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Stability Analysis for Linear Systems with a Differentiable Time-Varying Delay via Auxiliary Equation-Based Method

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Abstract: This paper concentrates on the stability problem for linear systems with a differentiable time-varying delay via an auxiliary equation-based method. By supposing that the second-order derivative of the system state is available, an auxiliary equation is obtained. On the basis of the system equation and the auxiliary equation, we define a suitable delay-product-type augmented Lyapunov-Krasovskii functional (LKF), under which more delay and system state information can be exploited. Based on the LKF, by utilizing some vital lemmas, adding zero terms, and the convex analysis method, we propose a new stability condition that is less conservative. Finally, to illustrate the merit of the obtained stability condition, two typical numerical examples are given.

Keywords: stability analysis; time-varying delay; auxiliary equation-based method; Lyapunov-Krasovskii functional (LKF)

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1. Introduction

In many practical application systems, such as rolling mills, chemical processes, power systems, neural networks, manufacturing systems, and networked control systems, a time-varying delay is known to widely exist [1–4]. The existence of a time-varying delay in a system often causes the oscillation, degradation of system performance, and even instability [5,6]. Therefore, stability analysis for the systems with time-varying delays is of great significance. Over the past two decades, the Lyapunov-Krasovskii functional (LKF) has been a powerful tool to analyze the systems since it can achieve delay-dependent stability criteria [7–11]. However, these criteria are only sufficient. There exists some inevitable conservatism. A stability theorem for the systems with time-varying delay is less conservative than another one if the stability theorem can achieve a greater delay upper bound than that of another one for a specific example. This notion has been introduced in [12–15]. It is hence a field of research that of introducing enhanced stability theorems to reduce conservatism. Generally speaking, it is very difficult, in theory, to prove the degree of conservatism of these stability criteria that are obtained by the LKF method. Therefore, the conservatism of these stability criteria is usually illustrated by the maximum delay upper bounds (MDUBs), which are obtained in some well-known numerical examples via different stability conditions formulated by different methods. For example, we can compare h_A obtained by Theorem A with h_B obtained by Theorem B in a well-known example, where h_A and h_B are the constant bounds of the time-varying delay, known as MDUBs. If Theorem A could have h_A larger than h_B of Theorem B, then we understand that Theorem A has less conservative compared with Theorem B for this example. In general, a conservative approach implies a reduced/limited/lower value of h, and a relaxed approach means a higher value of h, which corresponds to a non-conservative approach. Consequently, the MDUB has been a vital index to evaluate the conservatism of these stability conditions that are derived by various methods.

In general, there are several ways to reduce the conservatism of the obtained stability criteria. The first one is to define a suitable LKF involving more delay and system state information [14]. For example, by constructing an augmented type LKF in [16–18], a delay-product-type augmented LKF in [19,20], and a containing multiple integral terms LKF in [2,21,22], the conservatism of the stability conditions proposed by this literature has been significantly reduced. The second one is to estimate the integral terms from the derivative of LKF [17,23]. Many integral inequalities, such as, Jensen integral inequality [1], Wirtinger-based integral inequality [24], Bessel-Legendre integral inequality [25] and so on (see references [2,12,15,26–30]), have been formulated to directly estimate the integral terms. These integral inequalities play an important role to reduce the conservatism of stability criteria. The third one is the convex analysis method [5,19]. When the time-varying delay and its derivative belongs to a given interval, respectively, this method can obtain the sufficient and necessary condition such that a matrix is less than zero, and thus some considerable terms in the derivative of LKF can be effectively used. The fourth one is the method of adding zero terms in the derivative of LKF [17,21,31,32]. This method can also effectively reduce the conservatism of the obtained stability criteria by introducing some free matrices. By studying the literature mentioned above, and references therein, although many excellent methods and stability conditions have been proposed, there is still room for improvement.

In this paper, the stability problem for linear systems with a differentiable time-varying delay is concerned with an auxiliary equation-based method. The main contributions of this work are summarized as

(1) Motivated by the method in [33], the auxiliary equation $\ddot{x}(t) = A\dot{x}(t) + (1-\dot{h}(t))A_d\dot{x}(t-h(t))$ is utilized to investigate the stability of the systems with a differentiable time-varying delay, and thus the information of delay derivative can be captured well and be used to derive a less conservative stability condition.
(2) Inspired by the fact that $2\int_b^a \dot{x}^T(s)U\ddot{x}(s)ds = \dot{x}^T(a)U\dot{x}(a) - \dot{x}^T(b)U\dot{x}(b)$, two state augmented zero equalities are introduced, which can help reduce the conservatism of the obtained stability condition.
(3) On the basis of the system equation and the auxiliary equation, a new delay-product-type augmented LKF is constructed, which can utilize more system information, such as $\ddot{x}(t)$, $\ddot{x}(t-h(t))$ and $\ddot{x}(t-h)$. Then, based on the LKF and by employing some vital lemmas, adding zero terms, and the convex analysis method, a relaxed stability condition is proposed. Finally, to illustrate the merit of the obtained stability condition, two typical numerical examples are given.

Notations. *The notation diag*$\{\cdot\}$ *is the block diagonal matrix, and adiag*$\{\cdot\}$ *is anti-block diagonal matrix, i.e., adiag*$\{U_1, U_2\} = \begin{bmatrix} 0 & U_1 \\ U_2 & 0 \end{bmatrix}$. $\mathbb{R}^n$ *represents the n-dimensional Euclidean space and the set of real* $n \times m$ *matrices is denoted by* $\mathbb{R}^{n\times m}$. *The other notations are standard.*

2. Problem Statement and Preliminaries

Consider one class of linear systems with a differentiable time-varying delay as

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h(t)) \\ x(t) = \varepsilon(t), t \in [-h, 0] \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $\varepsilon(t)$ is the initial condition. To obtain the main results of this work, the following assumptions are necessary.

Assumption 1. *The matrices* $A \in \mathbb{R}^{n\times n}$ *and* $A_d \in \mathbb{R}^{n\times n}$ *in (1) are constant-coefficient matrices, and the time-varying delay* $h(t)$ *in (1) satisfies*

$$0 \le h(t) \le h,\ |\dot{h}(t)| \le \mu \le 1 \tag{2}$$

where h *and* μ *are known constant scalars.*

Assumption 2. *The second-order derivative of the system state is supposed to be available.*

Remark 1. *The Assumption 1 has been conducted in [33] to investigate the stability problem of systems with time-varying for the first time. And then the same issue is concerned in [34] by using new inequality based on the second-order derivative of $x(t)$. In addition, the second-order linear systems, in which the second-order derivative of $x(t)$ is available, have been studied in [35,36]. Above all, it is reasonable that the second-order derivative of $x(t)$ is supposed to be available.*

Based on Assumption 1, and according to the system dynamic Equation (1), the second order derivative of $x(t)$ is

$$\ddot{x}(t) = A\dot{x}(t) + (1 - \dot{h}(t))A_d\dot{x}(t - h(t)) \tag{3}$$

where Equation (3) is called auxiliary equation.

