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Bisection Constraint Method for Multiple-Loading Conditions in Structural Topology Optimization

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Abstract: Topology optimization (TO) is currently a focal point for researchers in the field of structural optimization, with most studies concentrating on single-loading conditions. However, real engineering structures often have to work under various loading conditions. Approaches addressing multiple-loading conditions often necessitate subjective input in order to determine the importance of each loading condition, aiming for a compromise between them. This paper proposes a so-called bisection constraint method (BCM), offering a unique, user-preference-independent solution for TO problems amidst multiple-loading conditions. It is well-known that minimizing the system's compliance is commonly used in TO as the objective. Generally, compliance is not as sufficient as stress to be used as a response to evaluate the performance of structures. However, formulations focusing on minimizing stress levels usually pose significant difficulties and instabilities. On the other hand, the compliance approach is generally simpler and more capable of providing relatively sturdy designs. Hence, the formulation of min–max compliance is used as the target problem formulation of the proposed method. This method attempts to minimize compliance under only one loading condition while compliances under the remaining loading conditions are constrained. During the optimization process, the optimization problem is automatically reformulated with a new objective function and a new set of constraint functions. The role of compliance under different loading conditions, i.e., whether it is to be treated as an objective or constraint function, might be changed throughout the optimization process until convergence. Several examples based on the solid isotropic material with penalization (SIMP) approach were conducted to illustrate the validity of the proposed method. Furthermore, the general effectiveness of the compliance approach in terms of stress levels is also discussed. The calculation results demonstrated that while the compliance approach is effective in several cases, it proves ineffective in certain scenarios.

Keywords: topology optimization; multiple-loading conditions; multi-objective optimization; min–max compliance; epsilon constraint; SIMP approach

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1. Introduction

1.1. Topology Optimization (TO)

Structural optimization has evolved over the years, with different approaches being developed, including sizing, shape, and topology optimization. These approaches address various aspects of structural design problems. Sizing optimization aims to determine certain quantities, such as the optimal diameter of a linear elastic bar or the optimal cross-sectional areas of members in a truss structure. The design variable is the diameter of the bar, and the state variable is its deflection. The design model is predetermined and remains fixed during the optimization process. Conversely, shape optimization seeks to find the optimal shape of the domain within the given design model and state variables. Unlike the above two approaches, the TO of solid structures consists of determining the ideal features, including the number, shape, and location of holes and the connectivity of the domain [1]. Each approach addresses specific challenges in structural design problems, and the appropriate optimization method depends on the specific design objectives. Sizing optimization is suitable when the goal is to optimize the size of individual components, whereas shape optimization focuses on determining the best shape for a given design model. In contrast, TO is most applicable when the objective is to determine the layout of a design model that will result in optimal performance.

Ever since the homogenization method for TO was proposed in 1988 [2], research on the TO of continuum structures has expanded rapidly. Numerous methods and approaches have been discussed and developed, including homogenization [2–6], evolutionary structural optimization (ESO) [7–9], bidirectional evolutionary structural optimization (BESO) [10–12], phase-field [13–17], level-set [18–24], the method of moving morphable components [25,26], and the solid isotropic material with penalization (SIMP) approach [27–30].

The most popular approach in TO is the SIMP approach due to its simplicity of implementation. It was initially introduced by Bendsøe in 1989 [27] as a method by which to tackle optimal shape design as a material distribution problem. The SIMP approach formulates TO problems by assigning a material density value to each element in the design domain, where the material density represents the amount of material present in that region. The material property, namely, stiffness, is penalized based on the density value, forcing the topology to converge toward a nearly black and white (0–1) solution. This approach has been extensively studied and developed in subsequent articles, further refining its formulation and application in various engineering disciplines. Sigmund's article in 1997 [28] focused on applying the SIMP approach to the design of compliant mechanisms. It explores the benefits of TO in generating optimized compliant structures and provides examples of compliant mechanism designs using the SIMP approach. In 2001, Sigmund [29] presented a concise implementation of the SIMP approach for TO. The code developed by Sigmund comprised only 99 lines in MATLAB and allowed for efficient and straightforward optimization of structures. The code was further improved in 2011 by Andreassen et al. [30] to become remarkably compact and efficient. Moreover, the SIMP approach has been widely used in a number of recent studies to address problems related to real engineering structures. These include Ma et al. (2023) [31], with an application of TO for ribbed slabs and shells; Golecki et al. (2023) [32], with a bridge TO; and Shah et al. (2023) [33], with a vehicle chassis TO. The SIMP approach is used in this article to solve various TO problems.

1.2. Topology Optimization Considering Multiple-Loading Conditions

Despite the extensive analysis, improvement, and application of various TO methods mentioned above, only a limited number of studies have focused on TO problems under multiple-loading conditions. Moreover, considering multiple-loading conditions is crucial when designing structures for complex systems. In reality, a single structure can experience various loading conditions, and accounting for each scenario is essential to ensuring structural integrity and safety.

A multi-load structural TO problem involves determining the optimal distribution of material within a given design space to maximize or minimize certain objective function(s) while considering each of the given loading conditions. Because there are numerous loading conditions that are of interest, the corresponding performance functions associated with each of these conditions must be considered. Therefore, many researchers refer to multi-load problems as multi-objective optimization problems (MOP) [34]. These problems entail seeking processes that aim to discover Pareto solutions [35]. In MOPs, Pareto-optimal solutions are the solutions where it is not possible to improve one objective without worsening one or more other objectives. For an MOP, there might be an infinite set of Pareto solutions that form a Pareto set (known as Pareto front, or Pareto frontier), representing the trade-offs between different objectives. Generally, when attempting to obtain Pareto solutions for TO problems with multiple-loading conditions, it is required for users to choose a single solution from the Pareto front based on their preferences.

When solving a MOP with all objective functions sharing the same unit of measurement, various methods can be used to transform it into a single-objective optimization problem. Thus, it is easier to obtain one or more optimal solutions by solving the transformed problem(s) [35]. Among these methods, the weighted sum (WS) method [29,34–42] for multi-objective optimization is widely utilized, which involves assigning weights to each objective and determining a compromised solution. This method not only allows for the generation of multiple solution points by adjusting the weights accordingly but also enables obtaining a single solution point reflecting preferences linked to a specific set of weights. Diaz et al., in 1992 [36], used a WS method to optimize the shapes of structures under multiple-loading conditions. They used a homogenization method to formulate and solve the TO problems. A WS method was employed by Bendsøe et al. in 1995 [37] to aggregate the objective functions and find the optimal design when considering simultaneous optimization of material properties and distribution under multiple-loading conditions. In 2000, Min et al. [34] applied the WS method to optimize the topology of structures for both static and dynamic loading conditions using a genetic algorithm (GA). Krog et al., (2004) [38] employed a WS method to optimize the topology of aircraft wing box ribs utilizing a sequential quadratic programming algorithm. Pedersen (2006) [39] discussed aspects of three-dimensional (3D) shape and TO with multiple loads. He presented several challenges that are associated with this type of optimization, such as the need to account for the interaction between load cases. In 2020, Li et al. [35] proposed a new method for multi-load TO based on the WS method. Their method considered the severity and ideality of each load case when assigning weights to the objectives. WS methods have also been widely used in recent studies, particularly in applied articles. For instance, Sun et al. [41] and Dämmer et al. [42], in 2023, used WS methods to develop 3D real robot structures. Guo et al. [43], in 2023, utilized WS methods in their studies to develop explicit TO for 3D geometrically nonlinear structures. Rong et al., in 2023 [44], employed WS methods to address TO in dual-material design problems with multiple load cases. Additionally, Chen et al., (2024) [45] used WS methods for solving TO problems of joints in truss structures considering multiple load cases. The advantages of the WS method include its simplicity and ease of implementation. However, the effectiveness of this method may depend on the accuracy of assigning severity degrees and ideality values to the load cases. In other words, WS methods can be sensitive to the subjective choice of weights and may not be able to obtain the global optimum solution for the original MOPs.

Other researchers adopted an approach when considering multiple load cases, referred to as the worst-case design [46–54]. In 1998, Achtziger [47] used this approach to carry out multiple-load truss topology and sizing optimization to determine the optimal discrete or discretized mechanical structures with respect to maximal stiffness. In the article, the feasible structures that are best able to withstand their worst loading condition were considered. The definition of the “worst loading” condition is given as the loading condition where maximal compliance occurs among all compliances, associated with each loading condition. However, in TO incorporated with multiple-loading conditions, defining the “worst loading” becomes challenging as it changes together with the change in the topology of the structure. Hence, the min–max concept was introduced in the same study to address the uncertain “worst loading” condition depending on the design itself. The min–max formulation of MOPs was first proposed in a linear model by Jutler and Solich and was further extended by Rao [48] and Tseng et al. [49]. In 1998, Coello [50] applied the min–max strategy to solve engineering optimization problems using a GA. The method includes giving a reasonably small number of weight combinations, making it a subjective approach that requires a better understanding of each problem. Consequently, the min–max formulation itself remains unsolved. The min–max compliance formulation was also introduced in Logo et al., (2018) [53] together with an alternative bound formulation. The alternative bound formulation was solved using a "parametric level" technique in Logo's paper. In 2020, Nowak et al. [54] adopted the min–max compliance formulation in a biomimetic approach, where it is combined with the weighted sum method to handle TO problems under multiple loads. However, the min–max compliance formulation was not directly handled in the aforementioned papers, and its true solution remains ambiguous.

Another method for solving MOPs is the epsilon-constraint method [55–58], which is mainly used to generate Pareto fronts for MOPs. This method consists of the optimization of only one objective as a single-objective function (single-objective problems), while the other remaining objectives are considered as constraints. The epsilon-constraint method was first introduced by Haimes et al. in 1971 [55]. The method was designed to address MOPs where one objective function was optimized and the remaining objective functions were constrained to avoid exceeding the given target values. This method was successfully implemented for various multi-objective mathematical programming problems in 2009 by Mavrotas [56]. In 2013, Chircop et al. [57] used the epsilon-constraint method to generate Pareto frontiers in MOPs. The application of this method to TO was proposed by Jaouadi in 2015 [58]. In a recent study from 2023, Hübner et al. [59] employed the epsilon-constraint method to solve multi-objective two-scale sizing optimization problems. However, in the epsilon-constraint method, the necessity of giving the epsilon values by users, which requires a certain understanding of all scenarios of the problems, leads to difficulty in method implementation, and the method remains somewhat subjective.

To overcome the aforementioned difficulties, this paper proposes a new method called the bisection constraint method (BCM). This method attempts to address the min–max compliance formulation of TO problems considering multiple-loading conditions. The BCM method offers a unique, user-preference-independent solution for TO problems amidst multiple-loading conditions. The paper is organized as follows: In Section 2, the basics of TO using the SIMP approach are explained, regarding single and multiple-loading conditions. Section 3 provides a detailed description of the proposed method. Representative examples are presented in Section 4 to demonstrate the effectiveness of the proposed method in solving the min–max compliance formulation of TO considering multiple load cases. Section 5 presents a comparative study discussing not only compliance minimization but also the stress levels arising in structures obtained via different methods. Finally, conclusions are provided in Section 6.

2. Topology Optimization under Single- and Multiple-Loading Conditions

In this section, we begin by discussing the TO formulation of an optimum design problem under a single-loading condition based on the SIMP approach. This discussion will provide a foundational understanding of how to state TO problems by choosing the objective function and design variables. Subsequently, we extend our discussion to consider multiple load cases. Towards the end of this section, the min–max compliance formulation of TO under multiple-loading conditions is explained. This formulation serves as a critical foundation for the method proposed in this article.

2.1. SIMP Approach

Originally, TO was first formulated to determine which points in the design space should be filled with material and which points should remain void. In other words, the optimization problem was interpreted using discrete variables, where a variable value of 0 represented a void and a value of 1 represented a solid material. Nevertheless, dealing with a discrete 0–1 problem causes many difficulties, especially for complex problems. It is worth noting that mathematical methods that use information about the derivatives of cost functions cannot be employed for discrete problems. Therefore, other approaches are commonly used to replace the integer discrete variables with continuous variables, including the SIMP approach [27] and the θ–type variables approach, where the density is given by a sigmoid function [60].

The SIMP approach is well-known as a simple and commonly used approach for solving TO problems. In the SIMP approach, it is assumed that the density of each element in the design space can have a value between 0 and 1, turning the original problem into a continuous one. However, only using continuous variables between 0 and 1, instead of the discreate integer 0 or 1 often, results in unreasonable structures containing many "gray" elements with intermediate densities. Consequently, there is a need to use a so-called power-law interpolation to interpret material properties. This power-law interpolation acts as a penalty, where elements with intermediate densities exhibit poor performance, i.e., lower stiffness. Thus, the algorithm is forced to yield a nearly black-and-white solution. Following this interpolation law, the stiffness tensor of a given isotropic material is expressed as follows:

$$E(x) = x^p E_0, p > 1, \tag{1}$$

where x is the density of the considered element; E_0 is the stiffness tensor of the element fulfilled with a solid material; and p is the penalty power in the SIMP approach. Usually, when using Equation (1) in the problem formulation, a lower bound on all design variables (densities), which is a sufficiently small positive value, is considered to prevent the singularity issue.

There is also another way to avoid singularity, that is, assuming

$$E(x) = E_{min} + x^p(E_0 - E_{min}),\ p > 1, \tag{2}$$

where E_{min} is a small positive value, which serves as the lower bound of material stiffness tensor $E(x)$. For simplicity, Equation (1) is used for the examples in this study.

