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Fuzzy Quality Evaluation Model Constructed by Process Quality Index

Chun-Min Yu [1], Chih-Feng Wu [2], Kuen-Suan Chen [1,3,4,*] and Chang-Hsien Hsu [5]

1 Department of Industrial Engineering and Management, National Chin-Yi University of Technology, Taichung 411030, Taiwan; march@ncut.edu.tw
2 Department of Digital Content Application and Management, Wenzao Ursuline University of Languages, Kaohsiung 80793, Taiwan; cfwu@mail.wzu.edu.tw
3 Department of Business Administration, Chaoyang University of Technology, Taichung 413310, Taiwan
4 Institute of Innovation and Circular Economy, Asia University, Taichung 41354, Taiwan
5 Department of Business Administration, Asia University, Taichung 41354, Taiwan; pci@asia.edu.tw
* Correspondence: kschen@ncut.edu.tw

Abstract: Many studies have pointed out that the-smaller-the-better quality characteristics (QC) can be found in many important components of machine tools, such as roundness, verticality, and surface roughness of axes, bearings, and gears. This paper applied a process quality index that is capable of measuring the level of process quality. Meanwhile, a model of fuzzy quality evaluation was developed by the process quality index as having a one-to-one mathematical relationship with the process yield. In addition to assessing the level of process quality, the model can also be employed as a basis for determining whether to improve the process quality at the same time. This model can cope with the problem of small sample sizes arising from the need for enterprises' quick response, which means that the accuracy of the evaluation can still be maintained in the case of small sample sizes. Moreover, this fuzzy quality evaluation model is built on the confidence interval, enabling a decline in the probability of misjudgment incurred by sampling errors.

Keywords: fuzzy hypothesis test; process quality index; membership function; quality level; normal process distribution

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1. Introduction

Process Capability Indices (PCIs) are the most popular tool for process quality evaluation in the industry of machining and manufacturing [1–4]. They are a convenient device for evaluating and analyzing the process capabilities of products as well as a good tool bridging the gap between sales departments and customers [5–13]. Taiwan's output value and export volume of machine tools both are at the top of the list in the world, especially in the middle of Taiwan, a stronghold of machine tools and machining factories [14–17]. Many studies have pointed out that the-smaller-the-better quality characteristics (QC) can be discovered in many important components of machine tools, such as roundness, verticality, and surface roughness of axes, bearings, and gears [18–22]. It is assumed that X denotes the related quality characteristic in the manufacturing process following a normal distribution with mean μ and standard deviation σ. Let random variable $Y = X/USL$, distributed as the normal distribution with mean $\delta = \mu/USL$ and standard deviation $\gamma = \sigma/USL$, in which USL is the upper specification limit. Chang et al. [23] put forward process quality index P_{QI} for the smaller-the-better (STB) quality characteristics, expressed as follows:

$$P_{QI} = \frac{1-\delta}{\gamma} \tag{1}$$

Chen and Huang [24] and Yu et al. [25] pointed out that in practice, due to cost considerations and limitations of processing technology, the measured value of the product is usually far away from the target value T_V and very close to USL, that is, the parameter δ is usually less than 1 but very close to 1. Therefore, γ is usually very small.

Moreover, in the case of normality, a one-to-one mathematical relationship exists between process quality index P_{QI} and process yield $Yield\%$ as described below:

$$Yield\% = p\{Y \leq 1\} = p\left\{Z \leq \frac{1-\delta}{\gamma}\right\} = \Phi(P_{QI}), \tag{2}$$

where $Z = (Y-\delta)/\gamma$ is distributed as standard normal distribution and $\Phi(\cdot)$ is the cumulative function of standard normal distribution. For instance, if $P_{QI} = 4.0$, then it is guaranteed that the process yield is $Yield\% = \Phi(4.0) = 99.9968\%$. The x-axis of Figure 1 is index P_{QI} and the y-axis is process yield $Yield\%$. Obviously, the larger the value of index P_{QI}, the larger the corresponding process yield $Yield\%$.

