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Fluid Flow Development in a Pipe as a Demonstration of a Sequential Change in Its Rheological Properties

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Abstract: The sequence of the process of changing the velocity profiles and the laws of resistance during the flow of a fluid in a pipe is considered. With the increasing of the Reynolds number, we obtain the transition of the flow regime from laminar to turbulent. In the presence of small additives of polymers, when the Toms effect is observed in the fluid flow, the turbulent regime changes with a further increase in the Reynolds number to another regime, the rheology of which leads to laminar velocity profiles and corresponding resistance laws. Then, with an increase in the Reynolds number for polymer solutions, the limiting Virk flow regime with its own rheology is reached. All the mentioned flow regimes and all types of rheology can be described using one rheological relation, which is a power-law generalization of Newton's formula, by changing the values of the power value in this ratio upon reaching the corresponding critical Reynolds numbers. This generalization can be extended to the spatial case of flow and the rheological relation can be represented in tensor form with a further system of differential equations for a fluid flow with an arbitrary rheology. After that, boundary value problems in fluid mechanics can be solved for any fluid flow regime.

Keywords: rheology; non-Newtonian fluids; power relation; resistance; Newton's formula; Blasius formula; Toms effect

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1. Introduction

In hydrodynamics, the description of a fluid's behavior depends on its flow regime [1,2]. At relatively low Reynolds numbers, a laminar flow regime occurs, described by the Navier–Stokes equations, which are based on Newton's rheological relation for a linearly viscous fluid. Then, with an increase in the flow velocity and at high Reynolds numbers, the rheology of the moving fluid changes: a turbulent flow regime is reached. There is no conventional uniform rheological relation for this flow regime, although work in this direction has been carried out since the times of Reynolds. Currently, modern turbulence models are based on the concept of turbulent viscosity, which has developed since the first theory of turbulence (L. Prandtl's mixing length theory), up to the present time in the form of modern differential models of turbulence such as the $k-\epsilon$ and $k-\omega$ models, as well as other models [3]. In addition, with an increase in the Reynolds number, the turbulent flow regime may change to another regime when small additives of polymers are added to the fluid flow, leading to the demonstration of the Toms effect. In this case, a "laminar-type" flow regime arises with its own rheological ratio [4,5]. Then, with an increase in the Reynolds number, the Virk flow reaches the limit regime [6] with its rheology.

All the mentioned flow regimes and all types of rheology can be approximately described using one rheological relation, which is a power-law generalization of Newton's formula for the flow of a viscous fluid, changing the power values in this ratio. Newton's formula for the shear stress in a longitudinal flow past a flat surface with the velocity u, depending on the y transverse coordinate is as follows:

$$\tau = \rho \nu \frac{du}{dy} \tag{1}$$

where ρ is the density, ν is the kinematic viscosity, and the formula can be generalized [7] to the following form

$$\tau = \rho \chi_n \left(\nu \frac{du^{2n-1}}{dy} \right)^{\frac{1}{n}} \tag{2}$$

In this expression χ_n is the non-dimensional coefficient, depending on the nth power value ($n \geq 1$). For $n = 1$ and when $\chi_n = 1$, Equation (2) leads to Newton's rheological relation and, as a result, to Poiseuille's formulas for laminar flow in a pipe. For $n = 4$ and when $\chi_n = 0.019746$, this formula leads to the rheological relation for turbulent fluid flow in a pipe and then to the Blasius formula for the resistance coefficient. Equation (2) can be generalized to the spatial case of flow and the corresponding rheological relation can be presented in tensor form [8]. As a result, a system of differential equations can be obtained, similar to the system of Navier–Stokes equations, which also allows for the solution of boundary value problems in fluid mechanics [7], even in relation to the turbulent flow regime.

2. Steady-State Flow in a Circular Cylindrical Pipe at an Arbitrary Value of the *n* Power

A circular pipe is the most common hydrodynamic object used for tests of theoretical solutions, since a huge amount of experimental material has been accumulated for the flow within this object, which makes it possible to evaluate the quality of the formulas obtained for resistance coefficients and velocity profiles. For the steady-state flow in a straight circular cylindrical pipe with radius R, y is the coordinate measured from the wall ($0 \leq y \leq R$). Let us introduce non-dimensional variables η (non-dimensional coordinate) and V (non-dimensional velocity) as follows:

$$\begin{cases} \eta = \frac{y}{R}, \ 0 \leq \eta \leq 1 \\ V = \frac{u}{V_*} \end{cases} \tag{3}$$

where V_* is the friction velocity, expressed in terms of shear stress τ_w on the wall:

$$|\tau_w| = \frac{d}{4} \frac{\Delta p}{l}, \ V_* = \sqrt{|\tau_w|/\rho} \tag{4}$$

where $d = 2R$ is the pipe diameter and Δp is the longitudinal pressure drop along the pipe of l length.

