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Asymptotic Analytical Solution on Lamb Waves in Functionally Graded Nano Copper Layered Wafer

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Abstract: In this study, the feasibility of using Lamb waves in functionally graded (FG) nano copper layered wafers in nondestructive evaluation is evaluated. The elastic parameters and mass densities of these wafers vary with thickness due to the variation in grain size. The power series technique is used to solve the governing equations with variable coefficients. To analyze multilayered structures, of which the material parameters are continuous but underivable, a modified transfer matrix method is proposed and combined with the power series method. Results show that multiple modes of Lamb waves exist in FG nano copper wafers. Moreover, the gradient property leads to a decrease in phase velocity, and the absolute value of the phase velocity variation is positively correlated with the gradient coefficient. The phase velocity variation and variation rate in Mode 2 are smaller than those in other modes. The findings indicate that Mode 4 is recommended for nondestructive evaluation. However, if the number of layers is greater than four, the dispersion curves of the Lamb waves in the multilayer structures tend to coincide with those in the equivalent uniform structures. The results of this study provide theoretical guidance for the nondestructive evaluation of FG nanomaterial layered structures.

Keywords: functionally graded nano copper; lamb wave; power series technique; modified transfer matrix method



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1. Introduction

Since the introduction of functionally graded (FG) nano copper in 2011 [1], this novel material with a graded grain size distribution has attracted increasing scientific interest due to its high strength and high ductility [2–4]. The interest in materials with graded grain size distributions is not limited to nano copper and also includes titanium [5,6], zirconium [7], metallic glass [8], Fe–Mn austenitic steel [9], and low-carbon steel [10]. The grain refinement mechanism, microstructure characteristics, and material properties of samples of a gradient nano/microstructured surface layer on pure copper are investigated through several experiments [11]. Copper rod samples with gradient grain structures are also developed, measured in terms of microhardness, analyzed through electron backscatter diffraction, and simulated numerically [12]. With the in-depth study on FG nano copper materials [13,14] and their superiority in terms of strength and ductility, these materials will be widely utilized in engineering fields. The nondestructive evaluation of structures made of these materials is an important research topic in laboratory and engineering applications.

The guided wave technique, which is one of the most popular nondestructive evaluation methods is used to analyze Lamb, shear horizontal, Love, and Rayleigh waves. Related previous studies focused on the waves in homogenous structures [15,16]. Scientists began to investigate waves in structures made of FG materials in the 1990s. Researchers proposed various analytical methods, including directly analytical method, special function solution, the Wentzel–Kramers–Brillouin (WKB) method, the Legendre series method, and the power series method. The directly analytical method is used to solve different wave propagation problems when the material parameters vary with the same exponential

function [17,18]. In this case, the governing differential equation with variable coefficients can be transformed into a differential equation with constant coefficients to obtain an analytical solution. If the governing equations can be decoupled and the material parameter variations follow specific laws, the special function method can be applied to solve wave propagation problems [19–21]. The WKB method is used to study the horizontal shear waves in different FG layered structures with one displacement component [22,23]. However, this approach is only applicable to wave propagation problems with large wave numbers [20]. As asymptotic methods with series form, the Legendre [24–26] and power series methods [27] are utilized to analyze waves in various FG structures. Especially, the Legendre series method has been applied, not only for solving wave propagation behavior of macroscopic structure [28], but also that of microstructure using the combined modified couple stress theory [29,30]. Numerical analyses of wave propagation in inhomogeneous media are also conducted. The main idea of the numerical solution is to divide the FG medium into a multilayer model. The material parameters in each layer of the model are assumed to be homogenous [31–33]. The transfer matrix method, which is based on the continuity of the stress and displacement in the interface, is used to solve the wave propagation problem in multilayered structures [15,34]. This numerical technique can also be applied to solve wave propagation problems in various FG layered structures [35,36]. Considering that Lamb waves have been widely used in nondestructive evaluation in engineering application, Kuznetsov analyzed and compared the similarity and discrepancy of dispersion properties of Lamb wave propagation in both functionally graded and homogeneous plates [37,38]. Considering the nondestructive evaluation of FG nano metal layered structure, we focused on different guided waves in a single layer FG nano metal wafer or a multilayered FG nano metal wafer. An early report has been published on horizontal shear (SH) waves in these structures [39].

In the present study, the Lamb waves in an FG nano copper layered structure are investigated analytically. The grain size of the FG nano copper is assumed to vary along the thickness direction, and the other material parameters are deduced using the Mori–Tanaka effective field result proposed by Wang et al. [40]. The governing differential equations for describing the Lamb waves in a simple FG nano wafer are solved using the power series method. In addition, the Lamb wave propagation problem in a multilayered FG nano metal wafer with continuous and non-differentiable material parameters is solved through a modified transfer matrix method combined with the power series technique. On the basis of the abovementioned methods, numerical examples are analyzed, and the propagation properties of the Lamb waves in an FG nano copper wafer are discussed.

2. Statement of the Problem

2.1. Governing Equations

In this study, the Lamb waves in two types of FG nano copper layered wafer are considered (Figure 1). The wave propagation direction is the positive direction of the x_1 axis, and the thickness is along the x_3 axis. If the grain size of the FG nano copper wafer varies with thickness and is a function of x_3 , then the corresponding material parameters, including the elastic parameters and mass density, are not constants, but are functions of x_3 . Figure 1a presents a simple FG nano copper wafer with thickness h and a monotonously changing grain size, whereas Figure 1b displays a multilayered FG nano copper wafer composed of several simple wafers with interfaces that have continuous grain sizes. The thickness of each simple wafer is h , and the total thickness is H . Therefore, the grain size is alternately increasing and decreasing.

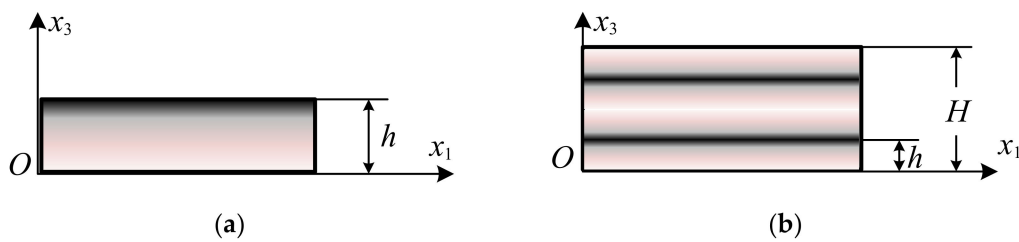


Figure 1. Two types of FG nano copper wafer. (a) Simple FG nano copper wafer; (b) multi-layered FG nano copper wafer.

Denoting the displacement component along the x_j direction as u_j , the relationship between the strain and the displacement is expressed as:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (1)$$

where ε_{ij} is the strain tensor, and the comma followed by subscript i indicates space differentiation with respect to the corresponding coordinate x_i .

