

Article

Active Sonar Target Classification Method Based on Fisher's Dictionary Learning

Tongjing Sun ¹, Jiwei Jin ¹, Tong Liu ¹ and Jun Zhang ^{2,*}

¹ Automation Department, Hangzhou Dianzi University, Hangzhou 310018, China; stj@hdu.edu.cn (T.S.); jiwei_jin@126.com (J.J.); jw_hdu@163.com (T.L.)

² Zhejiang Electronic Information Products Inspection and Research Institute, Hangzhou 310000, China

* Correspondence: zj@zdzjy.org.cn; Tel.: +86-18698559336

Abstract: The marine environment is complex and changeable, and the interference of noise and reverberation seriously affects the classification performance of active sonar equipment. In particular, when the targets to be measured have similar characteristics, underwater classification becomes more complex. Therefore, a strong, recognizable algorithm needs to be developed that can handle similar feature targets in a reverberation environment. This paper combines Fisher's discriminant criterion and a dictionary-learning-based sparse representation classification algorithm, and proposes an active sonar target classification method based on Fisher discriminant dictionary learning (FDDL). Based on the learning dictionaries, the proposed method introduces the Fisher restriction criterion to limit the sparse coefficients, thereby obtaining a more discriminating dictionary; finally, it distinguishes the category according to the reconstruction errors of the reconstructed signal and the signal to be measured. The classification performance is compared with the existing methods, such as SVM (Support Vector Machine), SRC (Sparse Representation Based Classification), D-KSVD (Discriminative K-Singular Value Decomposition), and LC-KSVD (label-consistent K-SVD), and the experimental results show that FDDL has a better classification performance than the existing classification methods.

Keywords: active sonar target classification; fisher criteria; dictionary learning; sparse representation classification

check for
updates

Citation: Sun, T.; Jin, J.; Liu, T.; Zhang, J. Active Sonar Target Classification Method Based on Fisher's Dictionary Learning. *Appl. Sci.* **2021**, *11*, 10635. <https://doi.org/10.3390/app112210635>

Academic Editor:

Antonio Fernández-Caballero

Received: 1 September 2021

Accepted: 5 November 2021

Published: 11 November 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Active sonar target classification [1,2] uses impulse acoustic signals transmitted by sonar, and makes a decision on the target category and attributes according to the characteristics of the received echo signal. The echo signal is a function of the target type, distance, and azimuth, including the echo broadening, amplitude, phase, reflection coefficient, target scale, energy spectrum, and other characteristic information of the target. Therefore, active sonar completely utilizes the information carried in the echo, which is conducive to classification and recognition, and reflects the essential characteristics of the target. This is an important and effective method of target recognition. The main problem is that active sonar works in a complex and changeable marine environment, which incurs various interferences, such as noise and reverberation [3,4]. Recently, stealth technology has been undergoing daily development, and various small targets—such as torpedoes, mines, and underwater unmanned equipment—have played an increasingly important role in underwater combat and defense. As a result, the echo signal of the active sonar target gradually weakens, and even completely submerges in the complex marine environment of noise and various other interferences. In particular, the classification and recognition of underwater targets, especially weak targets, are facing more severe challenges.

2. Related Work

The main concept of sparse representation classification [5,6] is to use the sparseness of the original signal and a set of over-complete dictionaries to linearly express the original signal to obtain a set of sparse coefficients with the essential characteristics of the original signal, and finally to combine the sparse coefficient and redundant dictionary to distinguish the category through the reconstruction error of the signal. Wright et al. [7] were the first to apply sparse representation classification (SRC) to facial recognition, and proposed sparse representation-based classification. This algorithm directly uses the training sample set to construct a redundant dictionary, and limits the sparsity of the sparse coefficient through the 1-norm minimization method. Subsequently, Yang et al. [8] proposed SRC based on dictionary learning, and Wang applied it to the facial recognition [9]. This algorithm no longer uses the training signal directly as a dictionary, but performs adaptive learning based on the input original signal to obtain a discriminating dictionary, and finally, according to the testing data and sparse coefficient, reconstructs the error to complete the classification. Zhang et al. [10] presented discriminative K-singular value decomposition (D-KSVD), which adds a linear classifier penalty function related to the category label information of the objective function. Jiang et al. [11] proposed a label-consistent K-SVD (LC-KSVD), which introduced the “recognition sparse coefficient error” item based on dictionary learning. This item assigns “label consistency” to the sparse coefficient obtained in the final training. Yang et al. [12] proposed the Fisher discriminant dictionary learning (FDDL) model, which introduces Fisher recognition criteria to learn a structured dictionary, and achieves relatively better results; it has achieved excellent results in fields such as facial recognition [13,14] and image denoising [15,16].

However, compared with the abovementioned fields, active sonar work in a complex marine environment, as well as its interference, includes not only noise but also reverberation generated by the sea surface, seabed, and other interference targets [17]. The problem of reverberation suppression has always been a key issue in active sonar research [18]. As a frequency sweeper, FRFT [19] can obtain the characteristics of the target signal in the time, frequency, and multi-order domains through angle transformation. The characteristic domain of the target signal has a strong energy concentration at an optimal angle. Therefore, for the active sonar target classification problem in a complex ocean environment, FRFT can determine the characteristic amplitude of the target signal under the optimal order, and can filter the reverberation of the random signal to a certain extent in order to suppress the reverberation [20,21]. Thus, this paper proposes an active sonar target classification method based on Fisher’s dictionary learning to realize the classification of active sonar targets in a low-signal-reverberation ratio. The paper’s contributions are to suppress the reverberation using the optimal-order domain features of the fractional Fourier transform, and to increase the separability between the similarity objects by limiting the sparse coefficient based on the Fisher criterion.

