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Article

On the Asymptotic Behavior of Advanced Differential Equations with a Non-Canonical Operator

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Abstract: In this paper, we aim to study the oscillatory behavior of a class of even-order advanced differential equations with a non-canonical operator. In addition, we present results on the asymptotic behavior of this type of equations and provide an example that illustrates our main results.

Keywords: oscillation; even-order; advanced differential equations; asymptotic behavior

1. Introduction

In recent decades, many authors have studied problems of a number of different classes of advanced differential equations including the asymptotic and oscillatory behavior of their solutions, see [1–8] and the references cited therein. For some more recent oscillation results, see [9–20]. The interest in studying advanced differential equations is also caused by the fact that they appear in models of several areas in science. In [21–23], singular systems of differential equations are used to study the dynamics and stability properties of electrical power systems. Some additional mathematical background on this can be found in [24]. Systems of differential equations with delays are used to study additional properties of electrical power systems in [25,26]. Non-linear advanced differential equations can be used to describe complex dynamical networks, see [27–29], and bring new insight to their stability. Furthermore, this type of equations can be also used in the modeling of dynamical networks of interacting free-bodies, see [30]. Finally, properties of advanced differential equations are used in the study of singular differential equations of fractional order, see [31,32]. Several other examples in Physics can be found in [33]. In this paper, we consider an even-order non-linear advanced differential equation with a non-canonical operator of the following type:

$$L_y + q(v)\, g(y(\eta(v))) = 0, \quad L_y := \left(a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta}\right)', \tag{1}$$

where $v \geq v_0$, κ is even and β is a quotient of odd positive integers. The operator L_y is said to be in canonical form if $\int_{v_0}^{\infty} a^{-1/\beta}(s)\, ds = \infty$; otherwise, it is called noncanonical. Throughout this work, we suppose that:

C1: $a \in C^1([v_0,\infty),\mathbb{R}),\ a(v) > 0,\ a'(v) \geq 0,$
C2: $q, \eta \in C([v_0,\infty),\mathbb{R}),\ q(v) \geq 0,\ \eta(v) \geq v,\ \lim_{v\to\infty} \eta(v) = \infty,$
C3: $g \in C(\mathbb{R},\mathbb{R})$ such that $g(x)/x^{\beta} \geq k > 0$, for $x \neq 0$ and under the condition

$$\zeta(v) = \int_{v_0}^{\infty} \frac{1}{a^{1/\beta}(s)} ds < \infty. \tag{2}$$

Definition 1. *The function* $y \in C^{\kappa-1}[v_y,\infty)$, $v_y \geq v_0$, *is called a solution of (1), if* $\left(y^{(\kappa-1)}(v)\right)^{\beta} \in C^1[v_y,\infty)$, *for* $a \in C^1([v_0,\infty),\mathbb{R})$, $a(v) > 0$ *and* $y(v)$ *satisfies (1) on* $[v_y,\infty)$.

Definition 2. *Let*

$$D = \{(v,s) \in \mathbb{R}^2 : v \geq s \geq v_0\} \text{ and } D_0 = \{(v,s) \in \mathbb{R}^2 : v > s \geq v_0\}.$$

A kernel function $H_i \in C(D,\mathbb{R})$ *is said to belong to the function class* $\Im$, *written by* $H \in \Im$, *if, for* $i = 1,2$,

(i) $H_i(v,s) > 0$, *on* D_0 *and* $H_i(v,s) = 0$ *for* $v \geq v_0$ *with* $(v,s) \notin D_0$;

(ii) $H_i(v,s)$ *has a continuous and nonpositive partial derivative* $\partial H_i/\partial s$ *on* D_0 *and there exist functions* $\tau,\vartheta \in C^1([v_0,\infty),(0,\infty))$ *and* $h_i \in C(D_0,\mathbb{R})$ *such that*

$$\frac{\partial}{\partial s}H_1(v,s) + \frac{\tau'(s)}{\tau(s)}H_1(v,s) = h_1(v,s)\,H_1^{\beta/(\beta+1)}(v,s) \tag{3}$$

and

$$\frac{\partial}{\partial s}H_2(v,s) + \frac{\vartheta'(s)}{\vartheta(s)}H_2(v,s) = h_2(v,s)\sqrt{H_2(v,s)}. \tag{4}$$

Next we will discuss the results in [34–36]. Actually, our purpose in this article is to complement and improve these results. Agarwal et al. in [34,35] studied the even-order nonlinear advanced differential equations

$$\left(\left(y^{(\kappa-1)}(v)\right)^{\beta}\right)' + q(v)\,y^{\beta}(\eta(v)) = 0. \tag{5}$$