Remark 2. *It can be found that the coefficient matrix of the delay-dependent term in the auxiliary Equation (3) is $(1 - \dot{h}(t))A_d$, which can effectively reflected the rate of change of delay. Moreover, some system states, such as $\ddot{x}(t)$, $\ddot{x}(t - h(t))$ and $\ddot{x}(t - h)$, also can be utilized to analyze the stability of system (1) with the help of the auxiliary Equation (3). Therefore, a less conservative stability criterion can be expected.*

Lemma 1. *([24]) For any positive definite matrix $R \in \mathbb{R}^{n\times n}$, scalars τ_1 and τ_2 with $\tau_1 < \tau_2$, and a differentiable function $\gamma(s) : [\tau_1, \tau_2] \to \mathbb{R}^n$, the following inequality holds*

$$(\tau_2 - \tau_1)\int_{\tau_1}^{\tau_2} \dot{\gamma}^T(s)R\dot{\gamma}(s)ds \geq \Phi_1^T R\Phi_1 + 3\Phi_2^T R\Phi_2$$

where $\Phi_1 = \gamma(\tau_2) - \gamma(\tau_1)$, $\Phi_2 = \gamma(\tau_2) + \gamma(\tau_1) - \frac{2}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\gamma(s)ds$.

Lemma 2. *([37]) For any vectors Λ_1, Λ_2, and symmetric matrices $R_1 > 0$, $R_2 > 0$, free matrix M, satisfying $\begin{bmatrix} R_1 & M \\ M^T & R_2 \end{bmatrix} > 0$, and a real scalar $\alpha \in (0, 1)$, the following inequality holds*

$$\frac{1}{\alpha}\Lambda_1^T R_1\Lambda_1 + \frac{1}{1-\alpha}\Lambda_2^T R_2\Lambda_2 \geq \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix}^T \begin{bmatrix} R_1 & M \\ M^T & R_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix}.$$

Lemma 3. *([38]) For an appropriate dimensional symmetric matrix $R > 0$, and matrices Ψ, Λ, Σ, the following (I) and (II) are equivalent*

$$\text{(I) } \Psi - \Lambda^T R\Lambda < 0, \text{ (II) } \begin{bmatrix} \Psi + \Lambda^T\Sigma + \Sigma^T\Lambda & \Sigma^T \\ \Sigma & -R \end{bmatrix} < 0.$$

3. Stability Conditions

In this section, with the help of the auxiliary Equation (3), our main aim is to utilize more system information to obtain a new stability criterion for the system (1), which can provide a larger MDUB compared with that of some existing ones in two well-known examples.

Theorem 1. *For given integer n, scalars $h > 0$, $\mu \leq 1$, matrices $A \in \mathbb{R}^{n\times n}$, $A_d \in \mathbb{R}^{n\times n}$, the system (1) subject to (2) is asymptotically stable if there exists a symmetric positive definite matrices $P_1 \in \mathbb{R}^{8n\times 8n}$, $P_2 \in \mathbb{R}^{8n\times 8n}$, $Q_1 \in \mathbb{R}^{3n\times 3n}$, $Q_2 \in \mathbb{R}^{3n\times 3n}$, $R_1 \in \mathbb{R}^{2n\times 2n}$, $R_2 \in \mathbb{R}^{2n\times 2n}$, and free matrices $S \in \mathbb{R}^{4n\times 4n}$, $M \in \mathbb{R}^{11n\times n}$, $N \in \mathbb{R}^{11n\times n}$, $\Sigma \in \mathbb{R}^{8n\times 11n}$, $U_1 \in \mathbb{R}^{n\times n}$, $U_2 \in \mathbb{R}^{n\times n}$, such that*

$$\begin{bmatrix} \Psi(0,\ \mu) + \Lambda(0)\Sigma + \Sigma^T\Lambda^T(0) & \Sigma^T \\ \Sigma & -\tilde{R} \end{bmatrix} < 0 \tag{4}$$

$$\begin{bmatrix} \Psi(0,\ -\mu) + \Lambda(0)\Sigma + \Sigma^T\Lambda^T(0) & \Sigma^T \\ \Sigma & -\tilde{R} \end{bmatrix} < 0 \tag{5}$$

$$\begin{bmatrix} \Psi(h,\ \mu) + \Lambda(h)\Sigma + \Sigma^T\Lambda^T(h) & \Sigma^T \\ \Sigma & -\tilde{R} \end{bmatrix} < 0 \tag{6}$$

$$\begin{bmatrix} \Psi(h,\ -\mu) + \Lambda(h)\Sigma + \Sigma^T\Lambda^T(h) & \Sigma^T \\ \Sigma & -\tilde{R} \end{bmatrix} < 0 \tag{7}$$

where

$$\begin{aligned}
\Psi(h(t),\ \dot{h}(t)) &= \Psi_1(h(t),\ \dot{h}(t)) + \Psi_2(\dot{h}(t)),\ \Lambda(h(t)) = [\Lambda_1(h(t)) \quad \Lambda_2(h(t))], \\
\Psi_1(h(t),\ \dot{h}(t)) &= 2T_1[h(t)P_1 + (h-h(t))P_2]T_2^T(\dot{h}(t)) + T_1\dot{h}(t)(P_1-P_2)T_1^T + T_3Q_1T_3^T - T_4(1- \\
&\quad \dot{h}(t))(Q_1-Q_2)T_4^T - T_5Q_2T_5^T + T_6h(1-\dot{h}(t))(h-h(t))(R_2-R_1)T_6^T + T_7h^2R_1T_7^T \\
&\quad + e_4^ThU_1e_4 - e_5^Th(U_1-U_2)e_5 - e_6^ThU_2e_6,\ \Psi_2(\dot{h}(t)) = 2eM\Gamma_1 + 2eN\Gamma_2(\dot{h}(t)), \\
\Lambda_1(h(t)) &= [e_1^T - e_2^T \quad e_4^T - e_5^T \quad h(t)(e_1^T + e_2^T) - 2e_{10}^T \quad h(t)(e_4^T + e_5^T) - 2(e_1^T - e_2^T)], \\
\Lambda_2(h(t)) &= [e_2^T - e_3^T \quad e_5^T - e_6^T \quad (h-h(t))(e_2^T + e_3^T) - 2e_{11}^T \quad (h-h(t))(e_5^T + e_6^T) - 2(e_2^T - e_3^T)], \\
e &= [e_1^T \quad e_2^T \quad e_3^T \quad e_4^T \quad e_5^T \quad e_6^T \quad e_7^T \quad e_8^T \quad e_9^T \quad e_{10}^T \quad e_{11}^T], \\
\Gamma_1 &= Ae_1 + A_de_2 - e_4,\ \Gamma_2(\dot{h}(t)) = Ae_4 + (1-\dot{h}(t))A_de_5 - e_7, \\
T_1 &= [e_1^T \quad e_2^T \quad e_3^T \quad e_4^T \quad e_5^T \quad e_6^T \quad e_{10}^T \quad e_{11}^T], \\
T_2(\dot{h}(t)) &= [e_4^T \quad (1-\dot{h}(t))e_5^T \quad e_6^T \quad e_7^T \quad (1-\dot{h}(t))e_8^T \quad e_9^T \quad e_1^T - (1-\dot{h}(t))e_2^T \quad (1-\dot{h}(t))e_2^T - e_3^T], \\
T_3 &= [e_1^T \quad e_4^T \quad e_7^T],\ T_4 = [e_2^T \quad e_5^T \quad e_8^T],\ T_5 = [e_3^T \quad e_6^T \quad e_9^T],\ T_6 = [e_5^T \quad e_8^T],\ T_7 = [e_4^T \quad e_7^T], \\
\bar{R}_1 &= R_1 + \bar{U}_1,\ \bar{U}_1 = adiag\{U_1, U_1\},\ \bar{R}_2 = R_2 + \bar{U}_2,\ \bar{U}_2 = adiag\{U_2, U_2\}, \\
\widehat{R}_1 &= diag\left\{\bar{R}_1, \frac{3\bar{R}_1}{h^2}\right\},\ \widehat{R}_2 = diag\left\{\bar{R}_2, \frac{3\bar{R}_2}{h^2}\right\},\ \tilde{R} = \begin{bmatrix} \widehat{R}_1 & S \\ S^T & \widehat{R}_2 \end{bmatrix}, \\
e_i &= [0_{n\cdot(i-1)n},\ I_n,\ 0_{n\cdot(11-i)n}] \in \mathbb{R}^{n\times 11n},\ i = 1,\ 2,\ \cdots,\ 11.
\end{aligned}$$