3. Active Sonar Target Classification Method Based on Fisher’s Dictionary Learning

In order to achieve the classification of similar targets under the condition of a low-signal-reverberation ratio, we propose an active sonar target classification method based on Fisher’s dictionary learning. We use the reverberation suppression ability of the optimal-order domain feature of FRFT and the separable performance of the Fisher discriminant method on the target. Based on the dictionary learning-based SRC, the optimal-order domain feature is used, and the Fisher criterion is used as the limiting item on the sparse coefficient.

3.1. Fisher Discriminant Model and Solution Method

3.1.1. Fisher Discriminant Model

$A = [A_1, A_2, \dots, A_k]$ denotes a set of training samples, where A_k is the subset of the training samples from class k , where $D = [D_1, D_2, \dots, D_k]$ denotes the matrix of the dictionary; D_k is the dictionary matrix corresponding to the training signal of class k . We can write $X = [X_1, X_2, \dots, X_k]$, where X is the sparse coefficient matrix of the training signal

A on dictionary D , and X_i is the sparse representation matrix A_i over D . This method can be expressed as follows:

$$J_{(D,X)} = \operatorname{argmin}_{(D,X)} \{r(A, D, X) + \lambda_1 \|X\|_1 + \lambda_2 f(X)\} \tag{1}$$

where $r(A, D, X)$ is the limited-data fidelity item, $\|X\|_1$ is the sparse penalty, $f(X)$ is the sparse coefficient limit term, λ_1 is the regularization parameter, λ_2 is the adjustment parameter, and $\lambda_1 > 0, \lambda_2 > 0$.

(1) The limited-data fidelity item $r(A, D, X)$

The dictionary items of similar signals have a greater degree of relevance to the information of this type of signal, and a smaller degree of relevance to other types of signal information. We write a limited-data fidelity item $r(A, D, X)$ to restrict the dictionary.

We denote the sparse representation matrix as $X_i = [X_i^1, \dots, X_i^j, \dots, X_i^k]$ over D , where X_i^k is the sparse coefficient of the training signal A_i with class i over the k -th sub-dictionary D_k . The model of the algorithm is given as

$$r(A_i, D, X_i) = \|A_i - DX_i\|_F^2 + \|A_i - D_i X_i\|_F^2 + \sum_{\substack{j=1 \\ j \neq i}}^K \|D_j X_i^j\|_F^2 \tag{2}$$

where $\|A_i - DX_i\|_F^2$ is the penalty of the training signal A_i over the entire dictionary D , $\|A_i - D_i X_i\|_F^2$ is the penalty of the training signal A_i over the i -th sub-dictionary D_i , and $\|D_j X_i^j\|_F^2$ is the penalty of the training signal A_i over the j -th sub-dictionary D_j . In order to make the sub-dictionary corresponding to each type of signal more recognizable to the corresponding type of signal, $\|A_i - D_i X_i^i\|_F^2$ and $\|D_j X_i^j\|_F^2$ are assigned smaller values.

(2) Sparse coefficient limit term: $f(X)$

In order to improve the classification and recognition of the dictionary, the sparse coefficient within-class divergence $S_w(X)$ is minimized and the inter-class divergence $S_b(X)$ is maximized by Fisher's criterion, and the sparse coefficient is restricted. The model can be expressed as follows:

$$S_w(X) = \sum_{i=1}^K \sum_{x_k \in X_i} (x_k - m_i)(x_k - m_i)^T \tag{3}$$

$$S_b(X) = \sum_{i=1}^K n_i (m_i - m)(m_i - m)^T \tag{4}$$

where m_i and m are the mean vectors of the sparse coefficients X_i and X , respectively, and n_i is the number of samples of class A_i . In order to determine the optimal solution, we intuitively define this item in the form of a convex optimization function:

$$f(X) = \operatorname{tr}(S_w(X)) - \operatorname{tr}(S_b(X)) + \eta \|X\|_F^2 \tag{5}$$

To this end, we propose the model $J_{(D,X)}$ of FDDL, as follows:

$$J_{(D,X)} = \operatorname{argmin}_{(D,X)} = \left\{ \sum_{i=1}^K r(A_i, D, X) + \lambda_1 \|X\|_1 + \eta \|X\|_F^2 + \lambda_2 [\operatorname{tr}(S_w(X)) - \operatorname{tr}(S_b(X))] \right\} \tag{6}$$

3.1.2. FDDL Model-Solving—The Solution of Cross-Iteration

In order to solve the FDDL model, we use the method of cross-iteration, and update the sparse coefficient and dictionary alternately. First, training signal A is used as the initial learning dictionary; then the learning dictionary and sparse coefficient are alternately fixed, and the sparse coefficient and dictionary are updated until the specified number of iterations is reached or the conditions are met. The specific solution steps are as follows.

Step 1: Once the dictionary D is fixed, solve for the sparse coefficient X .

When dictionary D is fixed, $J_{(D,X)}$ is a sparse coding problem. When calculating X_i , $X_j (i \neq j)$ is fixed. The model is simplified as follows:

$$\min_{X_i} \{r(A_i, D, X_i) + \lambda_1 \|X_i\|_1 + \lambda_2 f_i(X_i)\} \tag{7}$$

$$f_i(X_i) = \|X_i - M_i\|_F^2 - \sum_{k=1}^K \|M_k - M\|_F^2 + \eta \|X_i\|_F^2 \tag{8}$$

where M_k and M are the mean column vector matrices of all of the categories. $r(A_i, D, X_i) + \lambda_2 f_i(X_i)$ is a strictly convex optimization, and is differentiable to X_i . In order to solve Equation (7), we use the iterative projection method [22].

Step 2: Once the sparse coefficient X is fixed, solve for dictionary D .