By means of the Riccati transformation technique, the authors established some oscillation criteria of (5). Grace and Lalli [36] investigated the second-order neutral Emden–Fowler delay dynamic equations

$$y^{(\kappa)}(v) + q(v)\,y(\eta(v)) = 0, \tag{6}$$

and established some new oscillation for (5) under the condition

$$\int_{v_0}^{\infty} \frac{1}{a^{1/\beta}(s)}\mathrm{d}s = \infty. \tag{7}$$

To prove this, we apply the previous results to the equation

$$y^{(\kappa)}(v) + \frac{q_0}{v^{\kappa}}y(\lambda v) = 0,\ v \geq 1. \tag{8}$$

if we set $\kappa = 4$ and $\lambda = 2$, then by applying conditions in [34–36] on Equation (8), we find the results in [35] improves those in [36]. Moreover, the those in [34] improves results in [35,36]. Thus, the motivation in our paper is to complement and improve results in [34–36]. We will use the following methods:

- Integral averaging technique.
- Riccati transformations technique.
- Method of comparison with second-order differential equations.

We will also use the following lemmas from (1):

Lemma 1 ([3]). *If* $y^{(i)}(v) > 0$, $i = 0,1,...,\kappa$, *and* $y^{(\kappa+1)}(v) < 0$, *then*

$$\frac{y(v)}{v^{\kappa}/\kappa!} \geq \frac{y'(v)}{v^{\kappa-1}/(\kappa-1)!}.$$

Lemma 2 ([19]). *Suppose that $y \in C^{\kappa}([v_0,\infty),(0,\infty))$, $y^{(\kappa)}$ is of a fixed sign on $[v_0,\infty)$, $y^{(\kappa)}$ not identically zero and there exists a $v_1 \geq v_0$ such that*

$$y^{(\kappa-1)}(v)\, y^{(\kappa)}(v) \leq 0,$$

for all $v \geq v_1$. If we have $\lim_{v\to\infty} y(v) \neq 0$, then there exists $v_\theta \geq v_1$ such that

$$y(v) \geq \frac{\theta}{(\kappa-1)!} v^{\kappa-1} \left| y^{(\kappa-1)}(v) \right|,$$

for every $\theta \in (0,1)$ and $v \geq v_\theta$.

Lemma 3 ([2]). *Let β be a ratio of two odd numbers, $V > 0$ and U are constants. Then*

$$Ux - Vx^{(\beta+1)/\beta} \leq \frac{\beta^\beta}{(\beta+1)^{\beta+1}} \frac{U^{\beta+1}}{V^\beta},\ V > 0.$$

Lemma 4. *Suppose that y is an eventually positive solution of (1). Then, there exist three possible cases:*

$(\mathbf{S}_1)$ $\quad y(v) > 0,\ y'(v) > 0, y''(v) > 0,\ y^{(\kappa-1)}(v) > 0,\ y^{(\kappa)}(v) < 0,$
$(\mathbf{S}_2)$ $\quad y(v) > 0, y^{(r)}(v) > 0, y^{(r+1)}(v) < 0$ *for all odd integer* $r \in \{1,3,...,\kappa-3\}, y^{(\kappa-1)}(v) > 0, y^{(\kappa)}(v) < 0,$
$(\mathbf{S}_3)$ $\quad y(v) > 0,\ y^{(\kappa-2)}(v) > 0,\ y^{(\kappa-1)}(v) < 0,\ L_y \leq 0,$

for $v \geq v_1$, where $v_1 \geq v_0$ is sufficiently large.

2. Oscillation Criteria

Theorem 1. *Assume that (2) holds. If the differential equations*

$$\left(\frac{(\kappa-2)!\, a^{\frac{1}{\beta}}(v)}{(\theta v^{\kappa-2})^\beta} (y'(v))^\beta \right)' + kq(v)\, y^\beta(v) = 0, \quad \forall\theta \in (0,1), \tag{9}$$

$$y''(v) + y(v) \frac{1}{(\kappa-4)!} \int_v^\infty (\varsigma - v)^{\kappa-4} \left(\frac{1}{a(\varsigma)} \int_\varsigma^\infty q(s)\, \mathrm{d}s \right)^{1/\beta} \mathrm{d}\varsigma = 0, \tag{10}$$

and

$$\left(a(v) (y'(v))^\beta \right)' + y^\beta(v)\, kq(v) \left(\frac{\zeta(\eta(v))}{\zeta(v)} \right)^\beta \left(\frac{\theta_1}{(\kappa-2)!} \eta^{\kappa-2}(v) \right)^\beta = 0, \quad \theta_1 \in (0,1) \tag{11}$$

are oscillatory for every constant $\theta, \theta_1 \in (0,1)$, then every solution of (1) is either oscillatory or satisfies $\lim\limits_{v\to\infty} y(v) = 0$.