Proof. For the simplicity, let us first define

$$\begin{aligned}
\theta^T(t) &= [x^T(t) \quad x^T(t-h(t)) \quad x^T(t-h) \quad \dot{x}^T(t) \quad \dot{x}^T(t-h(t)) \quad \dot{x}^T(t-h) \quad \ddot{x}^T(t) \quad \ddot{x}^T(t-h(t)) \\
&\quad \ddot{x}^T(t-h) \quad \int_{t-h(t)}^{t} x^T(s)ds \quad \int_{t-h}^{t-h(t)} x^T(s)ds]. \\
\chi_1^T(t) &= [x^T(t) \quad x^T(t-h(t)) \quad x^T(t-h) \quad \dot{x}^T(t) \quad \dot{x}^T(t-h(t)) \quad \dot{x}^T(t-h) \\
&\quad \int_{t-h(t)}^{t} x^T(s)ds \quad \int_{t-h}^{t-h(t)} x^T(s)ds], \\
\chi_2^T(t) &= [x^T(t) \quad \dot{x}^T(t) \quad \ddot{x}^T(t)],\ \dot{\kappa}^T(t) = [\dot{x}^T(t) \quad \ddot{x}^T(t)].
\end{aligned}$$

Then, we choose a delay-product-type augmented LKF candidate as

$$V(t) = V_1(t) + V_2(t) + V_3(t) \tag{8}$$

where

$$V_1(t) = \chi_1^T(t)(h(t)P_1 + (h-h(t))P_2)\chi_1(t) \tag{9}$$

$$V_2(t) = \int_{t-h(t)}^{t} \chi_2^T(s)Q_1\chi_2(s)ds + \int_{t-h}^{t-h(t)} \chi_2^T(s)Q_2\chi_2(s)ds \tag{10}$$

$$V_3(t) = h\int_{t-h(t)}^{t} (h-t+s)\dot{\kappa}^T(s)R_1\dot{\kappa}(s)ds + h\int_{t-h}^{t-h(t)} (h-t+s)\dot{\kappa}^T(s)R_2\dot{\kappa}(s)ds \tag{11}$$

Along the trajectories of system (1), the derivative of $V(t)$ on time t is

$$\dot{V}(t) \;=\; \dot{V}_1(t)+\dot{V}_2(t)+\dot{V}_3(t) \tag{12}$$

where

$$\begin{aligned}
\dot{V}_1(t) &= \theta^T(t)[2h(t)T_1P_1T_2^T(\dot{h}(t))+2(h-h(t))T_1P_2T_2^T(\dot{h}(t))+T_1\dot{h}(t)(P_1-P_2)T_1^T]\theta(t) && (13)\\
\dot{V}_2(t) &= \theta^T(t)[T_3Q_1T_3^T-(1-\dot{h}(t))T_4(Q_1-Q_2)T_4^T-T_5Q_2T_5^T]\theta(t) && (14)\\
\dot{V}_3(t) &= \theta^T(t)\Delta(h(t),\ \dot{h}(t))\theta(t)-h\int_{t-h(t)}^{t}\dot{\kappa}^T(s)R_1\dot{\kappa}(s)ds-h\int_{t-h}^{t-h(t)}\dot{\kappa}^T(s)R_2\dot{\kappa}(s)ds && (15)
\end{aligned}$$

with $\Delta(h(t),\ \dot{h}(t)) = T_6h(1-\dot{h}(t))(h-h(t))(R_2-R_1)T_6^T+T_7h^2R_1T_7^T$. Then, let us consider the following two zero equalities

$$\begin{aligned}
0 &= h\Big[\dot{x}^T(t)U_1\dot{x}(t)-\dot{x}^T(t-h(t))U_1\dot{x}(t-h(t))\Big]-2h\int_{t-h(t)}^{t}\dot{x}^T(s)U_1\ddot{x}(s)ds && (16)\\
0 &= h\Big[\dot{x}^T(t-h(t))U_2\dot{x}(t-h(t))-\dot{x}^T(t-h)U_2\dot{x}(t-h)\Big]-2h\int_{t-h}^{t-h(t)}\dot{x}^T(s)U_2\ddot{x}(s)ds && (17)
\end{aligned}$$

where $U_1\in\mathbb{R}^{n\times n}$ and $U_2\in\mathbb{R}^{n\times n}$ are free matrices. Summing (16) and (17) yields

$$\begin{aligned}
0 \;=\;& \theta^T(t)h\Big[e_4^TU_1e_4-e_5^T(U_1-U_2)e_5-e_6^TU_2e_6\Big]\theta(t)\\
&-h\int_{t-h(t)}^{t}\dot{\kappa}^T(s)\bar{U}_1\dot{\kappa}(s)ds-h\int_{t-h}^{t-h(t)}\dot{\kappa}^T(s)\bar{U}_2\dot{\kappa}(s)ds
\end{aligned} \tag{18}$$

Adding (18) into (15), one has

$$\begin{aligned}
\dot{V}_3(t) \;=\;& \theta^T(t)\Big[\Delta(h(t),\ \dot{h}(t))+e_4^ThU_1e_4-e_5^Th(U_1-U_2)e_5-e_6^ThU_2e_6\Big]\theta(t)\\
&-h\int_{t-h(t)}^{t}\dot{\kappa}^T(s)\bar{R}_1\dot{\kappa}(s)ds-h\int_{t-h}^{t-h(t)}\dot{\kappa}^T(s)\bar{R}_2\dot{\kappa}(s)ds
\end{aligned} \tag{19}$$

By employing Lemma 1, if $\bar{R}_1>0$, based on $h(t)\le h$, one gets that

$$\begin{aligned}
&-h\int_{t-h(t)}^{t}\dot{\kappa}^T(s)\bar{R}_1\dot{\kappa}(s)ds \le -\frac{h}{h(t)}\Big[\Phi_1\bar{R}_1\Phi_1^T+\Phi_2 3\bar{R}_1\Phi_2^T\Big]\\
&\le -\frac{h}{h(t)}\bigg(\Phi_1\bar{R}_1\Phi_1^T+\Phi_2\frac{3\bar{R}_1}{h^2}h^2(t)\Phi_2^T\bigg) = -\frac{h}{h(t)}\theta^T(t)\Lambda_1(h(t))\widehat{R}_1\Lambda_1^T(h(t))\theta(t)
\end{aligned} \tag{20}$$

where

$$\Phi_1=\begin{bmatrix} x(t)-x(t-h(t)) \\ \dot{x}(t)-\dot{x}(t-h(t)) \end{bmatrix}^T,\ \Phi_2=\begin{bmatrix} x(t)+x(t-h(t))-\frac{2}{h(t)}\int_{t-h(t)}^{t}x(s)ds \\ \dot{x}(t)+\dot{x}(t-h(t))-\frac{2}{h(t)}(x(t)-x(t-h(t))) \end{bmatrix}^T.$$