Then, considering notation (3), the formula (2) for the flow in a pipe in non-dimensional form can be represented as follows:

$$\tau = \rho \chi_n \left[\nu \frac{{V_*}^{2n-1}}{R} \frac{dV^{2n-1}}{d\eta} \right]^{\frac{1}{n}} = \rho \chi_n V_*^2 \left[\frac{\nu}{V_* R} \frac{dV^{2n-1}}{d\eta} \right]^{\frac{1}{n}} \tag{5}$$

This non-negative expression can also be associated with the non-negative value of the shear stress $\tau = \tau_w (1 - y/R)$ from the motion equation of the continuous medium in stresses, i.e., $\rho V_*^2 (1 - \eta)$:

$$\rho \chi_n V_*^2 \left(\frac{\nu}{V_* R} \frac{dV^{2n-1}}{d\eta} \right)^{\frac{1}{n}} = \rho V_*^2 (1 - \eta) \tag{6}$$

Hence follows the differential equation:

$$\chi_n \left(\frac{1}{Re_*} \frac{dV^{2n-1}}{d\eta} \right)^{\frac{1}{n}} = (1 - \eta)$$

where $Re_* = V_* R/\nu$ is the Reynolds number, calculated from the friction velocity. Then:

$$\frac{dV^{2n-1}}{d\eta} = \frac{Re_*}{\chi_n^n}(1-\eta)^n \quad (7)$$

The boundary condition for this equation is the no-slip condition: $\eta = 0$, $V = 0$.

The integration of Equation (7) with this boundary condition leads to the following expression:

$$V^{2n-1} = \frac{Re_*}{\chi_n^n(n+1)}\left[1-(1-\eta)^{n+1}\right]$$

from which the non-dimensional velocity profile takes the form:

$$V = \left(\frac{Re_*}{(n+1)\chi_n^n}\right)^{\frac{1}{2n-1}}\left[1-(1-\eta)^{n+1}\right]^{\frac{1}{2n-1}} \quad (8)$$

The non-dimensional velocity averaged over the pipe's cross section is calculated using the formula [9,10]:

$$V_{av} = 2\int_0^1 V(\eta)(1-\eta)d\eta$$

Substitution of expression (8) for the velocity profile into this formula leads to the expression:

$$V_{av} = 2\left(\frac{Re_*}{(n+1)\chi_n^n}\right)^{\frac{1}{2n-1}} Y(n) \quad (9)$$

where $Y(n)$ has the following form for a flow in the pipe:

$$Y(n) = \int_0^1 \left[1-(1-\eta)^{n+1}\right]^{\frac{1}{2n-1}}(1-\eta)d\eta \quad (10)$$

This parameter is expressed through hyper-geometric functions; its values for different power values are shown in Table 1.

Table 1. $Y(n)$ values.

n	1	2	3	4	5	20
$Y(n)$	0.25	0.403067	0.447761	0.467138	0.477358	0.498156

For other n values, $Y(n)$ can be found in reference materials. It is useful to note that when $n \to \infty$, $Y(n) \to 0.5$. The true Reynolds number $Re = 2RV_{av}/\nu$, according to the average velocity V_{av}, can be calculated [2] in terms of the Re_* number using friction velocity:

$$Re = 2Re_* V_{av}$$

Then, considering Formula (9), the following can be written:

$$Re = 4Y(n)\left(\frac{1}{(n+1)\chi_n^n}\right)^{\frac{2n}{2n-1}} Re_*^{\frac{2n}{2n-1}} \quad (11)$$

Hence, if expressing Re_* inversely in terms of Re, one obtains

$$Re_* = \frac{((n+1)\chi_n^n)^{\frac{1}{2n}}}{2n-1} Re^{\frac{2n-1}{2n}} \quad (12)$$

The resistance coefficient $\lambda = 8|\tau_w|/\left(\rho V_{av}^2\right)$ can be represented in terms of the square of the non-dimensional average velocity as $\lambda = 8/V_{av}^2$, which, when considering Equation (9), leads to a formula for this coefficient at an arbitrary n value:

$$\lambda = 2^{\frac{n+2}{n}} \frac{(n+1)^{\frac{1}{n}}}{Y(n)^{\frac{2n-1}{n}}} \frac{\chi_n}{Re^{\frac{1}{n}}} \tag{13}$$

Formulas (9)–(13) make it possible to describe the velocity field and resistance coefficient for a flow in the pipe for any values of the nth power.

3. Fluid Flow Development in the Pipe with Increasing Reynolds Number

As the Reynolds number increases on the resistance curve, there is a transition in the values of resistance coefficient from one region to another due to a change in rheology. Figure 1 shows three schemes for the change in the resistance coefficient λ for three different cases of flow description implementations corresponding to different values of the nth power and the χ_n non-dimensional coefficient, respectively. These diagrams indicate: 1—Hagen–Poiseuille curve $n = 1$, $\chi_n = 1$; 2—Blasius curve, $n = 4$, $\chi_n = 0.019746$; 3—curve for $n = 6$, $\chi_n = 0.00910904$, 4—curve for $n = 1000$, $\chi_n = 0.02/m^{1/3}$, where m is the roughness parameter, equal to the ratio of the pipe radius R to the reduced height of the roughness bumps k ($m = R/k$). Figure 1a shows the transition from a laminar flow regime to a turbulent one, Figure 1b shows this transition to the Prandl–Nikuradze curve, which is approximated by curves 2 and 3 with different values of the nth power; and Figure 1c shows the transition of curve 2 to curve 4 with a horizontal region with its own roughness value.