The constitutive equation for elastic materials is written as:

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} \quad (2)$$

where σ_{ij} is the stress tensor, c_{ijkl} is the elastic parameter, and the repeated index in the subscript implies summation with respect to that index. The two independent elastic parameters for isotropic elastic materials can be expressed by the Lamé parameters λ and μ . The elastic parameters of the FG nano copper wafer that depend on the grain size are not constants, but are functions of position.

The motion equation is expressed as:

$$\sigma_{ij,j} = \rho \ddot{u}_i \quad (3)$$

where ρ is the mass density, which depends on the grain size and is a function of x_3 .

The displacement components of the Lamb waves propagating in the FG nano copper wafer are expressed as:

$$u_1 = u_1(x_1, x_3, t), u_2 = 0, u_3 = u_3(x_1, x_3, t) \quad (4)$$

and the governing equations for the mechanical displacements are defined as:

$$\begin{aligned} (\lambda + 2\mu) \frac{\partial^2 u_1}{\partial x_1^2} + \lambda \frac{\partial^2 u_3}{\partial x_1 \partial x_3} + \mu \left(\frac{\partial^2 u_3}{\partial x_1 \partial x_3} + \frac{\partial^2 u_1}{\partial x_3^2} \right) + \frac{d\mu}{dx_3} \left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \right) &= \rho \frac{\partial^2 u_1}{\partial t^2} \\ (\lambda + 2\mu) \frac{\partial^2 u_3}{\partial x_3^2} + \lambda \frac{\partial^2 u_1}{\partial x_1 \partial x_3} + \mu \left(\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + \frac{\partial^2 u_3}{\partial x_1^2} \right) + \frac{d\lambda}{dx_3} \frac{\partial u_1}{\partial x_1} + \frac{d(\lambda + 2\mu)}{dx_3} \frac{\partial u_3}{\partial x_3} &= \rho \frac{\partial^2 u_3}{\partial t^2} \end{aligned} \quad (5)$$

2.2. Boundary Conditions

The traction-free boundary conditions should be satisfied in the wave propagation in a simple or layered wafer. Moreover, the interface continuity conditions should be considered when investigating wave propagation in the latter. The boundary and the continuity conditions of a layered wafer composed of N single layers of wafer can be expressed as follows, where superscript $\langle l \rangle$ represents the l th wafer.

- (i) Lamb waves in a simple wafer Traction-free conditions: $\sigma_{33} = 0$ and $\sigma_{13} = 0$ at $x_3 = 0, h$
- (ii) Lamb waves in a multilayered wafer Traction-free conditions: $\sigma_{33}^{\langle 1 \rangle} = 0$ and $\sigma_{13}^{\langle 1 \rangle} = 0$ at $x_3 = 0$ and $\sigma_{33}^{\langle N \rangle} = 0$ and $\sigma_{13}^{\langle N \rangle} = 0$ at $x_3 = H$ Continuity conditions: $\sigma_{33}^{\langle l \rangle} = \sigma_{33}^{\langle l+1 \rangle}$, $\sigma_{13}^{\langle l \rangle} = \sigma_{13}^{\langle l+1 \rangle}$, $u_1^{\langle l \rangle} = u_1^{\langle l+1 \rangle}$, and $u_3^{\langle l \rangle} = u_3^{\langle l+1 \rangle}$ at $x_3 = lh$

2.3. Material Parameters

The nanocrystalline material can be considered as a two-phase composite that comprises the interface and the grain phases (Figure 2). By assuming that both phases are isotropic, Wang [40] derived the effective modulus of elasticity of nanocrystalline materials using the Mori–Tanaka effective field method. The results included the effective bulk modulus (or volume modulus) K , shear modulus G , and Young's modulus (or elastic modulus) E of nanocrystalline materials. In this study, the Lamé parameters λ and μ and mass density ρ of FG nano copper are derived on the basis of Wang's approach.

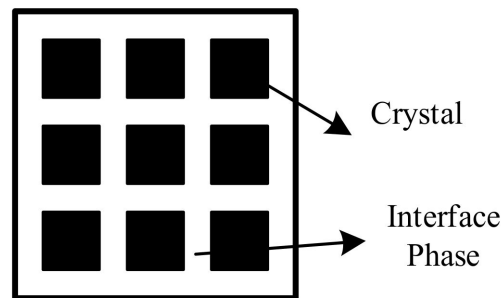


Figure 2. Microstructure model of nanocrystalline materials [40].

K and G can be expressed in terms of E and Poisson's ratio ν .

$$K = \frac{E}{3(1-2\nu)}, G = \frac{E}{2(1+\nu)} \quad (6)$$

$$K_m = \frac{E_m}{3(1-2\nu_m)}, K_c = \frac{E_c}{3(1-2\nu_c)}, G_m = \frac{E_m}{2(1+\nu_m)}, G_c = \frac{E_c}{2(1+\nu_c)} \quad (7)$$

where subscripts m and c represent the parameters of the interface phase and the crystal, respectively. In this study, the Poisson's ratio of the interface phase is assumed to be the same as that of the crystal, that is, $\nu_m = \nu_c$. Based on the Mori–Tanaka effective field method, the relationship between the elastic moduli of the interface phase and the crystal is defined as [40].

$$\frac{E_m}{E_c} = \frac{E(r)}{E(r_0)} = \frac{[(n+1)(\frac{r_0}{r})^{n+3} - (m+1)(\frac{r_0}{r})^{m+3}]}{(n-m)} \quad (8)$$

where m and n are material constants, r is the average atomic spacing, and r_0 is the equilibrium position. The relationship of the average atomic spacing and mass density of the interfacial phase ρ_m is expressed as [40].

$$\frac{r_0}{r} = \left(\frac{\rho_m}{\rho_c} \right)^{\frac{1}{3}} \quad (9)$$

The volume fraction of the crystal c_c is determined as [40].

$$c_c = \left(\frac{L^3}{L+d} \right)^3 = \left(1 + \frac{d}{L} \right)^{-3} \quad (10)$$

where L is the average radius of the grain and d is the average thickness of the interfacial phase.

From Reference [40], the parameter β_1 and β_2 are defined as:

$$\beta_1 = \frac{2(4-5\nu_m)}{15(1-\nu_m)} \text{ and } \beta_2 = 3-5\beta_1 \quad (11)$$

The shear and effective bulk moduli of nanocrystalline materials are obtained as:

$$K = K_m + \frac{c_c K_m (K_c - K_m)}{K_m + \beta_2 (1 - c_c) (K_c - K_m)}, G = G_m + \frac{c_c G_m (G_c - G_m)}{G_m + \beta_1 (1 - c_c) (G_c - G_m)} \quad (12)$$

The Lamé parameters λ and μ can be determined in terms of K and G as:

$$\lambda = K - \frac{2}{3}G, \mu = G \quad (13)$$

The mass density also depends on the volume fraction of the crystal.

$$\rho = \rho_c c_c + \rho_m (1 - c_c) \quad (14)$$

where ρ_c and ρ_m are the mass densities of the ideal crystal and the interfacial phase, respectively.