2.2. Single-Loading Condition

In a structural TO problem with a single-loading condition, the goal is to find the best distribution of material within a given design domain, which is referred to as the optimum design, minimizing (or maximizing) an objective function while satisfying a set of constraints. The most common objective in TO is maximizing its stiffness, which is equivalent to minimizing its compliance. Other objectives include minimizing the structural weight with consideration of stress constraints and minimizing the stress level within a given amount of material. The minimization of the maximum stress is a general concept of structural design. However, the formulations aimed at reducing the maximum stress or considered stress constraints usually cause very complex calculations of cost functions and sensitivities and, consequently, a high computational cost. In contrast, a formulation that considers compliance minimization is always simpler. Accordingly, the second formulation is preferable and widely used in the field of structural TO.

A simple compliance-minimizing TO problem under a single-loading condition using SIMP approach can be formulated as follows:

$$\min_{\boldsymbol{x}} c(\boldsymbol{x}) = \boldsymbol{u}^T \boldsymbol{K} \boldsymbol{u}, \tag{3a}$$

$$\text{subject to } \frac{V(\boldsymbol{x})}{V_0} = f, \tag{3b}$$

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}, \tag{3c}$$

$$\boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}, \tag{3d}$$

$$0 < x_{min} \leq x_i \leq 1,\ i = 1, 2, \ldots, n, \tag{3e}$$

where $c(\boldsymbol{x})$ is the compliance of the system; $\boldsymbol{u}$ is the global displacement vector, also known as the state variable; $\boldsymbol{K}$ is the global stiffness matrix; $\boldsymbol{g}(\boldsymbol{x})$ is a set of constraint functions, which can be performance (e.g., strain or stress), geometrical, or other types of restriction; $V(\boldsymbol{x})$ and V_0 are the material volume and the volume of the design domain, respectively; f is a volume fraction, given to constrain the amount of material that can be used for the structure; $\boldsymbol{F}$ is the global force vector; $\boldsymbol{x} = \{x_1 \quad x_2 \quad \cdots \quad x_n\}^T$ is the design variables vector; n is the number of design variables; and x_{min} is a positive small lower bound of the design variables employed to avoid the singularity issue.

Equation (3c) is the equilibrium equation of the system, where the force vector $\boldsymbol{F}$ is given as a constant; and the stiffness matrix $\boldsymbol{K}$ is defined with respect to the design variables, i.e., $\boldsymbol{K}$ is a function of $\boldsymbol{x}$, which can be written as $\boldsymbol{K} = \boldsymbol{K}(\boldsymbol{x})$. Therefore, when $\boldsymbol{x}$ is defined, $\boldsymbol{K}$ is defined, and $\boldsymbol{u}$ can be found by solving the equilibrium equation as follows:

$$\boldsymbol{u} = \boldsymbol{K}^{-1}\boldsymbol{F}. \tag{4}$$

During the optimization process, the equilibrium equation associated with the only loading condition is solved once at each iteration to find the vector of displacements $\boldsymbol{u}$, and all cost functions are evaluated using this state variable. The update of the design variables is decided based on the evaluated cost functions and their derivatives. The process is repeated for new design variables until convergence.

2.3. Multiple-Loading Conditions

In the context of multiple-loading conditions, choosing the objective function becomes a non-trivial task in terms of compliance minimization. This complexity arises because there exists a set of compliances under each loading condition that must be considered. Hence, it is common to use formulations of either a multi-objective problem or a single-objective problem, where the objective function represents an appropriate norm [36] of all compliances.

Consider TO under n_L loading conditions. When using a multi-objective optimization approach, by providing suitable guidelines for arranging and manipulating the set of objectives, it is possible to express the problem as an "unclear" vector argument corresponding to its minimization, as follows:

$$\min_{\boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}) = \{c_1(\boldsymbol{x}) \quad c_2(\boldsymbol{x}) \quad \cdots \quad c_{n_L}(\boldsymbol{x})\}, \tag{5a}$$

$$\text{subject to } \frac{V(\boldsymbol{x})}{V_0} = f, \tag{5b}$$

$$\boldsymbol{K}(\boldsymbol{x})\boldsymbol{u}_j(\boldsymbol{x}) = \boldsymbol{p}_j,\ j = 1, 2, \ldots, n_L \tag{5c}$$

$$\boldsymbol{g}(\boldsymbol{x}) \leq \mathbf{0B}, \tag{5d}$$

$$0 < x_{min} \leq x_i \leq 1,\ i = 1, 2, \ldots, n, \tag{5e}$$

where $\boldsymbol{x} = \{x_1 \quad x_2 \quad \cdots \quad x_n\}^T$ is the vector of design variables; n is the number of design variables; x_{min} is a positive small lower bound of the design variables; $\boldsymbol{K}$ is the stiffness matrix; $\boldsymbol{p}_j$ is the load vector of the jth load case; $\boldsymbol{u}_j$ is the displacement vector when only the j-th load case is applied; $c_j(\boldsymbol{x}) = \boldsymbol{p}_j^T \boldsymbol{u}_j(\boldsymbol{x})$ is the compliance under the j-th loading condition; V is the material volume; V_0 is the volume of the design domain; f is a given ratio limiting the amount of material to be used; and n_L is the number of loading conditions.

When addressing MOPs, it is uncommon for all objective functions to be simultaneously optimized. Moreover, due to the inherent conflicting nature of certain objectives, achieving a unique optimum is generally challenging [61]. In vector problems, the optimal solution lies within a set of Pareto-optimal designs (Pareto front). However, identifying the ideal solution within this set often requires additional information, such as weighting or a basis for making informed judgments [35]. This process can be intricate, which is why it is frequently preferred to interpret multi-objective problems as single-objective ones. Consequently, the function $\boldsymbol{f}(\boldsymbol{x})$ is chosen to be an appropriate norm [36], with the WS method being the most widely used, which involves computing the weighted average as follows:

$$f(\boldsymbol{x}) = \sum_{j=1}^{n_L} w_j c_j, \tag{6}$$

where w_j is the weight factor of the j-th loading condition, satisfying $\sum_{j=1}^{n_L} w_j = 1$ and $w_j > 0$.

Several strategies have been proposed to allocate weight factors for solving TO problems under multiple-loading conditions [35]. The simplest and most common strategy is to use the same weight factors for all the loading conditions. In this case, the objective function is typically the sum of all compliances under each loading condition [29], as follows:

$$f(\boldsymbol{x}) = \sum_{j=1}^{n_L} c_j. \tag{7}$$

This WS method with the same weight factor for all loading conditions is later referred to as the traditional equally weighted sum (TEWS) method in this article. In general, the TEWS method can solve various problems and provide relatively good solutions while its implementation is simple and cost-effective. This method was used in a recent article providing a free MATLAB code for educational purposes by Kumar in 2022 [62]. Nevertheless, assigning the same weights to different loading conditions implies that the role of the loading conditions is always the same by default. As a result, the method might not be able to provide a sufficiently good solution when there are severe load cases that require more attention than other loading conditions.

Returning to the original problem, when considering multiple-loading conditions, the goal of the problem should be to find a design that exhibits the best performance across all loading conditions, compared with other designs. From a compliance standpoint, the design should have the smallest compliance value considering all the loading conditions compared with the other designs [51,52]. At this point, the min–max compliance formulation of the problem is likely the best translation of the original problem. This formulation aims to find the design with the smallest maximum compliance among all compliances under each load case compared with other designs [51,52]. The min–max formulation is explained in detail in Section 2.4.

2.4. The Min–Max Compliance Problem in TO Considering Multiple-Loading Conditions

Conventionally, the min–max formulation can be used to represent various MOPs, where the objectives are either of the same or different types of quantities, i.e., the same or different units of measurement. When the objective functions are of the same type of quantity, e.g., interpreted via Problem (5), the min–max form can be used to convert it into a single-objective optimization problem, which minimizes the maximum compliance with respect to a given volume as follows:

$$\min_{\boldsymbol{x}} \left[\max_{j} \left(\boldsymbol{u}_j^T(\boldsymbol{x}) \boldsymbol{K} \boldsymbol{u}_j(\boldsymbol{x}) \right) \right], j = 1, \ldots, n_L, \tag{8a}$$

$$\text{subject to } \boldsymbol{K}(\boldsymbol{x}) \boldsymbol{u}_j(\boldsymbol{x}) = \boldsymbol{p}_j, j = 1, 2, \ldots, n_L \tag{8b}$$

$$\frac{V(\boldsymbol{x})}{V_0} = f, \tag{8c}$$

$$\boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}, \tag{8d}$$

$$0 < x_{min} \leq x_i \leq 1, \quad i = 1, 2, \ldots, n. \tag{8e}$$

Directly solving Problem (8) remains challenging due to the complex computation of the objective function and its derivative. Therefore, several researchers have continued to reform it using either a bounded form [53,63,64] or a WS method. While the WS method carries a few disadvantages, as mentioned in the Introduction (Section 1.2), the bounded method also faces a significant challenge, namely, in the optimization process using the bounded form, the value of the upper bound must be specified [53,63,64]; thus, the bounded method does not completely break out of the vicious circle of previous subjective methods, in which certain values must be subjectively specified by users.

As discussed in Section 1.2, the epsilon-constraint method [55–58] appears to be one of the promising methods for treating MOPs, including structural TO with respect to multi-load cases, e.g., given in the form of Problem (8), the min–max form. Nevertheless, the epsilon-constraint method still belongs to a group of methods that require additional assignments of users regarding the importance of each loading condition. In the present study, we propose a new method that can directly address Problem (8). The proposed method is explained in detail in Section 3.

3. Bisection Constraint Method for Topology Optimization Considering Multiple-Loading Conditions

In Section 3.1, the proposed bisection constraint method is explained. The starting point of the method is discussed in Section 3.2, and the termination criterion is introduced in Section 3.3.

3.1. The Bisection Constraint Method

Consider a structural TO problem under multiple-loading conditions $j \in N_L = \{1, 2, \ldots, n_L\}$. Here, N_L is the set of loading conditions; and n_L is the number of given loading conditions, $n_L \geq 2$. The essence of the problem is to find the material distribution within a given design domain such that the compliance under each load condition is as small as possible within the limited amount of material allowed (typically, a volume constraint is considered), as vaguely interpreted in Problem (5).

According to past approaches [46–50], the worst load case needs to be identified, and the compliance under which must be minimized. However, the worst load case depends on the design itself, which changes throughout the optimization process. Consequently, evaluating the worst load case should not be conducted only once from the beginning of the optimization process. Accordingly, a new method is proposed in this paper, consisting of nested loops, where each outer loop identifies the worst load case and the compliance associated with it, then a sub-TO problem is set up to minimize the compliance under the worst load case. Meanwhile, the inner iterations within each outer loop solve that sub-TO problem. The proposed method is implemented in a framework consisting of two stages, given in the flowchart in Figure 1.

For better understanding, the second stage of the proposed framework will be explained first (Stage 2). Stage 2 begins with the identification of the worst load case (with the maximum compliance) at step k for a definite design variable vector $\boldsymbol{x}_k$ (Step 4). For this purpose, a finite element analysis (FEA) is performed (in Stage 1 and Step 2), and the displacement vector $\boldsymbol{u}_j$ under the jth load case is found by solving the equilibrium equation:

$$\boldsymbol{K}\boldsymbol{u}_j = \boldsymbol{p}_j, \tag{9}$$

where $\boldsymbol{K}$ is the global stiffness matrix; and $\boldsymbol{p}_j$ is the load vector of the jth load case.

After defining all $\boldsymbol{u}_j$, in Step 3, the system compliance under each loading condition is calculated as follows:

$$\overline{c_j^k} = \boldsymbol{u}_j^T \boldsymbol{K} \boldsymbol{u}_j. \tag{10}$$

In Step 4, a comparison among all individual compliances $\overline{c_j^k}$ is made to determine their maximum compliance $\overline{c_{m_k}}$. Here, m_k is the index of the loading condition where the maximum compliance occurs at iteration k, $m_k \in N_L$. Hence, the function of the compliance under the m_kth loading condition is chosen as the objective function of the sub-TO problem in this outer loop.

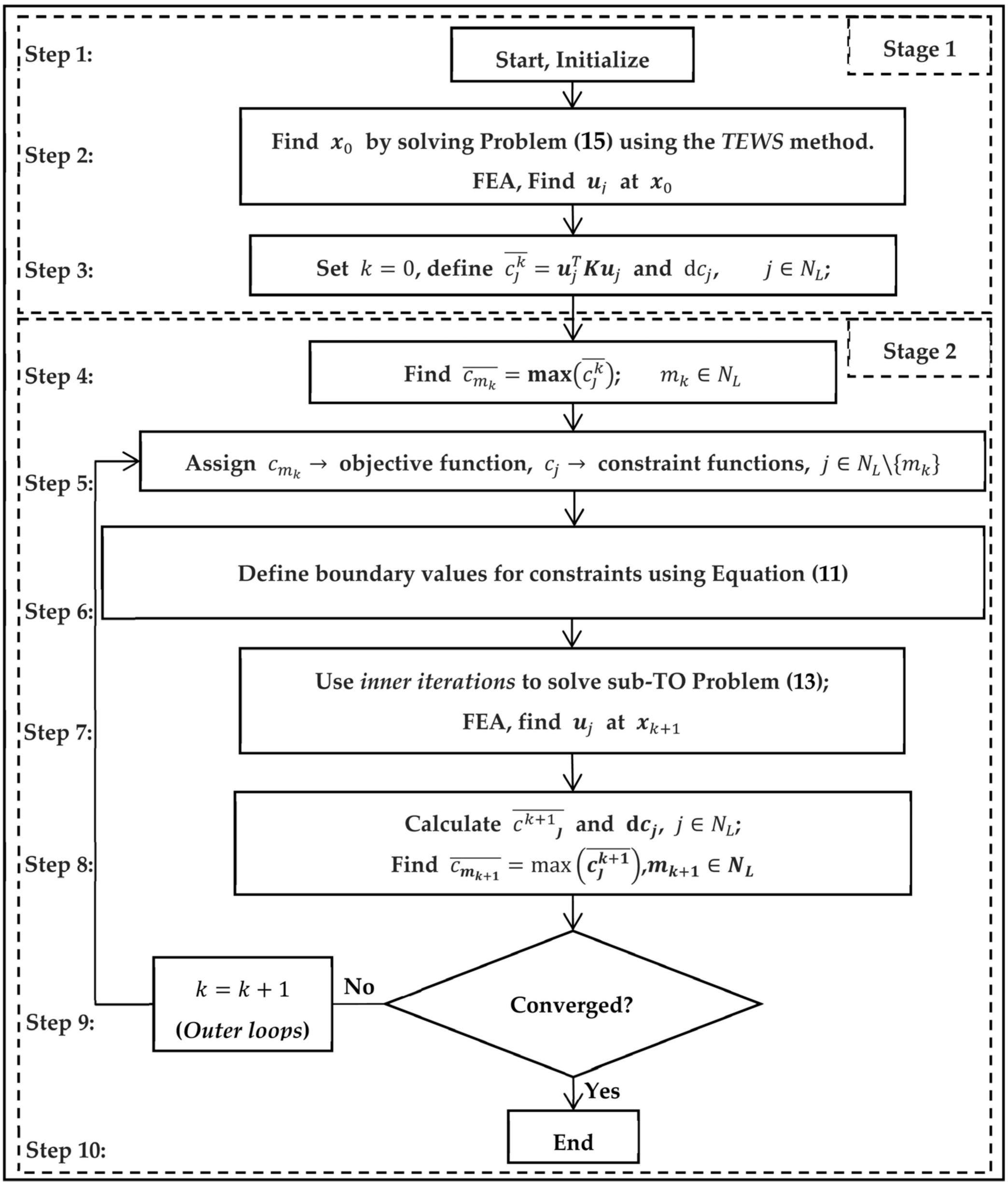


Figure 1. Flowchart of the framework to implement the bisection constraint method.

