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Criteria for the Nonexistence of Kneser Solutions of DDEs and Their Applications in Oscillation Theory

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Abstract: The objective of this study was to improve existing oscillation criteria for delay differential equations (DDEs) of the fourth order by establishing new criteria for the nonexistence of so-called Kneser solutions. The new criteria are characterized by taking into account the effect of delay argument. All previous relevant results have neglected the effect of the delay argument, so our results substantially improve the well-known results reported in the literature. The effectiveness of our new criteria is illustrated via an example.

Keywords: differential equations of fourth-order; Kneser solutions; oscillation



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1. Introduction

The issue of studying the oscillatory behavior of delay differential equations (DDEs) is one of the most important branches of qualitative theory. The oscillation theory of DDEs has captured the attention of many researchers for several decades. Recently, an active research movement has emerged to improve, complement and simplify the criteria for oscillations of many classes of differential equations of different orders; for second-order, see [1–9]; for third-order, see [10–13]; for fourth-order & higher-order, see [14–25]; and for special cases, see [26–38]. Fourth-order differential equations appear in models related to physical, biological and chemical phenomena, for example, elasticity problems, soil leveling and the deformation of structures; see, for example, [7,23,32]. It is also worth mentioning the oscillatory muscle movement model represented by a fourth-order delay differential equation, which can arise due to the interaction of a muscle with its inertial load [37].

In this paper we are concerned with the study of the asymptotic behavior of the fourth-order delay differential equation:

$$(a(l)(x'''(l))^\alpha)' + f(l, x(\tau(l))) = 0, \quad l \geq l_0. \quad (1)$$

Throughout the paper, we assume $\alpha \in \mathbb{Q}_{odd}^+ := \{\beta/\gamma : \beta, \gamma \in \mathbb{Z}^+ \text{ are odd}\}$, $a \in C^1(I_0, \mathbb{R}^+)$, $a'(l) \geq 0$, $\int^\infty a^{-1/\alpha}(q) dq < \infty$, $\tau \in C(I_0, \mathbb{R}^+)$, $\tau(l) < l$, $\lim_{l \rightarrow \infty} \tau(l) = \infty$, $I_q := [l_q, \infty)$, $f \in C(I_0 \times \mathbb{R}, \mathbb{R})$, $xf(l, x) > 0$ for all $x \neq 0$ and there exists a function $q \in C(I_0, [0, \infty))$ such that $f(l, x) \geq q(l)x^\alpha$.

If there exists a $l_x \geq l_0$ such that the real-valued function x is continuous, $a(x''')^\alpha$ is continuously differentiable and satisfies (1), for all $l \in I_x$, then x is said to be a solution of (1). We take into account these solutions x of (1) such that $\sup\{|x(s)| : s \geq l_x\} > 0$ for every l_x in I_* . A solution x of (1) is said to be a Kneser solution if $x(l)x'(l) < 0$ for all $l \geq l_*$, where l_* is large enough. The set of all eventually positive Kneser solutions of Equation (1) is denoted by \mathfrak{R} . A solution x of (1) is said to be non-oscillatory if it is positive or negative,

ultimately; otherwise, it is said to be oscillatory. The equation itself is said to be oscillatory if all its solutions oscillate.

Below, we mention specifically some related works that were the motivation for this paper.

Zhang et al. [25] studied the oscillatory behavior of (1) when $f(l, x) := q(l)x^\beta$. Results in [25] used an approach that leads to two independent conditions in comparison with first-order delay differential equations and a condition in a traditional form ($\limsup(\cdot) = +\infty$). However, to use (Lemma 2.2.3, [27]), they conditioned $\lim_{l \rightarrow \infty} x(l) \neq 0$. Thus, under the conditions of (Theorem 1, [25]), Equation (1) still has a non-oscillatory solution that tends to zero. To surmount this problem, Zhang, et al. [38] considered—by using (Lemma 2.2.1, [27])—three possible cases for the derivatives of the solutions, and they followed the same approach as in (Theorem 1, [25]). However, in the case where $x' > 0$, they ensured that $\lim_{l \rightarrow \infty} x(l) \neq 0$, so they ensured that every solution of (1) is oscillatory.

By comparing with one or a couple of first-order delay differential equations, Baculikova et al. [14] studied the oscillatory behavior of (1) under the conditions

$$f'(x) \geq 0 \text{ and } -f(-xy) \geq f(xy) \geq f(x)f(y), \text{ for } xy > 0.$$

In this study, we first create new criteria for the nonexistence of Kneser solutions of nonlinear fourth-order differential Equations (1). By using these new criteria, we introduce sufficient conditions for oscillation which take into account the effect of delay argument $\tau(l)$. All previous relevant results have neglected the effect of the delay argument, so our results substantially improve the well-known results reported in the literature. The effectiveness of our new criteria is illustrated via an example.

