

Percolation Distribution in Small-World Quantum Networks

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Abstract: Quantum networks have good prospects for applications in the future. Compared with classical networks, small-world quantum networks have some interesting properties. The topology of the network can be changed through entanglement exchange operations, and different network topologies will result in different percolation thresholds when performing entanglement percolation. A lower percolation threshold means that quantum networks require fewer minimum resources for communication. Since a shared singlet between two nodes can still be a limitation, concurrence percolation theory (ConPT) can be used to relax the condition. In this paper, we investigate how entanglement distribution is performed in small-world quantum networks to ensure that nodes in the network can communicate with each other by establishing communication links through entanglement swapping. Any node can perform entanglement swapping on only part of the connected edges, which can reduce the influence of each node in the network during entanglement swapping. In addition, the ConPT method is used to reduce the percolation threshold even further, thus obtaining a better network structure and reducing the resources required.

Keywords: small-world quantum networks; entanglement swapping; quantum entanglement percolation; concurrence percolation theory



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1. Introduction

With the continuous research on quantum communication, the structure and scale of quantum networks are changing dramatically. Quantum networks are no longer limited to regular network shapes but are developing toward large-scale, long-range, multi-user complex networks [1,2], which is determined by the future research and application prospects of quantum networks. The properties of quantum communication itself, such as Entanglement Swapping, also make it have better transmission performance than regular quantum networks under complex networks. Entanglement Swapping is the cornerstone of large-scale quantum networks. Relying on quantum repeaters in the network, Entanglement Swapping is able to overcome the exponential fading of quantum pairs over distance in the channel, allowing any two end nodes in the network to be connected. The connection between two nodes represents a pair of entangled qubits shared by these two nodes. Ideally, this pair of qubits would be in singlet state $|\varphi(\theta)\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle$, where $0 \leq \theta \leq \frac{\pi}{4}$ so that complete transmission of information ($p = 1$) is ensured [3]. However, in practical applications, the qubits in singlet state are affected by the environment, storage and channel noise, thus turning into a purely partially entangled state, making the transmission success probability $p < 1$, resulting in unstable transmission of information. In complex quantum networks, the above problems can be amplified geometrically, causing serious impact on communication. Therefore, how to convert partially entangled state to singlet state (maximally entangled state) via LOCC (Local Operations and Classical Communications) while building complex quantum networks is our key concern [4].

Similar to the traditional networks, quantum networks can be described in terms of graphs with similar properties. Specifically, a quantum network can be represented as $G(V, E)$, where $V = 1, \dots, N$ is the set of nodes and E is the set of edges contained in $V \times V$, representing the communication nodes (including quantum relay nodes and end

nodes) and the entangled connections between nodes in a quantum network, respectively. The entangled connections are in a singlet state in the ideal case and in a purely partially entangled state in the non-ideal case. For any node in the network, the number of nodes with which it has entangled connections is called the degree of the node, denoted as k [5]. Since we choose a small-world quantum network for research, the graphs under study also have a large clustering coefficient and a short average distance.

Percolation is a common phenomenon in complex networks: There is a certain value p_0 for the connection probability p between nodes in the network. When the connection probability is less than this value ($p < p_0$), the nodes are slowly connected to form numerous small-scale connection clusters, and when the connection probability exceeds this value ($p > p_0$), a Giant Connected Component (GCC), which basically runs through the whole network, will rapidly appear in the network. This certain value p_0 is called the percolation threshold. For complex quantum networks, the percolation phenomenon occurs during the conversion from a partially entangled state to a maximally entangled state. When the conversion probability exceeds the percolation threshold of the network, we can quickly build a nearly fully connected quantum network, and this process is called Classical Entanglement Percolation (CEP) [6]. Studies of percolation in classical complex networks have shown that the structure of the network affects the value of the percolation threshold to a large extent. Similarly, in complex quantum networks, as the edge reconnection occurs after entanglement swapping between the nodes and newly connected edges replace the original ones, both the structure of the network and the percolation threshold of the network are changed. Therefore, if we preprocess the network using entanglement swapping before the percolation occurs and lower the percolation threshold of the quantum network on purpose, we can establish the quantum communication network more efficiently. This is called Quantum Entanglement Percolation (QEP). After QEP, the change in the network structure is expressed as the change in the degree of nodes.