Similarly, if $\bar{R}_2>0$, based on $h(t)\ge 0$, we obtain

$$\begin{aligned}
&-h\int_{t-h}^{t-h(t)}\dot{\kappa}^T(s)\bar{R}_2\dot{\kappa}(s)ds \le -\tfrac{h}{h-h(t)}\big(\Phi_3\bar{R}_2\Phi_3^T+\Phi_4 3\bar{R}_2\Phi_4^T\big)\\
&\le -\tfrac{h}{h-h(t)}\Big(\Phi_3\bar{R}_2\Phi_3^T+\Phi_4\tfrac{3\bar{R}_2}{h^2}(h-h(t))^2\Phi_4^T\Big) = -\tfrac{h}{h-h(t)}\theta^T(t)\Lambda_2(h(t))\widehat{R}_2\Lambda_2^T\theta(t)
\end{aligned} \tag{21}$$

where

$$\Phi_3=\begin{bmatrix} x(t-h(t))-x(t-h) \\ \dot{x}(t-h(t))-\dot{x}(t-h) \end{bmatrix}^T,\ \Phi_4=\begin{bmatrix} x(t-h(t))+x(t-h)-\frac{2}{h-h(t)}\int_{t-h}^{t-h(t)}x(s)ds \\ \dot{x}(t-h(t))+\dot{x}(t-h)-\frac{2}{h-h(t)}(x(t-h(t)-x(t-h)) \end{bmatrix}^T.$$

Combining (20) and (21), for arbitrary matrix $S \in \mathbb{R}^{4n\times 4n}$, if $\tilde{R} = \begin{bmatrix} \widehat{R}_1 & S \\ S^T & \widehat{R}_2 \end{bmatrix} > 0$, by Lemma 2, we obtain

$$-\theta^T(t)\Big(\tfrac{h}{h(t)}\Lambda_1(h(t))\widehat{R}_1\Lambda_1^T(h(t)) + \tfrac{h}{h-h(t)}\Lambda_2(h(t))\widehat{R}_2\Lambda_2^T(h(t))\Big)\theta(t) \\ \leq -\theta^T(t)\Lambda(h(t))\tilde{R}\Lambda^T(h(t))\theta(t) \tag{22}$$

Moreover, when $h(t) = 0$, we have $\int_{t-h(t)}^{t} \dot{\kappa}^T(s)ds = \int_{t-h(t)}^{t}\int_{\theta}^{t} \dot{\kappa}^T(s)dsd\theta = 0$. when $h(t) = h$, we obtain $\int_{t-h}^{t-h(t)} \dot{\kappa}^T(s)ds = \int_{t-h}^{t-h(t)}\int_{\theta}^{t-h(t)} \dot{\kappa}^T(s)dsd\theta = 0$. Therefore, the inequality (22) still holds. From (13) to (22), we get an upper bound of $\dot{V}(t)$ as

$$\dot{V}(t) = \sum_{i=1}^{3}\dot{V}_i(t) \leq \theta^T(t)\Big(\Psi_1(h(t),\ \dot{h}(t)) - \Lambda(h(t))\tilde{R}\Lambda^T(h(t))\Big)\theta(t) \tag{23}$$

Now, we further rewrite the system (1) and the auxiliary Equation (3) as

$$0 = Ax(t) + A_d x(t-h(t)) - \dot{x}(t) \tag{24}$$
$$0 = A\dot{x}(t) + A_d(1-\dot{h}(t))\dot{x}(t-h(t)) - \ddot{x}(t) \tag{25}$$

As the zero equalities in (24) and (25) can be defined by the elements of $\theta(t)$, respectively, we get that $\Gamma_1\theta(t) = 0$ and $\Gamma_2(\dot{h}(t))\theta(t) = 0$. As a result, the following two zero equalities hold

$$0 = 2\theta^T(t)eM\Gamma_1\theta(t) \tag{26}$$
$$0 = 2\theta^T(t)eN\Gamma_2(\dot{h}(t))\theta(t) \tag{27}$$

for any matrices $M \in \mathbb{R}^{11n\times n}, N \in \mathbb{R}^{11n\times n}$. Taking (26) and (27) into (23), one gets that

$$\dot{V}(t) \leq \theta^T(t)\Big(\Psi(h(t),\ \dot{h}(t)) - \Lambda(h(t))\tilde{R}\Lambda^T(h(t))\Big)\theta(t) \tag{28}$$

For $\theta(t) \neq 0$, $h(t) \in [0,\ h], \dot{h}(t) \in [-\mu,\ \mu]$, if

$$\Psi(h(t),\ \dot{h}(t)) - \Lambda(h(t))\tilde{R}\Lambda^T(h(t)) < 0 \tag{29}$$

then $\dot{V}(t) < 0$. By utilizing Lemma 3, the inequality (29) is equivalent to

$$\Theta(h(t),\ \dot{h}(t)) = \begin{bmatrix} \Psi(h(t),\ \dot{h}(t)) + \Lambda(h(t))\Sigma + \Sigma^T\Lambda^T(h(t)) & \Sigma^T \\ \Sigma & -\tilde{R} \end{bmatrix} < 0 \tag{30}$$

for any matrix $\Sigma \in \mathbb{R}^{8n\times 11n}$. Since $\Theta(h(t),\ \dot{h}(t))$ is affine on $h(t) \in [0,\ h]$ and $\dot{h}(t) \in [-\mu,\ \mu]$, by the convex property, $\Theta(h(t),\ \dot{h}(t)) < 0$ for $h(t) \in [0,\ h]$ and $\dot{h}(t) \in [-\mu,\ \mu]$ if and only if inequalities (4)–(7) hold. In addition, if inequality (30) is satisfied, then the condition $\tilde{R} = \begin{bmatrix} \widehat{R}_1 & S \\ S^T & \widehat{R}_2 \end{bmatrix} > 0$ also holds. It can be further seen that $\widehat{R}_j = \begin{bmatrix} \bar{R}_j & 0 \\ 0 & \frac{3\bar{R}_j}{h^2} \end{bmatrix} > 0$, then, $\bar{R}_j > 0$, $j = 1, 2$. As a result, if the matrix inequalities (4)–(7) are satisfied, based on Lyapunov stability theory, the system (1) subject to (2) is asymptotically stable. This completes the proof. □

Remark 3. *Theorem 1 includes $h(t)$ and $\dot{h}(t)$, but it does not include the second order derivative $\ddot{x}(t)$, The proof of Theorem 1 uses it in Equations (16), (17) and (25), but the result stated in Theorem 1 does not use $\ddot{x}(t)$. Then, it is necessary for the accomplishment of the model (1), but the application of Theorem 1 would not need the knowledge on $\ddot{x}(t)$. Above all, it seems that Assumption 1 is not necessary. The question comes from the reviewer. In fact, if we do not make an assumption, then the* *auxiliary Equation (3) may not be obtained. Without the help of (3), the obtained stability condition based on the LKF (8) is infeasible, which can be checked by the LMI toolbox in Matlab. Above all, the Assumption 1 is necessary.*

