Supplementary Materials: Derivation of the Control Allocation Strategy

Jasan Zughaibi, Matthias Hofer and Raffaello D'Andrea

1. Derivation of the Control Allocation Strategy

The inverse direction of the control allocation strategy was presented in [1] and proven in [2]. In this section, we provide a compact derivation for completeness including a visualization to provide more intuition. Formally, we show the inverse direction of the control allocation strategy from virtual control inputs to actuator pressures,

$$(\Delta p_\alpha, \Delta p_\beta, \bar{p}) \rightarrow (p_\mathrm{A}, p_\mathrm{B}, p_\mathrm{C}), \tag{1}$$

and thereby show that the following equations hold,

$$\begin{aligned} p_\mathrm{A} &= \max\{\bar{p}, \bar{p} + \Delta p_\mathrm{AB}, \bar{p} + \Delta p_\mathrm{AB} + \Delta p_\mathrm{BC}\} \\ p_\mathrm{B} &= \max\{\bar{p}, \bar{p} + \Delta p_\mathrm{BC}, \bar{p} - \Delta p_\mathrm{AB}\} \\ p_\mathrm{C} &= \max\{\bar{p}, \bar{p} - \Delta p_\mathrm{BC}, \bar{p} - \Delta p_\mathrm{AB} - \Delta p_\mathrm{BC}\}, \end{aligned} \tag{2}$$

with $\Delta p_\mathrm{AB} = p_\mathrm{A} - p_\mathrm{B}$ and $\Delta p_\mathrm{BC} = p_\mathrm{B} - p_\mathrm{C}$ and the virtual control inputs Δp_α and Δp_β as defined by

$$\begin{bmatrix} \Delta p_\alpha \\ \Delta p_\beta \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{3}/2 \\ -1 & -1/2 \end{bmatrix} \begin{bmatrix} \Delta p_\mathrm{AB} \\ \Delta p_\mathrm{BC} \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} \Delta p_\mathrm{AB} \\ \Delta p_\mathrm{BC} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{3} & -1 \\ 2/\sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} \Delta p_\alpha \\ \Delta p_\beta \end{bmatrix}. \tag{3}$$

Proof. First, a visualization of the virtual control inputs in the absolute pressure space is provided in Fig. S1.

A point expressed by the actuator pressure constrained to the red plane, where $p_\mathrm{A} = \bar{p}$ holds, is given as a function of Δp_AB, Δp_BC as,

$$\begin{aligned} p_\mathrm{A} &= \bar{p} \\ \Delta p_\mathrm{AB} &= \bar{p} - p_\mathrm{B} \quad \Rightarrow \quad p_\mathrm{B} = \bar{p} - \Delta p_\mathrm{AB} \\ \Delta p_\mathrm{BC} &= p_\mathrm{B} - p_\mathrm{C} \quad \Rightarrow \quad p_\mathrm{C} = \bar{p} - \Delta p_\mathrm{AB} - \Delta p_\mathrm{BC}. \end{aligned} \tag{4}$$

Similarly for the green plane ($p_\mathrm{B} = \bar{p}$), we have

$$\begin{aligned} p_\mathrm{A} &= \bar{p} + \Delta p_\mathrm{AB} \\ p_\mathrm{B} &= \bar{p} \\ p_\mathrm{C} &= \bar{p} - \Delta p_\mathrm{BC}, \end{aligned} \tag{5}$$

and for the blue plane ($p_\mathrm{C} = \bar{p}$), we have

$$\begin{aligned} p_\mathrm{A} &= \bar{p} + \Delta p_\mathrm{AB} + \Delta p_\mathrm{BC} \\ p_\mathrm{B} &= \bar{p} + \Delta p_\mathrm{BC} \\ p_\mathrm{C} &= \bar{p}. \end{aligned} \tag{6}$$