Preprocessing in small-world quantum networks can have problems: The nodes have different degrees and neighboring nodes have influence on each other because they share an edge, and the connected edges cannot be exchanged again once entanglement swapping is performed. Moreover, a reliable connection composed of maximally entangled state can be difficult for real application scenario, so the ConPT [7] method is introduced and “sponge-crossing” probability is used to evaluate the percolation process instead of traditional size of clusters. In this paper, q-pswap is proposed to overcome these problems, and a central node is added when establishing the network in order to deal with the appearance of isolated nodes after entanglement swapping. Furthermore, the effects of ConPT methods are investigated so that the robustness of the network can be improved and the percolation threshold of the network can be reduced.

2. Percolation in Small-World Quantum Networks

A general quantum network composed of quantum nodes and connections are shown in Figure 1. We consider a Watts–Strogatz (WS) small-world quantum network, as shown in Figure 2, where any connected edge in the network is composed of a pure partially entangled state, where $|\varphi(\theta)\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle$ ($0 \leq \theta \leq \frac{\pi}{4}$). $|\psi(\theta)\rangle$ can be converted to a maximally entangled state $|\varphi(\frac{\pi}{4})\rangle$ with singlet conversion probability (SCP) $p = \min(1, 2 \sin^2 \theta)$ by LOCC. $|\varphi(\theta)\rangle^{\otimes 2}$ is composed of two pure partially entangled states $|\varphi(\theta)\rangle$, and its optimal SCP is $p_1 = \min(1, 2p - \frac{p^2}{2})$ [8]. Similar to the classical WS operation, the connected edges in the network are converted to maximally entangled state with a certain probability, and the connected edges (maximally entangled state) can be established between any two nodes in the connected cluster through entanglement swapping [3].

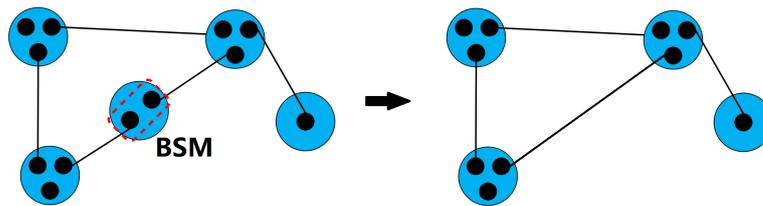


Figure 1. A general quantum network. Each black dot represents a qubit and each edge represents a pure partially entangled state. The blue circles, which include multiple qubits, are nodes of the network. By performing Bell State measurement (BSM), new entanglement can be constructed between two qubits and new edges are formed between two nodes.

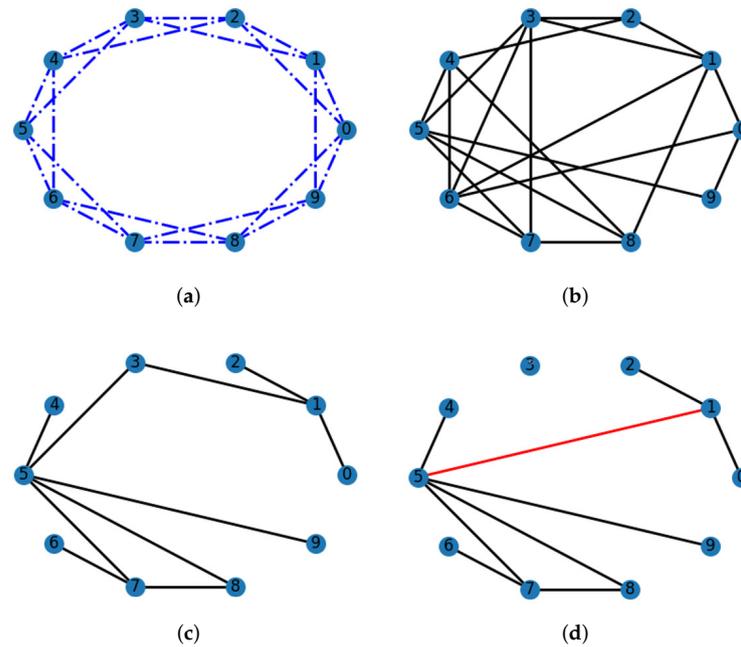


Figure 2. Entanglement percolation of a 10-node WS small-world quantum network. (a): A nearest-neighbor coupled network with four qubits per node, which are entangled with the qubits on the four neighbor nodes. Each edge in the network is a pure partially entangled state. (b): A WS small-world quantum network is established and a disconnect–reconnect operation is performed with probability $p_r = 0.6$. (c): The partially entangled states that form the connected edges in the WS small-world quantum network are converted to the maximally entangled states by LOCC with probability $p_r = 0.6$, and all the connected edges in the obtained network are maximally entangled state [6]. (d): An entanglement swapping is performed to disconnect the connected edges (3,5) (1,3) and connect (1,5). A communication link is established between nodes 1 and 5 (red line).