2. Main Results

Firstly, for simplicity's sake, we assume $\delta_0(l) := \int_l^\infty a^{-1/\alpha}(\varrho)d\varrho$ and $\delta_m(l) := \int_l^\infty \delta_{m-1}(\varrho)d\varrho$, for $m = 1, 2$. Moreover, we let

$$(H) \text{ there is a constant } h > 1 \text{ such that } \frac{\delta_2(\tau(l))}{\delta_2(l)} \geq h \text{ for } l \geq l_0.$$

When checking the behavior of positive solutions of DDE (1), we have—by using (Lemma 2.2.1, [27])—three cases:

- Case (1) : $x'(l) > 0, x'''(l) > 0$ and $x^{(4)}(l) < 0$;
- Case (2) : $x'(l) > 0, x''(l) > 0$ and $x'''(l) < 0$;
- Case (3) : $x'(l) < 0, x''(l) > 0$ and $x'''(l) < 0$.

Moreover, from (1), we have that $(a(l)(x'''(l))^\alpha)' \leq 0$, for $l \in I_1$. We note that if $x \in \mathfrak{R}$, then x satisfies Case (3).

Lemma 1. Assume that $x \in \mathfrak{R}$. If

$$\int_{l_0}^\infty \left(\frac{1}{a(v)} \int_{l_0}^v q(\varrho)d\varrho \right)^{1/\alpha} dv = \infty, \tag{2}$$

then

$$\lim_{l \rightarrow \infty} x(l) = 0. \tag{3}$$

Lemma 2. Assume that $x \in \mathfrak{R}$ and (2) hold. Then

$$\eta := \limsup_{l \rightarrow \infty} \delta_2 \left(\int_{l_0}^l q(\varrho)d\varrho \right)^{1/\alpha} \leq 1. \tag{4}$$

Proof. Suppose $x \in \mathfrak{R}$. Integrating (1) from l_1 to l and using that fact that $x'(l) < 0$, we get

$$\begin{aligned} -a(l)(x'''(l))^\alpha &\geq -a(l_1)(x'''(l_1))^\alpha + \int_{l_1}^l q(\varrho)x^\alpha(\tau(\varrho))d\varrho \\ &\geq -a(l_1)(x'''(l_1))^\alpha + x^\alpha(\tau(l)) \int_{l_0}^l q(\varrho)d\varrho - x^\alpha(\tau(l)) \int_{l_0}^{l_1} q(\varrho)d\varrho, \end{aligned} \tag{5}$$

for all $l \in I_1$. In view of (3), there is a $l_2 \in I_1$ such that

$$a(l_1)(x'''(l_1))^\alpha + x^\alpha(\tau(l)) \int_{l_0}^{l_1} q(\varrho)d\varrho < 0,$$

for $l \in I_2$. Thus, (5) becomes

$$-a(l)(x'''(l))^\alpha \geq x^\alpha(\tau(l)) \int_{l_0}^l q(\varrho)d\varrho \geq x^\alpha(l) \int_{l_0}^l q(\varrho)d\varrho. \tag{6}$$

Now, by using the monotonicity of $a^{1/\alpha}(l)x'''(l)$, we have

$$x''(\tau(l)) \geq x''(l) \geq \int_l^\infty \frac{1}{a^{1/\alpha}(\varrho)} \left(-a^{1/\alpha}(\varrho)x'''(\varrho)\right)d\varrho \geq -a^{1/\alpha}(l)x'''(l)\delta_0(l). \tag{7}$$

Integrating (7) twice from l to ∞ and using $\left(a^{1/\alpha}(l)x'''(l)\right)' \leq 0$, we get

$$-x'(l) \geq -a^{1/\alpha}(l)x'''(l)\delta_1(l) \tag{8}$$

and

$$x(l) \geq -a^{1/\alpha}(l)x'''(l)\delta_2(l). \tag{9}$$

From (9) and (6), we see that

$$-a(l)(x'''(l))^\alpha \geq -a(l)(x'''(l))^\alpha \delta_2^\alpha(l) \int_{l_0}^l q(\varrho)d\varrho,$$

and so

$$1 \geq \delta_2^\alpha(l) \int_{l_0}^l q(\varrho)d\varrho.$$

Taking the limsup on both sides of the inequality, we arrive at (4). The proof is complete. \square

Lemma 3. Assume that $x \in \mathfrak{R}$ and (2) hold. Then there exists a $l_\varepsilon \geq l_1$ such that

$$\frac{d}{dl} \left(\frac{x(l)}{\delta_2^{\eta-\varepsilon}(l)} \right) \leq 0,$$

for any $\varepsilon > 0$ and $l \in I_\varepsilon$. Moreover, if (H) holds, then

$$x(\tau(l)) \geq h^{\eta-\varepsilon}x(l) \text{ for } l \in I_\varepsilon. \tag{10}$$

Proof. Suppose $x \in \mathfrak{R}$. Then, there is a $l_1 \in I_0$ such that $x(\tau(l)) > 0$. Proceeding as in the proof of Lemma 2, we arrive at (6) and (8). Thus, for $l \geq l_2$, where $l_2 \in I_1$ is large enough, we have

$$-a(l)^{1/\alpha}x'''(l) \geq x(l) \left(\int_{l_0}^l q(\varrho)d\varrho \right)^{1/\alpha}.$$