In the most general case, the considered loading conditions have a high possibility of being in a trade-off relationship; i.e., a reduction in one compliance must be offset by an increase in one or many other compliances. An attempt to rearrange the material in the domain in order to reduce the cost function associated with the worst loading condition will likely increase the cost functions associated with the remaining loading conditions due to the aforementioned trade-off relationship. Hence, there is a need to let the cost functions under the remaining loading conditions increase to allow for the reduction of compliance under the worst loading condition. At the same time, the cost functions under the remaining loading conditions are not expected to become excessively large. Hence, in the sub-TO problem, the compliance functions under the remaining loading conditions are considered as constraints (Step 5). A set of boundary values for these compliance functions is calculated with respect to the defined maximum compliance, $\overline{c_{m_k}}$, as follows (Step 6):

$$\overline{c_{crj}^k} = \frac{\overline{c_{m_k}} + \overline{c_j^k}}{2},\ j \in N_L \backslash \{m_k\}, \tag{11}$$

where c_{crj}^{k} is the critical value (boundary value) of the jth loading condition. The constraint function is expressed as follows:

$$c_j(\boldsymbol{x}) \leq \overline{c_{crj}^{k}},\ j \in N_L \backslash \{m_k\}. \tag{12}$$

Equations (11) and (12) explain the name of the proposed bisection constraint method.

After assigning the objective function and constraints, a sub-TO problem is solved based on constrained TO techniques (see Step 7 in Figure 1). The objective is to minimize the compliance under the worst load case m_k at step k as follows:

$$\min_{\boldsymbol{x}} c_{m_k}(\boldsymbol{x}),\ m_k \in N_L, \tag{13a}$$

$$\text{subject to } \frac{V(\boldsymbol{x})}{V_0} = f, \tag{13b}$$

$$c_j(\boldsymbol{x}) \leq \overline{c_{crj}^{k}},\ \ j \in N_L \backslash \{m_k\}, \tag{13c}$$

$$\boldsymbol{K}(\boldsymbol{x})\boldsymbol{u}_j(\boldsymbol{x}) = \boldsymbol{p}_j,\ j = 1, 2, \ldots, n_L \tag{13d}$$

$$0 < x_{min} \leq x_i \leq 1,\ \ i = 1, 2, \ldots, n. \tag{13e}$$

The sub-TO Problem (13) is a typical single-objective TO problem under one volume constraint and $(n_L - 1)$ compliance constraints. Various methods can be used to solve this sub-TO problem. For simplicity, in the next section of this article, sequential linear programming (SLP) [65] is used to solve the example problems.

After solving the sub-TO Problem (13), a set of new cost function values c_j is defined at a new design variable vector $\boldsymbol{x}_{k+1}$. The processes of cost function comparison, assigning objective and constraint functions, calculating boundary values for the constraints, and sub-TO problem solving are repeated until convergence. An example of the compliances' evolution during Stage 2 of the optimization process is shown in Figure 2 (with three loading conditions), where the bar symbols $\overline{c_1^k}$, $\overline{c_2^k}$, and $\overline{c_3^k}$ represent the critical values (boundary values) of compliances associated with the compliance functions c_1, c_2, and c_3, respectively.

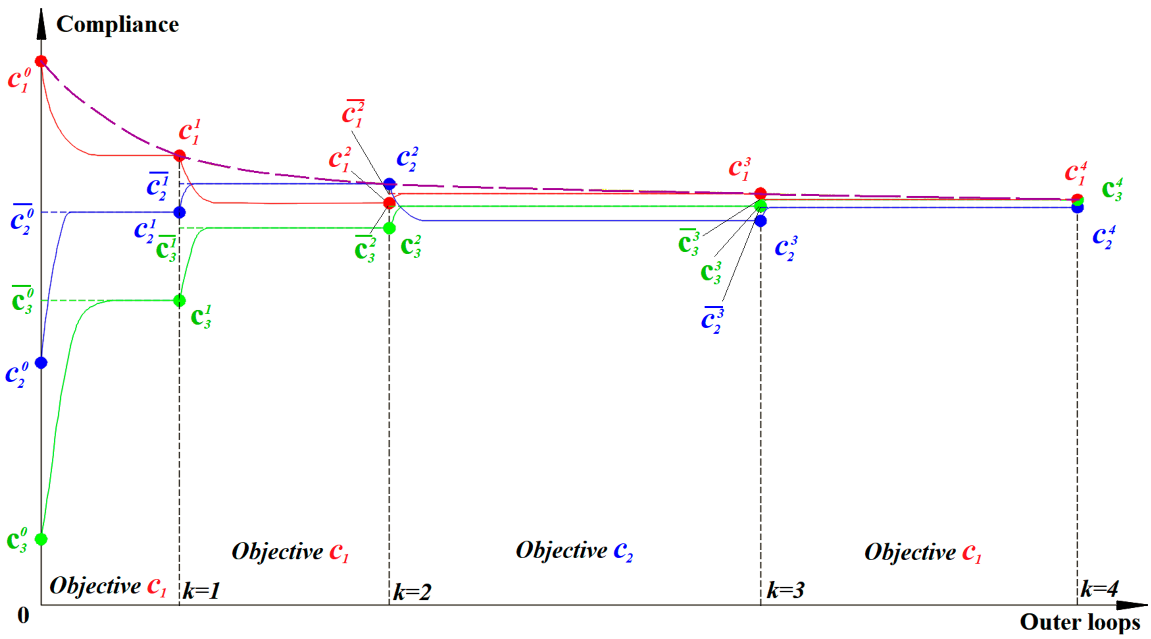


Figure 2. An example of evolution of compliances during the optimization process with three loading conditions using the proposed bisection constraint method.

By finding the maximum compliance at every outer loop and reducing it while keeping other compliances from exceeding the boundary values (constrained), the proposed method can obtain the true solution of the min–max compliance formulation.

3.2. Starting Point

All optimization algorithms require users to supply a starting point denoted by $\boldsymbol{x}_0$. Users with knowledge of the application and the dataset may be in a good position to choose $\boldsymbol{x}_0$ to reasonably close the true solution. Otherwise, the starting point must be selected by an algorithm, either systematically or arbitrarily. The choice of the starting point is an important task that significantly affects the robustness of the algorithm. A poor choice of the starting point might even lead to a convergence failure [66].

As shown in Figure 2, the result at the end of each outer loop is a solution that belongs to the Pareto set because it is obtained from one sub-TO problem, forming a trade-off relationship between compliances [58]. Therefore, this method is an algorithm for generating several Pareto solutions with different trade-off relationship setups for the compliances. However, after each outer loop, a Pareto solution is generated with better performance than the previous one (the maximum compliance becomes smaller). At this point, improving a structure that does not belong to the Pareto set holds no significance because it fails to represent any trade-off position of the compliances. In addition, starting at a point belonging to the Pareto front can increase the possibility of convergence to the true solution because it might be closer to the true solution than picking a point in an arbitrary manner. Thus, it is a good idea to start the algorithm at a point that belongs to the Pareto front, which is generated by an optimization algorithm.

In general, a WS method with positive weights can be used to generate a solution which is Pareto optimal [40]. In the current article, an optimization algorithm using the TEWS method is used to generate the starting point for the proposed method. The TEWS method has several advantages, including simple formulation, low computational cost, and low time consumption, and, more importantly, its solution might be reasonably close to the true solution.

As shown in Figure 1, the TEWS method is used in Stage 1 to generate the starting point for the main stage (Stage 2). Particularly, in Stage 1, a TO problem is solved using the objective function as the sum of the compliances under each loading condition:

$$C(\boldsymbol{x}) = \sum_{j=1}^{n_L} c_j(\boldsymbol{x}),\ j \in N_L = \{1, 2, \ldots, n_L\}. \tag{14}$$

The optimization problem in this stage is formulated as follows.

$$\min_{\boldsymbol{x}} C(\boldsymbol{x}), \tag{15a}$$

$$\text{subject to } \boldsymbol{K}(\boldsymbol{x})\boldsymbol{u}_j(\boldsymbol{x}) = \boldsymbol{p}_j,\ j = 1, 2, \ldots, n_L \tag{15b}$$

$$\frac{V(\boldsymbol{x})}{V_0} = f, \tag{15c}$$

$$0 < x_{min} \leq x_i \leq 1,\ \ i = 1, 2, \ldots, n. \tag{15d}$$

It is straightforward to solve Problem (15), where the initial point is usually set as all elements in the design domain having the same density, $x_0 = f$. Stage 2 further improves the design in the sense of minimizing the maximum compliance, as explained in Section 3.1.

3.3. Termination Criterion

The most commonly used termination criterion in optimization algorithms occurs when the reduction in the objective function is sufficiently small. Hence, in this study, the reduction in the objective function is checked at every outer loop to decide whether the optimization process should be continued or terminated. This condition is expressed as follows (Step 9 in Figure 1):

$$\left|\overline{c_{m_k}} - \overline{c_{m_{k+1}}}\right| \leq \epsilon, \tag{16}$$

where $\overline{c_{m^k}}$ is the maximum compliance at step k, $\overline{c_{m_{k+1}}}$ is the maximum compliance at step $k+1$; and ϵ is a small positive scalar. As shown in Figure 2, the reduction in the objective function is represented by the reduction in the purple dotted line, which is the line crossing maximum compliance at every outer loop.

Notably, the algorithm of the proposed method can also be terminated even before the reduction in the maximum compliance becomes sufficiently small. As explained in Section 3.1, during the optimization process, the most critical compliance is treated as the objective function and is reduced when all the trade-off compliances are relaxed. When the two largest compliances meet each other at the end of a sub-TO problem, it is not possible to reduce one of them without increasing the remaining one. Thus, this case creates a stable equilibrium position of the compliances. In other words, the algorithm converges.

In summary, two termination criteria are set for the proposed method: (1) the reduction in the maximum compliance is sufficiently small; and (2) the two most critical compliances are the same at the end of a sub-TO problem. The algorithm can be terminated when either condition is reached.

4. Application of the Proposed Method

To demonstrate the effectiveness of the proposed BCM method, a TO problem for a simple two-dimensional (2D) beam was considered with three loading conditions: Load Case 1 (LC1), Load Case 2 (LC2), and Load Case (LC3), as shown in Figure 3. Here, concentrated loads were used in LC1 and LC2, whereas a uniformly distributed load was used in LC3. The magnitude of the distributed loads in LC3 is given by the load sum F_3, and the load distribution (q_3) can be found from the following relationship:

$$F_3 = q_3 \cdot \frac{l}{3}, \tag{17}$$

where l is the length of the entire design domain; and $l/3$ is the range in which the distributed load is applied to the lower edge of the design domain. Usually, when considering a compliance minimizing problem, a volume constraint is used, which is set to 50% of the domain ($f = 0.5$). The material is assumed to have Young's modulus $E = 1$ and Poisson's ratio $\nu = 0.3$.

The problem was formulated based on the flowchart given in Figure 1 using the SIMP approach. MATLAB software (version R2019a) was used to develop the code, and the SLP method was employed to solve the optimization problems for simplicity. To use the SLP method, the cost function derivatives must be identified alongside the cost function values themselves. The derivatives of the compliance functions (cost functions) are then used to estimate them as linear functions, which is a fundamental step in the SLP method [65]. For FE analysis, the design domain was discretized using square 2D elements. A mesh of 120×40 (120 elements in the horizontal direction and 40 elements in the vertical direction) was used for this example.

Two case studies were conducted with different correlations of force magnitudes under different loading conditions to observe the performance of the proposed BCM method and, in particular, how it manages problems with different roles of objective and constraint functions. The calculation results are as follows.