When the sparse coefficient X is fixed, the method of updating dictionary D is the same as above; when updating the sub-dictionary D_i , $D_j (i \neq j)$ is fixed. The model can be simplified as follows:

$$\min_{D_i} \left\{ \left\| A_i - D_i X_i - \sum_{\substack{j=1 \\ j \neq i}}^k D_j X_j \right\|_F^2 + \left\| A - D_i X_i \right\|_F^2 + \sum_{\substack{j=1 \\ j \neq i}}^k \left\| D_j X_j \right\|_F^2 \right\} \text{st. } \|d_l\|_2 = 1, l = 1, \dots, p_i \tag{9}$$

In order to solve Equation (9) as a quadratic planning problem, we solve each column of the dictionary items.

3.2. Active Sonar Target Classification Method Based on Fisher’s Dictionary Learning

The flowchart of the classification method is shown in Figure 1, and the specific steps are as follows.

(1) We input the active sonar target echo signal under the reverberation background and use the iterative two-dimensional peak search method to determine the optimal order (P_1, P_2, \dots, P_i) (different types of signals have different orders, P , and there are small differences in the orders obtained from different signals in the same signal).

(2) To obtain the U-domain value of each signal, we use the optimal order $P_i (i = 1, 2, \dots, n)$ to perform an FRFT on the measured signal under the reverberation background.

(3) We divide the U-domain value of each signal into a training sample set and a test sample set.

(4) For a comparison with the final classification results, we calculate the accuracy of the classification model by attaching its own initial category label to each type of test sample set.

(5) We import the test sample sets serially, and calculate different sparse coefficients according to the dictionary of each category and the sparse coding algorithm (each testing signal generates four sparse coefficients).

(6) The four sparse coefficients generated by each testing signal are reconstructed according to the corresponding category dictionary to reconstruct the reconstructed data (each testing signal generates four reconstructed signals, which are based on the sparse reconstruction of the testing signal based on the four types of sparse dictionaries).

(7) We calculate the matching degree between each testing signal and the four reconstructed signals it generates, and mark them as m_1, m_2, m_3 , and m_4 .

(8) We find the reconstructed signal with the highest matching degree by using the testing signal (the maximum of the four matching degrees), and determine that the test signal category is the same as the dictionary category corresponding to the maximum matching degree (m_1 corresponds to category 1, m_2 corresponds to category 2, m_3 corresponds to category 3, and m_4 corresponds to category 4).

(9) After storing the judgment category, we return to Step 3 to enter the loop (the number of cycles is determined by the number of selected order P), and finally, all of the categories determined under the order P are obtained.

(10) We find the category with the largest number of occurrences among all of the results of the judgment category; that is, the final judgment is the category of the current test signal.

(11) We determine whether the initial category labels of all of the test signals are the same as those of the classification results. If they are the same, it means that the classification is correct, and the correct number is counted to obtain the accuracy rate.

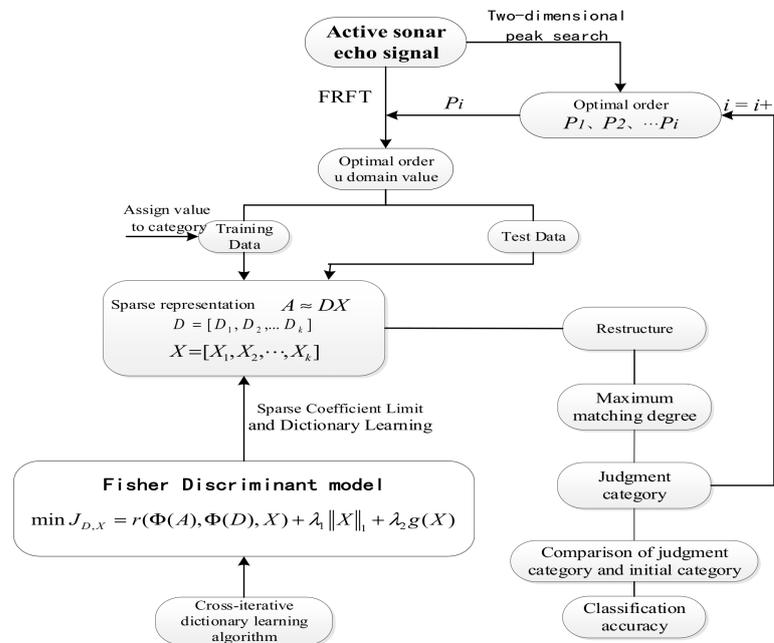


Figure 1. General block diagram of FRFT-DLSRC.

4. Method Performance Verification Based on Measured Data

In order to classify the lingual signals of the four types of active sonar targets (signal-to-mix ratios of -5 dB, -3 dB, 0 dB, 3 dB, and 5 dB), we use an active sonar target classification method based on FDDL, and validate its performance by comparing it with a support vector machine (SVM), SRC, D-KSVD, and LC-KSVD.

4.1. Overview of the Measured Data

We used an indoor pool as the test environment, as shown in Figure 2. Figure 3 is the test placement method. The depth of the transceiver transducer was the same as that of the targets. The distance between them was 4 m.



Figure 2. The indoor pool.

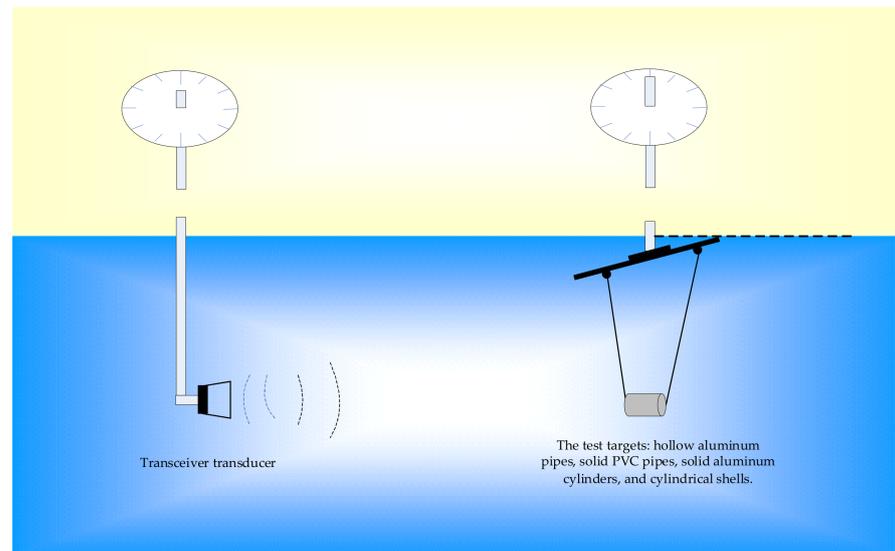


Figure 3. The test placement method.