From the definition of η , for every $\varepsilon > 0$, there exists a $l_3 \geq l_2$ such that

$$\delta_2(l) \left(\int_{l_0}^l q(\varrho) d\varrho \right)^{1/\alpha} > \eta_* := \eta - \varepsilon,$$

for $l \in I_3$. Hence, from (8), we have

$$\frac{d}{dl} \left(\frac{x(l)}{\delta_2^{\eta_*}(l)} \right) \leq \frac{\delta_2^{\eta_*}(l) a^{1/\alpha}(l) x'''(l) \delta_1(l) + \eta_* x(l) \delta_2^{\eta_*-1}(l) \delta_1(l)}{\delta_2^{2\eta_*}(l)},$$

which with (6) gives

$$\begin{aligned} \frac{d}{dl} \left(\frac{x(l)}{\delta_2^{\eta_*}(l)} \right) &\leq \frac{1}{\delta_2^{2\eta_*}(l)} \left(-x(l) \delta_2^{\eta_*}(l) \delta_1(l) \left(\int_{l_0}^l q(\varrho) d\varrho \right)^{1/\alpha} + \eta_* x(l) \delta_2^{\eta_*-1}(l) \delta_1(l) \right) \\ &\leq \frac{1}{\delta_2^{2\eta_*}(l)} x(l) \delta_2^{\eta_*-1}(l) \delta_1(l) \left(\eta_* - \delta_2(l) \left(\int_{l_0}^l q(\varrho) d\varrho \right)^{1/\alpha} \right) \leq 0. \end{aligned}$$

Using this fact, one can easily see that

$$x(\tau(l)) \geq x(l) \left(\frac{\delta_2(\tau(l))}{\delta_2(l)} \right)^{\eta_*} \geq h^{\eta_*} x(l).$$

The proof is complete. \square

Lemma 4. Assume that $x \in \mathfrak{R}$ and (H), (2) hold. Then

$$h^\eta \eta \leq 1. \tag{11}$$

Proof. Suppose $x \in \mathfrak{R}$. Using Lemma 3, we get that (10) holds. As in the proof of Lemma 2, we have that (6) holds. From (6) and (10), we have

$$-a(l) (x'''(l))^\alpha \geq x^\alpha(l) h^{\alpha\eta_*} \int_{l_0}^l q(\varrho) d\varrho, \tag{12}$$

which implies

$$-a(l) (x'''(l))^\alpha \geq -a(l) (x'''(l))^\alpha \delta_2^\alpha(l) h^{\alpha\eta_*} \int_{l_0}^l q(\varrho) d\varrho.$$

Taking the limsup on both sides of the latter inequality, we obtain $h^{\eta_*} \eta \leq 1$. Since ε is arbitrary, we obtain that (11) holds. The proof is complete. \square

Lemma 5. Assume that $x \in \mathfrak{R}$ and (H), (2) hold. Then

$$\tilde{\eta} := \liminf_{l \rightarrow \infty} \frac{h^\eta}{\delta_2(l)} \int_l^\infty \delta_2^{\alpha+1}(\varrho) q(\varrho) d\varrho \leq \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1}. \tag{13}$$

Proof. Suppose $x \in \mathfrak{R}$. Proceeding as in the proof of Lemma 2, we obtain (8) and (9). Define the function $\omega(l) \in C^1([l_0, \infty), R)$, such that

$$\omega(l) = \frac{a(l) (x'''(l))^\alpha}{x^\alpha(l)}.$$

Differentiating $\omega(l)$ and using (1), (8), (10) and the fact that $x'(l) < 0$, we have

$$\omega'(l) = \frac{(a(l)(x'''(l))^\alpha)'}{x^\alpha(l)} - \frac{\alpha a(l)(x'''(l))^\alpha x'}{x^{\alpha+1}(l)} \leq -h^{\eta*}q(l) - \alpha\delta_1(l)\omega^{(\alpha+1)/\alpha}(l). \tag{14}$$

Multiplying (14) by δ_2^α and integrating the resulting inequality from l_1 to l , we obtain

$$\begin{aligned} \delta_2^\alpha(l)\omega(l) - \delta_2^\alpha(l_1)\omega(l_1) + \int_{l_1}^l h^{\eta*}q(\varrho)\delta_2^\alpha(\varrho)d\varrho &\leq - \int_{l_1}^l \alpha\delta_2^{\alpha-1}(\varrho)\delta_1(\varrho)\omega(\varrho)d\varrho \\ &\quad - \int_{l_1}^l \alpha\delta_1(\varrho)\delta_2^\alpha(\varrho)\omega^{(\alpha+1)/\alpha}(\varrho)d\varrho. \end{aligned}$$

Using the inequality

$$-By + Ay^{(\alpha+1)/\alpha} \geq -\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}, \quad A, B > 0,$$

with $A = \delta_1(\varrho)\delta_2^\alpha(\varrho)$, $B = \delta_1(\varrho)\delta_2^{\alpha-1}(\varrho)$ and $y = -\omega(\varrho)$, we conclude that

$$\int_{l_1}^l \left(h^{\eta*}q(\varrho)\delta_2^\alpha(\varrho) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{\delta_1(\varrho)}{\delta_2(\varrho)} \right) d\varrho \leq \delta_2^\alpha(l_1)\omega(l_1) - \delta_2^\alpha(l)\omega(l). \tag{15}$$