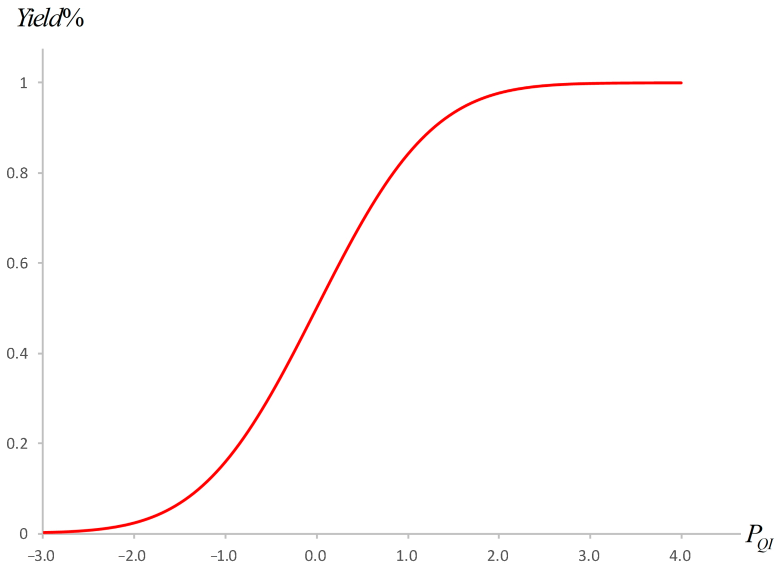

Figure 1. The relation between index P_{QI} and process yield $Yield\%$.

Not only can the process quality index P_{QI} measure the process quality level, but it also has a one-to-one mathematical relation with the process yield at the same time. Accordingly, this paper modified the fuzzy testing method created by Buckley [26] and Chen [27,28] to develop a fuzzy quality evaluation model. In addition to having a simpler calculation procedure, the model will employ this index to come up with a quality fuzzy evaluation model based on the upper confidence limit for the STB quality characteristics and use it as a basis to determine whether the process needs to improve.

Obviously, the model proposed in this paper is not only applicable to the evaluation of the small sample size and correspondent with the needs of enterprises for quick responses, but it is also capable of reducing the risk of misjudgment caused by sampling error for it is based on the confidence interval. There are many industries that are very suitable for the model described in this paper, such as the process quality evaluation of the-smaller-the-better quality characteristics produced by the machining process and the waste discharge of the-smaller-the-better characteristics regulated by many environmental regulations.

The other sections are arranged as follows. According to Boole's inequality and DeMorgan's theorem, we obtain the upper confidence limit of the process quality index in Section 2. In Section 3, we construct α-cuts of the half-triangular shaped fuzzy number and fuzzy membership function for index P_{QI}. In Section 4, we propose a fuzzy hypothesis testing method aiming to evaluate the process quality and determine whether the process needs to improve. In Section 5, an application example is taken to prove the applicability of the approach proposed in this study. Finally, conclusions are made in Section 6.

2. Confidence Limits of the Process Quality Index

Many studies have suggested that companies use control charts to perform process control. When the process is under statistical process control, the process capability will be evaluated [29–31]. Without loss of generality, it is assumed that each subsample contains n observations on quality characteristics, and m subsamples are available. For each subsample, let $\overline{Y}_h$ and S_h^2 be the sample mean and sample variance of the hth subsample and N be the total number of observations as displayed below:

$$\overline{Y}_h = \frac{1}{n}\sum\nolimits_{j=1}^{n} Y_{hj},\ S_h^2 = \frac{1}{n-1}\sum\nolimits_{j=1}^{n}\left(Y_{hj}-\overline{Y}_h\right)^2, \text{ and } N = \sum_{j=1}^{m} n = mn$$

The overall sample mean and the pooled sample variance are applied below. The estimates of δ and γ can be obtained as follows:

$$\delta^* = \frac{1}{m}\sum_{h=1}^{m}\overline{Y}_h, \text{ and } \gamma^* = \sqrt{\frac{1}{N-m}\sum_{h=1}^{m}(n-1)S_h^2}$$

Consequently, the estimator of the index P_{QI} is displayed as follows:

$$P_{QI}^* = \frac{1-\delta^*}{\gamma^*} \tag{3}$$

In the case of normality, let

$$T = \frac{\sqrt{N}(\delta^*-\delta)}{\gamma^*} \tag{4}$$

and

$$K = \frac{(N-m)\gamma^{*2}}{\gamma^2} \tag{5}$$

Then, T is distributed as t_{N-m} and K is distributed as χ^2_{N-m}, respectively. Where t_{N-m} is Student's t-distribution with $N-m$ degree of freedom and χ^2_{N-m} is chi-square distribution with $N-m$ degree of freedom. To derive the $(1-\alpha)\times 100\%$ upper confidence limits of index P_{QI}, some events are defined as follows:

$$E_{L\delta} = \{T \le t_{\alpha/2;N-m}\} \tag{6}$$

and

$$E_{U\gamma} = \left\{K \le \chi^2_{1-\alpha/2;N-m}\right\} \tag{7}$$

where $\chi^2_{1-\alpha/2;N-m}$ is the lower $1-\alpha/2$ quintile of χ^2_{N-m} and $1-\alpha$ represents the confidence level.