Remark 4. *It has been proved that Lemma 1 can get a tighter lower bound compared with Jensen integral inequality [24]. However, when we use Lemma 1 to derive Theorem 1, quadratic or higher order terms on $h(t)$ and $(h - h(t))$ will be encountered inevitably. Since the quadratic or higher-order terms are non-convex, the solving of non-convex matrix inequalities is not easy. To obtain an affine stability criterion on $h(t)$ and $\dot{h}(t)$, we need to avoid the emergence of quadratic or higher order terms on $h(t)$ and $\dot{h}(t)$ in Theorem 1. Therefore, based on $h(t) \le h$, $\Phi_2 3\bar{R}_1 \Phi_2^T$ is estimated as $h(t)\Phi_2 \frac{3\bar{R}_1}{h^2}\Phi_2^T h(t)$ in (20). And based on $h(t) \ge 0$, $\Phi_4 3\bar{R}_2 \Phi_4^T$ is estimated as $(h-h(t))\Phi_4 \frac{3\bar{R}_2}{h^2}\Phi_4^T (h-h(t))$ in (21). In addition, the method of adding zero terms is adopted to avoid directly calculating $\ddot{x}^T(t)R_1\ddot{x}(t)$ in $\dot{V}_3(t)$, and thus avoid the emergence of quadratic or higher order terms on $\dot{h}(t)$ in Theorem 1. Finally, a new stability criterion (Theorem 1), which is affine on $h(t)$ and $\dot{h}(t)$ and can be directly solved by the LMI toolbox, has been achieved by Lemma 3.*

Remark 5. *The LKF (8) is different from some existing ones. States vectors $\dot{x}^T(t), \dot{x}^T(t-h(t))$, $\dot{x}^T(t-h)$ are included in $\chi_1^T(t)$, and $\ddot{x}^T(t)$ is introduced in $\chi_2^T(t)$ and $\dot{\kappa}^T(t)$, which bring more helpful information on $\dot{h}(t)$ and system states into the derivative of the LKF. Moreover, the derivative of $V_3(t)$ includes the terms of $h(t)$ and $\dot{h}(t)$ and thus it can reflect the rate of change of delay well. Therefore, the relaxed stability condition (Theorem 1) has been achieved by the LKF (8).*

Remark 6. *Two new state augmented zero equalities i.e., (16) and (17), are introduced for the first time, which can help reduce the conservatism of the obtained stability condition.*

To further illustrate the function of the auxiliary Equation (3), the following Corollary 1 without the help of the auxiliary Equation (3) is provided.

Corollary 1. *For given integer n, scalars $h > 0$, $\mu \le 1$, matrices $A \in \mathbb{R}^{n\times n}$, $A_d \in \mathbb{R}^{n\times n}$, the system (1) subject to (2) is asymptotically stable if there exist symmetric positive definite matrices $\bar{P}_1 \in \mathbb{R}^{5n\times 5n}$, $\bar{P}_2 \in \mathbb{R}^{5n\times 5n}$, $\bar{Q}_1 \in \mathbb{R}^{2n\times 2n}$, $\bar{Q}_2 \in \mathbb{R}^{2n\times 2n}$, $X_1 \in \mathbb{R}^{n\times n}$, $X_2 \in \mathbb{R}^{n\times n}$, and free matrices $\bar{S} \in \mathbb{R}^{2n\times 2n}$, $\bar{M} \in \mathbb{R}^{8n\times n}$, $\bar{\Sigma} \in \mathbb{R}^{4n\times 8n}$, such that*

$$\begin{bmatrix} \bar{\Psi}(0,\ \mu) + \bar{\Lambda}(0)\bar{\Sigma} + \bar{\Sigma}^T\bar{\Lambda}^T(0) & \bar{\Sigma}^T \\ \bar{\Sigma} & -\tilde{X} \end{bmatrix} < 0 \tag{31}$$

$$\begin{bmatrix} \bar{\Psi}(0,\ -\mu) + \bar{\Lambda}(0)\bar{\Sigma} + \bar{\Sigma}^T\bar{\Lambda}^T(0) & \bar{\Sigma}^T \\ \bar{\Sigma} & -\tilde{X} \end{bmatrix} < 0 \tag{32}$$

$$\begin{bmatrix} \bar{\Psi}(h,\ \mu) + \bar{\Lambda}(h)\bar{\Sigma} + \bar{\Sigma}^T\bar{\Lambda}^T(h) & \bar{\Sigma}^T \\ \bar{\Sigma} & -\tilde{X} \end{bmatrix} < 0 \tag{33}$$

$$\begin{bmatrix} \bar{\Psi}(h,\ -\mu) + \bar{\Lambda}(h)\bar{\Sigma} + \bar{\Sigma}^T\bar{\Lambda}^T(h) & \bar{\Sigma}^T \\ \bar{\Sigma} & -\tilde{X} \end{bmatrix} < 0 \tag{34}$$

where

$$\begin{aligned}
\bar{\Psi}(h(t),\ \dot{h}(t)) &= 2\bar{T}_1[h(t)\bar{P}_1 + (h-h(t))\bar{P}_2]\bar{T}_2^T(\dot{h}(t)) + \bar{T}_1\dot{h}(t)(\bar{P}_1 - \bar{P}_2)\bar{T}_1^T + \bar{T}_3\bar{Q}_1\bar{T}_3^T - \bar{T}_4(1- \\
&\quad \dot{h}(t))(\bar{Q}_1 - \bar{Q}_2)\bar{T}_4^T - \bar{T}_5\bar{Q}_2\bar{T}_5^T + e_5^T h(1-\dot{h}(t))(h-h(t))(X_2 - X_1)e_5 \\
&\quad + e_4^T h^2 X_1 e_4 + 2\bar{e}\bar{M}(Ae_1 + A_d e_2 - e_4),\ \bar{\Lambda}(h(t)) = [\bar{\Lambda}_1(h(t)) \quad \bar{\Lambda}_2(h(t))], \\
\bar{\Lambda}_1(h(t)) &= [e_1^T - e_2^T \quad h(t)(e_1^T + e_2^T) - 2e_7^T],\ \bar{\Lambda}_2(h(t)) = [e_2^T - e_3^T \quad (h-h(t))(e_2^T + e_3^T) - 2e_8^T], \\
\bar{e} &= [e_1^T \quad e_2^T \quad e_3^T \quad e_4^T \quad e_5^T \quad e_6^T \quad e_7^T \quad e_8^T],\ \bar{T}_1 = [e_1^T \quad e_2^T \quad e_3^T \quad e_7^T \quad e_8^T], \\
\bar{T}_2(\dot{h}(t)) &= [e_4^T \quad (1-\dot{h}(t))e_5^T \quad e_6^T \quad e_1^T - (1-\dot{h}(t))e_2^T \quad (1-\dot{h}(t))e_2^T - e_3^T], \\
\bar{T}_3 &= [e_1^T \quad e_4^T],\ \bar{T}_4 = [e_2^T \quad e_5^T],\ \bar{T}_5 = [e_3^T \quad e_6^T],
\end{aligned}$$

$$\bar{X}_1 = diag\left\{X_1, \frac{3X_1}{h^2}\right\}, \bar{X}_2 = diag\left\{X_2, \frac{3X_2}{h^2}\right\}, \tilde{X} = \begin{bmatrix} \bar{X}_1 & \bar{S} \\ \bar{S}^T & \bar{X}_2 \end{bmatrix},$$
$$e_i = [0_{n\cdot(i-1)n}, I_n, 0_{n\cdot(8-i)n}] \in \mathbb{R}^{n\times 8n}, i = 1, 2, \cdots, 8.$$