From (9), one can easily see that $-1 \leq \omega(l)\delta_2^\alpha(l) < 0$, which with (15) gives

$$\int_{l_1}^l \left(h^{\eta*}q(\varrho)\delta_2^\alpha(\varrho) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{\delta_1(\varrho)}{\delta_2(\varrho)} \right) d\varrho < \infty.$$

Hence, there is a $l_\epsilon \geq l_1$ such that

$$\int_{l_1}^\infty \left(h^{\eta*}q(\varrho)\delta_2^\alpha(\varrho) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{\delta_1(\varrho)}{\delta_2(\varrho)} \right) d\varrho < \epsilon,$$

for any $\epsilon > 0$ and $l \in I_\epsilon$. Since δ_2 is decreasing, we get

$$\begin{aligned} \epsilon &> \frac{1}{\delta_2(l)} \int_{l_1}^\infty \left(h^{\eta*}q(\varrho)\delta_2^{\alpha+1}(\varrho) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \delta_1(\varrho) \right) d\varrho \\ &> \frac{h^{\eta*}}{\delta_2(l)} \int_{l_1}^\infty q(\varrho)\delta_2^{\alpha+1}(\varrho)d\varrho - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}. \end{aligned}$$

Taking the limsup on both sides of the inequality, we arrive at (13). The proof is complete. \square

From the previous results, the following theory can be inferred.

Theorem 1. Assume that (2) holds. If one of the following conditions holds:

- (C₁) $\eta > 1$;
- (C₂) $h^\eta \eta > 1$ and (H);
- (C₃) $\tilde{\eta} > (\alpha/(\alpha+1))^{\alpha+1}$ and (H),

then the set \mathfrak{R} is empty.

Proof. Suppose $x \in \mathfrak{R}$. Using Lemmas 2, 4 and 5, we have that (4), (11) and (13) hold. Then we obtain a contradiction with (C₁) – (C₃) respectively. The proof is complete. \square

Lemma 6. Assume that $M > 0, L$ and N are constants $\psi(\vartheta) = L\vartheta - M(\vartheta - N)^{(\alpha+1)/\alpha}$. Then,

$$\psi(\vartheta) = LN + \frac{\alpha^\alpha}{(\alpha + 1)^{(\alpha+1)}} \frac{L^{\alpha+1}}{M^\alpha}.$$

Proof. It is easy to see that the maximum value of ψ on R at $\vartheta^* = N + (\alpha L / ((\alpha + 1)M))^\alpha$ is

$$\max_{\vartheta \in R} \psi(\vartheta) = \psi(\vartheta^*) = LN + \frac{\alpha^\alpha}{(\alpha + 1)^{(\alpha+1)}} \frac{L^{\alpha+1}}{M^\alpha}. \tag{16}$$

Then, the proof is complete. \square

Theorem 2. Assume (H) and (2) hold. If

$$\limsup_{l \rightarrow \infty} \frac{\delta_2^\alpha(l)}{\rho(l)} \int_{l_1}^l \left(\rho(\zeta) h^{\alpha\eta} q(\zeta) - \frac{1}{(\alpha + 1)^{(\alpha+1)}} \frac{(\rho'(\zeta))^{\alpha+1}}{\rho^\alpha(\zeta) \delta_1^\alpha(\zeta)} \right) d\zeta > 1, \tag{17}$$

then the set \mathfrak{R} is empty.

Proof. Suppose $x \in \mathfrak{R}$. As in the proof of Lemma 2, we have that (8) and (9) hold. From (9), we obtain

$$\frac{a(l)(x'''(l))^\alpha}{x^\alpha(l)} \geq -\frac{1}{\delta_2^\alpha(l)}. \tag{18}$$

Thus, if we define the a generalized Riccati substitution as

$$w(l) := \rho(l) \left(\frac{a(l)(x'''(l))^\alpha}{x^\alpha(l)} + \frac{1}{\delta_2^\alpha(l)} \right), \tag{19}$$

where $\omega(l) \in C^1([l_0, \infty), R)$, then $w(l) > 0$ for all $l \geq l_1$. Differentiating $\omega(l)$, we have

$$\begin{aligned} w'(l) &= \frac{\rho'(l)}{\rho(l)} w(l) + \rho(l) \frac{(a(l)(x'''(l))^\alpha)'}{x^\alpha(l)} - \alpha \rho(l) \frac{a(l)(x'''(l))^\alpha}{x^{\alpha+1}(l)} x'(l) \\ &\quad - \frac{\alpha \delta_2'(l)}{\delta_2^{\alpha+1}(l)}. \end{aligned} \tag{20}$$

From (1), we see that

$$\begin{aligned} (a(l)(x'''(l))^\alpha)' &= -f(l, x(\tau(l))) \\ &\leq -q(l)x^\alpha(\tau(l)). \end{aligned} \tag{21}$$

Using (8) and (21), (20) becomes

$$\begin{aligned} w'(l) &\leq \frac{\rho'(l)}{\rho(l)} w(l) - \rho(l) q(l) \left(\frac{x(\tau(l))}{x(l)} \right)^\alpha \\ &\quad - \alpha \rho(l) a(l) \left(\frac{x'''(l)}{x(l)} \right)^{\alpha+1} a^{1/\alpha}(l) \delta_1(l) + \frac{\alpha \delta_1(l)}{\delta_2^{\alpha+1}(l)}. \end{aligned} \tag{22}$$