In fact, the probability of event $E_{L\delta}$ equal to the probability of event $E_{U\gamma}$ is $1-\alpha/2$, that is $P(E_{L\delta}) = P(E_{U\gamma}) = 1-(\alpha/2)$. Similarly, the probability of event $E^C_{L\delta}$ equal to the probability of event $E^C_{U\gamma}$ is $\alpha/2$, that is $P(E^C_{L\delta}) = P\left(E^C_{U\gamma}\right) = \alpha/2$, where $E^C_{L\delta}$ is the complement set of $E_{L\delta}$ and $E^C_{U\gamma}$ is the complement set of $E_{U\gamma}$. Based on Boole's inequality and DeMorgan's theorem, we have $P(E_{L\delta}\cap E_{U\gamma}) \ge 1-P(E^C_{L\delta})-P\left(E^C_{U\gamma}\right) = 1-\alpha$. Then,

$$P\left\{T \le t_{\alpha/2;N-m}, K \le \chi^2_{1-\alpha/2;N-m}\right\} = 1-\alpha \tag{8}$$

Equivalently,

$$P\left\{1-\delta < 1-\delta^* + t_{\alpha/2;N-m} \times \left(\frac{\gamma^*}{\sqrt{mn}}\right), \frac{1}{\gamma} \leq \frac{1}{\gamma^*}\sqrt{\frac{\chi^2_{1-\alpha/2;N-m}}{N-m}}\right\} = 1-\alpha \tag{9}$$

Therefore,

$$P\left\{P_{QI} < \left(P^*_{QI} + \frac{t_{\alpha/2;N-m}}{\sqrt{mn}}\right)\sqrt{\frac{\chi^2_{1-\alpha/2;N-m}}{N-m}}\right\} = 1-\alpha, \tag{10}$$

and the upper confidence limit of index P_{QI} is

$$UP_{QI} = \left(P^*_{QI} + \frac{t_{\alpha/2;N-m}}{\sqrt{mn}}\right)\sqrt{\frac{\chi^2_{1-\alpha/2;N-m}}{N-m}} \tag{11}$$

3. The Half-Triangular Shaped Fuzzy Number

Let $(y_{h1}, y_{h2}, \cdots, y_{hn})$ be the observed value of $(Y_{h1}, Y_{h2}, \cdots, Y_{hn})$ for the hth subsample, then the observed values of δ^* and γ^* are respectively expressed as follows:

$$\delta^*_0 = \frac{1}{m}\sum_{h=1}^{m} \overline{y}_h, \tag{12}$$

$$\gamma^*_0 = \sqrt{\frac{1}{N-m}\sum_{h=1}^{m}(n-1)s_h^2}, \tag{13}$$

where $\overline{y}_h$ is the observed value of $\overline{Y}_h$ and s_h^2 is the observed value of S_h^2. Consequently, then the observed value of P^*_{QI} is displayed as follows:

$$P^*_{QI0} = \frac{1-\delta^*_0}{\gamma^*_0}, \tag{14}$$

Consequently, the observed value of the upper confidence limit for index P_{QI} is the function of α as shown below:

$$UP_{QI0} = \left(P^*_{QI0} + \frac{t_{\alpha/2;N-m}}{\sqrt{mn}}\right)\sqrt{\frac{\chi^2_{1-\alpha/2;N-m}}{N-m}}. \tag{15}$$

Let

$$x = UP_{QI0} \times \sqrt{\frac{N-m}{\chi^2_{0.5;N-m}}}. \tag{16}$$

Inspired by Buckley's approach [26], the α-cuts of the half-triangular shaped fuzzy number $\tilde{x}$ is expressed as below:

$$\tilde{x}[\alpha] = \begin{cases} [x(1), x(\alpha)], \text{ for } 0.01 \leq \alpha \leq 1 \\ [x(1), x(0.01)], \text{ for } 0 \leq \alpha \leq 0.01 \end{cases}, \tag{17}$$

where

$$x(\alpha) = \left(P^*_{QI0} + \frac{t_{\alpha/2;N-m}}{\sqrt{mn}}\right)\sqrt{\frac{\chi^2_{1-\alpha/2;N-m}}{\chi^2_{0.5;N-m}}}, \tag{18}$$

and the starting point of α-cuts is 0.01. In fact, the α value is the significance level of the test and the risk of type I error.