3. Solution to the Problem

In this section, the governing equations are simplified to be a set of ordinary differential equations with variable coefficients. For a simple FG layer, the governing equations are solved based on the power series method. Furthermore, the modified transfer matrix method is proposed to solve the wave propagation equations in layered structures.

3.1. Ordinary Differential Equations with Variable Coefficients and Power Series Solution

The solutions to the governing equations for Lamb waves propagating in an FG nano copper wafer are expressed as:

$$u_1^{(1)} = U_1^{(1)}(x_3) \exp(ikx_1 - \omega t), u_3^{(1)} = U_3^{(1)}(x_3) \exp(ikx_1 - \omega t), \text{ and } u_2^{(1)} = 0 \quad (15)$$

where i is the imaginary unit, ω is the frequency and satisfies $\omega = ck$, k and c are the wave number and wave velocity, respectively, $U_1(z)$ and $U_3(z)$ are the unknown amplitudes of the displacement, and superscript $\langle l \rangle$ represents the l th wafer for the layered wafer ($l = 1 \sim N$). This superscript is not present in the simple wafer. The relationship between the wave number and wave length l is expressed as $kl = 2\pi$. Given that the form of the governing equation is similar, the superscript is ignored when solving the above equations.

Substituting Equation (15) to Equation (5) yields:

$$\begin{aligned} \mu \frac{d^2 U_1}{dx_3^2} + \frac{d\mu}{dx_3} \frac{dU_1}{dx_3} + i(\lambda + \mu)k \frac{dU_3}{dx_3} + (\rho c^2 - \lambda - 2\mu)k^2 U_1 + i \frac{d\mu}{dx_3} k U_3 &= 0 \\ (\lambda + 2\mu) \frac{d^2 U_3}{dx_3^2} + i(\lambda + \mu)k \frac{dU_1}{dx_3} + \frac{d(\lambda + 2\mu)}{dx_3} \frac{dU_3}{dx_3} + ik \frac{d\lambda}{dx_3} U_1 + (\rho c^2 - \mu)k^2 U_3 &= 0 \end{aligned} \quad (16)$$

Equation (6) is a set of a set of ordinary differential equations with variable coefficients. In this study, power series method, which both the coefficients and solution are expressed as power series form, is applied for solving the equations.

Considering that the Lamé parameters and mass density are functions of thickness, they can be expressed in power series form as:

$$\lambda = f_1\left(\frac{x_3}{h}\right) = \sum_{n=0}^{\infty} a_{n[1]} \left(\frac{x_3}{h}\right)^n, \mu = f_2\left(\frac{x_3}{h}\right) = \sum_{n=0}^{\infty} a_{n[2]} \left(\frac{x_3}{h}\right)^n, \text{ and } \rho = f_3\left(\frac{x_3}{h}\right) = \sum_{n=0}^{\infty} a_{n[3]} \left(\frac{x_3}{h}\right)^n \quad (17)$$

where $a_{n[1]}$, $a_{n[2]}$, and $a_{n[3]}$ are the n th coefficients of the Taylor series of λ , μ , and ρ , respectively.

To solve Equation (16) with variable coefficients, the solution is assumed to also follow the power series form.

$$U_1 = \sum_{n=0}^{\infty} s_n \left(\frac{x_3}{h}\right)^n, U_3 = -i \sum_{n=0}^{\infty} t_n \left(\frac{x_3}{h}\right)^n \quad (18)$$

By substituting Equations (17) and (18) to Equation (16) and equating the coefficient of $(x_3/h)^n$ to zero, the following recursive equations are obtained.

$$\begin{aligned}
& \left[\sum_{n=0}^{\infty} a_{n[2]} \left(\frac{x_3}{h} \right)^n \right] \left[\sum_{n=0}^{\infty} s_{n+2}(n+2)(n+1) \left(\frac{x_3}{h} \right)^n \right] + \left[\sum_{n=0}^{\infty} a_{n+1[2]}(n+1) \left(\frac{x_3}{h} \right)^n \right] \left[\sum_{n=0}^{\infty} s_{n+1}(n+1) \left(\frac{x_3}{h} \right)^n \right] \\
& + (kh) \left[\sum_{n=0}^{\infty} a_{n+1[2]}(n+1) \left(\frac{x_3}{h} \right)^n \right] \left[\sum_{n=0}^{\infty} t_n \left(\frac{x_3}{h} \right)^n \right] + (kh) \left[\sum_{n=0}^{\infty} (a_{n[1]} + a_{n[2]}) \left(\frac{x_3}{h} \right)^n \right] \left[\sum_{n=0}^{\infty} t_{n+1}(n+1) \left(\frac{x_3}{h} \right)^n \right] \\
& + (kh)^2 \left[\sum_{n=0}^{\infty} (a_{n[3]}c^2 - a_{n[1]} - 2a_{n[2]}) \left(\frac{x_3}{h} \right)^n \right] \left[\sum_{n=0}^{\infty} s_n \left(\frac{x_3}{h} \right)^n \right] = 0 \\
& \left[\sum_{n=0}^{\infty} (a_{n[1]} + 2a_{n[2]}) \left(\frac{x_3}{h} \right)^n \right] \left[\sum_{n=0}^{\infty} t_{n+2}(n+2)(n+1) \left(\frac{x_3}{h} \right)^n \right] - (kh) \left[\sum_{n=0}^{\infty} (n+1)a_{n+1[1]} \left(\frac{x_3}{h} \right)^n \right] \left[\sum_{n=0}^{\infty} s_n \left(\frac{x_3}{h} \right)^n \right] \\
& - (kh) \left[\sum_{n=0}^{\infty} (a_{n[1]} + a_{n[1]}) \left(\frac{x_3}{h} \right)^n \right] \left[\sum_{n=0}^{\infty} s_{n+1}(n+1) \left(\frac{x_3}{h} \right)^n \right] + (kh)^2 \left[\sum_{n=0}^{\infty} (a_{n[2]}c^2 - a_{n[2]}) \left(\frac{x_3}{h} \right)^n \right] \left[\sum_{n=0}^{\infty} t_n \left(\frac{x_3}{h} \right)^n \right] \\
& + \left[\sum_{n=0}^{\infty} (a_{n+1[1]} + 2a_{n+1[2]}) (n+1) \left(\frac{x_3}{h} \right)^n \right] \left[\sum_{n=0}^{\infty} t_{n+1}(n+1) \left(\frac{x_3}{h} \right)^n \right] = 0
\end{aligned} \tag{19}$$