Using Lemma 3, we have that (10) holds. Thus, (22) yields

$$\begin{aligned} w'(l) &\leq \frac{\rho'(l)}{\rho(l)} w(l) - \rho(l) h^{\alpha(\eta-\varepsilon)} q(l) - \alpha \rho(l) a(l) \left(\frac{x'''(l)}{x(l)} \right)^{\alpha+1} a^{1/\alpha}(l) \delta_1(l) \\ &\quad + \frac{\alpha \delta_1(l)}{\delta_2^{\alpha+1}(l)}. \end{aligned}$$

Hence, from the definition of w , we obtain

$$w'(l) \leq -\rho(l)h^{\alpha(\eta-\varepsilon)}q(l) + \frac{\rho'(l)}{\rho(l)}w(l) - \alpha \frac{1}{\rho^{1/\alpha}(l)} \left(w(l) - \frac{\rho(l)}{\delta_2^\alpha(l)} \right)^{1+1/\alpha} \delta_1(l) + \frac{\alpha\delta_1(l)}{\delta_2^{\alpha+1}(l)}. \tag{23}$$

Using inequality (16) with

$$L := \frac{\rho'(l)}{\rho(l)}, \quad M := \alpha \frac{\delta_1(l)}{\rho^{1/\alpha}(l)}, \quad N := \frac{\rho(l)}{\delta_2^\alpha(l)}$$

and $\vartheta := w$, we obtain

$$\frac{\rho'(l)}{\rho(l)}w(l) \leq \alpha \frac{1}{\rho^{1/\alpha}(l)} \left(w(l) - \frac{\rho(l)}{\delta_2^\alpha(l)} \right)^{1+1/\alpha} + \frac{1}{(\alpha+1)^{(\alpha+1)}} \frac{(\rho'(l))^{\alpha+1}}{\rho^\alpha(l)\delta_1^\alpha(l)} + \frac{\rho'(l)}{\delta_2^\alpha(l)},$$

which, with (23), gives

$$w'(l) \leq -\rho(l)h^{\alpha(\eta-\varepsilon)}q(l) + \frac{1}{(\alpha+1)^{(\alpha+1)}} \frac{(\rho'(l))^{\alpha+1}}{\rho^\alpha(l)\delta_1^\alpha(l)} + \frac{\rho'(l)}{\delta_2^\alpha(l)} + \frac{\alpha\delta_1(l)}{\delta_2^{\alpha+1}(l)}$$

or

$$w'(l) \leq -\rho(l)h^{\alpha(\eta-\varepsilon)}q(l) + \frac{1}{(\alpha+1)^{(\alpha+1)}} \frac{(\rho'(l))^{\alpha+1}}{\rho^\alpha(l)\delta_1^\alpha(l)} + \frac{d}{dl} \left(\frac{\rho(l)}{\delta_2^\alpha(l)} \right). \tag{24}$$

By integrating (24) from l_1 to l , we obtain

$$w(l) - w(l_1) \leq - \int_{l_1}^l \left(\rho(\zeta)h^{\alpha(\eta-\varepsilon)}q(\zeta) - \frac{1}{(\alpha+1)^{(\alpha+1)}} \frac{(\rho'(\zeta))^{\alpha+1}}{\rho^\alpha(\zeta)\delta_1^\alpha(\zeta)} \right) d\zeta + \frac{\rho(l)}{\delta_2^\alpha(l)} - \frac{\rho(l_1)}{\delta_2^\alpha(l_1)}.$$

From (19), we are led to

$$\begin{aligned} & \int_{l_1}^l \left(\rho(\zeta)h^{\alpha(\eta-\varepsilon)}q(\zeta) - \frac{1}{(\alpha+1)^{(\alpha+1)}} \frac{(\rho'(\zeta))^{\alpha+1}}{\rho^\alpha(\zeta)\delta_1^\alpha(\zeta)} \right) d\zeta \\ & \leq -\rho(l) \frac{a(l)(x'''(l))^\alpha}{x^\alpha(l)} + \rho(l_1) \frac{a(l_1)(x'''(l_1))^\alpha}{x^\alpha(l_1)} \\ & \leq -\rho(l) \frac{a(l)(x'''(l))^\alpha}{x^\alpha(l)}. \end{aligned}$$

In view of (18), we get

$$\int_{l_1}^l \left(\rho(\zeta)h^{\alpha(\eta-\varepsilon)}q(\zeta) - \frac{1}{(\alpha+1)^{(\alpha+1)}} \frac{(\rho'(\zeta))^{\alpha+1}}{\rho^\alpha(\zeta)\delta_1^\alpha(\zeta)} \right) d\zeta \leq \frac{\rho(l)}{\delta_2^\alpha(l)}$$

or

$$\frac{\delta_2^\alpha(l)}{\rho(l)} \int_{l_1}^l \left(\rho(\zeta)h^{\alpha(\eta-\varepsilon)}q(\zeta) - \frac{1}{(\alpha+1)^{(\alpha+1)}} \frac{(\rho'(\zeta))^{\alpha+1}}{\rho^\alpha(\zeta)\delta_1^\alpha(\zeta)} \right) d\zeta \leq 1.$$