Proof. Assume to the contrary that y is a positive solution of (1). Then, we can suppose that $y(v)$and $y(\eta(v))$ are positive for all $v \geq v_1$ sufficiently large. From Lemma 4, we have three possible cases $(\mathbf{S}_1)$, $(\mathbf{S}_2)$ and $(\mathbf{S}_3)$. Let case $(\mathbf{S}_1)$ hold. Using Lemma 2, we find

$$y'(v) \geq \frac{\theta}{(\kappa-2)!} v^{\kappa-2} y^{(\kappa-1)}(v), \tag{12}$$

for every $\theta \in (0,1)$ and for all large v. We set

$$\varphi(v) := \tau(v)\left(\frac{a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta}}{y^{\beta}(v)}\right), \tag{13}$$

and observe that $\varphi(v) > 0$ for $v \geq v_1$, where $\tau \in C^1([v_0,\infty),(0,\infty))$ and

$$\begin{aligned}\varphi'(v) &= \tau'(v)\frac{a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta}}{y^{\beta}(v)} + \tau(v)\frac{\left(a\left(y^{(\kappa-1)}\right)^{\beta}\right)'(v)}{y^{\beta}(v)} \\ &\quad -\beta\tau(v)\frac{y^{\beta-1}(v)\,y'(v)\,a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta}}{y^{2\beta}(v)}.\end{aligned}$$

Using (12) and (13), we obtain

$$\begin{aligned}\varphi'(v) &\leq \frac{\tau'_+(v)}{\tau(v)}\varphi(v) + \tau(v)\frac{\left(a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta}\right)'}{y^{\beta}(v)} \\ &\quad -\beta\tau(v)\frac{\theta}{(\kappa-2)!}v^{\kappa-2}\frac{a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta+1}}{y^{\beta+1}(v)} \\ &\leq \frac{\tau'(v)}{\tau(v)}\varphi(v) + \tau(v)\frac{\left(a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta}\right)'}{y^{\beta}(v)} \\ &\quad -\frac{\beta\theta v^{\kappa-2}}{(\kappa-2)!\,(\tau(v)\,a(v))^{\frac{1}{\beta}}}\varphi(v)^{\frac{\beta+1}{\beta}}.\end{aligned} \tag{14}$$

From (1) and (14), we obtain

$$\varphi'(v) \leq \frac{\tau'(v)}{\tau(v)}\varphi(v) - k\tau(v)\frac{q(v)\,y^{\beta}(\eta(v))}{y^{\beta}(v)} - \frac{\beta\theta v^{\kappa-2}}{(\kappa-2)!\,(\tau(v)\,a(v))^{\frac{1}{\beta}}}\varphi(v)^{\frac{\beta+1}{\beta}}.$$

Note that $y'(v) > 0$ and $\eta(v) \geq v$, thus, we find

$$\varphi'(v) \leq \frac{\tau'(v)}{\tau(v)}\varphi(v) - k\tau(v)\,q(v) - \frac{\beta\theta v^{\kappa-2}}{(\kappa-2)!\,(\tau(v)\,a(v))^{\frac{1}{\beta}}}\varphi(v)^{\frac{\beta+1}{\beta}}. \tag{15}$$

If we set $\tau(v) = k = 1$ in (15), then we find

$$\varphi'(v) + \frac{\beta\theta v^{\kappa-2}}{(\kappa-2)!\,a^{\frac{1}{\beta}}(v)}\varphi(v)^{\frac{\beta+1}{\beta}} + q(v) \leq 0.$$

From [37], we can see that Equation (9) is non-oscillatory, which is a contradiction.
Let case $(\mathbf{S}_2)$ hold. If we set

$$\psi(v) := \vartheta(v)\frac{y'(v)}{y(v)},$$

we see that $\psi(v) > 0$ for $v \geq v_1$, where $\vartheta \in C^1([v_0,\infty),(0,\infty))$. By differentiating $\psi(v)$, we find

$$\psi'(v) = \frac{\vartheta'(v)}{\vartheta(v)}\psi(v) + \vartheta(v)\frac{y''(v)}{y(v)} - \frac{1}{\vartheta(v)}\psi(v)^2. \tag{16}$$