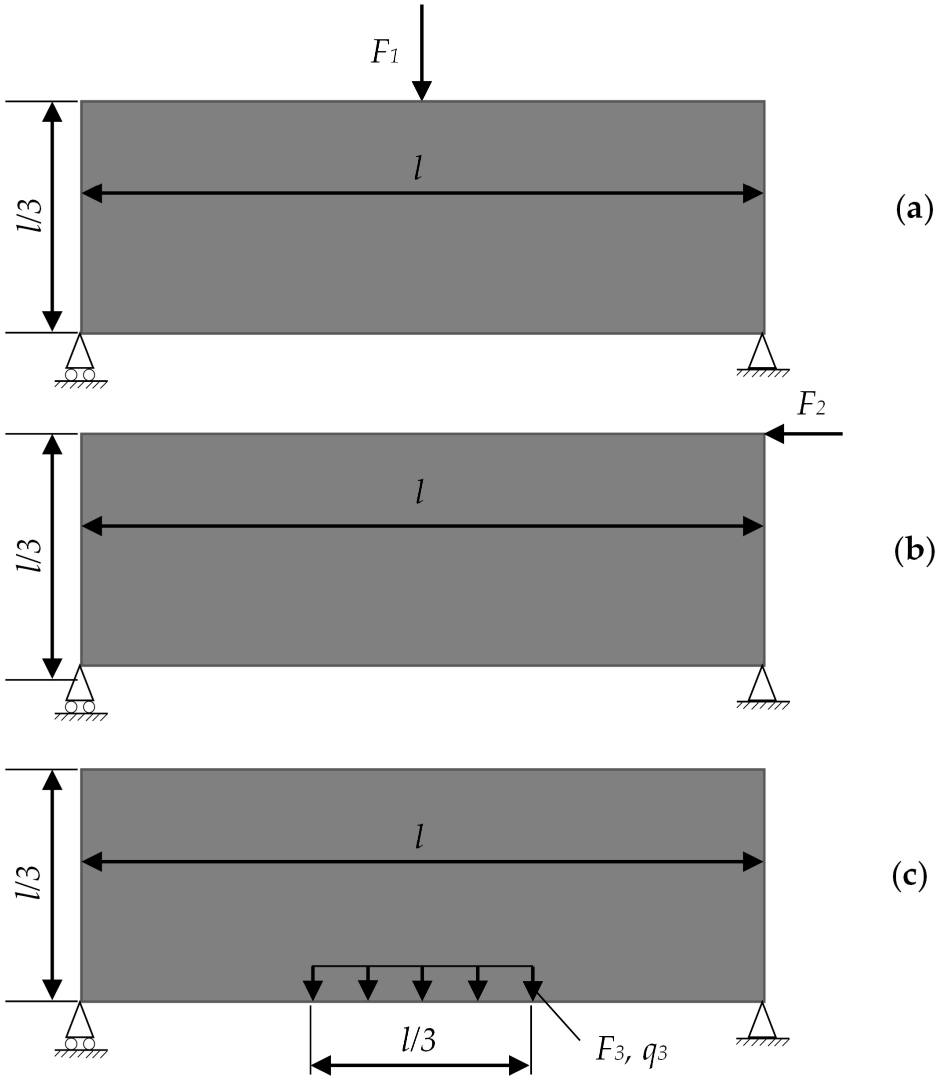
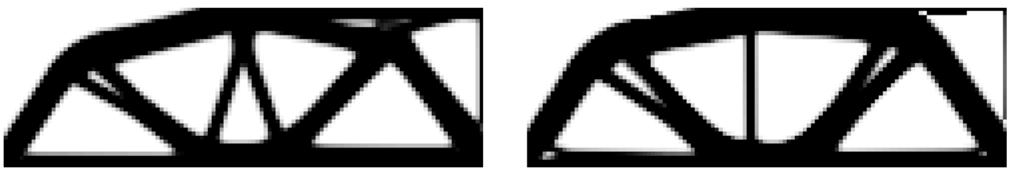


Figure 3. A 2D beam with three loading conditions: (**a**) Load case 1; (**b**) Load case 2; (**c**) Load case 3.

4.1. Case Study 1

In this case study, $F_1 = 1.5$, $F_2 = 1.2, F_3 = 2$ are given. The topology obtained from Stage 1 is presented in Figure 4a, following the minimization of the compliance sum (Equation (14)), when considering each of the loading conditions. Figure 5a shows the evolution of the compliances during Stage 1, where the TEWS method was used. It is observed that at this stage, all compliances are reduced.



(**a**) (**b**)

Figure 4. Topologies in Case Study 1: (**a**) after Stage 1 (traditional equally weighted sum method); (**b**) after Stage 2.

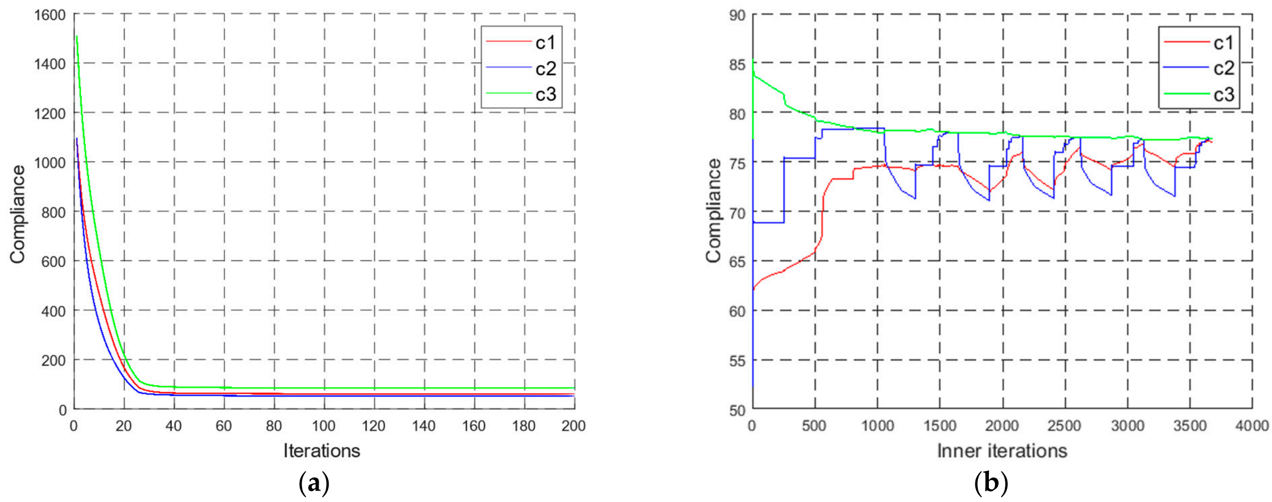


Figure 5. Evolution of compliances in Case Study 1: (**a**) during Stage 1 (traditional equally weighted sum method); (**b**) during Stage 2.

The topology obtained from Stage 2 is shown in Figure 4b, which was changed from Stage 1's topology in order to withstand the worst loading condition better. In this case study, the worst loading condition was LC3 most of the time. The performance of the structure was improved after Stage 2 in terms of reducing the most critical compliance, considering all the loading conditions individually, as shown in Figure 5b.

Figure 6 illustrates how the maximum compliance decreased during Stage 2, where the number shown in the graph is the index of the loading condition. For example, when number 3 is shown, the compliance function under LC3 is assigned as the objective. It can be observed in Figures 5b and 6 that the role of compliance, being the most critical, changed occasionally. Compliance under LC3 was assigned to be the objective function most frequently, whereas compliance under LC2 was chosen on only a few occasions. The algorithm was terminated when the compliances under LC2 and LC3 appeared to have the same values, as listed in Table 1. The table demonstrates that the maximum compliance was reduced from 85.39076 (TEWS) to 77.27681 (BCM), implying the effectiveness of the proposed BCM method.

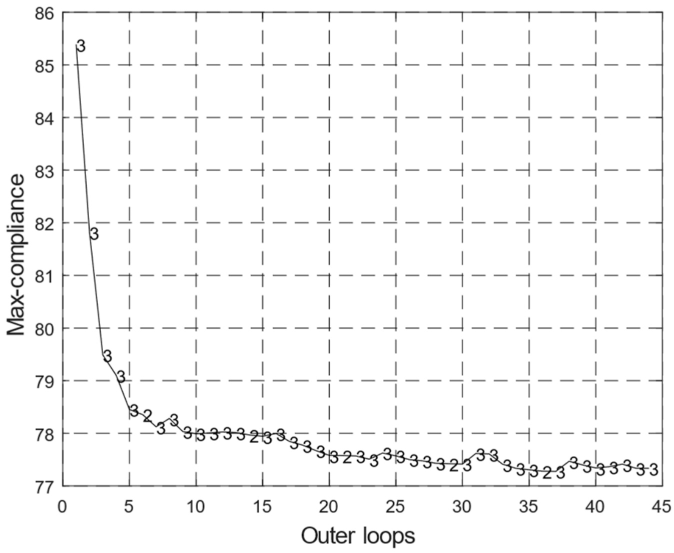


Figure 6. Reduction of the maximum compliance in Case Study 1 during Stage 2.

Table 1. Compliances using the traditional equally weighted sum (TEWS) method and bisection constraint method (BCM) in Case Study 1.

Method	c_1	c_2	c_3
TEWS	61.17191	52.23056	**85.39076**
BCM	76.93185	**77.27681**	77.27835

Bold font represents the maximum value.

4.2. Case Study 2

In this case study, $F1 = 1.0, F2 = 3, F3 = 2$ are given. The topology obtained from Stage 1 is shown in Figure 7a, which resulted from the TEWS method. Similar to the previous case study, all compliances were observed to reduce simultaneously in this stage.

Figure 7. Topologies in Case Study 2: (**a**) after Stage 1 (traditional equally weighted Sum method); (**b**) after Stage 2.

After Stage 2, a new topology was obtained, which is shown in Figure 7b. The appearance of the trusses directed towards the upper right corner helps bear the compression loads in LC2 better. At the same time, the arrangement of these trusses made the tie bars bearing LC3 disappear or become smaller, worsening the performance of the structure under LC3.

The evolution of compliances during Stage 2 is illustrated in Figure 8, where the compliance under LC2 was always selected as the objective function, explaining its importance compared to other loading conditions. The calculation results are listed in Table 2.

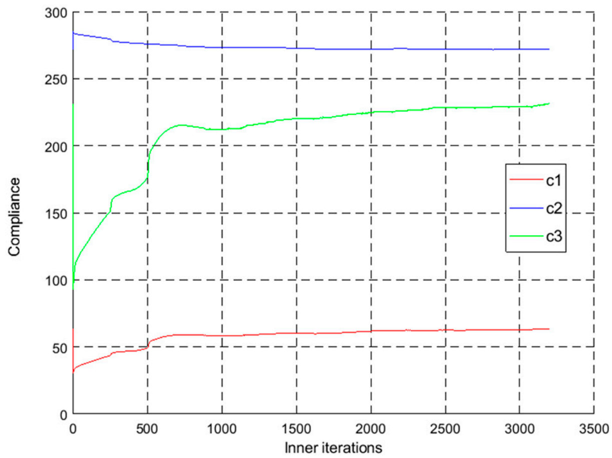


Figure 8. Evolution of compliances in Case Study 2 during Stage 2.

Table 2. Compliances using the traditional equally weighted sum (TEWS) method and bisection constraint method (BCM) in Case Study 2.

Method	c_1	c_2	c_3
TEWS	30.26961	**285.3740**	93.13086
BCM	63.27235	**271.6586**	231.3640

Bold font represents the maximum value.

Unlike the previous case study, all compliances are very different from each other in the final topology generated in Stage 2. This is because the terminal condition given in Equation (16) was activated before the two largest compliances coincided. However, the most critical compliance (c_2) was reduced from 285.374 (TEWS) to 271.6586 (BCM), whilst the compliances under LC1 and LC3 remained smaller. Hence, the effectiveness of the proposed method was demonstrated in this case study.

These two examples (Case Study 1 and Case Study 2) demonstrate that the solutions obtained using the TEWS method can be improved using the proposed method in terms of minimizing compliance despite the well-known TEWS method providing relatively good results. The proposed BCM method can be used for various TO problems considering multiple-loading conditions, including problems with different load correlations (load severities) between the given loading conditions. Thanks to the simple strategy of compliance constraining (relaxing) that allows for any possible reduction in the most critical compliance, the proposed method ensures the reliable yielding of the true solution of the min–max compliance formulation. Additionally, the effectiveness of convergence using two different terminal criteria is also demonstrated in these two examples, improving the stability of the proposed method.

5. Comparison of the Proposed Method with the Weighted Sum Methods

From Section 4, it is found that the proposed method yields good results by minimizing the maximum compliance when solving TO problems under multiple-loading conditions compared to the TEWS method. On the other hand, past research has considered that the proper optimization solution can be obtained for multiple objectives with different degrees of importance by changing the weight factors (WS methods, [29,34–42]). In this section, the features of the proposed method and its advantages are discussed by comparing the TO results between this method and the WS methods (including the TEWS method in Stage 1 of the proposed framework).

In this section, the aforementioned methods were used to solve the same TO problems. For better visualization using graphs, simple 2D problems with only two loading conditions were considered. During the optimization processes, the Von Mises (VM) stress in the structures was observed. Calculations were performed to determine the VM stress for each load case. The VM stress at the center of each element was computed by averaging the stress tensors at the Gauss points (integration points). For the FE analysis, we employed 2D square elements with four Gauss points because the design domains used in the examples were rectangular.

All the methods were formulated based on the SIMP approach and implemented in MATLAB. Similar to examples used in Section 4, the material was assumed to have Young's modulus $E = 1$ and Poisson's ratio $\nu = 0.3$. A volume fraction $f = 0.5$ was considered as a constraint. The derivative of the cost functions (compliance functions) is computed following Bendsøe and Sigmund [1].

To assess the performance of the obtained structures, a representative stress level was determined by calculating the average stress level of the 10 elements with the highest VM stress values observed throughout the structures. Subsequently, in this paper, the representative stress level is simply referred to as the stress level, indicating the performance of the structures. This approach effectively captures the stress level in a localized region where high stress occurs while avoiding reliance on outlier values.