Taking limit supremum, we obtain a contradiction with (17). This completes the proof. \square

Corollary 1. Assume (H) and (2) hold. If one of the following conditions holds:

$$\limsup_{l \rightarrow \infty} \delta_2^\alpha(l) \int_{l_1}^l h^{\alpha\eta} q(\zeta) d\zeta > 1 \tag{25}$$

or

$$\limsup_{l \rightarrow \infty} \delta_2^{\alpha-1}(l) \int_{l_1}^l \left(h^{\alpha\eta} q(\zeta) \delta_2(\zeta) - \frac{1}{(\alpha+1)^{(\alpha+1)}} \frac{\delta_1(\zeta)}{\delta_2^\alpha(\zeta)} \right) d\zeta > 1 \tag{26}$$

or

$$\limsup_{l \rightarrow \infty} \int_{l_1}^l \left(h^{\alpha\eta} q(\zeta) \delta_2^\alpha(\zeta) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{(\alpha+1)}} \frac{\delta_1(l)}{\delta_2(l)} \right) d\zeta > 1, \tag{27}$$

then the set \mathfrak{R} is empty.

Proof. By choosing $\rho(l) = 1$, $\rho(l) = \delta_2(l)$ or $\rho(l) = \delta_2^\alpha(l)$, the condition (17) in Theorem 2 becomes as (25), (26) or (27), respectively. \square

3. Discussion and Applications

Depending on the new criteria for the nonexistence of Kneser solutions, we introduced new criteria for oscillation of (1). When checking the behavior of positive solutions of DDE (1), we have three Cases (1)–(3). In order to illustrate the importance of the results obtained for Case (3), we recall an existing criterion for a particular case of (1) with $\alpha = \beta$:

Theorem 3 (Theorem 2.1 with $n = 4$, [25]). Assume that $\alpha = \beta$,

$$\liminf_{l \rightarrow \infty} \int_{\tau(l)}^l q(\varrho) \frac{\tau^{3\alpha}(\varrho)}{a(\tau(\varrho))} d\varrho > \frac{6^\alpha}{e} \tag{28}$$

and there exists a $\rho \in C^1(I_0, \mathbb{R}^+)$ such that

$$\limsup_{l \rightarrow \infty} \int_{l_0}^l \left(\delta_0^\alpha(\varrho) q(\varrho) \left(\frac{\lambda}{2!} \tau^2(\varrho) \right)^\alpha - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{(\alpha+1)}} \frac{1}{\delta(\varrho) a^{1/\alpha}(\varrho)} \right) d\varrho = \infty, \tag{29}$$

for some $\lambda \in (0, 1)$. Then every solution of (1) is oscillatory or tends to zero.

From the previous Theorems, we conclude under the assumptions of the Theorem that every positive solution x of (1) tends to zero, and hence x satisfies Case (3). Therefore, conditions (28) and (29) ensure (3) without verifying the extra condition (2). In view of Theorems 1 and 3, we obtain the following:

Corollary 2. Assume that (28) and (29) hold for some $\lambda \in (0, 1)$. If (C_1) , (C_2) or (C_3) holds, then (1) is oscillatory.

Proof. Suppose that x is a nonoscillatory solution of (1). Thus, we have three cases. From Theorem 3, we find (28) and (29) contradicts Case (1) and Case (2) respectively.

For Case (3), using Theorem 1, if one of the conditions (C_1) – (C_3) holds, then we obtain a contradiction. The proof is complete. \square

Corollary 3. Assume that (28) and (29) hold for some $\lambda \in (0, 1)$. If (25), (26) or (27) holds, then (1) is oscillatory.

Proof. Suppose the x is a nonoscillatory solution of (1). Thus, we have three cases. From Theorem 3, we find (28) and (29) contradict Case (1) and Case (2) respectively.

For Case (3), using Corollary 1, if one of the conditions (25)–(27) holds, then we obtain a contradiction. The proof is complete. \square

We state now an Example:

Example 1. We have the fourth-order DDE

$$\left(e^{\alpha l} x'''(l) \right)' + q_0 e^{\alpha l} x^\alpha(\tau_0 l) = 0, \tag{30}$$

where $l \geq 1, \tau_0 \in (0, 1 - 1/e)$ and $q_0 > 0$. Note that $a(l) = e^{\alpha l}, q(l) = q_0 e^{\alpha l}, \tau(l) = \tau_0 l$. It is easy to conclude that $\delta_m(l) = e^{-l}$ for $m = 0, 1, 2$. Then, we see that (28) and (29) are satisfied for all $q_0 > 0$.

For condition (C₁), we have

$$\eta = \left(\frac{q_0}{\alpha} \right)^{1/\alpha} > 1. \tag{31}$$

By using the fact that $e^y > ey$ for $y > 0$, we get

$$\frac{\delta_2(\tau(l))}{\delta_2(l)} = e^{(1-\tau_0)l} > e(1-\tau_0)l \geq e(1-\tau_0) := h > 1,$$

for $l \geq 1$. Hence, Conditions (C₂) or (C₃) reduce to

$$\left(\frac{q_0}{\alpha} \right)^{1/\alpha} (e(1-\tau_0))^{(q_0/\alpha)^{1/\alpha}} > 1 \tag{32}$$

and

$$q_0 (e(1-\tau_0))^{(q_0/\alpha)^{1/\alpha}} > \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1}, \tag{33}$$

respectively.