As a result, the half-triangular shaped fuzzy number of $\widetilde{x}$ is $\Delta\widetilde{x} = (x_M, x_R)$, where

$$x_M = P^*_{QI0}, \tag{19}$$

and

$$x_R = \left(P^*_{QI0} + \frac{t_{0.005;N-m}}{\sqrt{mn}}\right)\sqrt{\frac{\chi^2_{0.995;N-m}}{\chi^2_{0.5;N-m}}}. \tag{20}$$

Then, the fuzzy membership function of $\widetilde{x}$ is denoted below:

$$\eta_0(x) = \begin{cases} 0 \ \ if \ \ x < x_M \\ 1 \ \ if \ \ x = x_M \\ \alpha \ \ if \ \ x_M < x \le x_R \\ 0 \ \ if \ \ x_R < x \end{cases}, \tag{21}$$

where α is decided by

$$\left(P^*_{QI0} + \frac{t_{\alpha/2;N-m}}{\sqrt{mn}}\right)\sqrt{\frac{\chi^2_{1-\alpha/2;N-m}}{\chi^2_{0.5;N-m}}} = x. \tag{22}$$

4. Developing a Fuzzy Hypothesis Testing Method

Before the fuzzy hypothesis testing method is developed, the process of statistical hypothesis testing is examined. According to Chen et al. [20], we adopt the following hypotheses when judging if the quality level meets the required value k:

null hypothesis H_0:$P_{QI} \geq k$
versus
alternative hypothesis H_1:$P_{QI} < k$.

Next, the critical value is determined by

$$\begin{aligned} 1-\alpha &= p\left\{P^*_{QI} < C_0 \middle| P_{QI} \ge k\right\} \\ &= p\left\{\sqrt{N}P^*_{QI0} < \sqrt{N}C_0 \middle| P_{QI} \ge k\right\} \\ &= p\left\{t_{N-m}\left(\Delta = \sqrt{N}k\right) < \sqrt{N}C_0\right\}, \end{aligned} \tag{23}$$

where $t_{N-m}\left(\Delta = \sqrt{N}k\right)$ is the non-central t distribution with $N-m$ degrees of freedom and non-centrality parameter $\Delta = \sqrt{N}k$. Thus, the critical value C_0 is denoted as follows:

$$C_0 = \frac{t_{\alpha;N-m}\left(\Delta = \sqrt{N}k\right)}{\sqrt{N}}, \tag{24}$$

where $t_{\alpha;N-m}\left(\Delta = \sqrt{N}k\right)$ is the lower αth quantile of $t_{N-m}\left(\Delta = \sqrt{N}k\right)$. In applying the critical value C_0 to statistical hypothesis testing, the rules are illustrated below:

(1) When $P^*_{QI0} < C_0$, then reject H_0 and conclude that $P_{QI} < \boldsymbol{k}$.
(2) When $P^*_{QI0} \geq C_0$, then do not reject H_0 and conclude that $P_{QI} \geq k$.

On the basis of Equation (17), the α-cuts of the triangular fuzzy number $\widetilde{C}_0$ are expressed as follows:

$$\widetilde{C}_0[\alpha] = \begin{cases} [C_0(1), C_0(\alpha)], \text{ for } 0.01 \le \alpha \le 1 \\ [C_0(1), C_0(0.01)], \text{ for } 0 \le \alpha \le 0.01 \end{cases}, \tag{25}$$

where

$$C_0(1) = \left(C_0 - \frac{t_{0.5;N-m}}{\sqrt{mn}}\right)\sqrt{\frac{\chi^2_{0.5;N-m}}{\chi^2_{0.5;N-m}}} = C_0 \quad (26)$$

and

$$C_0(\alpha) = \left(C_0 + \frac{t_{\alpha/2;N-m}}{\sqrt{mn}}\right)\sqrt{\frac{\chi^2_{1-\alpha/2;N-m}}{\chi^2_{0.5;N-m}}} \quad (27)$$

Based on Equations (26) and (27), the half-triangle-shaped fuzzy number of $\widetilde{C}_0$ is $\Delta\widetilde{C}_0 = (C_M, C_R)$, where

$$C_M = C_0 \quad (28)$$

and

$$C_R = \left(C_0 + \frac{t_{0.005;N-m}}{\sqrt{mn}}\right)\sqrt{\frac{\chi^2_{0.995;N-m}}{\chi^2_{0.5;N-m}}}. \quad (29)$$