When using the WS methods, all compliance functions can be aggregated into one objective function using Equation (6). For problems with two loading conditions, only two compliance functions associated with the two loading conditions are considered. Hence, the trade-off relationship can be represented using only one parameter: w_1 (the weight factor of the first load case). The weight factor of the remaining loading condition can be determined automatically as follows:

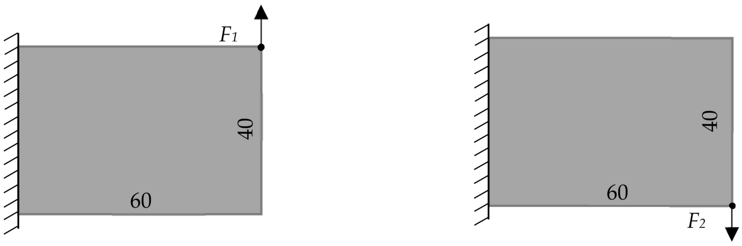
$$w_2 = 1 - w_1. \tag{18}$$

Additionally, in practice, it is common for loads to spread across the entire structure owing to interactions between adjacent elements. This implies that the loads are borne by the entire structure, and as a result, the majority of the elements within the structure contribute to its global stiffness to withstand the applied loads. In such cases, when the compliance sum of all elements is reduced, there is a possibility that the stress level will also reduce. However, in extreme scenarios in which only a limited region of the structure is affected by the applied loads, maintaining a small compliance may not effectively reduce the stress level throughout the structure (see also the discussion in Section 5.2). To evaluate the effectiveness of the proposed method in various cases, two scenarios for testing were considered: (1) the most common scenario, when loads are borne by the entire structure; and (2) the extreme scenario, when loads are borne by local regions of the structure.

5.1. Loads Are Borne by the Entire Structure

5.1.1. Problem 1

Consider a cantilever beam subjected to two simple loading conditions (concentrated loads), as shown in Figure 9. In this problem (Problem 1), we consider several load combinations (i.e., situations with different load correlations between the magnitudes of LC1 and LC2), in which the magnitude of LC2 remains the same in all load combinations, and the magnitude of LC1 is changed from small to large, representing the change in the importance of the two loading conditions. A mesh with 60 elements in the horizontal direction and 40 elements in the vertical direction was used in this problem.



(**a**) (**b**)

Figure 9. Loading conditions for Problem 1: (**a**) Load case 1; (**b**) Load case 2.

(a) When $F_1 = 0.2$ and $F_2 = 1.0$

For this load combination, the magnitude of loads in LC1 is significantly smaller than the magnitude of loads in LC2. In principle, the design should be strengthened to withstand the loads in LC2 more than the loads in LC1.

Again, it is noted that by using the TEWS method, where the weights of all conditions are the same ($w_1 = 0.5$ for the problems in Section 5), it is not possible to obtain a solution with minimized maximum compliance in most cases. The structures generated in Stage 1 (TEWS method) and Stage 2 are shown in Figure 10. A small change was made from Stage 1's structure to Stage 2's structure to strengthen it in the direction of withstanding the worst loading condition, which was LC2 in this load combination. By rearranging the material in the TEWS structure, the compliance under LC2 was reduced from 43.38 to 41.92 (Figure 11a) after applying the proposed BCM method.

(**a**) (**b**)

Figure 10. Structures obtained in Problem 1 with $F_1 = 0.2$ and $F_2 = 1.0$: (**a**) after Stage 1 (traditional equally weighted sum method); (**b**) after Stage 2.

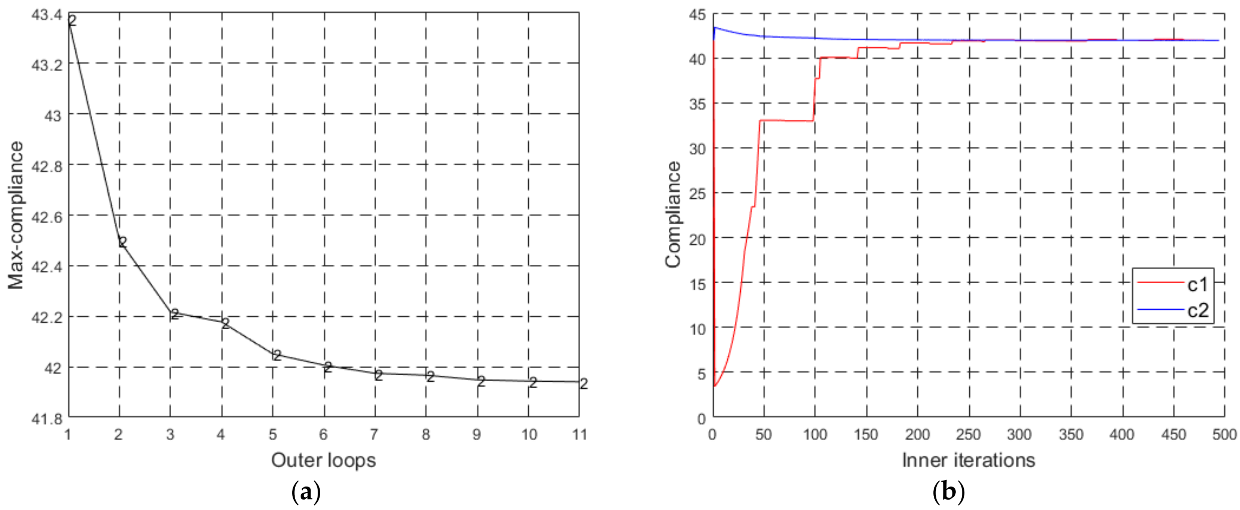


Figure 11. Evolution of compliances in Problem 1 with $F_1 = 0.2$ and $F_2 = 1.0$ during Stage 2: (**a**) the maximum compliance only, with the indices of the corresponding loading conditions; (**b**) all compliances under each loading condition.

As shown in Figure 11, the algorithm underwent 11 outer loops with approximately 500 inner iterations. During this process, compliance under LC2 was always treated as the objective function (the number shown on the graph in Figure 11a is the index of the maximum compliance occurred at every outer loop, namely, "2" for the compliance under LC2). Figure 11b also shows that a reduction in the compliance under LC2 (the most important loading condition) can be achieved when increasing the compliance under LC1. This is a good example of the trade-off relationship between different loading conditions.

When using the WS methods, by changing the weight (w_1), it is possible to find the weight where the compliances under LC1 and LC2 are the same (at the intersection of *c1_WS* and *c2_WS* solid curves in Figure 12a). The compliance at that point is the same as the maximum compliance in the structure obtained via the proposed method (denoted by *c1_BCM* and *c2_BCM* dotted lines in Figure 12a). In this case, the two dotted lines (*c1_BCM* and *c2_BCM*) overlap. It is noted that the advantage of the proposed method is that a solution with minimized maximum compliance can be obtained automatically, i.e., without any weighting of the importance of the load cases or other subjective manners.

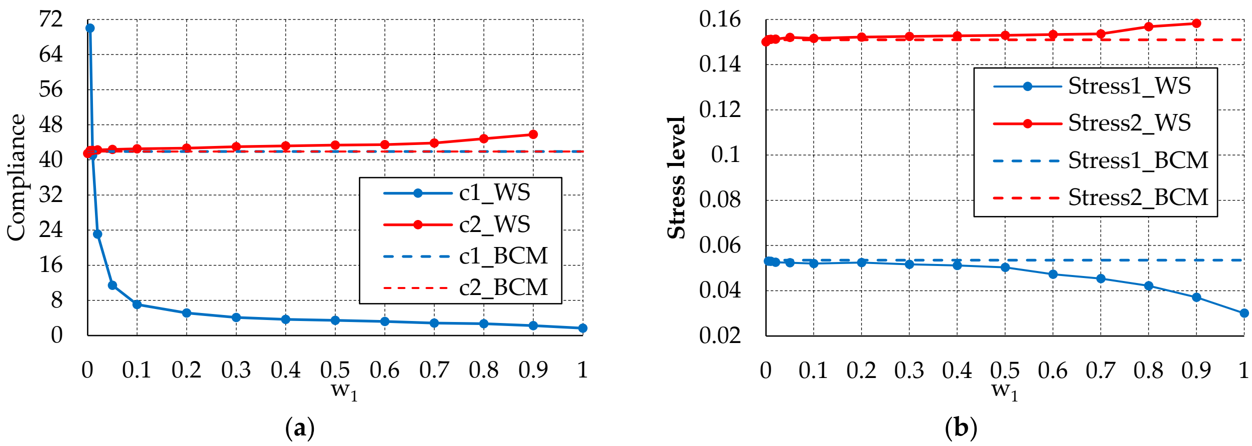


Figure 12. Structure performance using the weighted sum (WS) methods and the bisection constraint method (BCM) for Problem 1 with $F_1 = 0.2$ and $F_2 = 1.0$: (**a**) compliance; (**b**) stress level.

The stress levels in the structure considering the two loading conditions using the WS methods with different weights are plotted in Figure 12b (*Stress1_WS* and *Stress2_WS* solid curves). In this graph, the stress levels in the structure obtained using the proposed method can also be observed, as represented by the blue (*Stress1_BCM*) and red (*Stress2_BCM*) horizontal dotted lines. A comparison of the stress levels in structures obtained using the WS methods and the structure obtained using the proposed BCM method shows that the maximum stress level is minimized in the BCM structure (the *Stress2_BCM* line seems to intersect with the *Stress2_WS* curve near the lowest point of the *Stress2_WS* curve). This example illustrates that in this loading correlation, the proposed method can also provide a solution with a low stress level in addition to minimizing the maximum compliance.

Moreover, Figure 12 illustrates that in order to obtain a unique solution having whether minimized maximum compliance or minimized stress level, conventional WS method necessitates numerous trials with different weight combinations, whereas the proposed method requires only one single run.

(b) When $F_1 = 0.9$ and $F_2 = 1.0$

In this load combination, the magnitudes of loads in LC1 and LC2 are nearly the same; hence, the structure obtained using the TEWS method (Figure 13a) is nearly symmetric. The structure obtained via the BCM method (Figure 13b) has the same topology as the TEWS structure, except for the size of the ties inside the structure; i.e., the tie bearing LC2 became wider, and the tie bearing LC1 became thinner. Because of the size change, the compliance under LC2 was reduced from 46.96 (TEWS) to 44.43 (BCM) (see Table 3), slightly improving the performance of the structure.

(**a**) (**b**)

Figure 13. Structures in Problem 1 with $F_1 = 0.9$ and $F_2 = 1.0$: (**a**) using the traditional equally weighted sum method (Stage 1); (**b**) using the proposed method.

Table 3. Compliances and stress levels using the traditional equally weighted sum (TEWS) method and the bisection constraint method (BCM) for Problem 1 with $F_1 = 0.9$ and $F_2 = 1.0$.

Method	c_1	c_2	Stress Level 1	Stress Level 2
TEWS	40.36	46.96	0.1435	0.1642
BCM	44.42	44.43	0.1623	0.1514

The graph in Figure 14a shows that when changing the weights using the WS methods, it is also possible to find a design, where the compliance under LC1 is the same as the compliance under LC2 (at the intersection of *c1_WS* and *c2_WS* curves). The compliance value at that point is greater than the maximum compliance obtained via the BCM method (approximately 45.5 compared with 44.43). This indicates that the minimized maximum compliance may not be found by changing the weights in the WS methods.

The graph in Figure 14b illustrates that the structure with the lowest stress level can be obtained using the WS methods at around $w_1 \approx 0.34$ (the intersection of the *Stress1_WS* curve and the *Stress2_WS* curve, showing the same stress level when considering LC1 and LC2 individually), while the proposed BCM method resulted in a structure with not-so-different stress levels (*Stress1_BCM* and *Stress2_BCM*), which quite close to that intersection point. Note that in the proposed method, the structure can be obtained automatically and uniquely. This observation demonstrates the capability of the proposed method in terms of reducing the stress level in this load combination.

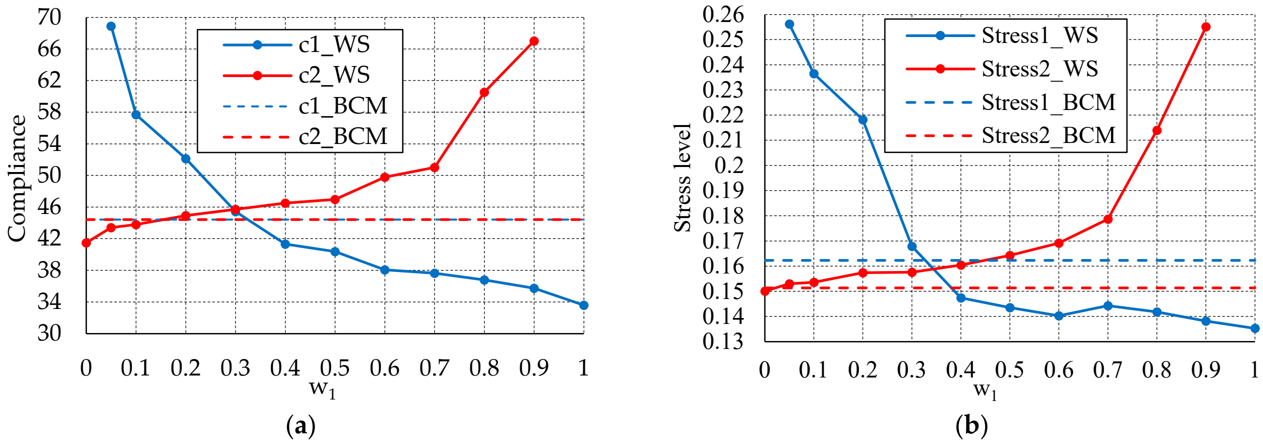


Figure 14. Structure performance using the weighted sum (WS) methods and the bisection constraint method (BCM) for Problem 1 with $F_1 = 0.9$ and $F_2 = 1.0$: (**a**) compliance; (**b**) stress level.