Thus, by Corollary, if (31), (32) or (33) holds, then (30) is oscillatory.

Remark 1. To the best of our knowledge, the known related sharp criterion for (30) based on (Theorem 2.1, [38]) gives

$$q_0 > \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1}. \tag{34}$$

Note firstly that our criteria (32) and (33) essentially take into account the influence of delay argument $\tau(l)$, which has been neglected in all previous results of fourth-order equations. Secondly, in the case where $\alpha = 1$ and $\tau_0 = 1/2$, we have

| Condition | (31) | (32) | (33) | (34) |
|-----------|--------------|---------------|---------------|-----------------|
| Criterion | $q_0 > 1.00$ | $q_0 > 0.786$ | $q_0 > 0.233$ | $q_0 > 0.250$. |

Condition (33) supports the most efficient and sharp criterion for oscillation of Equation (30).

4. Conclusions

We worked on extending and improving existing oscillation criteria for DDEs of the fourth order for the nonexistence of Kneser solutions. The new criteria that we proved are characterized by taking into account the effect of the delay argument.

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References

1. Agarwal, R.P.; Zhang, C.; Li, T. Some Remarks on oscillation of second order neutral differential equations. *Appl. Math. Comput.* **2016**, *274*, 178–181. [[CrossRef](#)]
2. Bazighifan, O.; Ruggieri, M.; Santra, S.S.; Scapellato, A. Qualitative Properties of Solutions of Second-Order Neutral Differential Equations. *Symmetry* **2020**, *12*, 1520. [[CrossRef](#)]
3. Bohner, M.; Grace, S.R.; Jadlovská, I. Oscillation criteria for second-order neutral delay differential equations. *Electron. J. Qual. Theory Differ. Equ.* **2017**, *2017*, 60. [[CrossRef](#)]
4. Chatzarakis, G.E.; Dzurina, J.; Jadlovská, I. New oscillation criteria for second-order half-linear advanced differential equations. *Appl. Math. Comput.* **2019**, *347*, 404–416. [[CrossRef](#)]
5. Chatzarakis, G.E.; Moaaz, O.; Li, T.; Qaraad, B. Some oscillation Theorems for nonlinear second-order differential equations with an advanced argument. *Adv. Differ. Eqs.* **2020**. [[CrossRef](#)]
6. Grace, S.R.; Dzurina, J.; Jadlovská, I.; Li, T. An improved approach for studying oscillation of second-order neutral delay differential equations. *J. Ineq. Appl.* **2018**, *2018*, 193. [[CrossRef](#)]
7. Graef, J.R.; Henderson, J. Double solutions of boundary value problems for 2nth-order differential equations and difference equations. *Comput. Math. Appl.* **2003**, *45*, 873–885. [[CrossRef](#)]
8. Dzurina, J.; Grace, S.R.; Jadlovská, I.; Li, T. Oscillation criteria for second-order Emden–Fowler delay differential equations with a sublinear neutral term. *Mathematische Nachrichten* **2020**, *293*, 910–922. [[CrossRef](#)]
9. Santra, S.; Dassios, I.; Ghosh, T. On the Asymptotic Behavior of a Class of Second-Order Non-Linear Neutral Differential Equations with Multiple Delays. *Axioms* **2020**, *9*, 34. [[CrossRef](#)]
10. Chatzarakis, G.E.; Dzurina, J.; Jadlovská, I. Oscillatory and asymptotic properties of third-order quasilinear delay differential equations. *J. Inequal. Appl.* **2019**, *2019*, 23. [[CrossRef](#)]
11. Chatzarakis, G.E.; Grace, S.R.; Jadlovská, I.; Li, T.; Tunc, E. Oscillation criteria for third-order Emden-Fowler differential equations with unbounded neutral coefficients. *Complexity* **2019**, *2019*, 5691758. [[CrossRef](#)]
12. Moaaz, O.; Chalishajar, D.; Bazighifan, O. Asymptotic behavior of solutions of the third order nonlinear mixed type neutral differential equations. *Mathematics* **2020**, *8*, 485. [[CrossRef](#)]
13. Moaaz, O.; Dassios, I.; Muhsin, W.; Muhib, A. Oscillation Theory for Non-Linear Neutral Delay Differential Equations of Third Order. *Appl. Sci.* **2020**, *10*, 4855. [[CrossRef](#)]
14. Baculikova, B.; Dzurina, J.; Graef, J.R. On The Oscillation of higher-order delay differential equations. *J. Math. Sci.* **2012**, *187*, 387–400. [[CrossRef](#)]
15. Moaaz, O.; Dassios, I.; Bazighifan, O. Oscillation Criteria of Higher-order Neutral Differential Equations with Several Deviating Arguments. *Mathematics* **2020**, *8*, 402. [[CrossRef](#)]
16. Bazighifan, O.; Elabbasy, E.M.; Moaaz, O. Oscillation of higher-order differential equations with distributed delay. *J. Inequalities Appl.* **2019**, *2019*, 55. [[CrossRef](#)]
17. Bazighifan, O.; Dassios, I. Riccati Technique and Asymptotic Behavior of Fourth-Order Advanced Differential Equations. *Mathematics* **2020**, *8*, 590. [[CrossRef](#)]
18. Dassios, I.; Bazighifan, O. Oscillation Conditions for Certain Fourth-Order Non-Linear Neutral Differential Equation. *Symmetry* **2020**, *12*, 1096. [[CrossRef](#)]
19. Elabbasy, E.M.; Cesarano, C.; Bazighifan, O.; Moaaz, O. Asymptotic and oscillatory behavior of solutions of a class of higher order differential equation. *Symmetry* **2019**, *11*, 1434. [[CrossRef](#)]
20. Elabbasy, E.M.; Moaaz, O.; Bazighifan, O. On the asymptotic behavior of fourth order nonlinear differential equations. *Adv. Differ. Equ.* **2017**, *2017*, 261.
21. El-Nabulsi, R.A.; Moaaz, O.; Bazighifan, O. New results for oscillatory behavior of fourth-order differential equations. *Symmetry* **2020**, *12*, 136. [[CrossRef](#)]
22. Dzurina, J.; Baculikova, B.; Jadlovská, I. Oscillatory solutions of fourth order advanced trinomial differential equations. *Mathematische Nachrichten* **2020**, *293*, 1110–1119. [[CrossRef](#)]
23. Graef, J.R.; Qian, C.; Yang, B. A three point boundary value problem for nonlinear fourth order differential equations. *J. Math. Anal. Appl.* **2003**, *287*, 217–233. [[CrossRef](#)]
24. Moaaz, O.; Dassios, I.; Bazighifan, O.; Muhib, A. Oscillation Theorems for Nonlinear Differential Equations of Fourth-Order. *Mathematics* **2020**, *8*, 520. [[CrossRef](#)]
25. Zhang, C.; Li, T.; Sun, B.; Thandapani, E. On the oscillation of higher-order half-linear delay differential equations. *Appl. Math. Lett.* **2011**, *24*, 1618–1621. [[CrossRef](#)]
26. Alzabut, J.; Manikandan, S.; Muthulakshmi, V.; Harikrishnan, S. Oscillation criteria for a class of nonlinear conformable fractional damped dynamic equations on time scales. *J. Nonlinear Funct. Anal.* **2020**, *2020*, 10.
27. Agarwal, R.P.; Grace, S.R.; O'Regan, D. *Oscillation Theory for Difference and Functional Differential Equations*; Marcel Dekker: New York, NY, USA; Kluwer Academic: Dordrecht, The Netherlands, 2000.