Then, the fuzzy membership function of $\widetilde{C}_0$ is defined below:

$$\eta(x) = \begin{cases} 0 & if \ \ x < C_M \\ 1 & if \ \ x = C_M \\ \alpha & if \ \ C_M < x \le C_R \\ 0 & if \ \ C_R < x \end{cases}, \quad (30)$$

where α is determined by

$$\left(C_0 + \frac{t_{\alpha/2;N-m}}{\sqrt{mn}}\right)\sqrt{\frac{\chi^2_{1-\alpha/2;N-m}}{\chi^2_{0.5;N-m}}} = x \quad (31)$$

Obviously, the critical value C_0 is the subject of the evaluation, and the estimated value of the index varies. Therefore, it is assumed that set A_T be the area in the graph of $\eta(x)$ and set A_R is the area in the graph of $\eta(x)$ but to the right of the vertical line $x = P^*_{QI0}$. Figure 2 displays a diagram of membership functions $\eta(x)$ and $\eta_0(x)$.

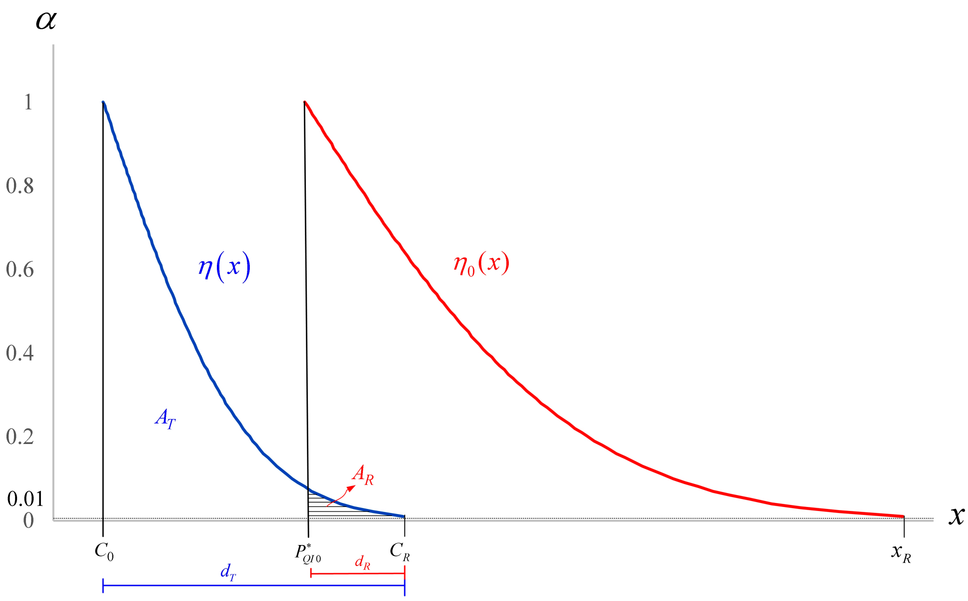

Figure 2. Diagram of membership functions $\eta(x)$ and $\eta_0(x)$.

Thus,

$$A_T = \{(x, \alpha) | C_0(1) \le x \le C_0(\alpha), 0 \le \alpha \le 1\} \quad (32)$$

and

$$A_R = \{(x,\alpha)|Q^*_{PI0} \le x \le C_0(\alpha), 0 \le \alpha \le a\}, \quad (33)$$

where $x(a) = P^*_{QI0}$. According to Buckley [26], the area of A_R is adopted as the numerator and the area of A_T as the denominator. Besides, the fuzzy test is performed based on the ratio of A_R/A_T. Some scholars think that the calculation of this ratio is relatively difficult and therefore can be replaced by the bottom-line length ratio [27,28]. First, let $d_T = C_R - C_M$ be the bottom length of $\eta(x)$. Based on Equations (19) and (20), d_T can be displayed as follows:

$$d_T = \left(C_0 + \frac{t_{0.005;N-m}}{\sqrt{mn}}\right)\sqrt{\frac{\chi^2_{0.995;N-m}}{\chi^2_{0.5;N-m}}} - C_0 \quad (34)$$

Then, let $d_R = C_R - Q^*_{PI0}$ be the bottom length placed between $x = P^*_{QI0}$ of $\eta_0(x)$ and $x = C_R$ of $\eta(x)$. According to Equations (20) and (28), d_R can be defined as follows:

$$d_R = \left(C_0 + \frac{t_{0.005;N-m}}{\sqrt{mn}}\right)\sqrt{\frac{\chi^2_{0.995;N-m}}{\chi^2_{0.5;N-m}}} - Q^*_{PI0} \quad (35)$$

