Lagrangian equations derivation process

The Lagrangian equation was used to calculate the equivalent joint torque dynamic equation of the SAM [1, 2]. Regardless of external interference and rigid-flexible coupling deformation, the kinetic energy of a SAM with a fixed base is mainly composed of the following parts: linkage kinetic energy, universal joint kinetic energy, and load kinetic energy. Therefore, the total kinetic energy of the SAM can be described as follows:

$$K = K_r + K_l + K_p \tag{S1}$$

where K_r is the kinetic energy of the connecting rod, K_l is the kinetic energy of the universal joint, and K_p is the kinetic energy of the end tool and load.

Assuming that the mass of any particle P on the connecting rod i is dm, its kinetic energy is as follows:

$$\begin{aligned} dK_i &= \frac{1}{2} {v_p}^2 dm \\ &= \frac{1}{2} Trace \left[\sum_{j=1}^{i} \sum_{k=1}^{i} \frac{\partial \boldsymbol{D_i}}{\partial q_j} \boldsymbol{r_p^i} (\boldsymbol{r_p^i})^T \, (\frac{\partial \boldsymbol{D_i}}{\partial q_k})^T \dot{q}_j \dot{q}_k \right] dm \end{aligned} \tag{S2}$$

where $\boldsymbol{D_i} = \boldsymbol{D_i^0}$ is the transformation matrix of the i-th joint, and $\boldsymbol{r_p^i}$ is the local coordinate system's position vector. We can then integrate the kinetic energy of the particle on connecting rod i to obtain the kinetic energy of connecting rod i:

$$\begin{aligned} K_i &= \int_{link\ i} dK_i \\ &= \frac{1}{2} Trace \sum_{j=1}^{i} \sum_{k=1}^{i} \frac{\partial \boldsymbol{D_i}}{\partial q_j} \left(\int_{link\ i} \boldsymbol{r_p^i} (\boldsymbol{r_p^i})^T dm \right) (\frac{\partial \boldsymbol{D_i}}{\partial q_k})^T \dot{q}_j \dot{q}_k \end{aligned} \tag{S3}$$

where $\int_{link\ i} \boldsymbol{r_p^i} (\boldsymbol{r_p^i})^T dm$ is the pseudo-inertia matrix of link i expressed by $\boldsymbol{I_i}$.

$$\begin{aligned} \boldsymbol{I_i} &= \int_{link\ i} \boldsymbol{r_p^i} (\boldsymbol{r_p^i})^T dm \\ &= \begin{bmatrix} \int_{link\ i} x_i^2 dm & \int_{link\ i} x_i y_i dm & \int_{link\ i} x_i z_i dm & \int_{link\ i} x_i dm \\ \int_{link\ i} x_i y_i dm & \int_{link\ i} y_i^2 dm & \int_{link\ i} y_i z_i dm & \int_{link\ i} y_i dm \\ \int_{link\ i} x_i z_i dm & \int_{link\ i} y_i z_i dm & \int_{link\ i} z_i^2 dm & \int_{link\ i} z_i dm \\ \int_{link\ i} x_i dm & \int_{link\ i} y_i dm & \int_{link\ i} z_i dm & \int_{link\ i} dm \end{bmatrix} \end{aligned} \tag{S4}$$

The total kinetic energy of each link of the SAM can be described as follows:

$$\begin{aligned} K_l &= \sum_{i=1}^{20} K_i \\ &= \frac{1}{2} \sum_{i=1}^{20} Trace \left[\sum_{j=1}^{i} \sum_{k=1}^{i} \frac{\partial \boldsymbol{D_i}}{\partial q_j} \boldsymbol{I_i} \, (\frac{\partial \boldsymbol{D_i}}{\partial q_k})^T \dot{q}_j \dot{q}_k \right] \end{aligned} \tag{S5}$$

Assuming that all external loads and the weight of the end tool are concentrated on a specific point on the end joint of the SAM, there would only be translational movement without rotational kinetic energy:

$$K_p = \frac{1}{2} m_{gp} Trace \left[\sum_{j=1}^{20} \sum_{k=1}^{20} \frac{\partial \boldsymbol{D_{20}}}{\partial q_j} \boldsymbol{r^{20}} (\boldsymbol{r^{20}})^T \left(\frac{\partial \boldsymbol{D_{20}}}{\partial q_k} \right)^T \dot{q}_j \dot{q}_k \right] \tag{S6}$$

where m_{gp} is the mass of the end tool and load.

Ignoring the rigid–flexible coupling elastic deformation of the cables, connecting rods, and joints, as well as any external interference, the SAM is only affected by the gravitational potential energy P:

$$P = P_{gl} + P_{gp} \tag{S7}$$

where P_{gl} is the gravitational potential energy of the connecting rod and the universal joint, and P_{gp} is the gravitational energy of the end tool and load.

The gravitational energy at position $\boldsymbol{r^i}$ on any connecting rod i of the SAM is as follows:

$$d_{pi} = -dm\boldsymbol{g^T}\boldsymbol{r^0} = -\boldsymbol{g^T}\boldsymbol{D_i}\boldsymbol{r^i}dm \tag{S8}$$

where $\boldsymbol{g^T} = [0, 0, 9.81, 1]$.