Now, by integrating (1) from v to m and using $y'(v) > 0$, we get

$$a(m)\left(y^{(\kappa-1)}(m)\right)^{\beta} - a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta} = -\int_v^m q(s)\,g(y(\eta(s)))\,ds.$$

By virtue of $y'(v) > 0$ and $\eta(v) \geq v$, we get

$$a(m)\left(y^{(\kappa-1)}(m)\right)^{\beta} - a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta} \leq -ky^{\beta}(v)\int_v^u q(s)\,ds.$$

Letting $m \to \infty$, we see that

$$a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta} \geq ky^{\beta}(v)\int_v^{\infty} q(s)\,\mathrm{d}s$$

and so

$$y^{(\kappa-1)}(v) \geq y(v)\left(\frac{k}{a(v)}\int_v^{\infty} q(s)\,\mathrm{d}s\right)^{1/\beta}.$$

Integrating again from v to ∞, $\kappa - 4$ times, we get

$$y''(v) + \frac{y(v)}{(\kappa-4)!}\int_v^{\infty}(\varsigma - v)^{\kappa-4}\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty} q(s)\,\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma \leq 0. \tag{17}$$

From (16) and (17), we obtain

$$\psi'(v) \leq \frac{\vartheta'(v)}{\vartheta(v)}\psi(v) - \frac{\vartheta(v)}{(\kappa-4)!}\varpi(s) - \frac{1}{\vartheta(v)}\psi(v)^2, \tag{18}$$

where

$$\varpi(s) = \int_v^{\infty}(\varsigma - v)^{\kappa-4}\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty} q(s)\,\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma.$$

If we now set $\vartheta(v) = k = 1$ in (18), then we obtain

$$\psi'(v) + \psi^2(v) + \frac{1}{(\kappa-4)!}\varpi(s)\,\varsigma \leq 0.$$

From [37], we see Equation (10) is non-oscillatory, which is a contradiction.

Let case $(\mathbf{S}_3)$ hold. By recalling that $a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta}$ is non-increasing, we obtain

$$a^{1/\beta}(s)\,y^{(\kappa-1)}(s) \leq a^{1/\beta}(v)\,y^{(\kappa-1)}(v),\ s \geq v \geq v_1.$$

Dividing the latter inequality by $a^{1/\beta}(s)$ and integrating the resulting inequality from v to u, we get

$$y^{(\kappa-2)}(u) \leq y^{(\kappa-2)}(v) + a^{1/\beta}(v)\,y^{(\kappa-1)}(v)\int_v^u a^{-1/\beta}(s)\,\mathrm{d}s.$$

Letting $u \to \infty$, we obtain

$$0 \leq y^{(\kappa-2)}(v) + a^{1/\beta}(v)\,y^{(\kappa-1)}(v)\,\zeta(v).$$

Thus,

$$\frac{-a^{1/\beta}(v)\,y^{(\kappa-1)}(v)\,\zeta(v)}{y^{(\kappa-2)}(v)} \leq 1. \tag{19}$$

Furthermore, we get

$$\left(\frac{y^{(\kappa-2)}(v)}{\zeta(v)}\right)' \geq 0, \tag{20}$$

due to (19). Now define

$$\phi(v) = \frac{a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta}}{\left(y^{(\kappa-2)}(v)\right)^{\beta}}, \tag{21}$$

we see that $\phi(v) < 0$ for $v \geq v_1$, and

$$\phi'(v) = \frac{\left(a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta}\right)'}{\left(y^{(\kappa-2)}(v)\right)^{\beta}} - \frac{\beta a(v)\left(y^{(\kappa-1)}(v)\right)^{\beta+1}}{\left(y^{(\kappa-2)}(v)\right)^{\beta+1}}.$$

It follows from (1) and (19) that

$$\phi'(v) = \frac{-kq(v)\,y^{\beta}(\eta(v))}{\left(y^{(\kappa-2)}(v)\right)^{\beta}} - \frac{\beta\phi^{\beta/\beta+1}(v)}{a^{1/\beta}(v)}.$$

From Lemma 2, we find

$$y(v) \geq \frac{\theta_1}{(\kappa-2)!} v^{\kappa-2} y^{(\kappa-2)}(v). \tag{22}$$