By equating the coefficient of $\frac{x_3}{h}$ to zero, the following recursive relationships are achieved.

$$\begin{aligned}
& \sum_{i=0}^n (i+2)(i+1)a_{n-i[2]}s_{i+2} + \sum_{i=0}^n (n-i+1)(i+1)a_{n-i+1[2]}s_{i+1} + (kh) \sum_{i=0}^n (n-i+1)a_{n-i+1[2]}t_i \\
& + (kh)^2 \sum_{i=0}^n (a_{n-i[3]}c^2 - a_{n-i[1]} - 2a_{n-i[2]})s_i + (kh) \sum_{i=0}^n (i+1)(a_{n-i[1]} + a_{n-i[2]})t_{i+1} \\
& = 0 \sum_{i=0}^n (i+2)(i+1)(a_{n-i[1]} + 2a_{n-i[2]})t_{i+2} + (kh)^2 \sum_{i=0}^n (a_{n-i[3]}c^2 - a_{n-i[2]})t_i \\
& - (kh) \sum_{i=0}^n (n-i+1)a_{n-i+1[1]}s_i - (kh) \sum_{i=0}^n (i+1)(a_{n-i[1]} + a_{n-i[2]})s_{i+1} \\
& + \sum_{i=0}^n (n-i+1)(i+1)(a_{n-i+1[1]} + 2a_{n-i+1[2]})t_{i+1} = 0
\end{aligned} \tag{20}$$

where s_0, s_1, t_0 , and t_1 are undetermined coefficients. For $i \geq 2$, all s_i and t_i values are linear functions of s_0, s_1, t_0 , and t_1 .

To decouple the undetermined coefficients, the following matrix is constructed.

$$\begin{pmatrix} s_{0j}, & s_{1j}, & t_{0j}, & t_{1j} \end{pmatrix} = \mathbf{I} \tag{21}$$

where $j = 1 - 4$ and $\mathbf{I}$ is a 4×4 unity matrix. The equivalent form of Equation (18) is written as:

$$U_1 = \sum_{j=1}^4 C_j \sum_{n=0}^{\infty} s_{nj} \left(\frac{x_3}{h} \right)^n, U_3 = -i \sum_{j=1}^4 C_j \sum_{n=0}^{\infty} t_{nj} \left(\frac{x_3}{h} \right)^n \tag{22}$$

where $C_j (j = 1 - 4)$ represents the undetermined constants. For $n \geq 2$, the values of s_{nj} and t_{nj} can be determined using Equation (20). The physical meaning of $C_j (j = 1 - 4)$ is the displacement components and the dimensionless derivatives of displacements $U_1, iU_3, h d \frac{U_1}{dx_3}$ and $i h d \frac{U_3}{dx_3}$ at $x_3 = 0$, respectively.

The boundary conditions should be considered when analyzing the Lamb waves propagating in a simple FG nano copper wafer. By substituting Equation (22) into the boundary conditions, the linear algebraic equations with respect to $C_j (j = 1 - 4)$ can be obtained. Considering the sufficient and necessary condition that a nontrivial solution should exist, the determinant of the coefficient matrix should be equal to zero. This condition leads to the following dispersion relationship for Lamb waves in a simple FG nano copper wafer.

$$|T_{ij}| = 0 \tag{23}$$

where:

$$T_{11} = \lambda^0 kh, T_{14} = -(\lambda^0 + 2\mu^0), T_{22} = 1, T_{23} = kh$$

$$T_{3j} = \sum_{n=0}^{\infty} \left[\lambda^h s_{nj} k h - (\lambda^h + 2\mu^h) (n+1) t_{(n+1)j} \right],$$

$$T_{4j} = \sum_{n=0}^{\infty} \left[t_{nj} k h + (n+1) s_{(n+1)j} \right], j = 1 - 4$$

All other terms are equal to zero. Superscripts 0 and h represent the material parameters of the lower and upper surfaces of the simple FG nano copper wafer, respectively.

3.2. Modified Transfer Matrix Method

The transfer matrix method is commonly used in the study of laminated structures. For example, when scientists investigate the wave propagation problem in FG structures, these structures can be simplified as multilayer structures with layers that are assumed to be homogeneous. Interface continuity conditions include stress and displacement continuity conditions. When multilayer structures are used to simulate FG structures, the strain in the interface is discontinuous because of the discontinuous properties of the elastic parameters in the interface.

The elastic parameters in the interface of the layered structure depicted in Figure 1b are continuous; thus, the strain components are continuous. The present study assumes that the displacement and strain components are continuous, the displacement components can be expressed as power series forms, and the strain components are related to the derivatives of the displacements with respect to the coordinates. Mathematically, the displacement components and their derivatives are continuous in the surface.

Let $x_3^{(l)} = x_3 - (l-1)h$, where l denotes the l th layer. The derivative of an arbitrary function with respect to x_3 is equal to the derivative of the function with respect to $x_3^{(l)}$. The governing equations for the Lamb waves in N -layered structures include l groups. These equations are similar to Equation (16), where the displacement amplitude components U_1 and U_3 ; material parameters λ , μ , and ρ ; and x_3 are replaced by $U_1^{(l)}$, $U_3^{(l)}$, $\lambda^{(l)}$, $\mu^{(l)}$, $\rho^{(l)}$, and $x_3^{(l)}$, respectively.

On the basis of the power series solution mentioned above, the solutions for the governing equations of each layer are obtained in the same form as Equation (22).

$$U_1^{(l)} = \sum_{j=1}^4 C_j^{(l)} \sum_{n=0}^{\infty} s_{nj}^{(l)} \left(\frac{x_3^{(l)}}{h} \right)^n, U_3^{(l)} = -i \sum_{j=1}^4 C_j^{(l)} \sum_{n=0}^{\infty} t_{nj}^{(l)} \left(\frac{x_3^{(l)}}{h} \right)^n \quad (24)$$

where:

$$\begin{pmatrix} s_{0j}^{(l)}, & s_{1j}^{(l)}, & t_{0j}^{(l)}, & t_{1j}^{(l)} \end{pmatrix} = I \quad (25)$$

The values of n , $s_{nj}^{(l)}$, and $t_{nj}^{(l)}$ can be determined using Equation (20). The material parameters for odd and even layers are different. Similarly, the coefficients of the equation and the obtained solutions are different.