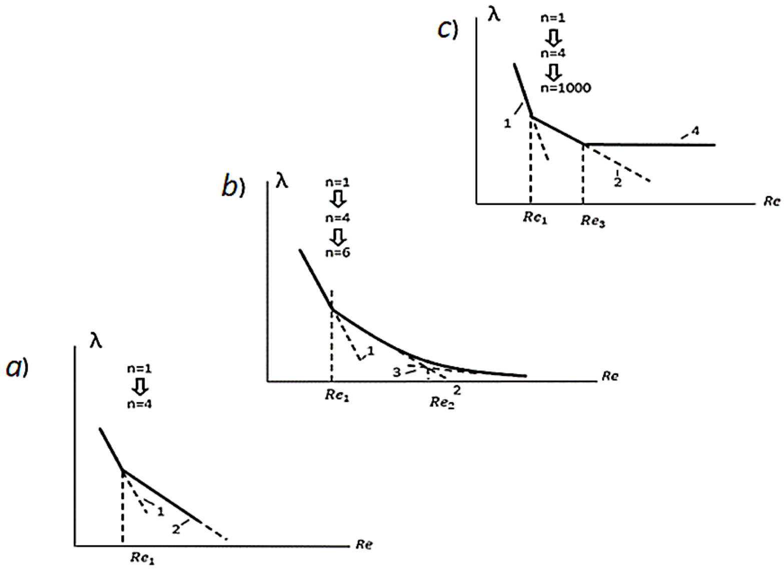

Figure 1. Resistance curves for different cases of flow of the nth power: (**a**) shows the transition from a laminar flow regime to a turbulent one; (**b**) shows this transition to the Prandl–Nikuradze curve; (**c**) shows the transition of curve 2 to curve 4 with a horizontal region.

The transition of the laminar flow regime with resistance $\lambda = 64/Re$ to turbulent one, when the Blasius formula takes place $\lambda = 0.3164/Re^{0.25}$ is shown in Figure 1a. From curve 1 corresponding to the $n = 1$ and $\chi_n = 1$, there is a transition to the Blasius curve 2, which corresponds to $n = 4$ and $\chi_n = 0.019746$. The velocity profile (9) takes the following form for $n = 1$:

$$V = \frac{Re_*}{2}\left[1 - (1-\eta)^2\right] \tag{14}$$

and it becomes a turbulent profile for $n = 4$:

$$V = \left(\frac{Re_*}{5\chi_n{}^4}\right)^{\frac{1}{7}} \left[1 - (1-\eta)^5\right]^{\frac{1}{7}} \quad (15)$$

In dimensional form, taking into account the notation (3) after the transition to cylindrical coordinates, when $\eta = y/r = (R-r)/R = 1 - r/R$, Formula (14) takes the form:

$$u = \frac{1}{4\mu}\left(-\frac{dp}{dz}\right)\left(R^2 - r^2\right) \quad (16)$$

and then (15) takes the form given below:

$$u = 0.93677\frac{1}{\nu^{1/7}}\left(\frac{1}{\rho}\frac{\Delta p}{l}\right)^{\frac{4}{7}}\left(R^5 - r^5\right)^{\frac{1}{7}} \quad (17)$$

The Reynolds number (Re_1 value) of the transition originating from the laminar regime to the turbulent one can be calculated as the result of the intersection of the Hagen–Poiseuille and Blasius curves:

$$\frac{64}{Re_1} = \frac{0.3164}{Re_1{}^{0.25}}$$

from which $Re_1 = 1187.4$. This value turns out to be less than 2300, indicated in the references [1,2], which is due to the disregard of the transition region from the laminar regime to the turbulent one. If we assume that this region exists, then it would be advisable to accept that $Re_1 = 2300$ and understand that there is a gap in the resistance curve at this value of the Reynolds number. Thus, for the case of the flow shown in the diagram in Figure 1a, one obtains:

$$\begin{array}{l} \mathrm{Re} < 2300, n = 1, \chi_n = 1 \\ \mathrm{Re} > 2300, n = 4, \chi = 0.019746 \end{array} \quad (18)$$

We shall not consider this case further.

Figure 1b shows a resistance curve, complying with the Prandl–Nikuradze formula:

$$\lambda = 0.0032 + \frac{0.221}{Re^{0.237}} \quad (19)$$

which is in good agreement with experimental data [9,11], and is approximated by two curves corresponding to $n = 4$ and $n = 6$. For $n = 4$ and $\chi_n = 0.019746$, the Blasius formula $\lambda = 0.3164/Re^{0.25}$ takes place, for $n = 6$ and $\chi_n = 0.00910904$ the resistance curve, according to the (13), corresponds to:

$$\lambda = \frac{0.1156}{Re^{1/6}} \quad (20)$$

The intersection point of these curves corresponds to the Reynolds number, which can be found according to the relation:

$$\frac{0.3164}{Re_2{}^{1/4}} = \frac{0.1156}{Re_2{}^{1/6}}$$

from which we obtain $Re_2 = 176{,}743.6$.