Before quantum percolation, quantum preprocessing will be performed [8,9], and the basic operation of quantum preprocessing is q-swap [10], which performs an entanglement swapping on nodes with a degree of q in the network (shown in Figure 3). If an edge $|\varphi\rangle$ has already undergone an entanglement swapping, then the newly generated edge is no longer $|\varphi\rangle$, so it cannot participate in the swapping operation again. Its SCP is identical to $|\varphi\rangle$. Different network structures change the percolation threshold of the network, therefore, quantum networks can change the network structure through quantum operations compared to classical ones [10].

In regular networks, such as two-dimensional square lattice networks, triangular networks, cellular networks, etc., the nodes that perform preprocessing have the same degree and have no influence on each other when performing QEP. Unlike regular networks [4], most of the nodes in a WS quantum network have different degrees, and since

the edges after q-swap cannot be operated again, the q-swap of one node has an impact on neighboring nodes. WS small-world quantum networks have shorter average distances and larger clustering coefficients [11].

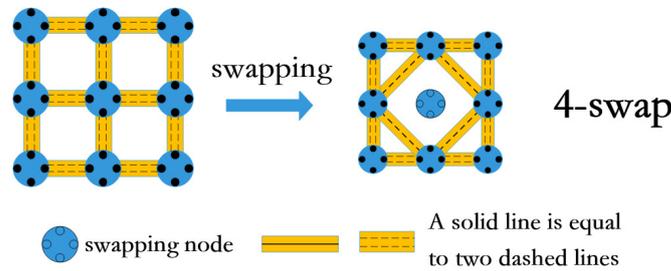


Figure 3. Effects of 4-swap on the structure of the underlying networks.

3. Percolation Optimization

In order to ensure that any node in the network can be connected through entanglement swapping after percolation, we add a central node in the center of the network, which is connected to other nodes in the network. Taking a hexagonal network in a cellular network as an example, as shown in Figure 4. In this case, fewer isolated nodes are made after entanglement percolation and most of the nodes in the network can communicate with others through entanglement swapping.

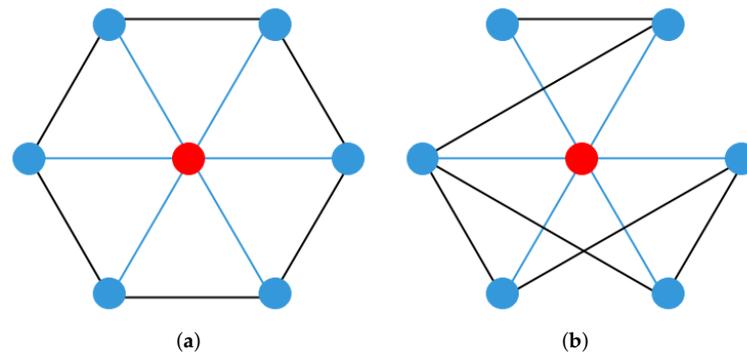


Figure 4. (a) Add the central node (red circle) to the network of 6 nodes (blue circle), the central node has connected edges with any other node in the network. (b) Perform a disconnect–reconnect operation on the network, no operation is performed on the connected edges of the central node with other nodes during the process.

Partial q-swap (q-pswap) operation is shown in Figure 5. For a node with a degree of four, entanglement swapping is only performed on part of the connected edges, and the rest of the connected edges can be swapped with the neighboring edges of other nodes, which can ensure that some important nodes do not become isolated nodes. At the same time, the impact of q-swap of one node on other nodes can be reduced.

The variation in GCC with edge conversion probability p is an important property in the process of entanglement percolation of the network. For small-world quantum networks, there are several important statistical properties [5]: clustering coefficient, degree distribution, and average shortest path. The clustering coefficient is a measure of the tendency of the nodes in a network to cluster together. The shorter the average shortest path, the fewer the number of entanglement swapping and resources are required by the nodes in the network to establish communication links.