Test parameters: a chirp signal was the incident signal (LFM), 100–200 kHz was the frequency range, 0.5 ms was the pulse width, and the incident angle was constant.

Test target: We set four types of targets—hollow aluminum pipes, solid PVC pipes, solid aluminum cylinders, and cylindrical shells.

The transducer sends the signals, and receives the signals from the objects. Thus, the echo signals are obtained. The four types of target echo signals are shown in Figures 4–6.

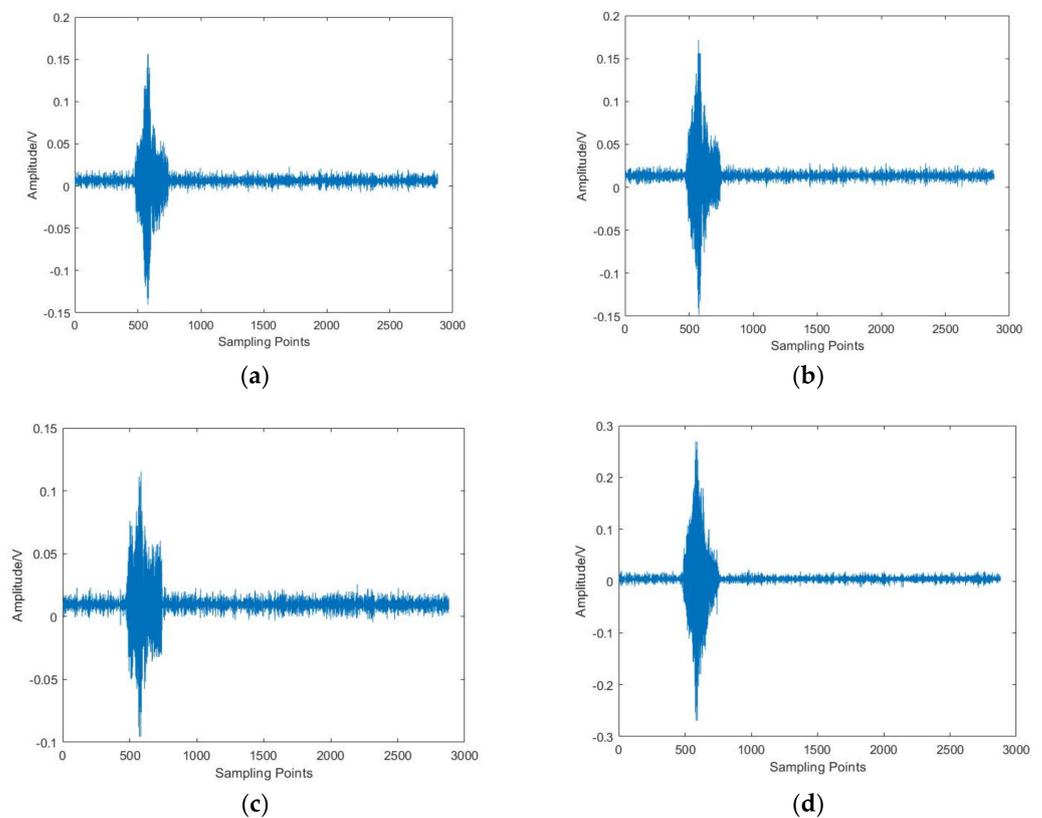


Figure 4. Four types of similar original target echo signal graphs. (a) Hollow aluminum tube echo signal; (b) solid PVC echo signal; (c) solid aluminum cylinder echo signal; (d) cylindrical shell echo signal.