The velocity profile, corresponding to the $n = 4$, complies with expression (15); for $n = 6$ it takes the following form, according to formula (8):

$$V = 10.868756\ Re^{\frac{1}{11}}\left[1 - (1-\eta)^7\right]^{\frac{1}{11}} \quad (21)$$

Figure 1c shows resistance curves for the flow in rough pipes with different values of the roughness parameter $m = R/k$. For a flow in rough pipes, one can take $n = 1000$, which gives an almost horizontal line in coordinates as follows: (Logarithm of Reynolds number − logarithm of resistance coefficient), i.e., as $(log_{10}(Re) - log_{10}(100\lambda))$.The rheological power relation (2) can be presented as follows for $n = 1000$:

$$\tau = \rho\chi_n \left[\nu \frac{du^{1999}}{dy}\right]^{\frac{1}{1000}}, \quad \chi_n = \frac{0.02}{m^{1/3}} \tag{22}$$

Hence, the velocity profile in non-dimensional coordinates, according to (8), can be presented in the form:

$$V = \left(\frac{Re_*}{101{\chi_n}^{1000}}\right)^{\frac{1}{1999}} \left[1 - (1-\eta)^{1005}\right]^{\frac{1}{1999}} \approx 1/\sqrt{\chi_n}\left[1 - (1-\eta)^{1001}\right]^{\frac{1}{1999}} \tag{23}$$

The average non-dimensional velocity value is

$$V_{av} \approx \frac{2\,Y(100)}{\sqrt{\chi_n}} = \frac{2}{\sqrt{\chi_n}} * 0.5 = 1/\sqrt{\chi_n}$$

taking into account the expression for χ_n, it takes the following form:

$$V_{av} = \frac{m^{1/6}}{\sqrt{0.02}} \tag{24}$$

For $m = 15; 60; 507$ values of V_{av} equal to 11.6, 14.4, and 20.2 can be obtained, which is in good agreement with experimental data [10]. The connection between Reynolds numbers for average and dynamic velocities for flow in rough pipes is as follows:

$$Re = 2Re_* V_{av} = \frac{2Re_*}{\sqrt{\chi_n}} \tag{25}$$

The resistance coefficient, according to (13), is equal to $\lambda = 8\chi_n/Re^{0.001}$, and can be rounded to the approximate value in this case:

$$\lambda \cong \frac{0.16}{m^{1/3}} \tag{26}$$

which is in satisfactory agreement with the experimental data.

The region of a hydraulically smooth pipe in Figure 1c transitions to the area with a constant value $\lambda = Const$, depending on the value of the roughness parameter m. The corresponding Reynolds number Re_3 is found as a result of the intersection of the corresponding resistance curves:

$$\frac{0.3164}{Re_3^{1/2}} = \frac{0.16}{m^{1/3}},$$

which leads to

$$Re_3 = 15.292\, m^{4/3} \tag{27}$$

The Reynolds number Re_3 values are presented in Table 2.

Table 2. Re_3 values.

m	60	100	507
Re_3	3592.0	7097.9	61,822.0
$log_{10}(Re_3)$	3.56	3.85	5.79

Resistance curves $\lambda = Const = 0.16/m^{1/3}$, when the quadratic law of resistance is satisfied, may have different positions according to each other. As a result, as the flow develops in rough pipes, and the nth power and χ_n coefficient change:

$$\begin{array}{l} if\ Re < Re_1,\ then\ n = 1000\ \ and\ \ \chi_n = 0.02/m^{1/3} \\ if\ Re_1 \leq Re \leq Re_3,\ then\ \ n = 1000\ \ and\ \chi_n = 0.02/m^{1/3} \\ if\ Re > Re_3,\ then\ n = 1000\ \ and\ \ \chi_n = 0.02/m^{1/3} \end{array}$$

Thus, with an increase in the values of the Reynolds number, the nth power changes, which leads to changes in the velocity profiles and resistance coefficients according to formulas (8) and (13), respectively.

4. The Flow of Weakly Concentrated Aqueous Solutions of Polymers

In the flow of weakly concentrated aqueous solutions of polymers, in which the Toms effect is observed [4,5], the resistance curves have the typical form shown in Figure 2.

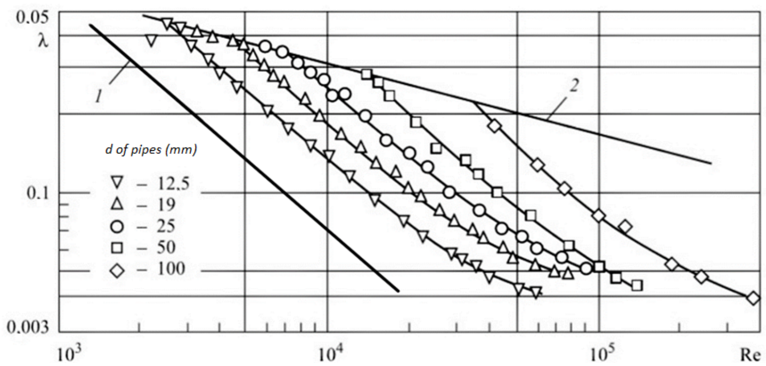

Figure 2. Change in resistance coefficients depending on the Reynolds number and pipe diameter at a constant concentration of WSR-301 (from [4]) polymer solution: 1—laminar flow; 2—turbulent flow.

It can be seen in Figure 2 that the Toms effect is associated with the deviation of the resistance curve from the Prandtl–Nikuradze law with access to an area, which is equidistant to the resistance curve $\lambda = 64/Re$ for the laminar flow regime.The case arising with the resistance curve can be schematically represented in Figure 3, which shows the "laminar-type" regions of the resistance curves during the flow of polymer solutions, where the Toms effect is observed.