The governing equation in the first layer is solved, and the displacement amplitude components $U_1^{(1)}$ and $U_3^{(1)}$ are calculated based on the solution. Then, the relationship between the displacement components and the dimensionless derivatives of the displacement concerning coordinates at $x_3^{(1)} = h$ and $C_j^{(1)}$ ($j = 1 - 4$) is obtained as:

$$\begin{pmatrix} U_1^{(1)} \Big|_{x_3^{(1)}=h}, & h \frac{dU_1^{(1)}}{dx_3^{(1)}} \Big|_{x_3^{(1)}=h}, & iU_3^{(1)} \Big|_{x_3^{(1)}=h}, & ih \frac{dU_3^{(1)}}{dx_3^{(1)}} \Big|_{x_3^{(1)}=h} \end{pmatrix}^T = A \begin{pmatrix} C_1^{(1)}, & C_2^{(1)}, & C_3^{(1)}, & C_4^{(1)} \end{pmatrix}^T \quad (26)$$

where superscript T represents the matrix transposition and A is a 4×4 matrix named the transfer matrix of odd layers.

By considering the physical meaning of $C_j^{(1)}$ ($j = 1 - 4$) and the continuous condition in the interface, we obtain:

$$\begin{pmatrix} U_1^{(1)} \Big|_{x_3^{(1)}=h}, & h \frac{dU_1^{(1)}}{dx_3^{(1)}} \Big|_{x_3^{(1)}=h}, & iU_3^{(1)} \Big|_{x_3^{(1)}=h}, & ih \frac{dU_3^{(1)}}{dx_3^{(1)}} \Big|_{x_3^{(1)}=h} \end{pmatrix}^T = \begin{pmatrix} C_1^{(2)}, & C_2^{(2)}, & C_3^{(2)}, & C_4^{(2)} \end{pmatrix}^T \quad (27)$$

Subsequently, the governing equation in the second layer is solved, and the transfer matrix of the even layer B , which can be used to describe the relationship between the displacement components and the dimensionless derivatives of the displacement concerning coordinates at $x_3^{(2)} = h$ and $C_j^{(2)}$ ($j = 1 - 4$), is calculated as:

$$\left(U_1^{(2)} \Big|_{x_3^{(2)}=h}, h \frac{dU_1^{(2)}}{dx_3^{(2)}} \Big|_{x_3^{(2)}=h}, iU_3^{(2)} \Big|_{x_3^{(2)}=h}, ih \frac{dU_3^{(2)}}{dx_3^{(2)}} \Big|_{x_3^{(2)}=h} \right)^T = B \left(C_1^{(2)}, C_2^{(2)}, C_3^{(2)}, C_4^{(2)} \right)^T. \quad (28)$$

Therefore, the relationship of $C_j^{(l)}$ ($j = 1 - 4$), where $l = 1 - N$, is expressed as:

$$\begin{pmatrix} C_1^{(2m+1)}, C_2^{(2m+1)}, C_3^{(2m+1)}, C_4^{(2m+1)} \end{pmatrix} = B \begin{pmatrix} C_1^{(2m)}, C_2^{(2m)}, C_3^{(2m)}, C_4^{(2m)} \end{pmatrix}^T \\ \begin{pmatrix} C_1^{(2m)}, C_2^{(2m)}, C_3^{(2m)}, C_4^{(2m)} \end{pmatrix} = A \begin{pmatrix} C_1^{(2m-1)}, C_2^{(2m-1)}, C_3^{(2m-1)}, C_4^{(2m-1)} \end{pmatrix}^T. \quad (29)$$

The displacement components and the dimensionless derivatives of the displacement concerning coordinates of the lower and upper surfaces can be expressed in terms of $C_j^{(1)}$ ($j = 1 - 4$).

$$\begin{pmatrix} U_1^{(1)} \Big|_{x_3=0}, h \frac{dU_1^{(1)}}{dx_3} \Big|_{x_3=0}, iU_3^{(1)} \Big|_{x_3=0}, ih \frac{dU_3^{(1)}}{dx_3} \Big|_{x_3=0} \end{pmatrix}^T = \begin{pmatrix} C_1^{(1)}, C_2^{(1)}, C_3^{(1)}, C_4^{(1)} \end{pmatrix}^T \\ \begin{pmatrix} U_1^{(N)} \Big|_{x_3=H}, h \frac{dU_1^{(N)}}{dx_3} \Big|_{x_3=H}, iU_3^{(N)} \Big|_{x_3=H}, ih \frac{dU_3^{(N)}}{dx_3} \Big|_{x_3=H} \end{pmatrix}^T, N \text{ is even.} \\ = \underbrace{B \bullet A \bullet \dots \bullet B \bullet A}_N \begin{pmatrix} C_1^{(1)}, C_2^{(1)}, C_3^{(1)}, C_4^{(1)} \end{pmatrix}^T \quad (30) \\ \begin{pmatrix} U_1^{(N)} \Big|_{x_3=H}, h \frac{dU_1^{(N)}}{dx_3} \Big|_{x_3=H}, iU_3^{(N)} \Big|_{x_3=H}, ih \frac{dU_3^{(N)}}{dx_3} \Big|_{x_3=H} \end{pmatrix}^T, N \text{ is odd.} \\ = \underbrace{A \bullet B \bullet \dots \bullet B \bullet A}_N \begin{pmatrix} C_1^{(1)}, C_2^{(1)}, C_3^{(1)}, C_4^{(1)} \end{pmatrix}^T$$

The traction-free boundary conditions in the lower and upper surfaces can be written as:

$$\begin{aligned} (\lambda_0 + 2\mu_0) \frac{dU_3^{(1)}}{dx_3^{(1)}} \Big|_{x_3^{(1)}=0} + ik\lambda_0 U_1^{(1)} \Big|_{x_3^{(1)}=0} = 0, ikU_3^{(1)} \Big|_{x_3^{(1)}=0} + \frac{dU_1^{(1)}}{dx_3^{(1)}} \Big|_{x_3^{(1)}=0} = 0 \\ (\lambda_H + 2\mu_H) \frac{dU_3^{(N)}}{dx_3^{(N)}} \Big|_{x_3^{(N)}=h} + ik\lambda_H U_1^{(N)} \Big|_{x_3^{(N)}=h} = 0, ikU_3^{(N)} \Big|_{x_3^{(N)}=h} + \frac{dU_1^{(N)}}{dx_3^{(N)}} \Big|_{x_3^{(N)}=h} = 0, \end{aligned} \quad (31)$$

where subscripts 0 and H represent the material parameters of the lower and upper surfaces, respectively.