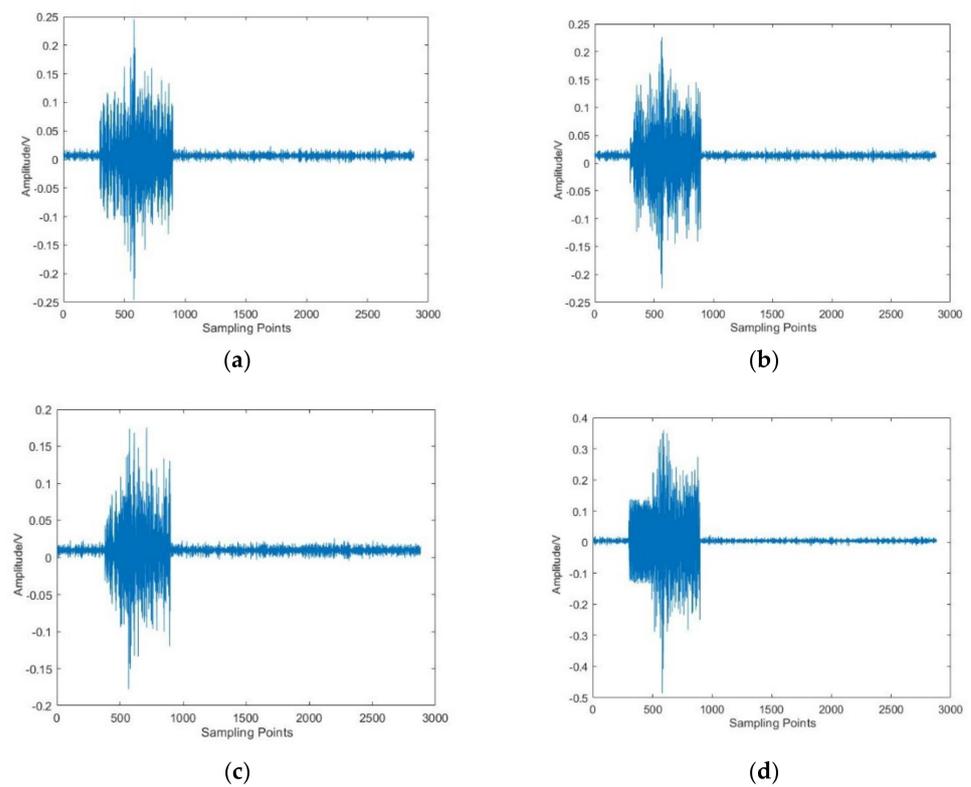


Figure 5. Signal diagram of the target echo with a signal mixing ratio $SRR = 0$ dB. (a) Hollow aluminum tube echo signal; (b) solid PVC echo signal; (c) solid aluminum cylinder echo signal; (d) cylindrical shell echo signal.

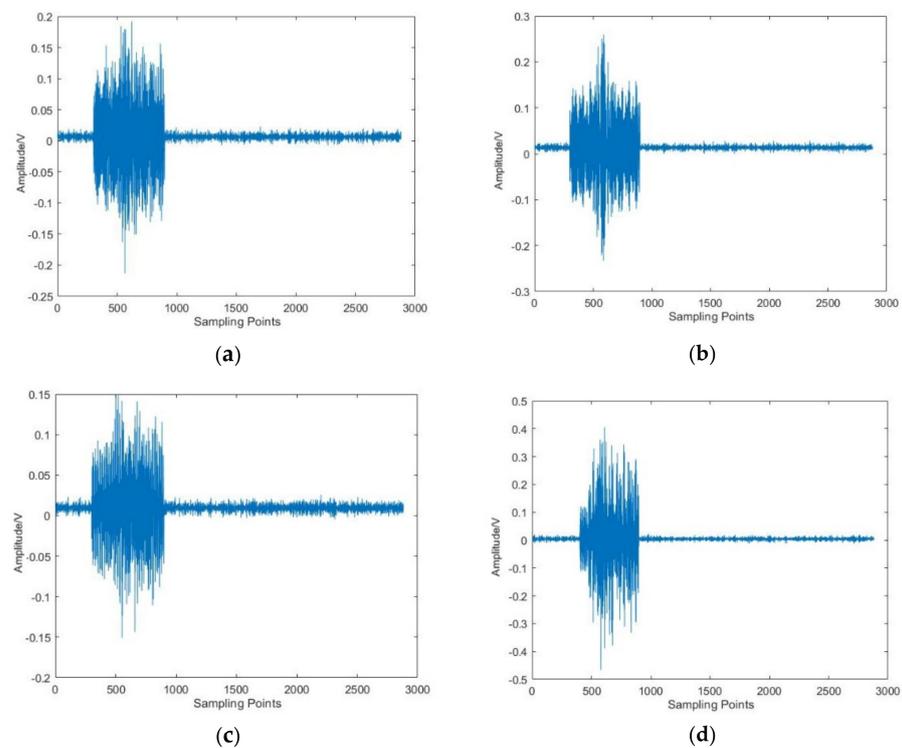


Figure 6. Signal diagram of the target echo with a signal mixing ratio $SRR = -3$ dB. (a) Hollow aluminum tube echo signal; (b) solid PVC echo signal; (c) solid aluminum cylinder echo signal; (d) cylindrical shell echo signal.

4.2. FRFT Optimal-Order Domain Characteristics of an Active Sonar Signal under a Reverberation Background

Taking the signal-mixing ratio equal to 0 dB as an example, the FRFT diagram of each signal type can be drawn as follows.

Figure 7 illustrates the two-dimensional distribution of each type of target signal after the FRFT, where a peak is observed in the two-dimensional distribution of the order P and the U -domain values. The peak value of each signal type has a certain difference in amplitude, the position corresponding to the U -domain, and the corresponding optimal order P . We first performed an FRFT on the four types of data, and then used the obtained signal U -domain values as input data to perform dictionary training and reconstruction classification based on these features.