The resistance curve for the flow of polymer solutions, immediately after deviating from the resistance law for a hydraulically smooth pipe, becomes similar to the resistance curve $\lambda = 64/Re$ (laminar regime). Therefore, this resistance law for flows of polymer solutions in the region after deviation from the resistance law for a hydraulically smooth pipe can be represented [12] in the form:

$$\lambda = \frac{A}{Re} \tag{28}$$

where A takes its value depending on the kind of polymer, its concentration, and the radius of the pipe. According to Figure 3, it is possible to estimate the values of this constant for the flow in pipes of different diameters at a constant volumetric concentration of the polymer solution $C = 15 \times 10^{-6}$ (Table 3).

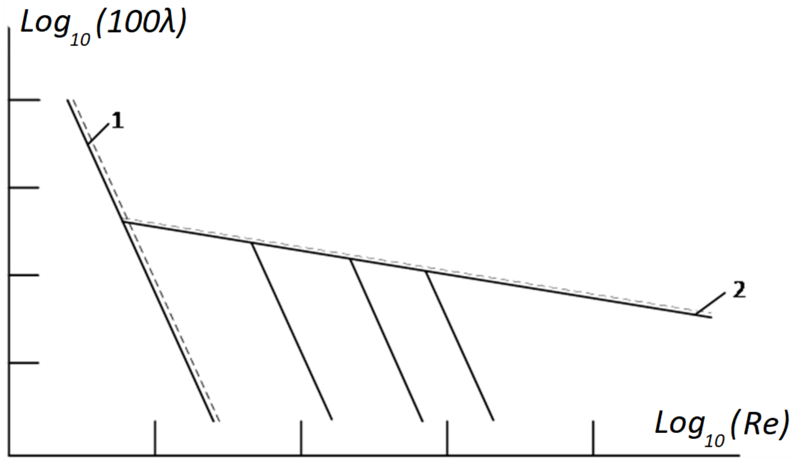

Figure 3. Deviation of the resistance curve from curve 2 for the turbulent regime into regions equidistant to curve 1 with the resistance law $\lambda = 64/Re$, corresponding to the laminar flow regime.

Table 3. The values of the "A" coefficient depending on the pipe diameter for a WSR-301 polymer solution concentration $C = 15 \times 10^{-6}$ according to Figure 3.

Experiment Number	d, mm	A
1	12.5	130
2	19	190
3	25	230
4	50	400
5	100	710

Then, with an increase in the Reynolds number, there is a deviation from the "laminar-type" resistance curve with a tendency to reach the Virk limit curve [11]. However, before beginning of the deviation from the resistance law for a hydraulically smooth pipe, the turbulent flow behaves in accordance with the flow laws of ordinary viscous Newtonian fluids (curve 2 in Figure 2).

With the development of the flow of weakly concentrated aqueous solutions of polymers, the resistance curve consists of several regions, shown schematically in Figure 4:

- The laminar a–1 region (beginning of the flow), which complies with the formula $\lambda = 64/Re$;
- The turbulent 1–2 region with an increase in the flow velocity, which complies with the Blasius formula $\lambda = 0.3164/Re^{0.25}$;
- The "laminar-type" 2–3 region with a further increase in speed, which complies with the formula $\lambda = A/Re$;
- The limit 3–b region, which is represented by the Virk curve [6] for the final stage of the flow, which complies with the formula $\lambda = 0.87/Re^{0.5}$.

The sequence of the process of the resistance change during the flow of a fluid with additives of polymers in a pipe is considered in detail as follows:

1. In the $Re < Re_1$ region, there is a laminar flow regime. Disregarding the transition region from the laminar regime to the turbulent one, we can approximately set the value as $Re_1 = 1187.2$;
2. In the 1–2 region, where $Re_1 \leq Re \leq Re_2$, a turbulent flow regime develops in a hydraulically smooth pipe. The Reynolds number Re_2 of the end of this flow regime is determined as the result of the intersection of the corresponding resistance curves for the Blasius law and the "laminar-type" resistance law: $\frac{0.3164}{Re_2^{0.25}} = \frac{A}{Re_2}$, and then

$$Re_2 = \left(\frac{A}{0.1364}\right)^{\frac{4}{3}};$$

3. In the "laminar-type" 2–3 region, where $Re_2 \leq Re \leq Re_3$, there is a deviation from the resistance curve of a hydraulically smooth pipe to the resistance curve for a polymer solution flow with a resistance law A/Re. The value of the "A" constant determines the type of polymer, its concentration in a solution and the pipe's diameter [4,5]. The Re_3 value can be found according to the intersection conditions of the "laminar-type" curve A/Re with the Virk limit curve $0.87/\sqrt{Re}$, i.e., $\frac{A}{Re} = \frac{0.87}{\sqrt{Re_3}}$, and then

$$Re_3 = \left(\frac{A}{0.87}\right)^2,$$

for different "A" parameter values; the values of Re_2 and Re_3 are given in Table 4;
4. With a further increase in the Reynolds number when $Re > Re_3$, there is a transition to the Virk resistance curve, complying with the law $0.87/\sqrt{Re}$.