28. Bazighifan, O.; Scapellato, A. Oscillatory properties of even-order ordinary differential equations with variable coefficients. *Miskolc Math. Notes* **2020**, *21*, 641–652.
29. Bazighifan, O.; Dassios, I. On the Asymptotic Behavior of Advanced Differential Equations with a Non-Canonical Operator. *Appl. Sci.* **2020**, *10*, 3130. [[CrossRef](#)]
30. Chatzarakis, G.E.; Jadlovská, I.; Li, T. Oscillations of differential equations with non-monotone deviating arguments. *Adv. Differ. Equ.* **2019**, *2019*, 1–20. [[CrossRef](#)]
31. Elabbasy, E.M.; El-Nabulsi, R.A.; Moaaz, O.; Bazighifan, O. Oscillatory Properties of Solutions of Even-Order Differential Equations. *Symmetry* **2020**, *12*, 212. [[CrossRef](#)]
32. Hale, J.K. *Theory of Functional Differential Equations*; Springer: New York, NY, USA, 1977.
33. Moaaz, O.; Elabbasy, E.M.; Muhib, A. Oscillation criteria for even-order neutral differential equations with distributed deviating arguments. *Adv. Differ. Equ.* **2019**, *2019*, 297. [[CrossRef](#)]
34. Moaaz, O. New criteria for oscillation of nonlinear neutral differential equations. *Adv. Differ. Equ.* **2019**, *2019*, 484. [[CrossRef](#)]
35. Moaaz, O.; Elabbasy, E.M.; Qaraad, B. An improved approach for studying oscillation of generalized Emden–Fowler neutral differential equation. *J. Inequal. Appl.* **2020**, *2020*, 69. [[CrossRef](#)]
36. Moaaz, O.; Park, C.; Muhib, A.; Bazighifan, O. Oscillation criteria for a class of even-order neutral delay differential equations. *J. Appl. Math. Comput.* **2020**, *63*, 607–617. [[CrossRef](#)]
37. Oguztöreli, M.N.; Stein, R.B. An analysis of oscillations in neuro-muscular systems. *J. Math. Biol.* **1975**, *2*, 87–105. [[CrossRef](#)]
38. Zhang, C.; Agarwal, R.P.; Bohner, M.; Li, T. New results for oscillatory behavior of even-order half-linear delay differential equations, *Appl. Math. Lett.* **2013**, *26*, 179–183. [[CrossRef](#)]