We choose a WS quantum network with the number of nodes $N = 2000$, the reconnection probability of the network $p_r = 0.4$, and the average network degree $\bar{k} = 4$. After adding the central node, the average degree of the network becomes 6 ($\bar{k} = 6$). In the

process of disconnection–reconnection and quantum preprocessing, the connected edges of the central node are not involved in any operation. The central node is only used as an auxiliary node to ensure that there are not too many isolated nodes in the network and to reduce the average shortest path of the network and the resources required to establish connections. In the preprocessing process, we perform 4-swap as well as 4-pswap operations on nodes with a degree of 4. The degree distribution is shown in Figure 6. Due to the existence of the central node, no isolated nodes will appear in the network. However, in the process of percolation, the connected edges of the central node will participate in the process, resulting in isolated nodes in the network. However, compared to the classical WS quantum networks, the addition of the central node can still reduce the appearance of isolated nodes. After 4-pswap, the degree distribution of the network is more uniform, and the probability of nodes with large degree values appears higher. In the process of percolation, nodes with larger degree values are more likely to percolate successfully.

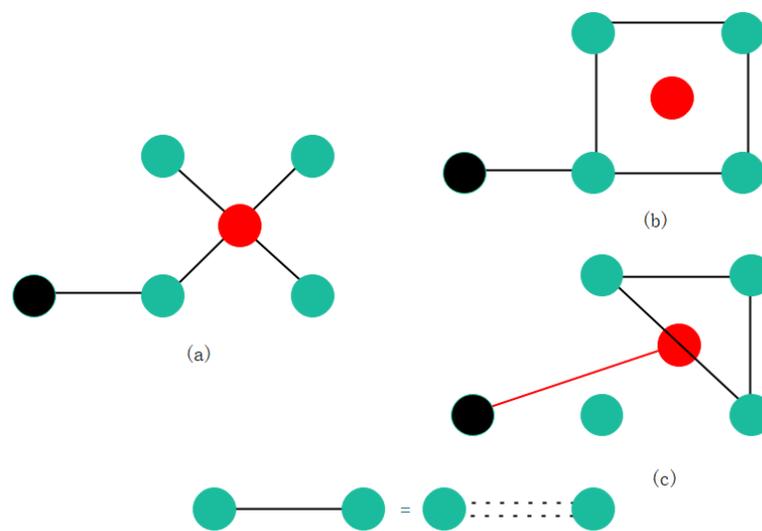


Figure 5. Different quantum preprocessing methods. (a) The central node (red circle) with a degree of 4 can perform entanglement swapping on 4 connected edges when doing quantum preprocessing. (b) The network structure obtained by doing q-swap on the central node, and the peripheral nodes (black circles) are not involved in the quantum preprocessing. (c) The central node operates on only 3 of its connected edges, and the last one is entangled and swapped with the connected edges of the peripheral nodes to obtain a new connected edge (red line). All nodes in the network are involved in the preprocessing process.

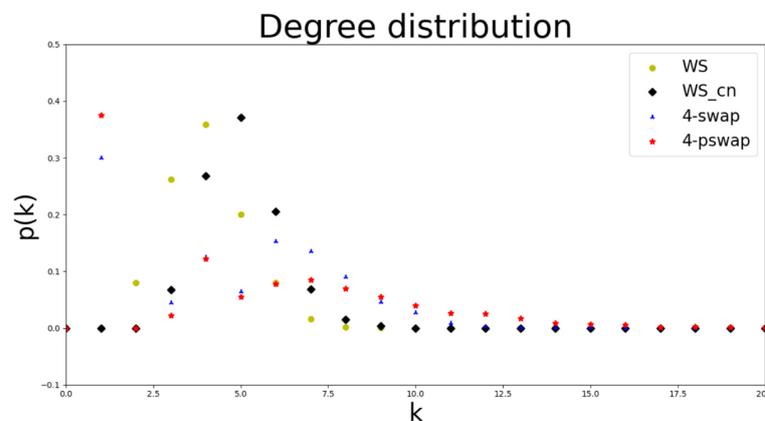


Figure 6. The degree distribution of the network before and after quantum preprocessing. “WS” and “WS_{cn}” represent the degree distribution in the original WS network and in the network with the central node, respectively. “WS_{4swap}” and “WS_{4pswap}” represent the degree distribution in networks with the central node after 4-swap and 4-pswap, respectively.