Thus, we have

$$\phi'(v) = \frac{-kq(v)\,y^{\beta}(\eta(v))}{\left(y^{(\kappa-2)}(\eta(v))\right)^{\beta}} \frac{\left(y^{(\kappa-2)}(\eta(v))\right)^{\beta}}{\left(y^{(\kappa-2)}(v)\right)^{\beta}} - \frac{\beta\phi^{\beta/\beta+1}(v)}{a^{1/\beta}(v)}.$$

From (22), we obtain

$$\phi'(v) \leq -kq(v)\left(\frac{\theta_1\eta^{\kappa-2}(v)}{(\kappa-2)!}\right)^{\beta}\left(\frac{\zeta(\eta(v))}{\zeta(v)}\right)^{\beta} - \frac{\beta\phi^{\beta/\beta+1}(v)}{a^{1/\beta}(v)}. \tag{23}$$

From [37], we can see that Equation (11) is non-oscillatory, which is a contradiction. Theorem 1 is proved. □

Remark 1. *It is well known (see [15]) that if*

$$\int_{v_0}^{\infty}\frac{1}{a(v)}\mathrm{d}v < \infty, \text{ and } \liminf_{v\to\infty}\left(\int_{v_0}^{v}\frac{1}{a(s)}\mathrm{d}s\right)^{-1}\int_{v}^{\infty}\left(\int_{v_0}^{v}\frac{1}{a(s)}\mathrm{d}s\right)^{2} q(s)\,\mathrm{d}s > \frac{1}{4},$$

then Equations (9)–(11) with $\beta = 1$ are oscillatory.

Based on the above results and Theorem 1, we can easily obtain the following Hille and Nehari type oscillation criteria for (1) with $\beta = 1$.

Theorem 2. *Let $\beta = k = 1$ and assume that (2) holds. If for $\theta, \theta_1 \in (0,1)$*

$$\liminf_{v\to\infty}\left(\int_{v_0}^{v}\frac{\theta s^{\kappa-2}}{(\kappa-2)!a(s)}\mathrm{d}s\right)^{-1}\int_{v}^{\infty}\left(\int_{v_0}^{v}\frac{\theta s^{\kappa-2}}{(\kappa-2)!a(s)}\mathrm{d}s\right)^{2} q(s)\,\mathrm{d}s > \frac{1}{4}, \tag{24}$$

with

$$\int_{v_0}^{\infty}\frac{\theta v^{\kappa-2}}{(\kappa-2)!a(v)}\mathrm{d}v < \infty,$$

and if

$$\liminf_{v\to\infty} v \int_{v_0}^{v} \frac{1}{(\kappa-4)!} \int_{v}^{v} (\varsigma - v)^{\kappa-4} \left(\frac{1}{a(\varsigma)} \int_{\varsigma}^{v} q(s)\,\mathrm{d}s \right)^{1/\beta} \mathrm{d}\varsigma \mathrm{d}v > \frac{1}{4}, \tag{25}$$

$$\liminf_{v\to\infty} \left(\int_{v_0}^{v} \frac{1}{a(s)} \mathrm{d}s \right)^{-1} \int_{v}^{\infty} \left(\int_{v_0}^{v} \frac{1}{a(s)} \mathrm{d}s \right)^{2} \frac{\theta_1 \zeta(\eta(s))\, \eta^{\kappa-2}(s)\, q(s)}{\zeta(s)(\kappa-2)!} \mathrm{d}s > \frac{1}{4}, \tag{26}$$

then every solution of (1) is either oscillatory or satisfies $\lim_{v\to\infty} y(v) = 0$.

In the next theorem, we employ the integral averaging technique to establish a Philos-type oscillation criteria for (1):

Theorem 3. *Let (2) holds. If there exist positive functions* $\tau, \vartheta \in C^1([v_0,\infty), \mathbb{R})$ *such that*

$$\limsup_{v\to\infty} \frac{1}{H_1(v,v_1)} \int_{v_1}^{v} \left(H_1(v,s)\, k\tau(s)\, q(s) - \pi(s) \right) \mathrm{d}s = \infty, \tag{27}$$

$$\limsup_{v\to\infty} \frac{1}{H_2(v,v_1)} \int_{v_1}^{v} \left(H_2(v,s) \frac{\vartheta(s)}{(\kappa-4)!} \varpi(s) - \frac{\vartheta(s)\, h_2^2(v,s)}{4} \right) \mathrm{d}s = \infty, \tag{28}$$