Proof. The LKF candidate is chosen as

$$\bar{V}(t) = \bar{V}_1(t) + \bar{V}_2(t) + \bar{V}_3(t) \tag{35}$$

where

$$\bar{V}_1(t) = \bar{\chi}_1^T(t)(h(t)\bar{P}_1 + (h-h(t))\bar{P}_2)\bar{\chi}_1(t) \tag{36}$$
$$\bar{V}_2(t) = \int_{t-h(t)}^{t} \bar{\chi}_2^T(s)\bar{Q}_1\bar{\chi}_2(s)ds + \int_{t-h}^{t-h(t)} \bar{\chi}_2^T(s)\bar{Q}_2\bar{\chi}_2(s)ds \tag{37}$$
$$\bar{V}_3(t) = h\int_{t-h(t)}^{t} (h-t+s)\dot{x}^T(s)X_1\dot{x}(s)ds + h\int_{t-h}^{t-h(t)} (h-t+s)\dot{x}^T(s)X_2\dot{x}(s)ds \tag{38}$$
$$\bar{\chi}_1^T(t) = [x^T(t) \quad x^T(t-h(t)) \quad x^T(t-h) \quad \int_{t-h(t)}^{t} x^T(s)ds \quad \int_{t-h}^{t-h(t)} x^T(s)ds],$$
$$\bar{\chi}_2^T(t) = [x^T(t) \quad \dot{x}^T(t)].$$

Then, we define

$$\bar{\theta}^T(t) = [x^T(t) \quad x^T(t-h(t)) \quad x^T(t-h) \quad \dot{x}^T(t) \quad \dot{x}^T(t-h(t)) \quad \dot{x}^T(t-h) \quad \int_{t-h(t)}^{t} x^T(s)ds \quad \int_{t-h}^{t-h(t)} x^T(s)ds],$$

The rest of the proof processes are similar to that of Theorem 1. This completes the proof. □

Remark 7. *As stated in Remark 3, the obtained stability condition based on the LKF (8) is infeasible without the help of (3). To further illustrate the function of the auxiliary Equation (3), we provide Corollary 1, which is derived based on the LKF (35). The LKF (35) is obtained by removing the $\ddot{x}(t)$ dependent terms in LKF (8). In this case, the zero equalities in (16) and (17) are not suitable for LKF (35) due to the system state $\ddot{x}(t)$ is not included.*

Remark 8. *On the one hand, for the case that the upper bound of the delay derivative is unknown, the auxiliary Equation (3) cannot be employed to analyze the systems. The reason is that one cannot effectively handle the function $\dot{h}(t)$ that appears in the auxiliary equation. Thus, the system state $\ddot{x}(t)$ also cannot be included in the LKF (8). Otherwise, the obtained stability condition based on the LKF (8) is infeasible, which can be checked by the LMI toolbox in Matlab. In other words, the obtained stability conditions in this manuscript are just suitable for the case that the upper bound of the delay derivative is available, but not for the case that the upper bound of the delay derivative is unknown. On the other hand, for the case that the upper bound μ of the delay derivative is larger than 1, the obtained stability condition is equivalent to the ones, which are obtained under $\mu = 1$. The proof can be found in Theorem 2.8 in Section 2.2.3 of Chapter 2 in [39].*

Remark 9. *The stabilization problem of system is not considered in this paper. As we all know, to achieve a less conservative stabilization condition, it is very important to first achieve improved stability conditions for the systems with time-varying delays. In fact, if system states are available for the state feedback control, the stabilization conditions can be easily derived based on the obtained stability conditions, and the method of designing a controller gain is similar to the ones in [40,41].*

In the next section, we will use two numerical examples that have been extensively studied in the literature to illustrate the effectiveness of the new stability conditions. The goal is to compute the MDUB h, under which the system is still stable. Based on Theorem 1 and Corollary 1, the Algorithm 1 is given.

Algorithm 1: Obtaining the optimal value of h based on Theorem 1 or Corollary 1.

Step 1. Given an integer $n > 0$, constants $\mu \leq 1$, $h > 0$ and a smaller $\epsilon > 0$, where ϵ is one step increment of h, and input the known constant coefficient matrices $A \in \mathbb{R}^{n\times n}$ and $A_d \in \mathbb{R}^{n\times n}$.
Step 2. Solve the LMIs in Theorem 1 or Corollary 1 by using the feasp solver in the Matlab/LMI Toolbox. If the LMIs are feasible, go to Step 3, else go to Step 4.
Step 3. Set $h = h + \epsilon$, and solve the LMIs again. If the LMIs are feasible, repeat Step 3, else go to Step 5.
Step 4. Set $h = h - \epsilon$, and solve the LMIs again. If the LMIs are infeasible, repeat Step 4, else go to Step 5.
Step 5. Output the MDUB h and exit.

4. Numerical Examples

Example 1. *Consider the system (1) with*

$$Ex1: \quad A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}. \tag{39}$$

$$Ex2: \quad A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, A_d = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}. \tag{40}$$

and suppose that the second-order derivatives of system states are available.

For different μ, based on Algorithm 1, Tables 1 and 2 list, respectively, the maximum delay upper bound(MDUB) h obtained by various methods and the number of decision variables(NoDVs) of these methods are also calculated in the last column, where if a matrix is a symmetric matrix, then the NoDV of the matrix is $\frac{n(n+1)}{2}$, and if a matrix is a free matrix, then the NoDV of the matrix is n^2. The n represents the dimension of the system.

Table 1. The MDUB h for different μ in Ex1.

Methods/μ	0.1	0.5	0.8	NoDVs
Theorem 3 [23]	4.8562	3.1831	2.7391	$59.5n^2 + 14.5n$
Theorem 1 [14]	4.867	3.12	–	$53.5n^2 + 8.5n$
Theorem 2(C1) [42]	4.940	3.304	2.877	$69n^2 + 12n$
Theorem 1 [43]	4.945	3.314	2.882	$100.5n^2 + 8.5n$
Corollary 1(II) [44]	4.966	3.395	2.983	$85n^2 + 15n$
Theorem 1 [15]	4.996	3.251	2.867	$38n^2 + 9n$
Theorem 8 (N = 4) [45]	5.01	3.19	2.70	$146.5n^2 + 9.5n$
Corollary 1	4.8662	3.3349	2.9886	$66n^2 + 8n$
Theorem 1	5.0213	3.6032	3.2235	$205n^2 + 13n$

Table 2. The MDUB h for different μ in Ex2.

Methods/μ	0.2	0.5	0.8	NoDVs
Theorem 1 [46]	4.5179	2.4158	1.8384	$142n^2 + 18n$
Theorem 3 [23]	4.6380	2.5898	2.0060	$59.5n^2 + 14.5n$
Corollary 1(II) [44]	4.947	2.801	2.137	$85n^2 + 15n$
Corollary 2 [3]	4.969	2.774	2.117	$235n^2 + 34n$
Theorem 2 (N = 5) [8]	4.985	2.806	2.148	$103.5n^2 + 15.5n$
Theorem 2 [17]	4.997	2.814	2.149	$307n^2 + 13n$
Theorem 1 [2]	5.0035	2.8096	2.1499	$249.5n^2 + 15.5n$
Corollary 1	4.9481	3.1531	2.7024	$66n^2 + 8n$
Theorem 1	5.1073	3.3984	2.9053	$205n^2 + 13n$

For the two well-known examples, as can be seen in Tables 1 and 2, the MDUB h obtained by Theorem 1 are larger than those of the existing ones listed in Tables 1 and 2, respectively. Moreover, the larger the μ is, the more obvious the improvement (see $\mu = 0.5$ or $\mu = 0.8$) is. Therefore, compared with these stability conditions proposed by the literature listed in Tables 1 and 2, in this work, a less conservative stability criterion (Theorem 1) has been obtained. Especially, the MAUB h obtained by Theorem 1 is larger than that of Corollary 1 for these two examples, which illustrates that taking the auxiliary Equation (3) into consideration is very important and efficient for the stability analysis of the systems with a differentiable time-varying delay.