After the quantum preprocessing, the CEP operation is carried out. While changing the value of the edge conversion probability, the change in GCC in the network is observed ($GCC = N_G/N$, where N_G is the number of nodes contained in the giant connected cluster). As shown in Figure 7, there is a significant decrease in the percolation threshold of the network due to the addition of the central node. So is the case after quantum preprocessing. The percolation threshold of the network after 4-pswap is lower compared to that after 4-swap. The clustering coefficient $\langle C \rangle$ implies the “cliquishness” of a small-world quantum network and is expressed as follows [12]:

$$\langle C \rangle = \frac{3(K - 1)}{2(2K - 1)} (1 - P_r)^3, \quad \text{where } K = \frac{\bar{k}}{2}. \tag{1}$$

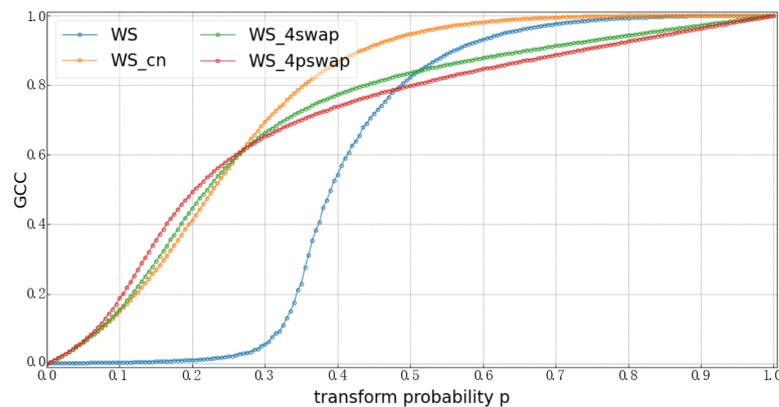


Figure 7. The relationship between GCC and conversion probability in the network. “WS” represents the percolation result in original WS network. “WS_{cn}” represents the result of CEP after adding the central node. “WS_{4swap}” and “WS_{4pswap}” represent the percolation result in networks with the central node after 4-swap and 4-pswap, respectively.

The relationship between the clustering coefficients of the networks obtained by different swap operations and the of edge conversion is shown in Figure 8. It can be seen that the clustering coefficients $\langle C \rangle$ of the networks after (3, 2)-swap are larger compared to one after 4-swap, and the networks are more clustered. A (3, 2)-swap can be seen in Figure 5c, in which swapping is performed on only three connected edges of the node with a degree of four, and the left one is swapped with peripheral nodes.

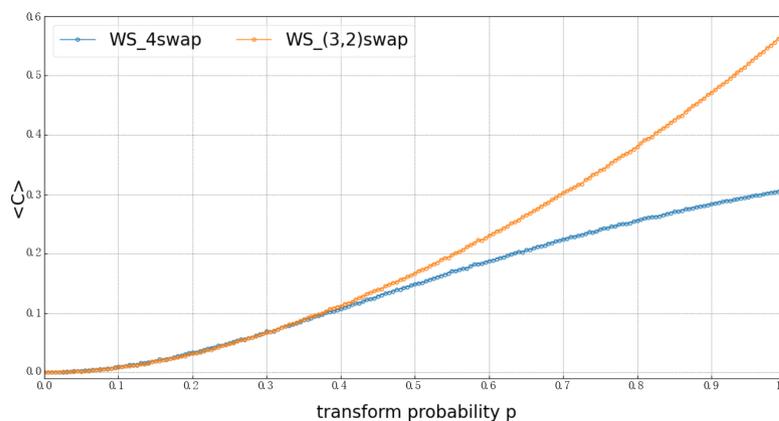


Figure 8. Distribution of network clustering coefficients obtained with different swaps.

4. Concurrence Percolation in Small-World Quantum Networks

Whether using CEP or optimized QEP, a link composed of a singlet is required to accomplish communication between any two nodes in the network, and concurrence

percolation [13] can reduce the percolation threshold of the network while relaxing this condition. Concurrence is a metric of pure state entanglement, denoted as $c := \sin 2\theta$ (c is defined as $c(\psi) = |\sum_i \alpha_i^2|$ for a pure state $|\psi\rangle = \sum_i \alpha_i |e_i\rangle$, where $|e_i\rangle$ is the basis state) [7]. Compared with traditional percolation analysis, concurrence percolation theory (ConPT) discards the measurement of cluster size and uses the “sponge-crossing” [14] probability P_{SC} , which is the probability that two distant nodes are interconnected by an open path. P_{SC} can be calculated by combining the occupation probability of each path that connects two distant nodes using connectivity rules [13]. The concurrency of “sponge-crossing” is denoted as C_{SC} . According to the thermodynamic limit, both P_{SC} and C_{SC} should jump from 0 to 1 once approaching the percolation threshold P_{th} and C_{th} respectively [13]. In the analysis of the network, the original network can be transformed by the following three rules to obtain the P_{SC} and C_{SC} of the network between two nodes.