and,

$$\limsup_{v\to\infty} \frac{1}{H_3(v,v_1)} \int_{v_1}^{v} \left(H_3(v,s)\, kq(s) \left(\frac{\theta_1 \eta^{\kappa-2}(s)}{(\kappa-2)!} \right)^{\beta} \zeta^{\beta}(\eta(s)) - \tilde{\pi}(s) \right) \mathrm{d}s = \infty,$$

where

$$\pi(s) = \frac{h_1^{\beta+1}(v,s)\, H_1^{\beta}(v,s)}{(\beta+1)^{\beta+1}} \frac{((\kappa-2)!)^{\beta}\, \tau(s)\, a(s)}{(\theta s^{\kappa-2})^{\beta}}$$

and

$$\tilde{\pi}(s) = \frac{\beta^{\beta+1} H_3(v,s)}{(\beta+1)^{\beta+1}} \frac{1}{a^{1/\beta}(s)\, \zeta(s)}.$$

Then every solution of (1) is either oscillatory or satisfies $\lim_{v\to\infty} y(v) = 0$.

Proof. Assume to the contrary that y is a positive solution of (1). Then, we can suppose that $y(v)$and $y(\eta(v))$ are positive for all $v \geq v_1$ sufficiently large. From Lemma 4, we have three possible cases $(\mathbf{S}_1)$, $(\mathbf{S}_2)$ and $(\mathbf{S}_3)$. Assume that $(\mathbf{S}_1)$ holds. From Theorem 1, we get that (15) holds. Multiplying (15) by $H_1(v,s)$ and integrating the resulting inequality from v_1 to v we find that

$$\begin{aligned} \int_{v_1}^{v} H_1(v,s)\, k\tau(s)\, q(s)\,\mathrm{d}s \quad \leq \quad & \varphi(v_1) H_1(v,v_1) + \int_{v_1}^{v} \left(\frac{\partial}{\partial s} H_1(v,s) + \frac{\tau'(s)}{\tau(s)} H_1(v,s) \right) \varphi(s)\,\mathrm{d}s \\ & - \int_{v_1}^{v} \frac{\beta\theta s^{\kappa-2}}{(\kappa-2)!\,(\tau(s)\,a(s))^{\frac{1}{\beta}}} H_1(v,s)\, \varphi^{\frac{\beta+1}{\beta}}(s)\,\mathrm{d}s. \end{aligned}$$

From (3), we get

$$\begin{aligned} \int_{v_1}^{v} H_1(v,s)\, k\tau(s)\, q(s)\,\mathrm{d}s \quad \leq \quad & \varphi(v_1) H_1(v,v_1) + \int_{v_1}^{v} h_1(v,s)\, H_1^{\beta/(\beta+1)}(v,s)\, \varphi(s)\,\mathrm{d}s \\ & - \int_{v_1}^{v} \frac{\beta\theta s^{\kappa-2}}{(\kappa-2)!\,(\tau(s)\,a(s))^{\frac{1}{\beta}}} H_1(v,s)\, \varphi^{\frac{\beta+1}{\beta}}(s)\,\mathrm{d}s. \end{aligned} \tag{29}$$

Using Lemma 3 with $V = \beta\theta s^{\kappa-2} / \left((\kappa-2)!\,(\tau(s)\,a(s))^{\frac{1}{\beta}} \right) H_1(v,s)$, $U = h_1(v,s)\, H_1^{\beta/(\beta+1)}(v,s)$

And $y = \varphi(s)$, we get

$$h_1(v,s)\,H_1^{\beta/(\beta+1)}(v,s)\,\varphi(s) - \frac{\beta\theta s^{\kappa-2}}{(\kappa-2)!\,(\tau(s)\,a(s))^{\frac{1}{\beta}}}H_1(v,s)\,\varphi^{\frac{\beta+1}{\beta}}(s)$$
$$\leq \frac{h_1^{\beta+1}(v,s)\,H_1^{\beta}(v,s)}{(\beta+1)^{\beta+1}}\frac{((\kappa-2)!)^{\beta}\,\tau(s)\,a(s)}{(\theta s^{\kappa-2})^{\beta}},$$

which, with (29) gives

$$\frac{1}{H_1(v,v_1)}\int_{v_1}^{v}(H_1(v,s)\,k\tau(s)\,q(s) - \pi(s))\,\mathrm{d}s \leq \varphi(v_1),$$