The same conclusion can be drawn from Figure 14 regarding the single run required when using the proposed method compared to the numerous trials with different combinations of weights using the WS method to obtain a unique solution for the considered TO problem under multiple-loading conditions. This emphasizes the distinct ability of the proposed method to automatically generate solutions for designated problems.

(c) When $F_1 = 1.0$ and $F_2 = 1.0$

This is a special load combination, referred to as symmetric loading conditions, for which it is expected that the final topology should be symmetric. In fact, in the proposed framework, the optimization algorithm stops after Stage 1, which consists of TO using the TEWS method. This is because after Stage 1, a symmetric structure is obtained (Figure 15) with the same compliance when applying either LC1 or LC2. Thus, there is no greater compliance to minimize.

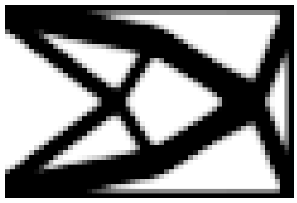


Figure 15. Structure obtained using the traditional equally weighted sum (TEWS) method (Stage 1) and the bisection constraint method (BCM) for Problem 1 with $F_1 = 1.0$ and $F_2 = 1.0$ (same result).

The graph in Figure 16a shows that a structure with the same compliance values under LC1 and LC2 obtained via WS methods can be found at the intersection of the two compliance curves (*c1* and *c2* curves) at $w_1 = 0.5$, which is the TEWS method. As previously mentioned, this structure is the same as the one obtained using the proposed framework. By adopting the TEWS method in Stage 1, the proposed method is applicable for this symmetric load scenario.

The stress levels in the structures using the WS methods with different weights (w_1) and using the BCM method can be observed in Figure 16b. This graph illustrates that a structure with minimized stress levels can be found uniquely using the WS methods at $w_1 = 0.5$, which is the same point as in the TEWS method (or Stage 1 of the proposed framework). Moreover, this point is the same as that of the design with minimized compliance as regarding Figure 16a. Hence, in this case, both the TEWS and BCM methods yielded a true and unique solution.

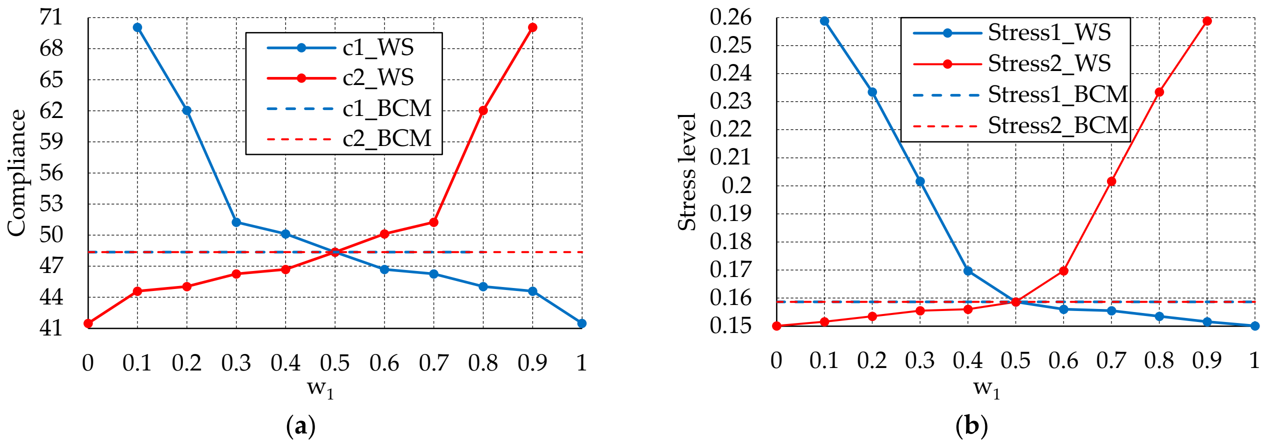


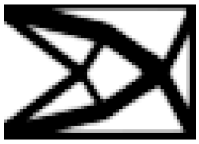
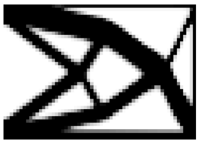
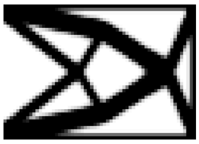
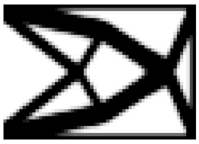
Figure 16. Structure performance using the weighted sum (WS) methods and the bisection constraint method (BCM) for Problem 1 with $F_1 = 1.0$ and $F_2 = 1.0$: (**a**) compliance; (**b**) stress level.

The results for Problem 1 are summarized in Table 4 to provide a broad view of this problem. In this table, the results of Stage 1 (the TEWS method) and Stage 2 of the proposed framework are presented, including the obtained topologies, compliances under LC1 and LC2, and stress levels under LC1 and LC2. By showing the results of the two stages, an improvement of compliance (maximum compliance being reduced) can be seen from Stage 1 to Stage 2, and the change in stress levels can also be observed. For example, when $F_1 = 0.5$ and $F_2 = 1.0$, the compliance under LC1 and LC2 in the TEWS structure is 15.64 and 45.07, respectively. After Stage 2, the compliances under the two loading conditions became the same and equal to 42.43, which means the maximum compliance was reduced from 45.07 (TEWS) to 42.43 (BCM) (an approximately 6% reduction). At the same time, the stress levels in the TEWS structure under LC1 and LC2 are 0.118 and 0.158, respectively. After Stage 2, these stress levels were 0.135 and 0.152, respectively. These results show the effectiveness of the proposed method not only in reducing the maximum compliance but also the stress level, considering both loading conditions.

Table 4. Calculation results for Problem 1 using the traditional equally weighted sum (TEWS) method and the bisection constraint method (BCM).

Load Combination	Method	Topology	c_1	c_2	Stress Level 1	Stress Level 2
$F_1 = 0.2$, $F_2 = 1.0$	TEWS (Stage 1)		3.44	**43.37**	0.050	**0.153**
	BCM (Stage 2)		**41.94**	**41.94**	0.053	**0.151**

Table 4. *Cont.*

Load Combination	Method	Topology	c_1	c_2	Stress Level 1	Stress Level 2
$F_1 = 0.5, F_2 = 1.0$	TEWS (Stage 1)		15.64	**45.07**	0.118	**0.158**
	BCM (Stage 2)		**42.43**	**42.43**	0.135	**0.152**
$F_1 = 0.9, F_2 = 1.0$	TEWS (Stage 1)		40.36	**46.96**	0.143	**0.164**
	BCM (Stage 2)		44.42	**44.43**	**0.162**	0.151
$F_1 = 1.0, F_2 = 1.0$	TEWS (Stage 1)		**48.37**	**48.37**	**0.159**	**0.159**
	BCM (Stage 2)		**48.37**	**48.37**	**0.159**	**0.159**

Bold font represents the maximum value(s)

Overall, in most cases of Problem 1, the proposed BCM method was more effective in finding structures with better performance (low stress level) than the TEWS method. In addition, by integrating the TEWS method in Stage 1, the proposed framework can result in a symmetric topology in a special loading scenario, that is, symmetric loading conditions.

5.1.2. Problem 2

Consider a cantilever structure with two distributed loading conditions, as shown in Figure 17. This problem differs from Problem 1 in the sense that the load type used is uniformly distributed instead of concentrated. The trend of applying the loads in the two load cases is also different; that is, LC2 is not a bending load but an axial load.

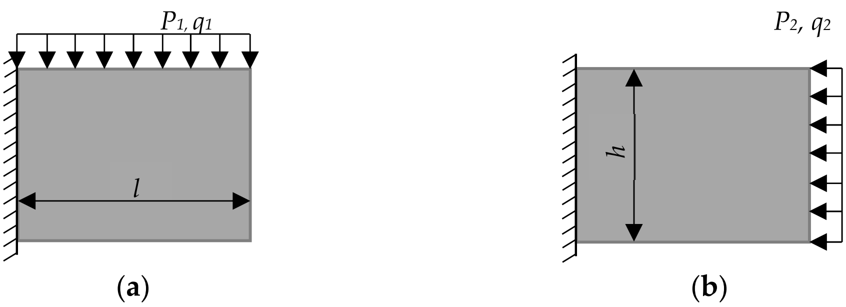


Figure 17. Loading conditions for Problem 2: (**a**) Load case 1; (**b**) Load case 2.

The magnitude (P_1) of the loads in LC1 is defined as the sum of the distributed loads applied on the upper edge of the structure in LC1, and it can be calculated as follows:

$$P_1 = q_1 \cdot l, \tag{19}$$

Similarly, the magnitude (P_2) of the loads in LC2 is

$$P_2 = q_2 \cdot h, \tag{20}$$

where l and h denote the length and height of the considered structure, respectively.

Like in the previous example, different load combinations were considered, in which the correlation between the two loading conditions is gradually changed in this example.

(a) When $P_1 = 0.5$ and $P_2 = 9.5$

This situation is characterized by disproportionately different force magnitudes under the two loading conditions. Such a load combination is highly uncommon in reality, and when considering this scenario in a TO context, it often results in unreasonable structures, e.g., the gray regions in the structures obtained for this particular load combination (Figure 18). Some researchers have addressed this issue using a load increment to intensify "ill-loading" conditions, thereby aiming to obtain a more reasonable topology [35]. Alternative approaches are available to subjectively intervene in the obtained structures to make them more realistic. These include the so-called fine-tuning process [67–69] (also known as refining, post-treatment, and post-processing), which involves modifying the structure to align it with the desired specifications. Typically, during the fine-tune process, a certain small amount of material is added to the structures to turn "gray" regions into "black" regions. At the same time, the process can also reduce stress concentration.

(**a**) (**b**)

Figure 18. Obtained structures for Problem 2 when $P_1 = 0.5$ and $P_2 = 9.5$: (**a**) using the traditional equally weighted sum method; (**b**) using the bisection constraint method.

In this article, we only consider different original load correlations and assume that in certain cases, it might be necessary to perform the "fine-tuning" process to the obtained structures to render them more realistic. In addition, it is straightforward to fine-tune such structures to enhance their ability to withstand the ill-loading conditions.

Figure 18 illustrates the structures obtained using the TEWS and BCM methods. The two main bars in the BCM structure moved closer to each other and toward the horizontal central axis of the structure to withstand LC2 better. Thus, the BCM structure has advantages when bearing the two loading conditions compared with the structure obtained using the TEWS method, as its maximum compliance is lower (328.7 vs. 339.1) and the maximum stress level is also lower (0.16400 vs. 0.16844) (see Table 5). Both structures in Figure 18 necessitate a fine-tuning process because they both contain unreasonable regions (regions with gray elements) to bear the loads in LC1.

Figure 19a shows that by using the WS methods, it is possible to find a structure with the lowest maximum compliance at weight w_1, which is very close to 0, at the intersection of the two compliance curves *c1_WS* and *c2_WS*. The compliance in the structure obtained using the proposed method is slightly lower than the compliance at the intersection point, emphasizing the effectiveness of the proposed method in determining the minimized maximum compliance.

Table 5. Compliances and stress levels using the traditional equally weighted sum (TEWS) method and the bisection constraint method (BCM) for Problem 2 when $P_1 = 0.5$ and $P_2 = 9.5$.

Method	c_1	c_2	Stress Level 1	Stress Level 2
TEWS	18.005	339.063	0.08307	0.16844
BCM	328.697	328.702	0.11578	0.16400

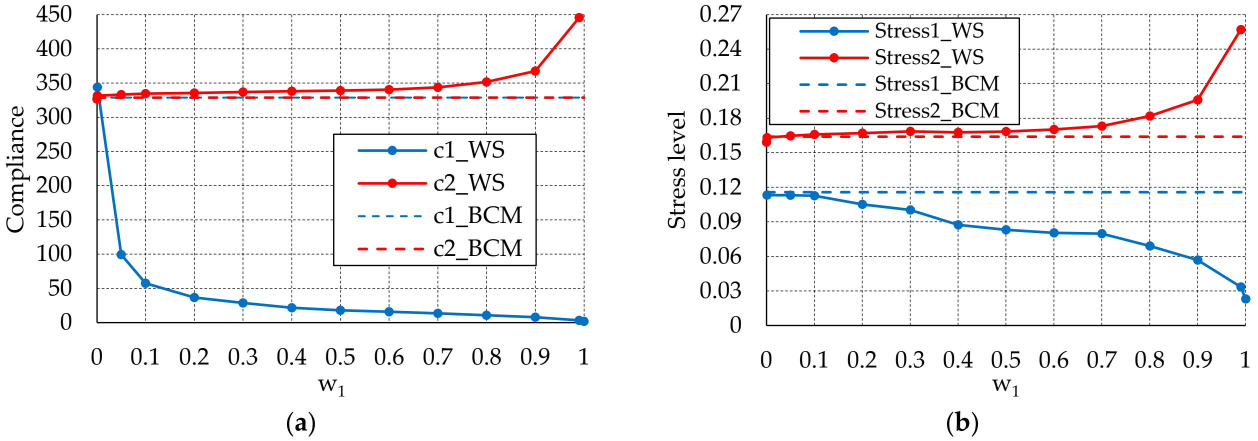


Figure 19. Structure performance using the weighted sum (WS) method and the bisection constraint method (BCM) for Problem 2 when $P_1 = 0.5$ and $P_2 = 9.5$: (**a**) compliance; (**b**) stress level.

As shown in Figure 19b, in this case, the two stress curves of the WS methods (*Stress1_WS* and *Stress2_WS* curves) do not intersect with each other. Therefore, the best structure can be roughly estimated at the bottom of the *Stress1_WS* curve, which is close to the point where $w_1 = 0$. It is not difficult to realize that the BCM structure possesses satisfactory properties as an optimal design, where the maximum stress level when considering both loading conditions is approximately at the minimum value (see Figure 19b). It should again be noted that the structure can be automatically obtained using the BCM method.