1. Serial rule: for any 3 nodes A, B, J in the quantum network, A and J are connected by a link of concurrency c_{AJ} , while B and J are connected by a link of concurrency c_{BJ} . By performing a projection measurement on J (XZ is chosen as the measurement base), the final average concurrency obtained by the measurement is

$$C = c_{AJ}c_{BJ}. \tag{2}$$

2. Parallel rule: Nodes A and B are connected by two parallel links, which consist of product state, specified as:

$$|\varphi_{AB}(\theta_1)\rangle \otimes |\varphi_{AB}(\theta_1)\rangle = (\cos \theta_1|00\rangle + \sin \theta_1|11\rangle)((\cos \theta_2|00\rangle + \sin \theta_2|11\rangle)). \tag{3}$$

when $\cos \theta_1 \cos \theta_2 \leq \frac{1}{\sqrt{2}}$, according to Nielsen’s theorem, the maximally entangled state can be obtained by conversion. The average concurrency obtained at this point $C = \sum_k \omega_k C_k$ is optimal, where ω_k is the probability that concurrency C_k obtained after measurement is in pure state.

3. Star-mesh (SM) transform: The star graph of size n is transformed into a complete graph of size $n - 1$, and then one node in the complete graph of $n - 1$ is selected as the root node to obtain a star graph of $n - 1$. After several iterations, any two nodes in the connected cluster of the network share a connection. The series-parallel rule can be represented by Table 1 [13].

Table 1. Connectivity rules.

Rules	Classical	ConPT
series	$p = p_1 p_2 \dots$	$c = c_1 c_2 \dots$
parallel	$p = 1 - (1 - p_1)(1 - p_2) \dots$	$\frac{1 + \sqrt{1 - c^2}}{2} = \max\{\frac{1}{2}, \frac{1 + \sqrt{1 - c_1^2}}{2}, \frac{1 + \sqrt{1 - c_2^2}}{2}, \dots\}$

The above three methods are applied to the WS quantum network to analyze the percolation threshold of the network. The number of nodes in the WS quantum network are selected as $N = 10$ and $N = 20$, the probability of reconnecting the network with disconnected edges $p_{re} = 0.5$, the average degree of the network $\bar{k} = 6$, both ConPT and classical methods are simulated. As shown in Figure 9, the percolation threshold obtained by the ConPT method is smaller: $C_{th} = 0.253$ and $p_{th} = 0.406$.

Unlike the structure of regular networks (e.g., cellular networks, Bethe lattice networks, etc.), WS quantum networks can have different topologies when different p_{re} are selected. Even if the same p_{re} is selected, the topologies obtained may be different as well. As shown in Figure 10, the number of nodes, reconnection probability, and average degree are the same for both WS quantum networks. It is assumed that all connected edges $|\varphi(\theta)\rangle$ in these two networks have $\theta = 0.2\frac{\pi}{4}$ and percolation from any node in the network will have little effect on the final obtained. Take the ConPT method as an example: We start from ConPT $0 \rightarrow 2$ in sequence with series-parallel rule and SM rule. a obtained $(1, 3)\theta_{SC} = 0.063\frac{\pi}{4}$,

while b obtained $\theta_{SC} = 0.016\frac{\pi}{4}$. For the same parameters, different network topologies also obtain different θ_{SC} , which affects the percolation threshold.

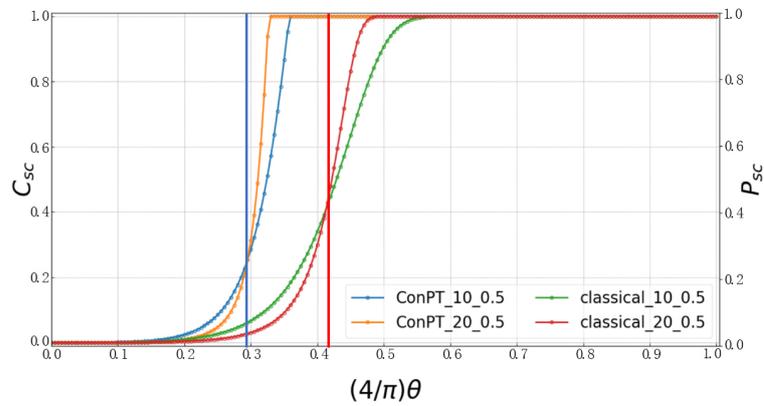


Figure 9. Comparison between classical percolation and ConPT: the percolation threshold using classical method (red vertical line) and the percolation threshold using ConPT method (blue vertical line) can be obtained by Monte Carlo simulation.