The total transfer matrix is defined as:

$$F = \begin{cases} \underbrace{A \bullet B \bullet \dots \bullet B \bullet A}_N & N \text{ is odd} \\ \underbrace{B \bullet A \bullet \dots \bullet B \bullet A}_N & N \text{ is even} \end{cases} \quad (32)$$

where F is a 4×4 matrix with component F_{ij} , ($i, j = 1-4$). Substituting Equations (30) and (32) to Equation (31) yields:

$$\begin{aligned} \lambda_0 kh C_1^{(1)} - (\lambda_0 + 2\mu_0) C_4^{(1)} = 0, C_2^{(1)} + kh C_3^{(1)} = 0 \\ (\lambda_H + 2\mu_H) \sum_{j=1}^4 F_{4j} C_j^{(1)} - kh \lambda_H \sum_{j=1}^4 F_{1j} C_j^{(1)} = 0 \\ kh \sum_{j=1}^4 F_{3j} C_j^{(1)} + \sum_{j=1}^4 F_{2j} C_j^{(1)} = 0 \end{aligned} \quad (33)$$

Equation (33) is a set of linear algebraic equations with respect to C_j^1 ($j = 1 - 4$). In accordance with the sufficient and necessary condition that a nontrivial solution should exist, the determinant of the coefficient matrix should be equal to zero. This condition leads to the following dispersion relationship for Lamb waves in a layered FG nano copper wafer:

$$|\bar{T}_{ij}| = 0 \quad (34)$$

where:

$$\begin{aligned} \bar{T}_{11} &= \lambda_0 kh, \bar{T}_{14} = \lambda_0 + 2\mu_0, \bar{T}_{22} = 1, \bar{T}_{23} = kh \\ \bar{T}_{3j} &= (\lambda_H + 2\mu_H)F_{4j} - kh\lambda_H F_{1j}, \bar{T}_{4j} = khF_{3j} + F_{2j}, j = 1 - 4 \end{aligned}$$

All other terms are equal to zero.

4. Numerical Results and Discussion

4.1. Materials

The material parameters of the FG nano copper materials are as follows.

$$E_C = 115 \text{ GPa}, \rho_C = 8900 \frac{\text{kg}}{\text{m}^3}, \nu_m = \nu_C = 0.3, m = 3, \text{ and } n = 8 \quad (35)$$

The mass density of the interface region ρ_m is taken as 80% of ρ_c [40]. The corresponding Young's modulus, which is determined based on Equation (8), is $E_m = 0.322E_C$.

The grain size of the FG nano copper in a simple FG nano wafer varies exponentially along the thickness direction.

$$d/L = 0.001e^{p(x_3/h)}$$

where p is the gradient coefficient. When p is equal to 0, the wafer is a homogenous structure. The variations of grain size, λ and μ , and ρ with thickness for different p values are plotted in Figure 3.

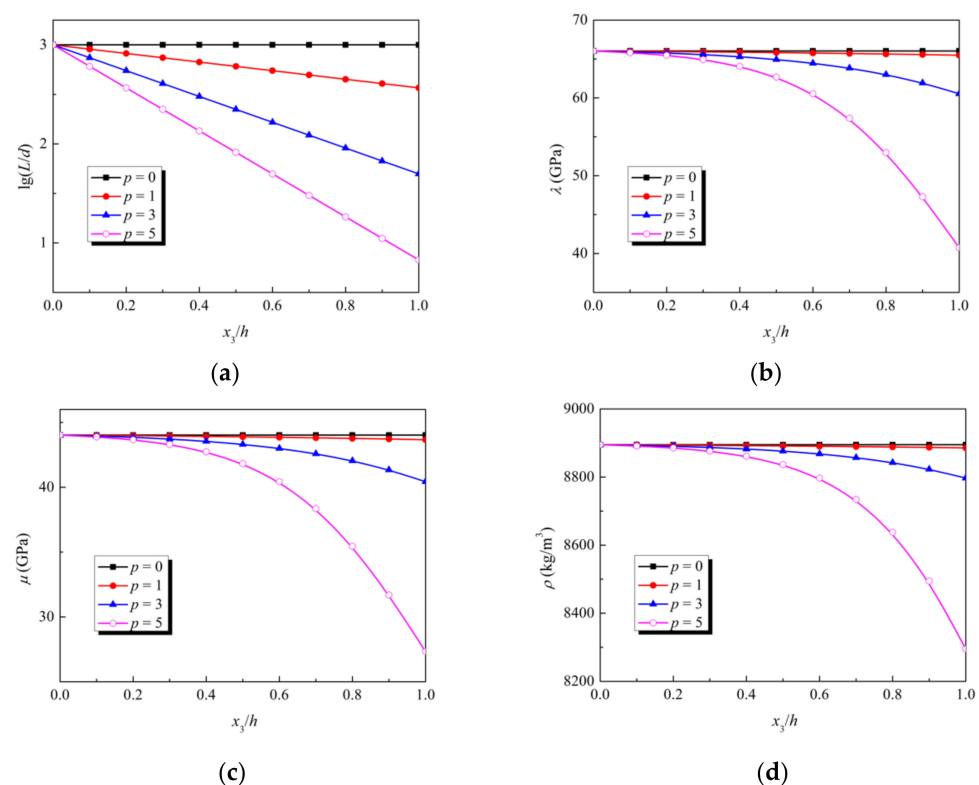


Figure 3. Variations of the material parameters with thickness for different p values: (a) grain size, (b) λ , (c) μ , and (d) ρ .

The grain size of the FG nano copper in a multilayered FG nano copper wafer is defined as:

$$d/L = \begin{cases} 0.001e^{p(\bar{x}_3/h)} & \text{Odd layer} \\ 0.001e^{p(1-\bar{x}_3/h)} & \text{Even layer} \end{cases}$$

As an illustration, the variations of the grain size, λ and μ , and ρ of a three-layer structure with thickness for different p values are shown in Figure 4. The differences among the material parameters at $p = 1$ and $p = 0$ (homogenous wafer) are minimal (Figures 3 and 4). A larger value of p signifies a higher inhomogeneity.