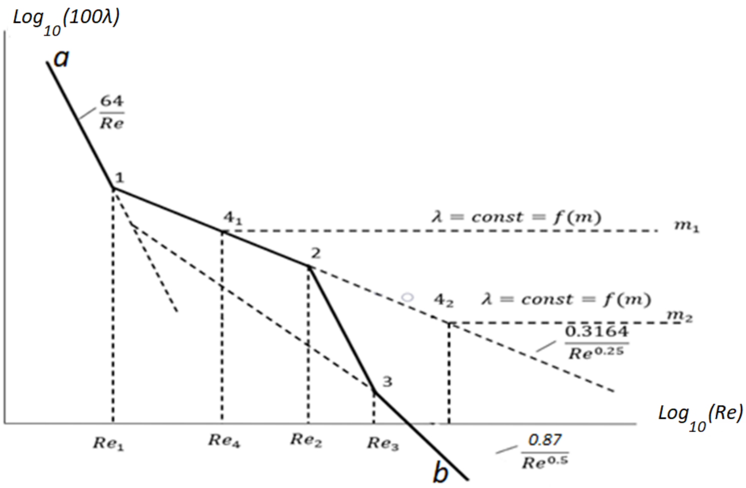

Figure 4. Resistance change scheme for fluid flow in a pipe.

Table 4. Values of Re_2 and Re_3 for different A parameters.

A	130	400	400
Re_2	3054.5	13,669.9	29,378.5
Re_3	22,311	211,230	665,508

Thus, with the development of the flow of weakly concentrated aqueous solutions of polymers, each region has its own values of the nth power, non-dimensional parameters χ_n and $Y(n)$:

1. In region a–1, when $Re < Re_1$, there are $n = 1$, $\chi_n = 1$, $Y(1) = 0.25$
2. In region 1–2, when $Re_1 \leq Re < Re_2$, there are $n = 4$, $\chi_n = 0.019746$, $Y(4) = 0.467138$
3. In region 2–3, when $Re_2 \leq Re \leq Re_3$, there are $n = 1$, $\chi_n = A/64$, $Y(n) = 0.25$
4. In region 3–b, when $Re > Re_3$, there are $n = 2$, $\chi_n = 0.032146$, $Y(n) = 0.403067$

In accordance with these values, for each region expressions for the resistance coefficients can be written according to (13), velocity profiles can be written in terms of universal coordinates according to (3), and Reynolds numbers can be written in terms of friction velocity. These expressions for each region are as follows:

1. Region a–1:

$$V = \frac{Re_*}{2}\left[1-(1-\eta)^2\right]; \quad Re_* = 1.41Re^{1/2}, \quad \lambda = 64/Re \tag{29}$$

2. Region 1–2:

$$\begin{gathered} V = \left(\frac{Re_*}{5\chi_n{}^4}\right)^{\frac{1}{7}}\left[1-(1-\eta)^5\right]^{\frac{1}{7}} = 7.484\ Re_*^{\frac{1}{7}}\left[1-(1-\eta)^5\right]^{\frac{1}{7}} \\ Re_* = 0.099436\ Re^{7/8}; \quad \lambda = 0.3164/Re^{1/4} \end{gathered} \tag{30}$$

3. Region 2–3:

$$V = 32\ \frac{Re_*}{A}\left[1-(1-\eta)^2\right]; \quad Re_* = 3.535534\ Re^{1/2}; \quad \lambda = A/Re \tag{31}$$

4. Region 3–b:

$$\begin{gathered} V = \left(\frac{Re_*}{3\chi_n{}^3}\right)^{\frac{1}{3}}\left[1-(1-\eta)^3\right]^{\frac{1}{3}} = 6.858\ Re_*^{\frac{1}{3}}\left[1-(1-\eta)^3\right]^{\frac{1}{3}}; \\ Re_* = 0.164917\ Re^{3/4}; \quad \lambda = 0.87/Re^{1/2} \end{gathered} \tag{32}$$

For the flow of polymer solutions in rough pipes with a roughness parameter m, the Reynolds number $Re_4 = 15.292m^{4/3}$ determines the beginning of the transition from the Blasius curve to the region with a constant resistance coefficient, according to (27). If $Re_4 \le Re_2$, then there will be no transition to the flow regime of the "laminar-type" region and polymer additives will have no effect on the flow (Figure 4). If it turns out that $Re_2 < Re_4$, then the roughness does not have time to influence the flow, and the resistance curve during the flow of polymer solutions will contain the same four regions (29)–(32) as a hydraulically smooth pipe.

For the resistance coefficients at points 1, 2, and 3 in Figure 4, there is a transition from one flow regime to another. Moreover, this transition, within the framework of this paper, is shown to be an abrupt one. In fact, there is always a smooth change in the resistance coefficients, and there are always transient flow regimes. The same goes for velocity profiles that transition smoothly from one shape to another. However, these transient processes occur in relatively narrow ranges of Reynolds numbers and in the first approximation they can be disregarded.