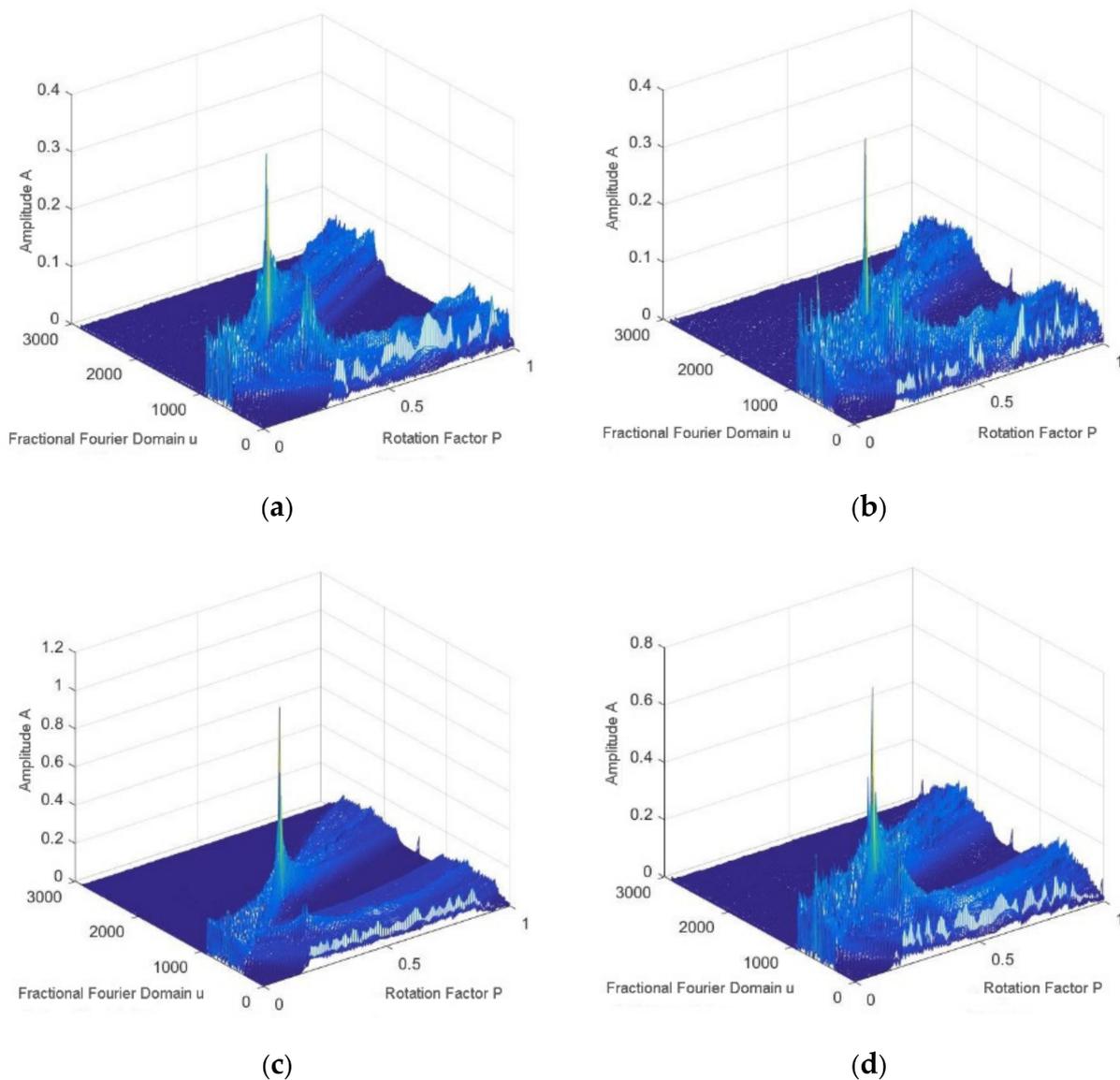


Figure 7. FRFT of each target signal's U -domain three-dimensional map. (a) U -domain of a hollow aluminum tube signal; (b) U -domain of a solid PVC pipe signal; (c) U -domain of a solid aluminum cylinder signal; (d) U -domain of a ribbed cylindrical shell signal.

4.3. SRC Based on FDDL

In order to verify the advantages of the Fisher criterion, we compared the results of using and not using the Fisher criterion to restrict the sparse coefficient when training the dictionary for $SRR = 0$ dB, as shown in Figure 8.

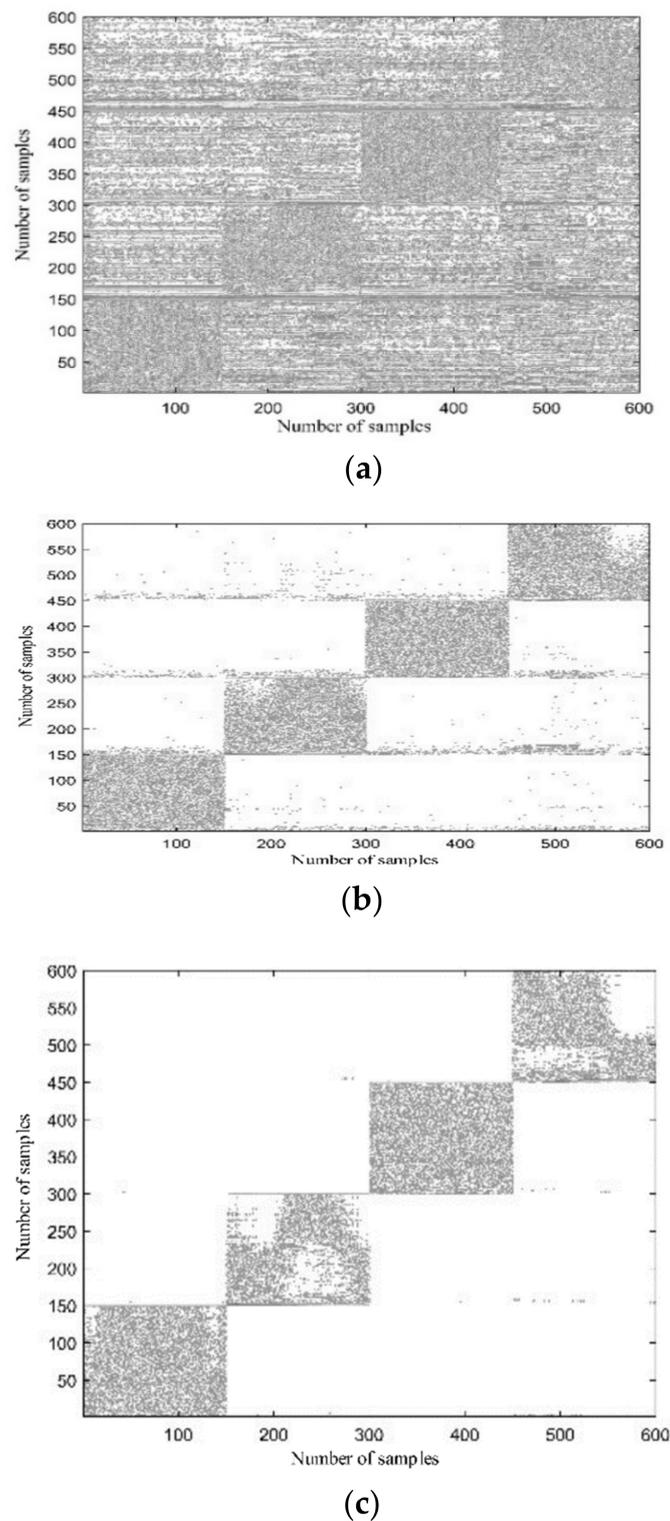


Figure 8. Diagram of the sparsity coefficient. (a) Contour map of sparse coefficients without Fisher restriction; (b) contour map of sparse coefficients using only the Fisher inter-class divergence restrictions; (c) contour map of sparse coefficients using the Fisher limit.

