{\nu_{i_j}}(\varrho)))\right)^{\frac{1}{\text{ɰ}}}\right)^{(1-\prod_{j=1}^{\text{ɰ}}\mathfrak{N}_{i_j})}\right)\right)^{\frac{1}{C_n^{\text{ɰ}}}}\right)} \end{pmatrix}$$

From property 4 we have:

$$I^- \le CPFWHM^{\text{ɰ}}(I_1, I_2, \ldots, I_n) \le I^+$$

□

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