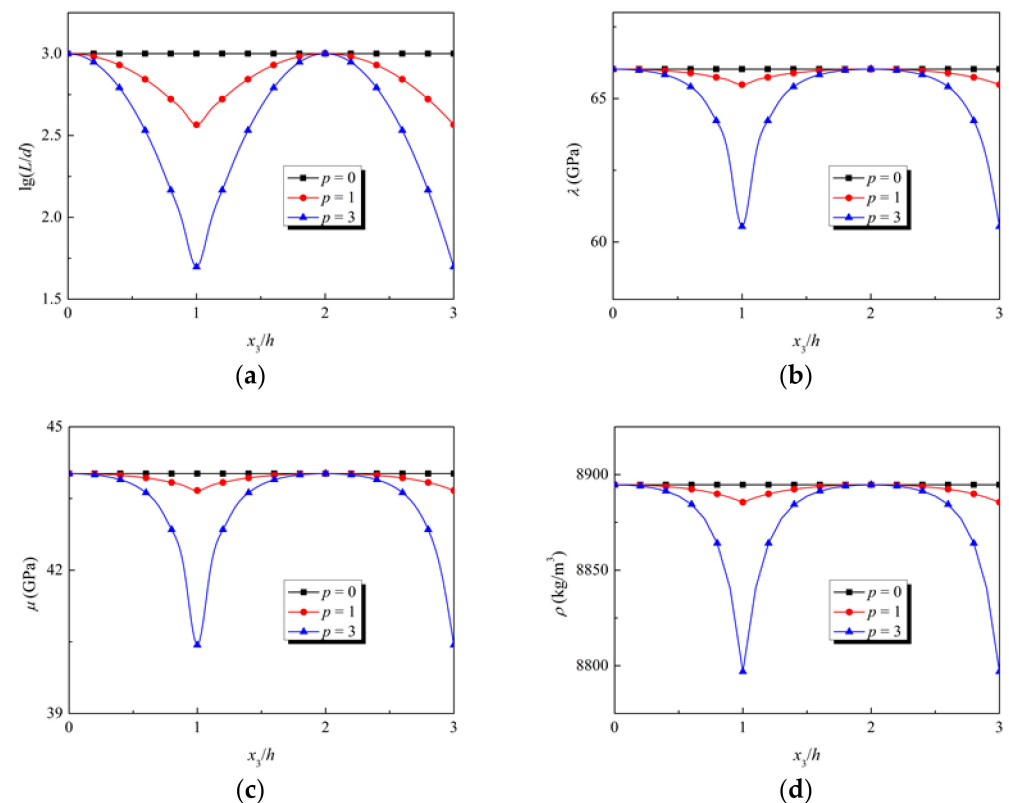


Figure 4. Variations of the material parameters of a three-layer structure with thickness for different p values: (a) grain size, (b) λ , (c) μ , and (d) ρ .

4.2. Lamb Waves in a Simple FG Nano Copper Wafer

The dispersion curves of Lamb waves propagating in a simple FG nano copper wafer are plotted in Figure 5. Similar to Lamb waves in a homogeneous wafer, Lamb waves in the FG nano wafer have many modes. For homogeneous ($p = 0$) and FG nano copper wafers ($p = 1, 3$), the phase velocity of the first mode increases with the increase in the dimensionless wave number. In other words, the first mode is the abnormal dispersion mode. Conversely, the phase velocities of other modes decrease with the increase in wave number, thereby representing a normal dispersion. Compared with the same order modes, the larger the value of p , the smaller the phase velocity. The dispersion curves of $p = 0$ and $p = 1$ almost coincide. The most obvious phase velocity change is observed at $p = 5$, followed by $p = 3$.

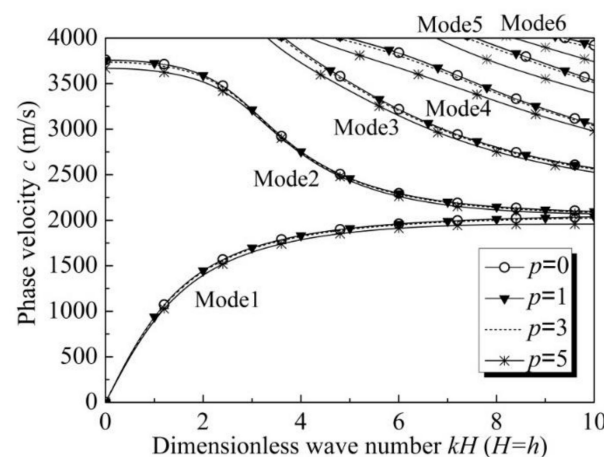


Figure 5. Dispersion curves of Lamb waves in a single FG nano copper wafer.

The gradient properties of the FG nano copper structure lead to the variation in the phase velocity. The relationship between this variation and p can be applied to nondestructive evaluation. We select $kh = 2\pi$ (i.e., the thickness is equal to the wave length) and calculate the change in phase velocity ($\Delta c = c - c_0$, where c_0 and c are the phase velocities in the homogenous and FG wafers, respectively) for the first four modes. The change (Δc) and the relative change rate ($\frac{\Delta c}{c_0}$) of the phase velocity are plotted in Figure 6. The figure shows that Δc is negative, and $-\Delta c$ increases as p increases. The most obvious phase velocity change is observed in Mode 4 (Figure 6a). The relative change rate of the phase velocity increases with the increase in p , thereby indicating that the former is positively correlated with the latter (Figure 6b). The most obvious relative change rate of the phase velocity ($\frac{|\Delta c|}{c_0}$) is detected in Mode 4, whereas the lowest is observed in Mode 2. The relative change rates of Modes 1 and 3 are similar. In conclusion, higher order modes should be selected to measure p .

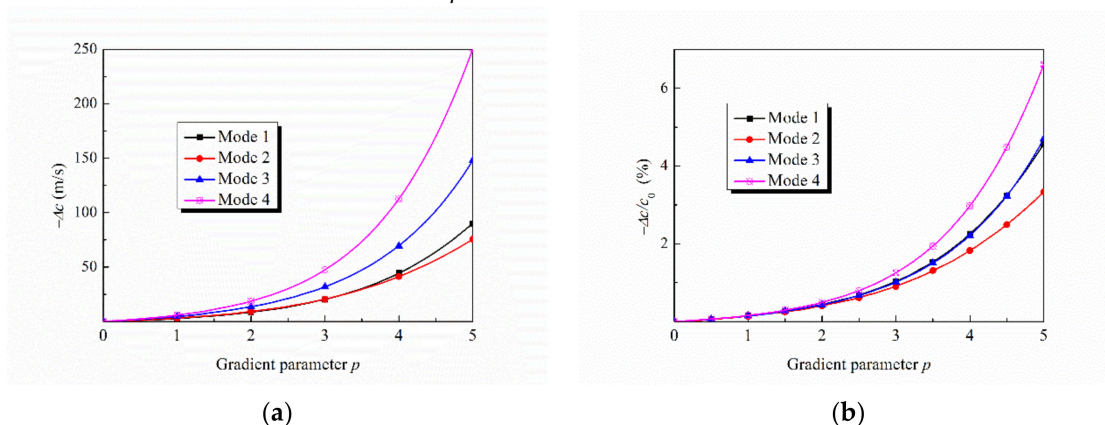


Figure 6. Relationship between the change in phase velocity and p in a simple FG nano copper wafer: (a) change in phase velocity and (b) relative change rate of the phase velocity.

4.3. Lamb Waves in a Multilayered FG Nano Copper Wafer

The dispersion curves of the Lamb waves in a three-layer wafer are calculated (Figure 7). Numerous modes of Lamb wave propagation exist in a multilayered FG nano copper wafer. The dispersion curves are similar to that of a simple FG nano copper wafer. For each mode, the larger the value of p , the smaller the phase velocity. The dispersion curves for $p = 0$ and $p = 1$ almost coincide, and the phase velocity at $p = 3$ slightly decreases. To provide a theoretical basis for nondestructive evaluation, the variations of the change and relative change rate of the phase velocity with p at $kh = 2\pi$ are plotted in Figure 8a,b, respectively.