The above figure shows that, due to the limitation of the sparse coefficients in the divergence between the classes in the Fisher restriction criterion, the sparse coefficients are gathered or separated to different degrees according to the similarity of the signal characteristics, which indirectly assigns strong recognizability to the learned dictionary. This shows that Fisher’s restriction of sparse coefficients helps to improve the classification performance of the algorithm.

4.4. Comparative Analysis of the Classification Results

In order to compare the classification performance of FDDL with those of SVM, SRC, D-KSVD, and LC-KSVD, we conducted experiments with signal-to-mix ratios of -5 dB, -3 dB, 0 dB, 3 dB, and 5 dB. The results are shown in Figure 9 and Table 1.

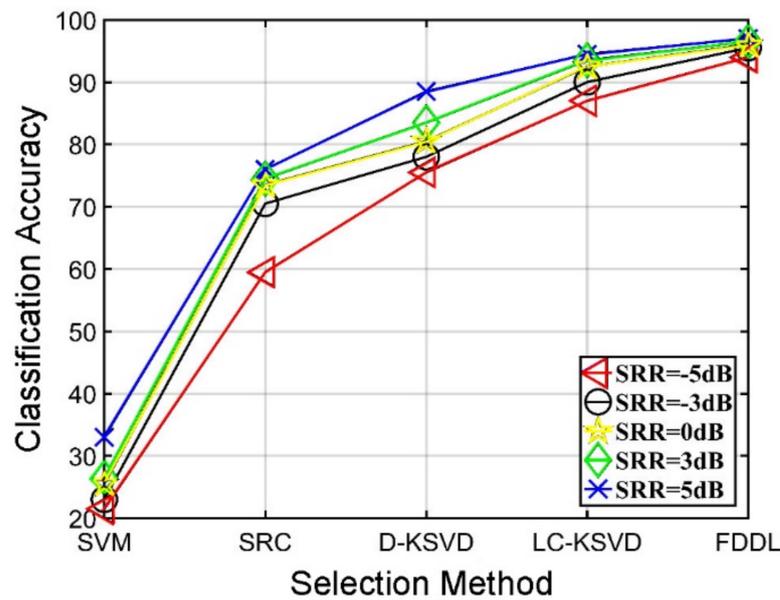


Figure 9. Contrast line chart with different reverberation methods.

Table 1. Comparison table of the methods under different reverberations.

SRR /dB	SVM /%	SRC /%	D-KSVD /%	LC-KSVD /%	FDDL /%
-5	21.5	59.5	75.5	87	94
-3	23	70.5	78	90	95.5
0	25.5	73.5	80.5	92.5	96
3	26	74.5	83.5	93.5	96.5
5	33	76	88.5	94.5	97

Figure 9 illustrates a comparison trend line graph of the recognition rate of different methods in different reverberation environments. Table 1 compares the specific classification recognition rates of the different methods in different reverberation environments. SVM has a low recognition rate for similar targets in a reverberant environment, as shown in Figure 9. This is because it is mainly suitable for two-class recognition; however, there are difficulties in multi-classification problems. Thus, we conclude that SVM is not suitable for underwater multi-target classification and recognition based on a reverberation environment.

Compared with the traditional classification method, SRC exhibits a relatively high overall recognition rate owing to the accurate extraction of essential information regarding the signal and the adaptive learning of the redundant dictionary. Therefore, this method is more suitable for multi-target classification and the recognition of similar features in a more complex reverberation environment. In the sparse classification algorithm, SRC directly classifies the training signal with reverberation as a redundant dictionary, and considerable

noise information is added to the dictionary, which leads to a poor classification recognition rate. The Fisher dictionary learning classification algorithm has a stronger recognition ability owing to the strong restriction of the Fisher criterion on the sparse coefficient; thus, the classification recognition rate under each signal-to-noise ratio is higher, and it has a better classification effect. The above experiments verify that FDDL has a high classification accuracy rate under various signal–mix ratios and still exhibits a suitable classification effect in underwater conditions with low signal–mix ratios; moreover, it exhibits suitable classification performance and anti-reverberation performance.

5. Conclusions

This paper proposes an active sonar target classification method based on Fisher’s dictionary learning to solve the similar target classification problem of active sonar in reverberant environments. In order to improve the reverberation suppression ability, we used fractional Fourier optimal-order domain features. In order to enhance the recognition of the learning dictionary, we added the Fisher restriction criterion to restrict the sparse coefficient on the basis of dictionary learning. The experimental results showed that the accuracy of 200 test data types at signal-to-reverberation ratios of -5 dB, -3 dB, 0 dB, 3 dB, and 5 dB were 94%, 95.5%, 96%, 96.5%, and 97%, respectively; the proposed method exhibited better classification performance than the existing classification methods. Moreover, this method has the features of anti-reverberation and similar target identification, and can effectively address the problem of active sonar target classification under reverberation conditions, especially low signal-to-reverberation ratio conditions. This provides strong support for the improvement of target detection, positioning, and recognition in a complex marine environment. In future work, two aspects will be further studied: one is the fusion of multi-order fractional Fourier domain features; the other is the classification of observed target data at sea.