Let us change the velocity profiles at points 1, 2 and 3 in Figure 4 in universal coordinates, which can be considered as follows:

$$y_* = \frac{yv_*}{\nu}; \quad v = \frac{u}{v_*},$$

where y is the coordinate measured from the pipe wall, v_* is friction velocity, ν is kinematic viscosity, and u is the Reynolds-averaged longitudinal velocity. In this case, the non-dimensional coordinate η, appearing in the expressions for the velocity profiles, will be represented in terms of the Reynolds number for the friction velocity as $\eta = \frac{y_*}{Re_*}$. At point 1 at the value of the Reynolds number $Re_1 = 1187$ by the average velocity, the laminar flow regime ($n = 1$) changes to a turbulent one ($n = 4$), and the velocity profile (14) becomes a turbulent one (15). So, for $n = 1$ and $n = 4$, the Reynolds number in terms of friction velocity according to formula (12) takes the same value, equal to $Re_{*1} = 48.7$ (for this transitional Reynolds number $Re_1 = 1187$). These profiles are shown in Figure 5.

It can thus be seen how the laminar profile 1 changes dramatically to the turbulent profile 2 when $Re_{*1} = 48.7$. The corresponding value of the non-dimensional average velocity according to (9) for both profiles will be the same: $V_{av} = 11.98$. Then, with an increase in the Reynolds number, there are power-law profiles 3 and 4 of the turbulent flow regime ($n = 4$) according to (15) for $Re_* = 100$ and $Re_* = 413$, respectively.

When small additives of polymers, for which the Toms effect is observed, are introduced into the flow, point 2 appears on the resistance curve shown in Figure 4, in which the turbulent velocity profile ($n = 4$) changes to a "laminar-type" one ($n = 1$) with the corresponding A value for this polymer. Figure 6 at this point shows the change in the turbulent profile (15) to the profile (31) when $Re_2 = 13{,}669.9$ for $A = 400$. In both cases for this Reynolds number, the dynamic Reynolds number is the same and equal to $Re_{*2} = 413$.

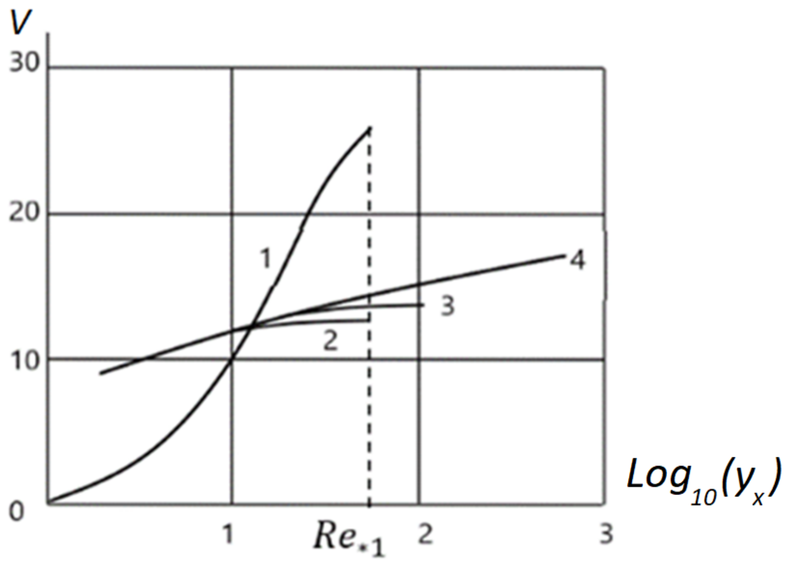

Figure 5. Velocity profiles: 1—laminar regime when $Re_* = 48.7$; 2—turbulent regime when $Re_* = 48.7$; 3, 4—turbulent ones when Re_* equals 100 and 413, respectively.

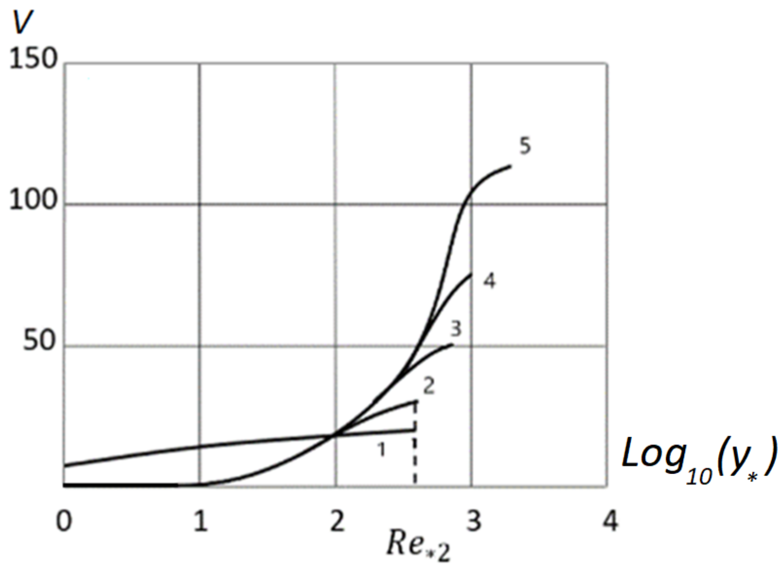

Figure 6. Velocity profiles: 1—turbulent one when $Re_* = 413.2$; 2—"laminar-type" one when $Re_* = 413.2$; 3, 4, 5—"laminar-type" ones when $Re_* = 650$, 1000, and 1625, respectively.