(b) When $P_1 = 1.0$ and $P_2 = 9.0$

In this load combination, we encounter the same issue regarding the realism of the obtained structures when using both the TEWS and BCM methods, requiring post-optimization fine-tuning (Figure 20). The material was rearranged from the TEWS structure to the BCM structure; hence, its compliance was reduced from 316.71 to 300.23, as shown in Table 6.

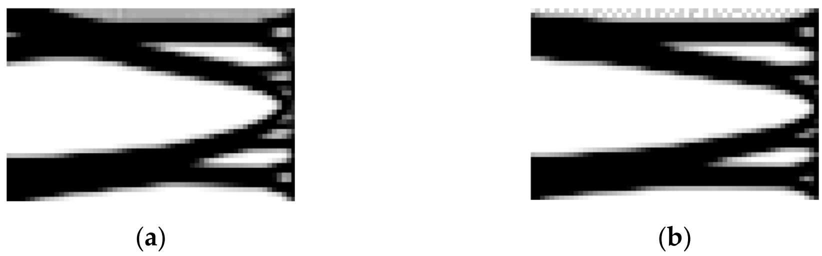


Figure 20. Structures obtained in Problem 2 when $P_1 = 1.0$ and $P_2 = 9.0$: (**a**) using the traditional equally weighted sum method; (**b**) using the bisection constraint method.

Table 6. Compliances and stress levels using the traditional equally weighted sum (TEWS) method and the bisection constraint method (BCM) for Problem 2 when $P_1 = 1.0$ and $P_2 = 9.0$.

Method	c_1	c_2	Stress Level 1	Stress Level 2
TEWS	41.03794	316.71307	0.13203	0.17372
BCM	300.23040	300.23161	0.21627	0.15325

The graph in Figure 21a shows that the structure with minimized maximum compliance using the WS methods is at some weight w_1 between 0.01 and 0.05. That compliance value is almost the same as the compliance of the structure obtained using the BCM method (*c1_BCM* and *c2_BCM* lines seem to pass through the intersection between *c1_WS* and *c2_WS* curves). This emphasizes the effectiveness of the proposed method in determining the minimized maximum compliance.

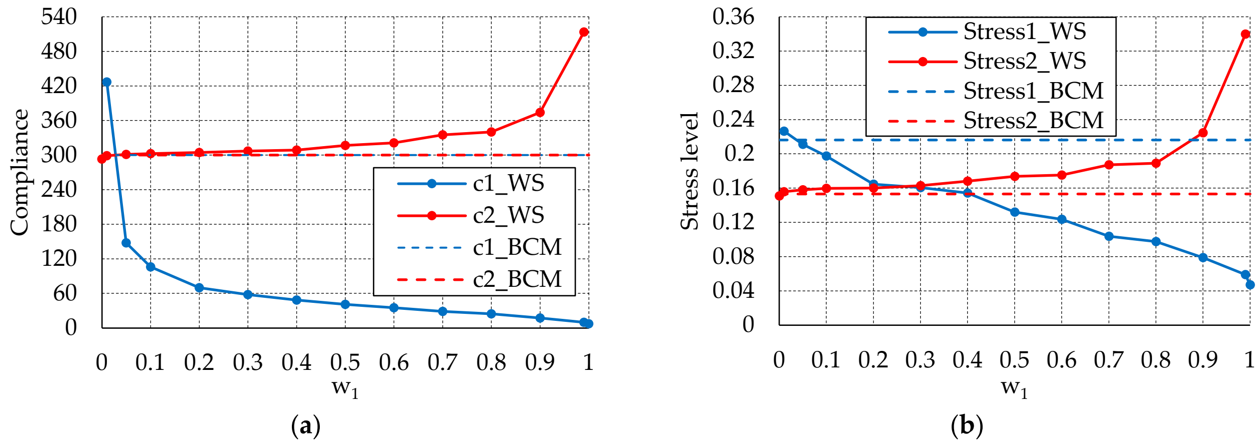


Figure 21. Structure performance using the weighted sum (WS) methods and the bisection constraint method (BCM) for Problem 2 when $P_1 = 1.0$ $P_2 = 9.0$: (**a**) compliance; (**b**) stress level.

Figure 21b illustrates the stress levels in the structures obtained via the WS methods with different weight combinations and the structure obtained via the proposed BCM method. The graph shows that the structure with the lowest stress level can be obtained using the WS methods at the weight $w_1 \approx 0.27$, where the stress levels are expected to be around 0.16 (the intersection of the *Stress1_WS* and *Stress2_WS* curves). Note that when using the WS methods, the weight combination must be assigned such that the stress level is the same in all loading cases, which is generally challenging, especially when there are more than two loading conditions. Additionally, the stress level in the structure obtained via the BCM method when applying LC2 is lower than the stress level at the intersection point mentioned above (the red dotted line is lower than the intersection between *Stress1_WS* and *Stress2_WS* curves in Figure 21b). However, the stress level under LC1 in this structure is significantly higher (0.21627 compared with 0.15325, Table 6). In this load combination, LC1 is generally less severe than LC2 due to the significantly smaller load magnitude.

As explained in Section 3, the BCM method focuses on improving the performance of the structure considering both loading conditions in terms of minimizing compliance. Therefore, after Stage 1 (the TEWS method), the compliance under LC2 reduced, which was the more severe loading condition. Hence, the stress level under LC2 also reduced (from 0.17372 to 0.15325, (Table 6)). Simultaneously, the stress level under LC1 increased (from 0.13203 to 0.21627, (Table 6)), even when the compliance under LC1 did not exceed that under LC2.

This situation necessitates examining the distribution of the Von Mises stress in the structures obtained using the TEWS and BCM methods. As shown in Figure 22, the VM stress distributions in both the TEWS and BCM structures under LC2 are nearly uniform. However, the VM stress distributions in the two structures under LC1 are uneven. At the same time, the maximum VM stress under LC1 in the BCM structure occurs near the left upper corner (circled regions in Figure 22b). These are local regions where it is still possible to strengthen the structure to improve the withstanding of LC1; i.e., there is still space to add material around the stress concentration areas. Note that LC1 is an ill-loading condition in this case, and it is straightforward to fine-tune the structures to better withstand ill-loading conditions. Based on the above discussion, the BCM method can be utilized in this case; however, the obtained structure must be fine-tuned.

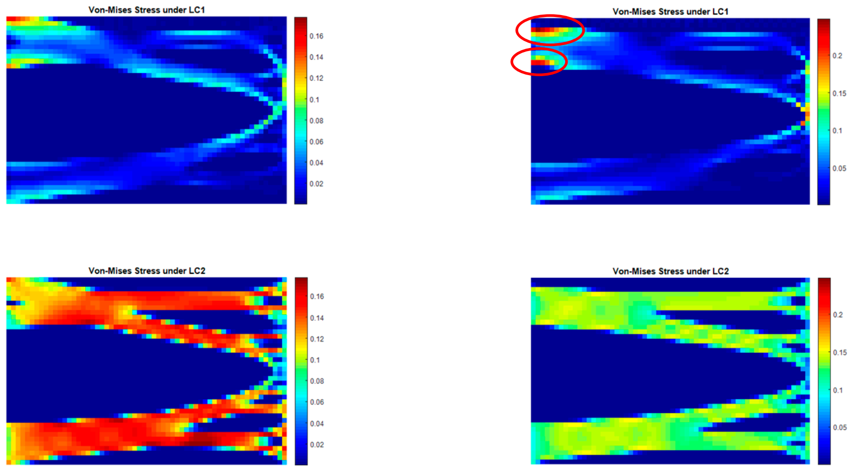


Figure 22. Von Mises stress distribution in structures obtained in Problem 2 when $P_1 = 1.0$ and $P_2 = 9.0$: (**a**) using the traditional equally weighted sum method; (**b**) using the bisection constraint method. (Red circles indicate regions with high stress concentration).

The calculation results for other load combinations are listed in Table 7. The compliances and the stress levels under LC1 and LC2 in the structures obtained via the TEWS and BCM methods are shown in this table. The cases marked in red are scenarios in which the stress level appears to be higher in the structures obtained via the BCM method (Stage 2) than in the structures obtained via the TEWS method (Stage 1). However, the key point is that the stress level under the more severe load case in the BCM structure is lower than that in the TEWS structure. The maximum stress level occurs when applying the less severe load case and can be easily addressed by fine-tuning the structures, which is essential due to the appearance of unreasonable regions in the structure, represented by elements with intermediate densities.

Table 7. Calculation results for Problem 2 using the traditional equally weighted sum (TEWS) method and the bisection constraint method (BCM).

Load Combination	Method	Topology	c_1	c_2	Stress Level 1	Stress Level 2
$P_1 = 0.5, P_2 = 9.5$	TEWS (Stage 1)		18.00	**339.06**	0.083	**0.168**
	BCM (Stage 2)		**328.70**	**328.70**	0.116	**0.164**
$P_1 = 1.0, P_2 = 9.0$	TEWS (Stage 1)		41.04	**316.71**	0.132	**0.173**
	BCM (Stage 2)		**300.23**	**300.23**	**0.216**	0.153

Table 7. *Cont.*

Load Combination	Method	Topology	c_1	c_2	Stress Level 1	Stress Level 2
$P_1 = 3.0, P_2 = 7.0$	TEWS (Stage 1)		129.94	**235.84**	**0.223**	0.190
	BCM (Stage 2)		196.70	**196.70**	**0.291**	0.143
$P_1 = 5.0, P_2 = 5.0$	TEWS (Stage 1)		**254.30**	154.19	**0.299**	0.183
	BCM (Stage 2)		**206.89**	206.841	0.262	**0.274**
$P_1 = 6.0, P_2 = 4.0$	TEWS (Stage 1)		**341.27**	121.18	**0.344**	0.178
	BCM (Stage 2)		**285.97**	285.76	0.302	**0.337**
$P_1 = 7.0, P_2 = 3.0$	TEWS (Stage 1)		**448.23**	91.66	**0.399**	0.180
	BCM (Stage 2)		379.85	**379.86**	**0.350**	0.321
$P_1 = 8.0, P_2 = 2.0$	TEWS (Stage 1)		**546.24**	77.81	**0.424**	0.187
	BCM (Stage 2)		**481.98**	**481.98**	**0.391**	0.184

Bold font represents the maximum value(s).

It is also noted that in Problem 2, the stress level may be higher under the loading condition possessing lower compliance compared with the loading condition possessing higher compliance. For example, when $P_1 = 3.0$ and $P_2 = 7.0$, the compliance in the structure obtained via the TEWS method under LC1 is much lower than that under LC2 (129.94 compared with 235.84). However, the stress level under LC1 is higher than that under LC2 (0.223 compared with 0.190) according to Table 7. Stage 2 helped reduce the compliance under LC2. At the same time, the compliance under LC1 increased, worsening the stress level under this loading condition; i.e., the stress level became much higher in

LC1 than in LC2 (0.291 vs. 0.143). This situation shows that controlling the compliance does not always help control the stress level arising in the structure. As previously discussed, this high stress level occurs in local regions where a fine-tuning operation can be applied to reduce the stress concentration phenomenon.

To recapitulate Problem 2, in most load combinations, the proposed BCM method is more advantageous than the TEWS method; i.e., not only is the maximum compliance minimized; the stress level is also reduced after Stage 2. However, in certain cases, the stress level under the less severe loading condition becomes higher than that under the most critical loading condition, including cases marked in red in Table 7. In addition, some load combinations result in designs with gray regions, which require fine-tuning of the obtained structures. During the fine-tune process, local regions with stress concentrations can be modified. Thus, the stress level (under the less severe loading condition) can be reduced. By utilizing post-optimization fine-tuning operation, the proposed method can be used to minimize the maximum compliance first, and then the local stress concentration can be reduced after the fine-tuning process, obtaining the final design.

5.2. Loads Are Borne by a Local Region of the Structure

To represent the loads borne by a local region of the structure, we consider a 2D structure within a rectangular design domain with support on the lower edge, and loads are applied on the upper edge. In this section, two problems with the same design domain and support are considered; however, different types of loads are applied to the structure.

5.2.1. Problem 3

In this problem, two loading conditions with concentrated loads (LC1 and LC2) are applied to the upper-left and upper-right corners, as shown in Figure 23. Similar to the previous problems, the correlation between the two loading conditions is gradually changed by changing the magnitudes of the forces P_1 and P_2.

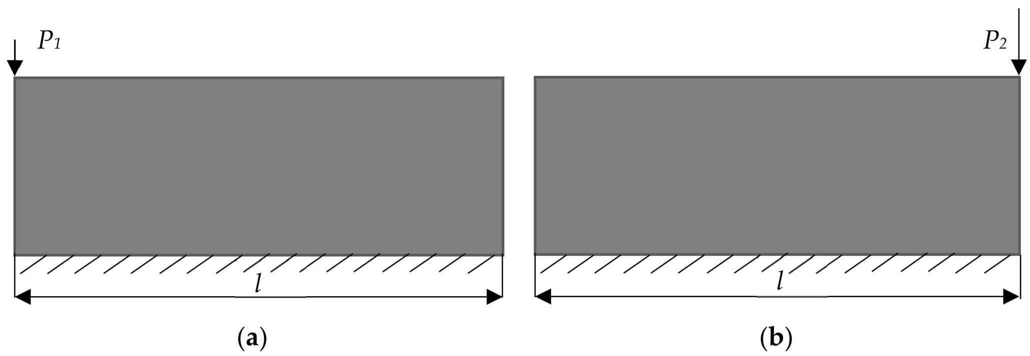


Figure 23. Loading conditions for Problem 3: (**a**) Load case 1; (**b**) Load case 2.