Then, we define $d_R/(2d_T)$ as follows:

$$d_R/(2d_T) = \begin{cases} 0.5, \ if\ P^*_{QI0} \le C_0 \\ \dfrac{\left(P^*_{QI0} + \frac{t_{0.005;N-m}}{\sqrt{mn}}\right)\sqrt{\frac{\chi^2_{0.995;N-m}}{\chi^2_{0.5;N-m}}} - C_0}{2\left(\left(P^*_{QI0} + \frac{t_{0.005;N-m}}{\sqrt{mn}}\right)\sqrt{\frac{\chi^2_{0.995;N-m}}{\chi^2_{0.5;N-m}}} - P^*_{QI0}\right)}, \ if\ x_R \le P^*_{QI0} \le C_0 \ . \\ 0, \ if\ x_R \le P^*_{QI0} \end{cases} \quad (36)$$

Let $0 \le \phi_1 < \phi_2 \le 0.5$. In addition, based on Buckley [26] and Chen et al. [20], the two numbers are taken into account as follows:

(1) When $\phi_2 \le d_R/(2d_T) \le 0.5$, then H_0 will be rejected and $P_{QI} < k$ will be concluded.
(2) When $\phi_1 < d_R/(2d_T) < \phi_2$, then the decision regarding whether to reject or not to reject will not be made.
(3) When $0 \le d_R/(2d_T) < \phi_1$, then reject H_0 will not be rejected and $P_{QI} \ge k$ will be concluded.

5. A Practical Application

When the customer's requirement for the quality level of the process is 4-sigma ($k = 4$), then we adopt the following hypotheses:

null hypothesis H_0: $P_{QI} \ge 4$
versus
alternative hypothesis H_1:$P_{QI} < 4$.

In order to prove the practical application of our methodology, we have the observed value $(y_{h,1}, y_{h,2}, \cdots, y_{h,11})$ of hth subsample ($h = 1, 2, \ldots, 25$) with $n = 11$, $m = 25$, and $N = 275$ from in-control chart data. Then, the critical value can be denoted as follows:

$$C_0 = \frac{t_{0.01;250}\left(\Delta = 4\sqrt{275}\right)}{\sqrt{275}} = 3.599 \quad (37)$$

Thus, the half-triangular shaped fuzzy number of $\widetilde{C}_0$ is denoted as $\Delta\widetilde{C}_0 = (3.599, 4.197)$, where

$$C_M = C_0 = 3.599 \quad (38)$$

and

$$C_R = \left(C_0 + \frac{t_{0.005;N-m}}{\sqrt{mn}}\right)\sqrt{\frac{\chi^2_{0.995;N-m}}{\chi^2_{0.5;N-m}}} = 4.197. \tag{39}$$

Then, the fuzzy membership function of $\widetilde{C}_0$ is defined as follows:

$$\eta(x) = \begin{cases} 0 & if \quad x < 3.599 \\ 1 & if \quad x = 3.599 \\ \alpha & if \quad 3.599 < x \leq 4.197 \\ 0 & if \quad 4.197 < x \end{cases}. \tag{40}$$

Based on Equation (31), α is determined by

$$\left(3.599 + \frac{t_{\alpha/2;250}}{\sqrt{275}}\right)\sqrt{\frac{\chi^2_{1-\alpha/2;250}}{\chi^2_{0.5;250}}} = x. \tag{41}$$

In addition, δ_0^* and γ_0^* are the observed values of δ^* and γ^*, respectively, as follows:

$$\delta_0^* = \frac{1}{m}\sum_{h=1}^{m} \overline{y}_h = \frac{1}{25}\sum_{h=1}^{25} \overline{y}_h = 0.691 \tag{42}$$

and

$$\gamma_0^* = \sqrt{\frac{1}{N-m}\sum_{h=1}^{m} (n-1)s_h^2} = \sqrt{\frac{1}{250}\sum_{h=1}^{25} 10 \times s_h^2} = 0.085 \tag{43}$$

Therefore, the observed value of P^*_{QI0} is as follows:

$$P^*_{QI0} = \frac{1-\delta_0^*}{\gamma_0^*} = 3.635 \tag{44}$$

Thus, the observed value of the upper confidence limit of index P_{QI} with $\alpha = 0.01$ can be shown as follows:

$$x_R = \left(P^*_{QI0} + \frac{t_{0.005;250}}{\sqrt{275}}\right)\sqrt{\frac{\chi^2_{0.995;250}}{\chi^2_{0.5;250}}} = \left(3.635 + \frac{2.596}{\sqrt{275}}\right)\sqrt{\frac{311.346}{249.334}} = 4.237 \tag{45}$$