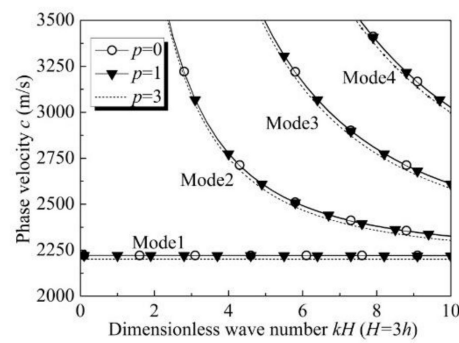


Figure 7. Dispersion curves of the Lamb waves in a three-layer FG nano copper wafer.

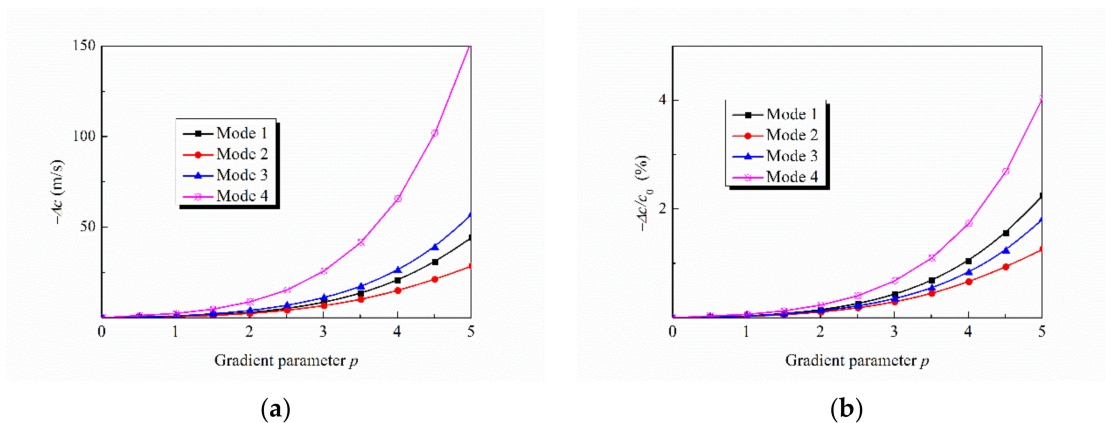


Figure 8. Relationship between the change in phase velocity and p in a three-layer FG nano copper wafer: (a) change in phase velocity and (b) relative change rate of the phase velocity.

When the thickness of the wafer is equal to the wavelength, the absolute values of the change in phase velocity $|\Delta c|$ and the relative change rate $\frac{|\Delta c|}{c_0}$ increase with the increase in p . If the wafer thickness is 5 mm and p is 3, the absolute values of the change in phase velocity for the first four modes are 8.349, 6.503, 10.839, and 25.781 m/s, respectively. The frequencies of Lamb waves with a given wavelength vary with different phase velocities. The frequency reduction is defined as $\Delta\omega = -\Delta ck$, and the frequency reduction values for the first four modes are 3497.23, 2723.97, 4540.24, and 10,799.15 Hz, respectively. Although the relative variable of phase velocity is limited to less than 1.2% (Figure 8b), the absolute value of the frequency change is expressed in kilohertz. The gradient properties lead to the significant changes in frequency. The most obvious phase velocity change with the gradient parameters is observed in Mode 4, followed by Modes 3, 1, and 2 (Figure 8a). The $\frac{|\Delta c|}{c}$ values of Modes 1 and 3 almost coincide, whereas those of Modes 4 and 2 are the largest and lowest values among the four, respectively.

To reveal the Lamb wave properties in a multilayered FG nano copper wafer, the Lamb waves in an effective homogenous wafer are selected for comparison. The material parameters of the latter, which are selected as the average material parameters of the FG nano copper wafer, satisfy the following conditions.

$$\bar{\lambda} = \frac{\int_0^h \lambda dx_3}{h}, \bar{\mu} = \frac{\int_0^h \mu dx_3}{h}, \bar{\rho} = \frac{\int_0^h \rho dx_3}{h} \quad (36)$$

where $\bar{\lambda}$ and $\bar{\mu}$ are the Lamé coefficients and $\bar{\rho}$ is the mass density of the effective homogenous wafer. At $p = 3$,

$$\bar{\lambda} = 64.41 \text{ GPa}, \bar{\mu} = 42.97 \text{ GPa}, \text{ and } \bar{\rho} = 8866.72 \frac{\text{kg}}{\text{m}^3}$$

The dispersion curves of the first four modes of the two-layer, four-layer, and effective homogenous wafers are displayed in Figure 9. The dispersion curve of the four-layer FG nano copper wafer is closer to the curve of the effective homogenous wafer than that of the two-layer FG nano copper wafer. The findings suggest the difference in the Lamb wave propagation in a multilayered FG nano copper wafer weakens with the increase in the number of layers.

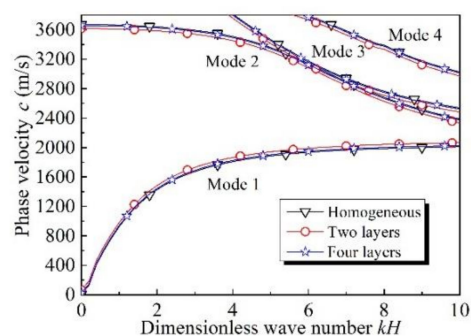


Figure 9. Dispersion curves of the Lamb waves in a two-layer, four-layer, and effective homogenous wafers.

5. Conclusions

In this study, Lamb waves' propagation in an FG nano copper single wafer and in an FG nano copper layered wafer are investigated analytically. The power series method is employed for solving the ordinary differential equations with variable coefficients. The modified transfer matrix method based on the power series solution is then proposed to solve the wave propagation problem in a multilayered wafer. Compared with the Lamb waves in a regular homogenous copper wafer, those in the FG nano copper lead to the decreased in phase velocity. It can be obtained from the theoretical results that the gradient coefficients can be measured by using the change of phase velocity of Lamb waves. For both a single layer structure and a multilayered structure, the findings show that the variation in the phase velocity in Mode 2 is the least obvious among the four modes considered in the analysis, whereas that in Mode 4 is the most obvious. Therefore, Mode 4 is recommended to be used in nondestructive evaluation. For a multilayered FG nano copper wafer, if the number of layers is greater than four, the dispersion curves of the Lamb waves in this structure tend to coincide with those in an effective homogenous structure. It is suggested that Lamb waves' technique is not suitable for testing a multilayered FG nano copper wafer in which the number of sub-layers is larger than four. These results provide theoretical guidance for the nondestructive evaluation of FG nano layered structures.

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