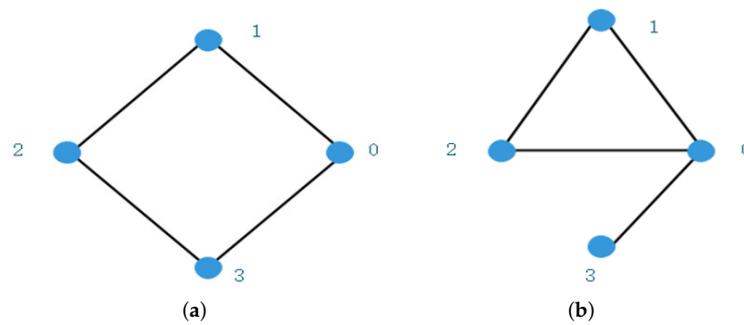


Figure 10. Two kinds of topology networks ($N = 4$, $p_{re} = 0.5$, $\bar{k} = 2$). (a) Ring topology; (b) Mesh topology.

In order to analyze the effect of disconnection–reconnection probability p_{re} on the percolation threshold in WS quantum networks, $p_{re} = 0.3, 0.5, 0.8$ are selected. As shown in Figure 11, when $p_{re} = 0.3$, $C_{th} = 0.347$, $p_{th} = 0.55$ and when $p_{re} = 0.8$, $C_{th} = 0.233$, $p_{th} = 0.39$. With the increase in p_{re} , the percolation threshold of the network decreases gradually whether the network is percolated using the ConPT or classical method.

The effect of the average degree on percolation threshold in a WS quantum network is also considered. Take a WS quantum network with an average degree of 4 ($\bar{k} = 4$) as an example, as shown in Figure 12, $C_{th} = 0.405$, $p_{th} = 0.589$. Compared to a WS quantum network with an average degree of 6 ($\bar{k} = 6$):

$$\Delta C = \frac{C_{th(\bar{k}=4)} - C_{th(\bar{k}=6)}}{C_{th(\bar{k}=4)}} = 37.5\%, \Delta p = \frac{p_{th(\bar{k}=4)} - p_{th(\bar{k}=6)}}{p_{th(\bar{k}=4)}} = 31.1\%. \quad (4)$$

It can be seen that for WS quantum networks, the percolation threshold obtained after ConPT method has a greater rate of change compared to that after classical method.

For an Erdős–Rényi (ER) network, an average degree of 6 ($\bar{k} = 6$) is selected. By simulation, we can restore the percolation threshold using the classical method $p_{th} = \frac{1}{\bar{k}} = 0.167$. While using the ConPT method, $C_{th} = 0.085$ is obtained, and the percolation threshold is reduced by $\gamma = \frac{p_{th} - C_{th}}{p_{th}} = 49\%$. It is shown in Figure 13 that the ConPT method can take advantage of concurrency and perform entanglement metric on quantum states to effectively reduce the percolation threshold of the network.