Remark 10. *It should be pointed out that Theorem 1 involves a larger number of NoDVs. In other words, it is time-consuming to solve the matrix inequalities in Theorem 1. Fortunately, the high-performance computer can easily make up for this shortcoming in the rapid development of technology.*

To system (39) and system (40), we set the initial condition $\varepsilon(t) = [-1, 1]^T$, the time-varying delay $h(t) = \frac{h}{2}(1 + sin(\frac{2\mu t}{h}))$. The state responses of system (39) with $h = 5.0213$ and $\mu = 0.1$ are plotted in Figure 1, and of system (40) with $h = 5.1073$ and $\mu = 0.2$ are depicted in Figure 2. From Figures 1 and 2, we can clearly see that the two systems with given parameters are stable at their respective equilibrium point.

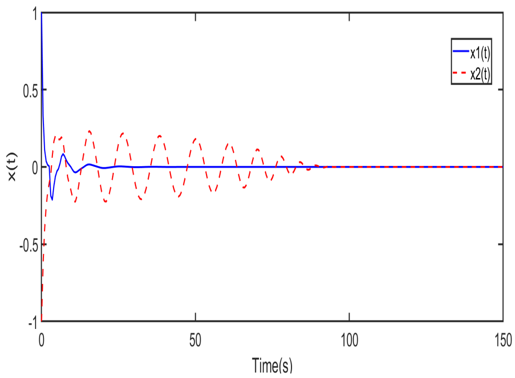

Figure 1. The trajectories of the system states in Ex1.

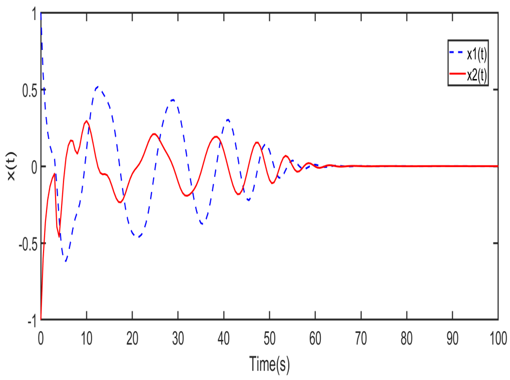

Figure 2. The trajectories of the system states in Ex2.

Remark 11. *The auxiliary equation may be complex when the original systems have uncertainties. For this case, how to deal with the uncertainties in the auxiliary equation is a difficult task, and it will be a work in our future study.*

5. Conclusions

The stability problem for the linear systems with a differentiable time-varying delay has been considered by an auxiliary equation-based method in this paper. According to the system (1) and the auxiliary Equation (3), an appropriate delay-product-type augmented

LKF has been constructed, which can utilize more delay and system state information, Then, based on the LKF, a relaxed stability condition has been derived by employing Lemmas 1–3, adding new zero terms and convex analysis method. Finally, two typical numerical examples have been given to illustrate the usefulness of the proposed method.

The stability condition proposed in this paper has some limitations. For example, the coefficient matrices $A \in \mathbb{R}^{n\times n}$ and $A_d \in \mathbb{R}^{n\times n}$ in (1) are needed to be assumed to be known, and the time-varying delay $h(t)$ in (1) need to be assumed to satisfies $0 \le h(t) \le h$, $|\dot{h}(t)| \le \mu \le 1$, where h and μ are known constants. On the other hand, without the help of an auxiliary Equation (3), the obtained result is infeasible by the LKF (8). Therefore, some future works are to solve these problems mentioned above.

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References

1. Gu, K.; Kharitonov, V.L.; Chen, J. *Stability of Time-Delay Systems*; Birkhäuser: Boston, MA, USA, 2003.
2. Zhang, B.L.; Cheng, L.H.; Pan, K.J.; Zhang, X.M. Reducing conservatism of stability criteria for linear systems with time-varying delay using an improved triple-integral inequality. *Appl. Math. Comput.* **2020**, *380*, 125254. [CrossRef]
3. Oliveira, D.; Fulvia, S.S.; Souza, F.O. Further refinements in stability conditions for time-varying delay systems. *Appl. Math. Comput.* **2020**, *380*, 124866. [CrossRef]
4. Zhang, A.L.; Lin, Z.Y.; Wang, B.; Han, Z.M. Nonlinear model predictive control of single-link flexible-joint robot using recurrent neural network and differential evolution optimization. *Electronics* **2021**, *10*, 2426. [CrossRef]
5. Zhang, X.M.; Han, Q.L.; Seuret, A.; Gouaisbaut, F. An improved reciprocally convex inequality and an augmented Lyapunov-Krasovskii functional for stability of linear systems with time-varying delay. *Automatica* **2017**, *84*, 221–226. [CrossRef]
6. Sun, J.; Liu, G.P.; Chen, J. Improved delay-range-dependent stability criteria for linear systems with time-varying delays. *Automatica* **2010**, *46*, 466–470. [CrossRef]
7. Zhang, X.M.; Han, Q.L.; Seuret, A.; Gouaisbaut, F.; He, Y. Overview of recent advances in stability of linear systems with time-varying delays. *IET Control Theory Appl.* **2019**, *13*, 1–19. [CrossRef]
8. Zeng, H.B.; Zhai, Z.L.; Wang, W. Hierarchical stability conditions of systems with time-varying delay. *Appl. Math. Comput.* **2021**, *404*, 126222. [CrossRef]
9. Wang, Y.Q.; Liu, H.B.; Li, X. A novel method for stability analysis of time-varying delay systems. *IEEE Trans. Autom. Control* **2021**, *66*, 1422–1428. [CrossRef]
10. Sun, Y.; Li, N.; Shen, M.; Wei, Z.; Sun, G. Robust H_∞ control of uncertain linear system with interval time-varying delays by using Wirtinger inequality. *Appl. Math. Comput.* **2018**, *335*, 1–11. [CrossRef]
11. Chen, G.L.; Xia, J.W.; Zhang, Z.W.; Zhao, J.S. Improved delay-dependent stabilization for a class of networked control systems with nonlinear perturbations and two delay components. *Appl. Math. Comput.* **2018**, *316*, 1–17. [CrossRef]
12. Park, M.J.; Kwon, O.M.; Park, J.H.; Lee, S.M.; Cha, E.J. Stability of time-delay systems via Wirtinger-based double integral inequality. *Automatica* **2015**, *55*, 204–208. [CrossRef]
13. Jiang, X.F.; Yin, Z.M.; Wu, J.J. Stability analysis of linear systems under time-varying samplings by a non-standard discretization method. *Electronics* **2018**, *7*, 278. [CrossRef]
14. Li, Z.C.; Yan, H.C.; Peng, Y.; Park, J.H. Stability analysis of linear systems with time-varying delay via intermediate polynomial-based functions. *Automatica* **2020**, *113*, 108756. [CrossRef]
15. Li, Z.C.;Yan, H.C.; Zhang, H.; Zhan, X.S. Huang, C.Z. Improved inequality based functions approach for stability analysis of time delay system. *Automatica* **2019**, *108*, 108416. [CrossRef]