The half-triangle-shaped fuzzy number of $\widetilde{x}$ is $\Delta\widetilde{x} = (3.635, 4.237)$. Then, the fuzzy membership function of $\widetilde{x}$ is illustrated below:

$$\eta_0(x) = \begin{cases} 0 & if \quad x < 3.635 \\ 1 & if \quad x = 3.635 \\ \alpha & if \quad 3.635 < x \leq 4.237 \\ 0 & if \quad 4.237 < x \end{cases}, \tag{46}$$

where α is determined by

$$\left(3.635 + \frac{t_{\alpha/2;250}}{\sqrt{275}}\right)\sqrt{\frac{\chi^2_{1-\alpha/2;250}}{\chi^2_{0.5;250}}} = x. \tag{47}$$

Figure 3 is a diagram of membership function $\eta_0(x)$ with $\Delta\widetilde{x} = (3.635, 4.237)$ and $\eta(x)$ with $\Delta\widetilde{C}_0 = (3.599, 4.197)$.

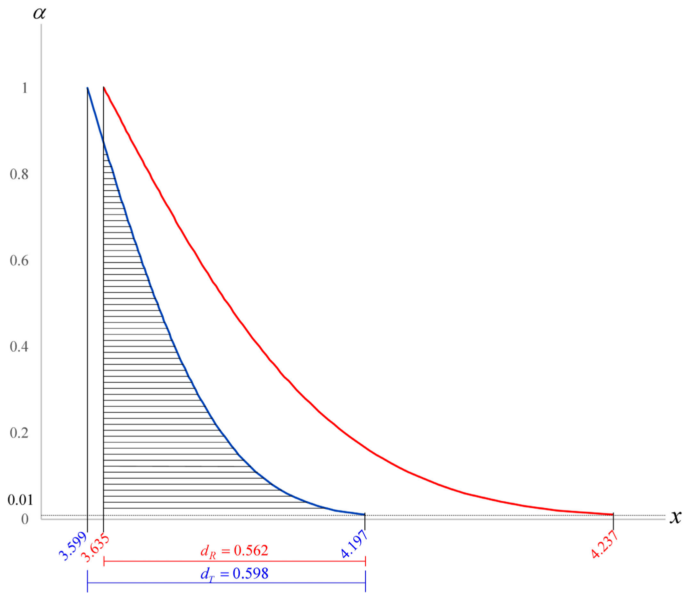

Figure 3. Diagram of $\eta_0(x)$ with $\Delta\tilde{x} = (3.635, 4.237)$ and $\eta(x)$ with $\Delta\tilde{C}_0 = (3.599, 4.197)$.

Then,

$$d_R = \left(C_0 + \frac{t_{0.005;250}}{\sqrt{275}}\right)\sqrt{\frac{\chi^2_{0.995;250}}{\chi^2_{0.5;250}}} - P^*_{QI0} = 4.197 - 3.635 = 0.562 \quad (48)$$

and

$$d_T = \left(C_0 + \frac{t_{0.005;250}}{\sqrt{275}}\right)\sqrt{\frac{\chi^2_{0.995;250}}{\chi^2_{0.5;250}}} - C_0 = 4.197 - 3.599 = 0.598 \quad (49)$$

Then,

$$\frac{d_R}{2d_T} = \frac{\left(P^*_{QI0} + \frac{t_{0.005;250}}{\sqrt{275}}\right)\sqrt{\frac{\chi^2_{0.995;250}}{\chi^2_{0.5;250}}} - C_0}{2\left(\left(P^*_{QI0} + \frac{t_{0.005;N-m}}{\sqrt{mn}}\right)\sqrt{\frac{\chi^2_{0.995;N-m}}{\chi^2_{0.5;N-m}}} - x_M\right)} = \frac{0.562}{2 \times 0.598} = 0.469 \quad (50)$$

According to Chen and Chang [32], this paper takes $\phi_1 = 0.2$ and $\phi_2 = 0.4$. Based on the fuzzy test rules in Section 4, the value of $d_R/(2d_T)$ is 0.469, so that H_0 will be rejected and $P_{QI} < 4$ will be concluded. Since the estimated value of the index is $P^*_{QI0} = 3.635$, greater than the critical value $C_0 = 3.599$, H_0 will not be rejected and $P_{QI} \geq 4$ will be concluded according to the statistic testing rules. In fact, $P^*_{QI0} = 3.635$ is much smaller than the required value of the index $P_{QI} \geq 4$. As noted above, the decision based on fuzzy hypothesis testing automatically uses more information about the statistical character of the decision problem than any traditional statistical hypothesis testing based on a single significance interval.