It can be seen that profile 1 is abrupt at $Re_{*2} = 413$ and transitions to profile 2. The corresponding value of the dimensionless average velocity according to (9) for both profiles will be the same, and equal to $V_{av} = 16.55$. Then, with an increase in the Reynolds number, there are "laminar-type" profiles 3, 4 and 5, according to (31) for Re_* = 650, 100, and 1625, respectively.

At point 3, the "laminar-type" flow regime ($n = 1$, $A = 400$) for $Re_3 = 146{,}861$ changes to the Virk flow regime ($n = 2$), and the velocity profile (31) changes to profile (32). For both profiles at this Reynolds number, the Reynolds number in terms of the friction velocity according to formula (12) takes the same value, $Re_{*3} = 1625$. These profiles are shown in Figure 7.

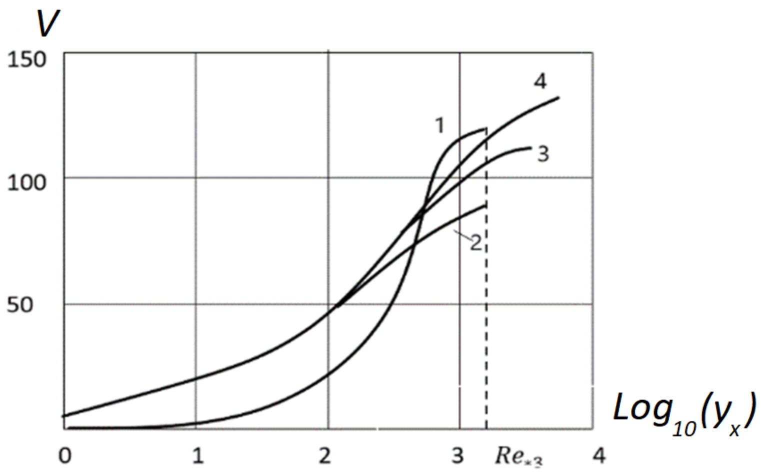

Figure 7. Velocity profiles: 1—"laminar-type" one for $Re_* = 1625$; 2—Virk profile for $Re_* = 1625$; 3, 4—Virk profiles for $Re_* = 5000$ and 8000, respectively.

It can be seen that the "laminar-type" velocity profile 1 abruptly changes to the Virk profile 2. The corresponding value of the non-dimensional average velocity according to (9) for both profiles will be the same and equal to $V_{av} = 45.19$. Then, with an increase in the Reynolds number, one obtains Virk profiles 3 and 4, according to (31) for $Re_* = 5000$ and $Re_* = 7500$, respectively.

5. Conclusions

In the present research, a power-law generalization of Newton's formula for the flow of a viscous fluid is used, in which several variants of rheological relations arise when the value of the exponent changes to be in accordance with the description of various fluid flow regimes, including non-Newtonian ones. This generalization, followed by expressing the corresponding rheological relation in tensor form and substituting it into the equation of motion of a continuous medium under stresses, leads to a system of differential equations of fluid motion, which can be used to describe flows with different rheologies and in different flow regimes (in turbulent ones as well). It should be noted that there are no exact analytical solutions to the problems of simple shear flows of an incompressible fluid in modern papers on the turbulent regime. There are no differential equations for this flow regime in the known papers that allow us to consider the problems of calculating turbulent flows as problems of mathematical physics. By using a system of equations (obtained on the basis of the considered power-law generalization of Newton's formula), it makes it possible to interpret the problems of calculating turbulent flows as problems of mathematical physics and obtain accurate analytical solutions to problems of simple shear flows. This system can be used in studies of both plane-parallel and three-dimensional flows, which was demonstrated to a certain extent in [7], which contains a solution for the problem of longitudinal flow around a flat plate and provides estimates of the boundary layer type. In this paper, a system of equations for a 2D fluid flow in the boundary layer of a flat plate is proposed and this system is reduced to one ordinary third-order equation, similarly to the work of Blasius in relation to the laminar boundary layer. The method of the direct reduction of the boundary value problem to the Cauchy problem was used to solve this equation. The results of this solution made it possible to determine the expressions for the resistance coefficients, as well as the thicknesses of the boundary layer, displacement, and momentum loss. These values are comparable with the available experimental data.

The present research shows that the fluid flow development in a pipe and all the considered flow regimes within it, along with their rheology, can be described by means of one rheological relation, which is a power-law two-parameter generalization of Newton's formula for a viscous fluid. In this case, the change in the flow movement characteristics can be described by changing the values of the nth power and the χ_n coefficient in this ratio when the corresponding critical Reynolds numbers are reached. This generalization makes it possible to research the behavior of flows with small concentrations of polymers, in which the Toms effect is observed. For such flows, the turbulent regime changes with an increase in the Reynolds number to another one, leading to "laminar-type" velocity profiles and the corresponding resistance laws. Then, the Virk flow reaches its limit regime with its own rheology with an increase in the Reynolds number. The results of this research allow us to carry out practical calculations of the velocity profiles and resistance coefficients of flows with arbitrary rheology based on a unified approach.

Subsequently, we plan to compare the proposed model with modern approaches such as DNS, and also perform calculations for more complex geometries on the basis of the proposed approach.

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