Author Contributions: Conceptualization, Methodology, Writing—Original Draft Preparation, T.S. and J.J.; Validation, T.L.; Writing—Review and Editing, Funding Acquisition, T.S. and J.Z.; Formal Analysis, T.S. and J.J. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Extension Fund from the Underwater Test and Control Technology Key Laboratory, Grant No. YS24071802, and the Open Fund from Zhejiang Electronic Information Products Inspection and Research Institute, Grant No. KF201901.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Dalian Test and Control Institute (China) provided the experimental data support.

Acknowledgments: This article used experimental data collected at Dalian Test and Control Institute (China), and we gratefully acknowledge our colleagues for their experimental skills. Thanks are given for the support of the open funding from Zhejiang Province Information Security Key Laboratory. Thanks are given for the support of the funding from the Extension Fund from Underwater Test and Control Technology Key Laboratory.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Young, V.W.; Hines, P.C. Perception-based Automatic Classification of Impulsive-source Active Sonar Echoes. *J. Acoust. Soc. Am.* **2007**, *122*, 1502–1517. [[CrossRef](#)] [[PubMed](#)]
2. Philips, S.; Pitton, J.; Atlas, L. Perceptual Feature Identification for Active Sonar Echoes. In Proceedings of the OCEANS 2006, Boston, MA, USA, 18–21 September 2006; pp. 1–6. [[CrossRef](#)]
3. Zhang, M.H.; Sun, H.; Zhu, G.P. Modeling and verification of transceiver composite reverberation intensity in irregular sea area. *Tech. Acoust.* **2011**, *30*, 489–492.
4. Li, Y.Q.; Xu, Y.; Zhou, S.Z. Analysis of ocean reverberation characteristics and modeling and simulation research. *Ship Electron. Eng.* **2018**, *38*, 86–88+182.

5. Ye, M.J.; Hu, C.H.; Wan, L.G.; Lei, G.H. Fast single sample face recognition based on sparse representation classification. *Multimed. Tools Appl.* **2021**, *80*, 1–23. [[CrossRef](#)]
6. Song, X.N.; Shao, C.B.; Yang, X.; Wu, X. Sparse representation-based classification using generalized weighted extended dictionary. *Soft Comput.* **2017**, *21*, 4335–4348. [[CrossRef](#)]
7. Wright, J.; Yang, A.Y.; Ganesh, A.; Sastry, S.S.; Ma, Y. Robust Face Recognition via Sparse Representation. *Trans. Pattern Anal. Mach. Intell.* **2009**, *31*, 210–227. [[CrossRef](#)] [[PubMed](#)]
8. Yang, M.; Zhang, L.; Yang, J.; Zhang, D. Metaface Learning for Sparse Representation Based Face Recognition. In Proceedings of the 2010 IEEE International Conference on Image Processing, Hong Kong, China, 26–29 September 2010; pp. 1601–1604.
9. Wang, W.; Zhu, Z.; Lu, J. Sparse Representation for Face Recognition Based on Dictionary Learning and Locality Constraint. *Comput. Knowl. Technol.* **2018**, *14*, 200–202.
10. Zhang, Q.; Li, B. Discriminative K-SVD for dictionary learning in face recognition. In Proceedings of the 2010 IEEE Computer Society Conference on Computer Vision and Pattern Recognition, San Francisco, CA, USA, 13–18 June 2010; pp. 2691–2698.
11. Jiang, Z.; Lin, Z.; Davis, L.S. Label Consistent K-SVD: Learning a Discriminative Dictionary for Recognition. *IEEE Trans. Pattern Anal. Mach. Intell.* **2013**, *35*, 2651–2664. [[CrossRef](#)] [[PubMed](#)]
12. Yang, M.; Zhang, L.; Feng, X.; Zhang, D. Fisher Discrimination Dictionary Learning for sparse representation. In Proceedings of the IEEE International Conference on Computer Vision, Barcelona, Spain, 6–13 November 2011; pp. 543–550.
13. Wang, Y.; Wang, C.; Liang, L. Sparse representation theory and its application for face recognition. *Int. J. Smart Sens. Intell. Syst.* **2015**, *8*, 107–124. [[CrossRef](#)]
14. Wang, W.; Lu, J. Face Recognition Based on FDDL and Hybrid Sparse. *Sci. Technol. Vis.* **2017**, *26*, 87+67.
15. Wang, R.; Shen, M.; Li, Y.; Gomes, S. Multi-task Joint Sparse Representation Classification Based on Fisher Discrimination Dictionary Learning. *Comput. Mater. Contin.* **2018**, *57*, 25–48. [[CrossRef](#)]
16. Yang, M.; Chang, H.; Luo, W. Fisher discrimination dictionary pair learning for image classification. *Neurocomputing* **2016**, *269*, 68–73. [[CrossRef](#)]
17. Wang, D.; Shang, E. *Hydroacoustics*, 1st ed.; Science Press: Beijing, China, 1984; pp. 298–350.
18. Liu, B.; Lei, J. *Principles of Hydroacoustics*, 1st ed.; Harbin Engineering University Press: Harbin, China, 2009; pp. 187–210.
19. Namias, V. The Fractional Order Fourier Transform and its Application to Quantum Mechanics. *IMA J. Appl. Math.* **1980**, *25*, 241–265. [[CrossRef](#)]
20. Deng, B.; Tao, R.; Qi, L.; Liu, F. A Study on Anti-reverberation Method Based on Fractional Fourier Transform. *Acta Armamentarii* **2005**, *6*, 43–47.
21. Wang, B.; Tang, J.; Yu, F. Identification of Sonar Detection Signal Based on Fractional Fourier Transform. *Pol. Marit. Res.* **2018**, *25*, 125–131.
22. Mairal, J.; Ponce, J.; Sapiro, G.; Zisserman, A.; Bach, F. Supervised Dictionary Learning. In Proceedings of the Advances in Neural Information Processing Systems, Proceedings of the 2008 Conference, Paris, France, September 2008; pp. 1033–1040.