16. Zhang, X.M.; Han, Q.L.; Ge, X.H. Novel stability criteria for linear time-delay systems using Lyapunov-Krasovskii functionals with a cubic polynomial on time-varying delay. *IEEE/CAA J. Autom. Sin.* **2021**, *8*, 77–85. [CrossRef]
17. Lin, H.C.; Zeng, H.B.; Wang, W. New Lyapunov-Krasovskii functional for stability analysis of linear systems with time-varying delay. *J. Syst. Sci. Complex.* **2021**, *34*, 632–641. [CrossRef]
18. Kwon, O.M.; Lee, S.H.; Park, M.J. Some novel results on stability analysis of generalized neural networks with time-varying delays via augmented approach. *IEEE Trans. Cybern.* **2022**, *52*, 2238–2248. [CrossRef]
19. Zhang, C.K.; He, Y.; Jiang, L.; Wu, M. Notes on stability of time-delay systems: Bounding inequalities and augmented Lyapunov-Krasovskii functional. *IEEE Trans. Autom. Control* **2017**, *62*, 5331–5336. [CrossRef]
20. Lian, Z.; He, Y.; Zhang, C.K.; Shi, P.; Wu, M. Robust H_∞ control for T-S fuzzy systems with state and input time-varying delays via delay-product-type functional method. *IEEE Trans. Autom. Control* **2019**, *27*, 1917–1930. [CrossRef]
21. Lee, S.Y.; Lee, W.; Park, P.G. Improved stability criteria for linear systems with interval time-varying delays: Generalized zero equalities approach. *Appl. Math. Comput.* **2017**, *292*, 336–348. [CrossRef]
22. Xiong, L.X.; Cheng, J.; Cao, J.D.; Liu, Z.X. Novel inequality with application to improve the stability criterion for dynamical systems with two additive time-varying delays. *Appl. Math. Comput.* **2018**, *321*, 672–688. [CrossRef]
23. Zhang, R.M.; Zeng, D.Q.; Park, J.H.; Zhong, S.M.; Liu, Y.J.; Zhou, X. New approaches to stability analysis for time-varying delay systems. *J. Frankl. Inst.* **2019**, *356*, 4174–4189. [CrossRef]
24. Seuret, A.; Gouaisbaut, F. Wirtinger-based integral inequality: Application to time-delay systems. *Automatica* **2013**, *49*, 2860–2866. [CrossRef]
25. Seuret, A.; Gouaisbaut, F. Hierarchy of LMI conditions for the stability analysis of time-delay systems. *Syst. Control Lett.* **2015**, *81*, 1–7. [CrossRef]
26. Park, P.; Lee, W.I.; Lee, S.Y. Auxiliary function-based integral inequalities for quadratic functions and their applications to time-delay systems. *J. Frankl. Inst.* **2015**, *352*, 1378–1396. [CrossRef]
27. Zeng, H.B.; He, Y.; Wu, M.; She, J.H. Free-matrix-based integral inequality for stability analysis of systems with time-varying delay. *IEEE Trans. Autom. Control* **2015**, *60*, 2768–2772. [CrossRef]
28. Chen, J.; Xu, S.Y.; Zhang, B.Y. Single/multiple integral inequalities with applications to stability analysis of time-delay systems. *IEEE Trans. Autom. Control* **2017**, *62*, 3488–3493. [CrossRef]
29. Zhang, X.M.; Han, Q.L.; Zeng, Z.G. Hierarchical type stability criteria for delayed neural networks via canonical Bessel-Legendre inequalities. *IEEE Trans. Cybern.* **2018**, *48*, 1660–1671. [CrossRef]
30. Zhang, C.K.; He, Y.; Jiang, L.; Wu, M.; Wang, Q.C. An extended reciprocally convex matrix inequality for stability analysis of systems with time-varying delay. *Automatica* **2017**, *85*, 481–485. [CrossRef]
31. Kwon, O.M.; Lee, S.H.; Park, M.J.; Lee, S.M. Augmented zero equality approach to stability for linear systems with time-varying delay. *Appl. Math. Comput.* **2020**, *381*, 125329. [CrossRef]
32. Park, M.J.; Kwon, O.M.; Ryu, J.H. Advanced stability criteria for linear systems with time-varying delays. *J. Frankl. Inst.* **2018**, *355*, 520–543. [CrossRef]
33. Zhi, Y.L.; He, Y.; Wu, M. Improved free matrix-based integral inequality for stability of systems with time-varying delay. *IET Control Theory Appl.* **2017**, *11*, 1571–1577. [CrossRef]
34. Zhao, X.; Lin, C.; Chen, B.; Wang, Q.G. Stability analysis for linear time-delay systems using new inequality based on the second-order derivative. *J. Frankl. Inst.* **2019**, *356*, 8770–8784. [CrossRef]
35. Souza, F.O. Critical frequencies of controlled second-order linear systems with delayed feedback. *Mech. Syst. Signal Pr.* **2022**, *166*, 108384. [CrossRef]
36. Zhang, J.F.; Ouyang, H.J.; Zhang, K.E.; Liu, H.M. Stability test and dominant eigenvalues computation for second-order linear systems with multiple time-delays using receptance method. *Mech. Syst. Signal Pr.* **2020**, *137*, 106180. [CrossRef]
37. Park, P.G.; Wan, K.J.; Changki, J. Reciprocally convex approach to stability of systems with time-varying delays. *Automatica* **2011**, *47*, 235–238. [CrossRef]
38. Briat, C. Convergence and equivalence results for the Jensen's inequality-application to time-delay and sampled-data systems. *IEEE Trans. Autom. Control* **2011**, *56*, 1660–1665. [CrossRef]
39. Zhang, X.M. *Study on Delay-Dependent Robust Control Based on an Integral Inequality Approach*; Central South University: Changsha, China, 2006.
40. Yin, Z.M.; Jiang, X.F.; Tang, L.; Wu, J.J. On stability and stabilization of T-S fuzzy systems with multiple random variables dependent time-varying delay. *Neurocomputing* **2020**, *412*, 91–100. [CrossRef]
41. Yin, Z.M.; Jiang, X.F. Delay-derivative/distribution dependent stability and stabilization criteria for T-S fuzzy systems with random time-varying delay. *IEEE Trans. Fuzzy Syst.* 2022, *to be published*. [CrossRef]
42. Long, F.; Zhang, C.H.; Jiang, L.; He, Y.; Wu, M. Stability analysis of systems with time-varying delay via improved Lyapunov-Krasovskii functionals. *IEEE Trans. Syst. Man Cybern. Syst.* **2021**, *51*, 2457–2466. [CrossRef]
43. Long, F.; Jiang, L.; He, Y.; Wu, M. Stability analysis of systems with time-varying delay via novel augmented Lyapunov-Krasovskii functionals and an improved integral inequality. *Appl. Math. Comput.* **2019**, *357*, 325–337. [CrossRef]
44. Zeng, H.B.; Lin, H.C.; He, Y.; Zhang, C.K.; Teo, K.L. Improved negativity condition for a quadratic function and its application to systems with time-varying delay. *IET Control Theory Appl.* **2020**, *14*, 2989–2993. [CrossRef]

45. Seuret, A.; Gouaisbaut, F. Stability of linear systems with time-varying delays using Bessel-Legendre inequalities. *IEEE Trans. Autom. Control* **2018**, *63*, 225–232. [CrossRef]
46. Lee, T.H.; Park, J.H.; Xu, S. Relaxed conditions for stability of time-varying delay systems. *Automatica* **2017**, *75*, 11–15. [CrossRef]