Computational results for Problem 3 are provided in Table 8, where c_1 and c_2 are the compliances under LC1 and LC2, respectively, and Stress level 1 and Stress level 2 represent the stress levels under LC1 and LC2, respectively.

The results given in Table 8 show that for all load combinations of this problem, the maximum compliance is always reduced after applying the BCM method (Stage 2) to the TEWS structures (Stage 1). This demonstrates the efficiency of the proposed method in terms of minimizing the maximum compliance. In addition, the stress levels of the BCM structures (Stage 2) are always lower than those of the TEWS structures (Stage 1). More importantly, the maximum stress levels arising under the worst load case in the BCM structures are lower than those in the TEWS structures, implying the effectiveness of the proposed method.

Table 8. Calculation results for Problem 3 using the traditional equally weighted sum (TEWS) method and the bisection constraint method (BCM).

Load Combination	Method	Topology	c_1	c_2	Stress Level 1	Stress Level 2
$P_1 = 2.0$, $P_2 = 8.0$	TEWS		49.56	**662.47**	0.285	**1.119**
	BCM		635.91	**654.71**	0.614	**1.118**
$P_1 = 3.0$, $P_2 = 7.0$	TEWS		104.04	**511.08**	0.423	**0.9789**
	BCM		503.04	503.04	0.853	**0.9787**
$P_1 = 5.0$, $P_2 = 5.0$	TEWS		271.911	271.911	0.7007	0.7007
	BCM		271.911	271.911	0.7007	0.7007
$P_1 = 8.0$, $P_2 = 2.0$	TEWS		**662.47**	49.56	**1.119**	0.285
	BCM		**654.71**	635.91	**1.118**	0.614
$P_1 = 9.0$, $P_2 = 1.0$	TEWS		**833.06**	14.75	**1.2584**	0.1626
	BCM		**827.99**	806.55	**1.2583**	0.3133

Bold font represents the maximum value(s).

One important observation in Problem 3 is that whenever the load magnitudes are different, the stress level under the more severe loading condition is consistently much higher than that under the less severe loading condition, no matter which method is used (e.g., Figure 24b). Consequently, it is challenging to find a structure with the same stress level under the two loading conditions. For instance, in the case of $P_1 = 3.0$ and $P_2 = 7.0$, despite the load magnitudes in the two loading conditions not differing significantly, the stress level under the more severe load case (i.e., LC2) is significantly higher than the stress level under the less severe load case (i.e., LC1). In addition, Figure 24b shows that the stress level under LC2 is nearly the same for most weight factors (w_1) in the WS methods (from $w_1 = 0$ to $w_1 = 0.7$). Most of the time, the compliance under LC2 is also significantly higher than that under LC1, and the two compliances can only be the same at a very small w_1 in the WS methods (see Figure 24a). On the other hand, the BCM method generated a structure in which not only was the compliance the same in both loading conditions (503.04 (see Figure 24a and Table 8)); the stress level under LC2 was also minimized while keeping the stress level under LC1 small.

In Problem 3, it is also noticed that the stress level is reduced by minimizing compliance, even though the loads are borne by local regions of the structure. In other words, when concentrated loads are applied to the structure, it is possible to control the stress level by controlling the compliance of the structures.

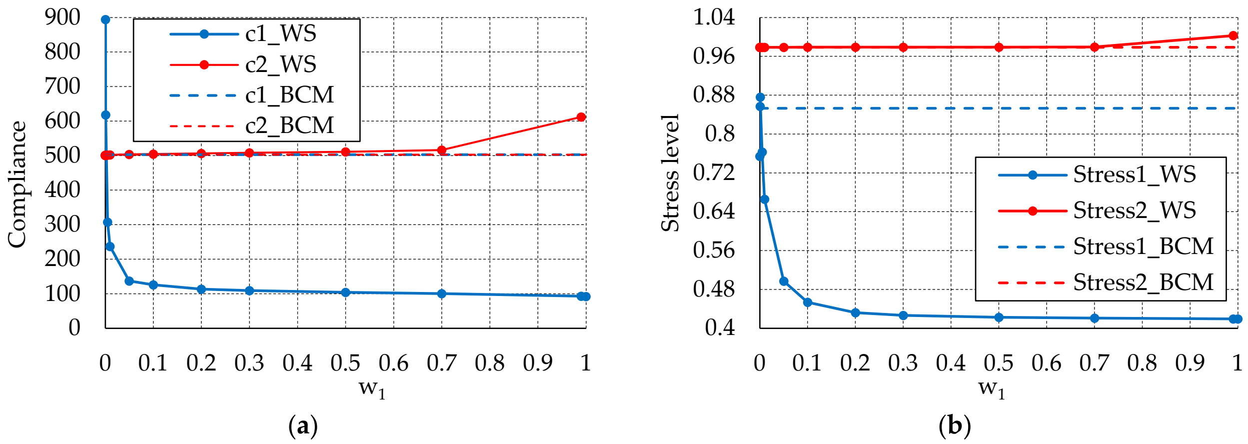


Figure 24. Structure performance using the weighted sum (WS) methods and the bisection constraint method (BCM) for Problem 3 when $P_1 = 3.0$ and $P_2 = 7.0$: (**a**) compliance; (**b**) stress level.

5.2.2. Problem 4

Problem 4 considered two loading conditions with distributed loads (LC1 and LC2), as illustrated in Figure 25, where the load magnitudes are gradually changed by changing P_1 and P_2.

In this problem,

$$P_1 = q_1 \cdot l/4, \tag{21}$$

and

$$P_2 = q_2 \cdot l/4. \tag{22}$$

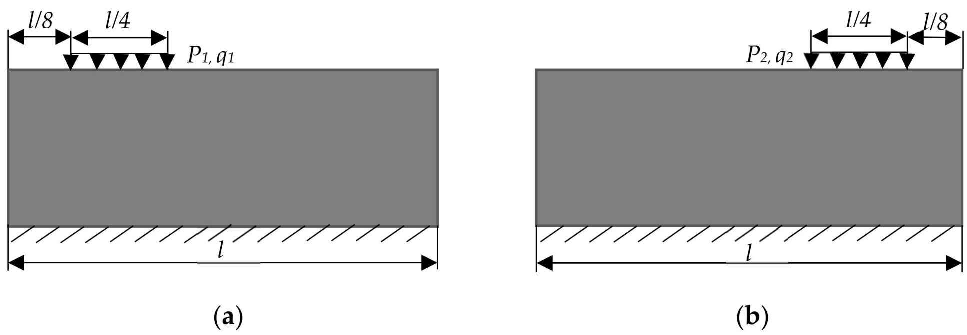


Figure 25. Loading conditions for Problem 4: (**a**) Load case 1; (**b**) Load case 2.

The computational results for Problem 4 are listed in Table 9 with compliances c_1 and c_2 under LC1 and LC2, respectively. Stress Level 1 and Stress Level 2 are stress levels under LC1 and LC2, respectively. From the results given in Table 9, it can be seen that in Problem 4, the maximum compliance was always reduced from Stage 1 (TEWS method) to Stage 2 (BCM), because minimizing maximum compliance is the objective of the proposed method. For example, for $P_1 = 2.0$ and $P_2 = 8.0$, the maximum compliance reduced from 65.029 (under LC2) to 59.568 (under LC1) after applying the BCM method on the TEWS structure (see Table 9).

In Problem 4, the stress level under more severe loading conditions was always reduced after Stage 2 of the proposed framework, along with a reduction in the maximum compliance. For example (see Table 9), when $P_1 = 2.0$ and $P_2 = 8.0$, the stress level under LC2 reduced from 0.06162 (TEWS) to 0.05960 (BCM); when $P_1 = 3.0$ and $P_2 = 7.0$, the stress level under LC2 reduced from 0.05567 (TEWS) to 0.05380 (BCM); and when $P_1 = 9.0$ and $P_2 = 1.0$, the stress level under LC1 reduced from 0.06717 (TEWS) to 0.06630 (BCM). This illustrates that controlling the compliance can help control the stress level under the same loading conditions.

Table 9. Calculation results for Problem 4 using the traditional equally weighted sum (TEWS) method and the bisection constraint method (BCM).

Load Combination	Method	Topology	c_1	c_2	Stress Level 1	Stress Level 2
$P_1 = 2.0$, $P_2 = 8.0$	TEWS		11.425	**65.029**	0.04171	**0.06162**
	BCM		58.655	**59.568**	**0.22641**	0.05960
$P_1 = 3.0$, $P_2 = 7.0$	TEWS		18.469	**53.520**	0.04483	**0.05567**
	BCM		**46.853**	46.844	**0.13768**	0.05380
$P_1 = 4.0$, $P_2 = 6.0$	TEWS		25.433	**43.472**	0.04648	**0.04942**
	BCM		**37.698**	37.665	**0.08742**	0.04704
$P_1 = 5.0$, $P_2 = 5.0$	TEWS		33.863	**33.863**	0.04952	**0.04952**
	BCM		33.863	**33.863**	0.04952	**0.04952**
$P_1 = 5.5$, $P_2 = 4.5$	TEWS		**38.400**	29.768	**0.048445**	0.04744
	BCM		**34.779**	34.776	0.04519	**0.06684**
$P_1 = 9.0$, $P_2 = 1.0$	TEWS		**77.637**	5.200	**0.06717**	0.03589
	BCM		**75.793**	70.078	0.06630	**0.18600**

Bold font represents the maximum value(s).

One important observation in Problem 4 is that not only are the loads borne by the local regions; the load types in both loading conditions are distributed loads. Under the effect of distributed loads, the structure is divided into two column-like separate parts, as shown in Table 9. This table also shows that the stress levels in structures obtained via the TEWS method under LC1 and LC2 appear to be not so different for most load combinations (e.g., 0.04648 compared to 0.04942 for $P_1 = 4.0$ and $P_2 = 6.0$; or 0.04845 compared to 0.04744 for $P_1 = 5.5$ and $P_2 = 4.5$). At the same time, the compliances of the TEWS structures are approximately proportional to the load magnitudes applied to the structure (e.g., $c_2/c_1 = 43.472/25.433$ is approximately proportional to $P_2/P_1 = 6/4$ (see Table 9)). When pursuing the minimization of the maximum compliance (e.g., using the BCM method), the stress level under the less severe loading condition increases conversely. For example, for $P_1 = 3.0$ and $P_2 = 7.0$, the stress level under LC1 increased from 0.04483 (TEWS) to 0.13768 (BCM), even though the maximum compliance was reduced from 53.520 to 46.853. This observation implies that when the load/response is completely divided into two regions, the TEWS method generally provides better results, and the application of the BCM method becomes irrelevant from the perspective of the stress level. Additionally, in most load combinations, the BCM structures contain gray regions and sometimes even checkerboard patterns (see Table 9). Therefore, a post-optimization fine-tune operation is essential in order to obtain satisfactory results.

It must be emphasized that the scenarios presented in Problem 3 and Problem 4 are highly uncommon because, in reality, loads are usually borne by the entire structure. Nevertheless, the proposed method can be employed in such scenarios with caution, considering less severe loading conditions after the design is obtained. In addition, these two problems show that the type of load might also play a key role in the effectiveness of the compliance approach in TO, i.e., in the case of concentrated loads, controlling compliance may directly help control the stress level in the structure, whereas in the case of distributed loads, stress concentration might occur when controlling compliance. In any case, a fine-tuning operation after the optimization process may be necessary when gray regions exist in the structure. During the fine-tune process, the stress concentration phenomenon can be effectively addressed, particularly the stress caused by less severe loading conditions.

6. Conclusions

This paper proposes a new method named the bisection constraint method for handling TO problems with multiple-loading conditions. In addition, a framework that includes the TEWS method is introduced to generate the starting point for the BCM method. The use of the TEWS method in Stage 1 of the proposed framework helps to find a starting point that might be relatively close to the true solution, thereby increasing the possibility of convergence at the true solution.

The proposed method offers automatic generations of a unique solution for every TO problem considering multiple-loading conditions, which sufficiently minimizes the maximum compliance amidst all given loading conditions. By gradually minimizing the maximum compliance and sectioning the gaps between the maximum compliance and the remaining ones, the proposed method can provide a unique and sufficient solution for the min–max compliance formulation of TO problems with respect to multiple-load cases, as evidenced by the numerical examples in this study. In contrast, conventional methods such as the WS method often require users to provide more information based on their preferences for the given loading conditions, and consequently, the results tend to reflect user preferences. WS methods also become impractical for visualizing stress levels using graphs with various weight combinations when dealing with a large number of loading conditions (i.e., more than three). This limitation makes it challenging to achieve a comprehensive overview and identify solutions with the lowest stress levels. Additionally, attempting numerous weight combinations requires significant effort to solve these problems using the WS method.

In addition, when tracking the stress levels of structures obtained using different methods, calculation results also showcase the effectiveness of the proposed method in providing solutions with low stress levels for common scenarios, although the stress level is not the object of interest in the proposed method. In particular, when the loads are applied to common regions among different load cases, the reduction of the compliance under the most severe load case leads to a reduction in the maximum stress. Therefore, the BCM method also works effectively in terms of the stress level.

Despite many advantages and promises in solving problems with multiple load cases, this BCM method still has a number of disadvantages, including the high computational cost owing to many nested loops. Furthermore, based on observation of the examples, it appears that when the loads are applied to geometrically different regions among different load cases, the TEWS method might give better results in general. This is because when the objective function is the sum of all compliances under each load case, it can capture all the load cases' impacts on different regions of the structure. The use of the proposed method for these cases must be considered with caution and with the post-optimization fine-tuning operation.

Through the examination of numerous simple problems in this paper, we have gained insights into both the advantages and limitations of the proposed method. This provides an important basis from which to apply the proposed method to practical engineering problems in our future studies in our narrow field, namely, marine engineering.

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