6. Conclusions

Many important components of machine tools have the-smaller-the-better quality characteristics (QC). In practice, due to cost consideration and limitations of processing technology, the measured value of the product is usually very far from the target value T and very close to USL, that is, parameter δ is usually less than 1 but very close to 1. Therefore, γ is usually very small. Not only can the process quality index P_{QI} measure the quality level of the process, but it also has a one-to-one mathematical relation with the process yield. Hence, this study adopted P_{QI} to come up with a quality fuzzy evaluation model for the smaller-the-better quality characteristics and used it as a decision-making basis for improvement. First, the upper confidence limit of the process quality index was derived from Boole's inequality and DeMorgan's theorem. Next, the upper confidence limit was adopted to construct a fuzzy membership function, and then a fuzzy testing method was built. To conclude, this method can incorporate past expert experience and accumulated data. Also, the accuracy of the evaluation can still be maintained under the condition of small sample sizes. Furthermore, the provided case can be convenient for readers and the industry to follow and apply.

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Nomenclature

X	a random sample
μ	process mean
σ	process standard deviation
USL	upper specification limit
Y	X/USL
δ	μ/USL
γ	σ/USL
P_{QI}	process quality index
T_V	the target value
$Yield\%$	process yield
Z	the standard normal distribution
$\Phi(\cdot)$	the cumulative function of the standard normal distribution
$\overline{Y}_h$	the sample mean of the hth subsample
S_h^2	the sample variance of the hth subsample
N	the total number of observations
n	the number of observations of each subsample
m	the number of subsample
δ^*	the estimates of δ
γ^*	the estimates of γ
P_{QI}^*	the estimator of index P_{QI}
T	the distributed as t_{N-m}
K	the distributed as χ^2_{N-m}
t_{N-m}	the Student's t-distribution with $N-m$ degree of freedom
χ^2_{N-m}	the chi-square distribution with $N-m$ degree of freedom
$\chi^2_{1-\alpha/2;N-m}$	the lower $1-\alpha/2$ quintile of χ^2_{N-m}
$1-\alpha$	the confidence level
$E_{L\delta}$;$E_{U\gamma}$	Events
$P(E_{L\delta})$	the probability of event $E_{L\delta}$
$P(E_{U\gamma})$	the probability of event $E_{U\gamma}$
$P(E_{L\delta}^C)$	the probability of event $E_{L\delta}^C$
$P\left(E_{U\gamma}^C\right)$	the probability of event $E_{U\gamma}^C$
UP_{QI}	the upper confidence limit of index P_{QI}
$(y_{h1}, y_{h2}, \ldots, y_{hn})$	the observed value of $(Y_{h1}, Y_{h2}, \ldots, Y_{hn})$ for hth subsample

$\overline{y}_h$	the observed value of $\overline{Y}_h$
s_h^2	the observed value of S_h^2
P_{QI0}^*	the observed value of P_{QI}^*
UP_{QI0}	the observed value of the upper confidence limit for index P_{QI}
$\widetilde{x}[\alpha]$	the α-cuts of the half-triangular shaped fuzzy number $\widetilde{x}$
$\Delta\widetilde{x} = (x_M, x_R)$	the half-triangular shaped fuzzy number of $\widetilde{x}$
$\eta_0(x)$	the fuzzy membership function of $\widetilde{x}$
k	the value of required level
H_0	null hypothesis
H_1	alternative hypothesis
C_0	the critical value
$t_{N-m}\left(\Delta = \sqrt{N}k\right)$	the non-central t distribution with $N-m$ degrees of freedom
$\Delta = \sqrt{N}k$	the non-centrality parameter
$t_{\alpha;N-m}\left(\Delta = \sqrt{N}k\right)$	the lower αth quantile of $t_{N-m}\left(\Delta = \sqrt{N}k\right)$
$\widetilde{C}_0[\alpha]$	the α-cuts of the triangular shaped fuzzy number $\widetilde{C}_0$
$\Delta\widetilde{C}_0 = (C_M, C_R)$	the half-triangular shaped fuzzy number of $\widetilde{C}_0$
$\eta(x)$	the fuzzy membership function of $\widetilde{C}_0$
A_T	the area in the graph of $\eta(x)$
A_R	the area in the graph of $\eta(x)$ but to the right of the vertical line $x = P_{QI0}^*$
d_T	the bottom length of $\eta(x)$
d_R	the bottom length placed between $x = P_{QI0}^*$ of $\eta_0(x)$ and $x = C_R$ of $\eta(x)$

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