which contradicts (27). Assume that $(\mathbf{S}_2)$ holds. From Theorem 1, we get that (18) holds. Multiplying (18) by $H_2(v,s)$ and integrating the resulting inequality from v_1 to v, we obtain

$$\begin{aligned}\int_{v_1}^{v}H_2(v,s)\,\frac{\vartheta(s)}{(\kappa-4)!}\varpi(s)\,\mathrm{d}s \leq\; & \psi(v_1)\,H_2(v,v_1)\\ &+\int_{v_1}^{v}\left(\frac{\partial}{\partial s}H_2(v,s)+\frac{\vartheta'(s)}{\vartheta(s)}H_2(v,s)\right)\psi(s)\,\mathrm{d}s\\ &-\int_{v_1}^{v}\frac{1}{\vartheta(s)}H_2(v,s)\,\psi^2(s)\,\mathrm{d}s.\end{aligned}$$

Thus, from (4), we obtain

$$\begin{aligned}\int_{v_1}^{v}H_2(v,s)\,\frac{\vartheta(s)}{(\kappa-4)!}\varpi(s)\,\mathrm{d}s \leq\; & \psi(v_1)\,H_2(v,v_1)+\int_{v_1}^{v}h_2(v,s)\sqrt{H_2(v,s)}\psi(s)\,\mathrm{d}s\\ &-\int_{v_1}^{v}\frac{1}{\vartheta(s)}H_2(v,s)\,\psi^2(s)\,\mathrm{d}s\\ \leq\; & \psi(v_1)\,H_2(v,v_1)+\int_{v_1}^{v}\frac{\vartheta(s)\,h_2^2(v,s)}{4}\mathrm{d}s\end{aligned}$$

and so

$$\frac{1}{H_2(v,v_1)}\int_{v_1}^{v}\left(H_2(v,s)\,\frac{\vartheta(s)}{(\kappa-4)!}\varpi(s)-\frac{\vartheta(s)\,h_2^2(v,s)}{4}\right)\mathrm{d}s \leq \psi(v_1),$$

which contradicts (28). Assume that $(\mathbf{S}_3)$ holds. Using (19) and (21), we see that

$$-\phi(v)\,\zeta^{\beta}(v) \leq 1 \tag{30}$$

due to (30). Multiplying this inequality by $\zeta^{\beta}(v)$ and integrating the resulting inequality from v_1 to v, we get

$$\zeta^{\beta}(v)\,\phi(v)-\zeta^{\beta}(v_1)\,\phi(v_1)+\beta\int_{v_1}^{v}a^{-1/\beta}(s)\,\zeta^{\beta-1}(s)\,\phi(s)\,ds$$
$$\leq\; -\int_{v_1}^{v}kq(s)\left(\frac{\theta_1\eta^{\kappa-2}(s)}{(\kappa-2)!}\right)^{\beta}\zeta^{\beta}(\eta(s))\,ds-\beta\int_{v_1}^{v}\frac{\phi^{\beta/\beta+1}(s)}{a^{1/\beta}(s)}\zeta^{\beta}(s)\,ds. \tag{31}$$

Multiplying (31) by $H_3(v,s)$, we find that

$$\begin{aligned}\int_{v_1}^{v} H_3(v,s)\,kq(s)\left(\frac{\theta_1\eta^{\kappa-2}(s)}{(\kappa-2)!}\right)^{\beta}\zeta^{\beta}(\eta(s))\,\mathrm{d}s \quad&\leq\quad \zeta^{\beta}(v_1)\,\phi(v_1)\,H_3(v,v_1)-\zeta^{\beta}(v)\,\phi(v)\,H_3(v,v_1)\\&\quad+\int_{v_1}^{v}\beta a^{-1/\beta}(s)\,\zeta^{\beta-1}(s)\,\phi(s)\,H_3(v,s)\,\mathrm{d}s\\&\quad-\int_{v_1}^{v}\frac{\beta\phi^{\beta/\beta+1}(s)}{a^{1/\beta}(s)}\zeta^{\beta}(s)\,H_3(v,s)\,\mathrm{d}s.\end{aligned}$$