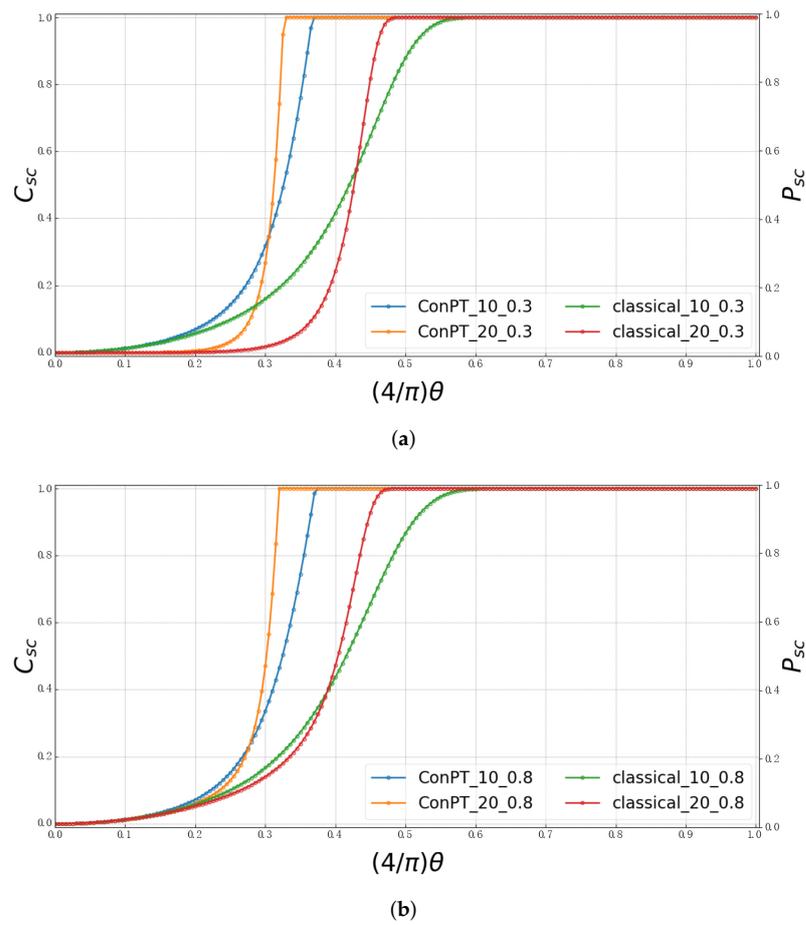


Figure 11. Effects of (a) $p_{re} = 0.3$ and (b) $p_{re} = 0.8$ on the percolation threshold of WS quantum networks.

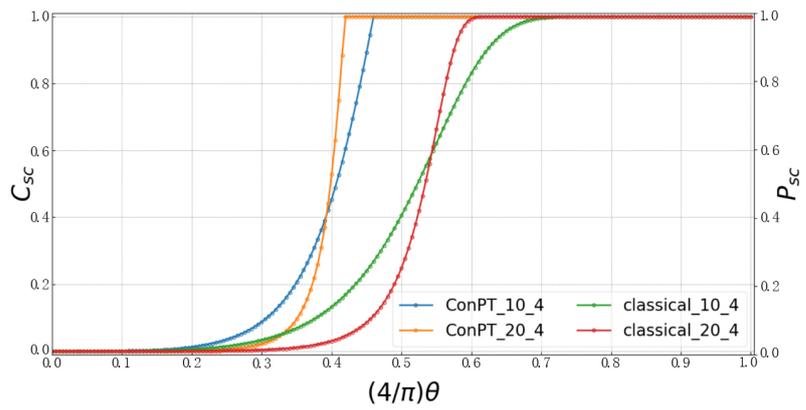


Figure 12. ConPT and classical percolation in a WS quantum network with average degree $\bar{k} = 4$.

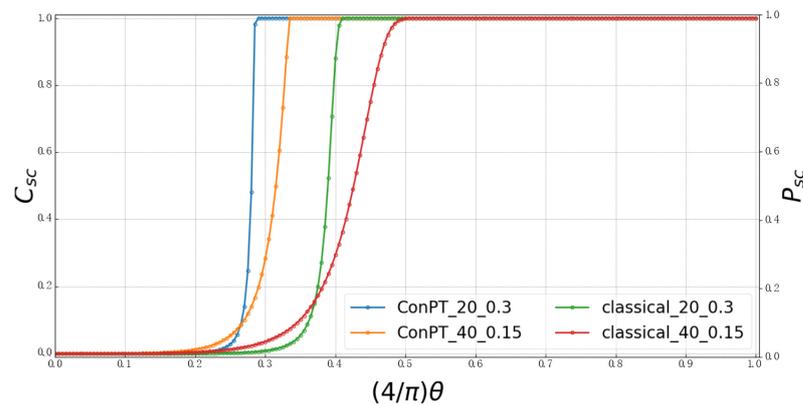


Figure 13. ConPT and classical percolation in an ER network with average degree $\bar{k} = 6$.

5. Conclusions

In this paper, we analyze the problems that arise in the quantum preprocessing of small-world quantum networks and propose solutions that allow the maximally entangled states to be distributed in the network so that communication can be ensured between nodes. Meanwhile, the obtained network has a lower percolation threshold and requires fewer minimum resources to carry out communication using q-pswap. In WS quantum networks, the percolation threshold of both the ConPT method and the CEP method is related to the reconnection probability and the average degree of the network. Compared with CEP and QEP, concurrence percolation theory can not only lower the percolation threshold of the network but can also relax the condition that there must be a link composed of singlet between two nodes in the network to complete the communication. Further research may focus on applying ConPT to multi-body entangled states.

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