Integrating the Eq (S8), the total gravitational energy of the connecting rod can be obtained as follows:

$$P_{gl} = \sum_{i=1}^{20} -\boldsymbol{g^T}\boldsymbol{D_i}\int_{link\ i}\boldsymbol{r^i}dm = -\sum_{i=1}^{20} m_i\boldsymbol{g^T}\boldsymbol{D_i}\boldsymbol{r_l^i} \tag{S9}$$

where m_i is the mass of the connecting rod i (if i is an even number, it is the mass of the universal joint), and $\boldsymbol{r_l^i}$ is the center of gravity of the connecting rod i relative to the previous joint coordinate system.

The gravitational energy of the loading mass of the end tool can be described as follows:

$$P_{gp} = -m_{gp}\boldsymbol{g^T}\boldsymbol{D_{20}}\boldsymbol{r_g^{20}} \tag{S10}$$

where m_{gp} is the mass of the end tool and load, and $\boldsymbol{r_g^{20}}$ is the coordinate of the center of gravity relative to the previous joint.

Derivation of dynamics equation

The cable traction is equivalent to the joint torque, and the SAM becomes a typical serial multi-joint manipulator, with the final dynamic model expressed as follows:

$$\boldsymbol{T} = \boldsymbol{M}(q)\ddot{\boldsymbol{q}} + \boldsymbol{H}(q,\dot{q})\dot{\boldsymbol{q}} + \boldsymbol{G}(q) + \boldsymbol{D} \tag{S11}$$

where $\ddot{\boldsymbol{q}} \in R^{20}$ is the joint angular acceleration vector, $\dot{\boldsymbol{q}} \in R^{20}$ is the joint velocity vector, and $\boldsymbol{T} \in R^{20}$ is the input torque vector, $\boldsymbol{M}(q) \in R^{20\times 20}$ is a nonsingular positive definite inertial force matrix, $\boldsymbol{H}(q,\dot{q}) \in R^{20\times 20}$ is the term of centrifugal force and Coriolis force. $\boldsymbol{G}(q)$ is the gravitational term (including the gravity of the connecting rod, the universal joint, the end tools and the load), and $\boldsymbol{D}$ represents the unknown bounded disturbance of the unstructured unbuilt dynamic model.

Using the Lagrangian method to expand Eq (S11), the torque of any joint p can be obtained as follows:

$$\begin{aligned}
\boldsymbol{T_p} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_p} - \frac{\partial L}{\partial q_p} &= \sum_{i=p}^{20}\sum_{k=1}^{i} Trace\left(\frac{\partial \boldsymbol{D_i}}{\partial q_k}\boldsymbol{I_i}(\frac{\partial \boldsymbol{D_i}}{\partial q_p})^T\right)\ddot{q}_k \\
&+ \sum_{i=p}^{20}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 \boldsymbol{D_i}}{\partial q_j \partial q_k}\boldsymbol{I_i}(\frac{\partial \boldsymbol{D_i^T}}{\partial q_p})^T\right)\dot{q}_j\dot{q}_k \\
&+ m_{gp}\sum_{j=p}^{20} Trace\left[\frac{\partial \boldsymbol{D_{20}}}{\partial q_j}\boldsymbol{r^{20}}(\boldsymbol{r^{20}})^T\frac{\partial \boldsymbol{D_{20}}}{\partial q_p}\right]\ddot{q}_j \\
&+ m_{gp}\sum_{j=p}^{20}\sum_{k=p}^{20} Trace\left[\frac{\partial \boldsymbol{D_{20}}}{\partial q_j \partial q_k}\boldsymbol{r^{20}}(\boldsymbol{r^{20}})^T\frac{\partial \boldsymbol{D_{20}}}{\partial q_p}\right]\dot{q}_j\dot{q}_k \\
&- \sum_{i=p}^{20} m_i\boldsymbol{g^T}\frac{\partial \boldsymbol{D_i}}{\partial q_p}\boldsymbol{r_l^i} - m_{gp}\boldsymbol{g^T}\frac{\partial \boldsymbol{D_{20}}}{\partial q_p}\boldsymbol{r_g^{20}} + \boldsymbol{D_p}
\end{aligned} \tag{S12}$$

where the Lagrangian function L is as follows:

$$
\begin{aligned}
L = K - P \\
&+ \frac{1}{2}\sum_{i=1}^{20}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left[\frac{\partial \boldsymbol{D_i}}{\partial q_j}\boldsymbol{I_i}(\frac{\partial \boldsymbol{D_i}}{\partial q_k})^T \dot{q}_j \dot{q}_k\right] \\
&+ \frac{1}{2} m_{gp} Trace\left[\sum_{j=1}^{20}\sum_{k=1}^{20}\frac{\partial \boldsymbol{D_{20}}}{\partial q_j}\boldsymbol{r^{20}}(\boldsymbol{r^{20}})^T\left(\frac{\partial \boldsymbol{D_{20}}}{\partial q_k}\right)^T \dot{q}_j \dot{q}_k\right] \\
&+ \sum_{i=1}^{20} m_i \boldsymbol{g^T}\boldsymbol{D_i}\boldsymbol{r_l^i} + m_{gp}\boldsymbol{g^T}\boldsymbol{D_{20}}\boldsymbol{r_g^{20}}
\end{aligned}
\tag{S13}
$$

References

1. Spong, M. W., Hutchinson, S., Vidyasagar, M.; Robot modeling and control, *John Wiley & Sons.* **2020**, pp. 163–214. [CrossRef]
2. Kurfess, T. R.; Lagrangian dynamics. *Robotics and Automation Handbook*. CRC Press, **2018**, pp. 73–88. [CrossRef]