Using Lemma 3 with $V=\zeta^{\beta}(s)\,H_3(v,s)/a^{1/\beta}(s)$, $U=a^{-1/\beta}(s)\,\zeta^{\beta-1}(s)\,H_3(v,s)$ and $y=\phi(s)$, we get

$$\begin{aligned}&\beta a^{-1/\beta}(s)\,\zeta^{\beta-1}(s)\,\phi(s)\,H_3(v,s)-\frac{\beta\phi^{\beta/\beta+1}(s)}{a^{1/\beta}(s)}\zeta^{\beta}(s)\,H_3(v,s)\\\leq\quad&\frac{\beta^{\beta+1}H_3(v,s)}{(\beta+1)^{\beta+1}}\frac{1}{a^{1/\beta}(s)\,\zeta(s)}\end{aligned}$$

and easily, we find that

$$\frac{1}{H_3(v,v_1)}\int_{v_1}^{v}\left(H_3(v,s)\,kq(s)\left(\frac{\theta_1\eta^{\kappa-2}(s)}{(\kappa-2)!}\right)^{\beta}\zeta^{\beta}(\eta(s))-\tilde{\pi}(s)\right)\mathrm{d}s\leq\zeta^{\beta}(v_1)\,\phi(v_1)+1,$$

which contradicts (27). This completes the proof. □

Example 1. *We consider the equation*

$$\left(v^5y'''(v)\right)'+vq_0y(3v)=0,\ v\geq1, \tag{32}$$

where $q_0>0$ is a constant. Note that $\beta=1,\ \kappa=4,\ a(v)=v^5,\ q(v)=vq_0$ and $\eta(v)=3v$. If we set $k=1$, then condition (24) becomes

$$\begin{aligned}&\liminf_{v\to\infty}\left(\int_{v_0}^{v}\frac{\theta s^{\kappa-2}}{(\kappa-2)!a(s)}\mathrm{d}s\right)^{-1}\int_{v}^{\infty}\left(\int_{v_0}^{v}\frac{\theta s^{\kappa-2}}{(\kappa-2)!a(s)}\mathrm{d}s\right)^{2}q(s)\,\mathrm{d}s\\=\quad&\liminf_{v\to\infty}\left(4v^2\right)\int_{v}^{\infty}\frac{q_0}{16s^3}ds=\liminf_{v\to\infty}\left(4v^2\right)\left(\frac{q_0}{32v^2}\right)\\=\quad&\frac{q_0}{8}>\frac{1}{4},\end{aligned}$$

while condition (25) becomes

$$\begin{aligned}\liminf_{v\to\infty}v\int_{v_0}^{v}\frac{1}{(\kappa-4)!}\int_{v}^{v}(\varsigma-v)^{\kappa-4}\left(\frac{1}{a(\varsigma)}\int_{\varsigma}^{v}q(s)\,\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma\mathrm{d}v\quad&=\quad\liminf_{v\to\infty}v\left(\frac{q_0}{4v}\right)\\&=\quad\frac{q_0}{4}>\frac{1}{4},\end{aligned}$$

and hence condition (26) is satisfied. Therefore, from Theorem 2, all solutions of Equation (32) are oscillatory if $q_0>2$.

Remark 2. *One can easily see that the results obtained in [18,19] cannot be applied to conditions in Theorem 2, so our results are new.*

Remark 3. *We can generalize our results by studying the equation in the form*

$$\left(a\left(v\right)\left(y^{\left(\kappa-1\right)}\left(v\right)\right)^{\beta}\right)'+\sum_{i=1}^{j}q_{i}\left(v\right)y^{\beta}\left(\eta_{i}\left(v\right)\right)=0,\ where\ v\geq v_{0},\ j\geq 1.$$

For this we leave the results to researchers interested .

3. Conclusions

In this article we studied we provided three new Theorems on the oscillatory and asymptotic behavior of a class of even-order advanced differential equations with a non-canonical operator in the form of (1).

For researchers interested in this field, and as part of our future research, there is a nice open problem which is finding new results in the following cases:

$$\begin{aligned}
(\mathbf{S}_1)\quad & y\left(v\right)>0,\ y'\left(v\right)>0, y^{\left(\kappa-2\right)}\left(v\right)>0,\ \ y^{\left(\kappa-1\right)}\left(v\right)\leq 0,\ \ \left(a\left(v\right)\left(y^{\left(\kappa-1\right)}\left(v\right)\right)^{\beta}\right)'\leq 0,\\
(\mathbf{S}_2)\quad & y\left(v\right)>0, y^{(r)}(v)<0, y^{(r+1)}(v)>0, \forall r\in\{1,3,...,\kappa-3\},\\
& \text{and } y^{(\kappa-1)}(v)<0,\ \left(a\left(v\right)\left(y^{\left(\kappa-1\right)}\left(v\right)\right)^{\beta}\right)'\leq 0.
\end{aligned}$$

For all this there is some research in progress.

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