The sets introduced in Fig. S1 are defined as,

$$\begin{aligned} S_\mathrm{A} &:= \{(\Delta p_\mathrm{AB}, \Delta p_\mathrm{BC}) \in \mathbb{R}^2 \mid \Delta p_\mathrm{AB} \leq -\Delta p_\mathrm{BC},\ \Delta p_\mathrm{AB} \leq 0\}, \\ S_\mathrm{B} &:= \{(\Delta p_\mathrm{AB}, \Delta p_\mathrm{BC}) \in \mathbb{R}^2 \mid \Delta p_\mathrm{AB} \geq 0,\ \Delta p_\mathrm{BC} \leq 0\}, \\ S_\mathrm{C} &:= \{(\Delta p_\mathrm{AB}, \Delta p_\mathrm{BC}) \in \mathbb{R}^2 \mid \Delta p_\mathrm{AB} \geq -\Delta p_\mathrm{BC},\ \Delta p_\mathrm{BC} \geq 0\}. \end{aligned} \tag{7}$$

Considering a point $(\Delta p_{\mathrm{AB}}, \Delta p_{\mathrm{BC}})$ that lies in S_{A}, we can conclude that $p_{\mathrm{A}} = \bar{p}$ (see Fig. S1) and p_{B} and p_{C} follow from (4). Applying the same reasoning for a point $(\Delta p_{\mathrm{AB}}, \Delta p_{\mathrm{BC}})$ that lies in S_{B} or S_{C}, and combining the cases for p_{A}, we conclude that,

$$p_{\mathrm{A}} = \begin{cases} \bar{p} & (\Delta p_{\mathrm{AB}}, \Delta p_{\mathrm{BC}}) \in S_{\mathrm{A}}, \\ \bar{p} + \Delta p_{\mathrm{AB}} & (\Delta p_{\mathrm{AB}}, \Delta p_{\mathrm{BC}}) \in S_{\mathrm{B}}, \\ \bar{p} + \Delta p_{\mathrm{AB}} + \Delta p_{\mathrm{BC}} & (\Delta p_{\mathrm{AB}}, \Delta p_{\mathrm{BC}}) \in S_{\mathrm{C}}. \end{cases} \tag{8}$$

Similarly for p_{B} it holds,

$$p_{\mathrm{B}} = \begin{cases} \bar{p} - \Delta p_{\mathrm{AB}} & (\Delta p_{\mathrm{AB}}, \Delta p_{\mathrm{BC}}) \in S_{\mathrm{A}}, \\ \bar{p} & (\Delta p_{\mathrm{AB}}, \Delta p_{\mathrm{BC}}) \in S_{\mathrm{B}}, \\ \bar{p} + \Delta p_{\mathrm{BC}} & (\Delta p_{\mathrm{AB}}, \Delta p_{\mathrm{BC}}) \in S_{\mathrm{C}}, \end{cases} \tag{9}$$

and for p_{C},

$$p_{\mathrm{C}} = \begin{cases} \bar{p} - \Delta p_{\mathrm{AB}} - \Delta p_{\mathrm{BC}} & (\Delta p_{\mathrm{AB}}, \Delta p_{\mathrm{BC}}) \in S_{\mathrm{A}}, \\ \bar{p} - \Delta p_{\mathrm{BC}} & (\Delta p_{\mathrm{AB}}, \Delta p_{\mathrm{BC}}) \in S_{\mathrm{B}}, \\ \bar{p} & (\Delta p_{\mathrm{AB}}, \Delta p_{\mathrm{BC}}) \in S_{\mathrm{C}}. \end{cases} \tag{10}$$

The three cases for each actuator pressure can be combined by the maximum function as,

$$\begin{aligned} p_{\mathrm{A}} &= \max\{\bar{p}, \bar{p} + \Delta p_{\mathrm{AB}}, \bar{p} + \Delta p_{\mathrm{AB}} + \Delta p_{\mathrm{BC}}\} \\ p_{\mathrm{B}} &= \max\{\bar{p}, \bar{p} + \Delta p_{\mathrm{BC}}, \bar{p} - \Delta p_{\mathrm{AB}}\} \\ p_{\mathrm{C}} &= \max\{\bar{p}, \bar{p} - \Delta p_{\mathrm{BC}}, \bar{p} - \Delta p_{\mathrm{AB}} - \Delta p_{\mathrm{BC}}\}. \end{aligned} \tag{11}$$

It remains to shows that this is indeed true. We execute the proof for p_{A}.

For $(\Delta p_{\mathrm{AB}}, \Delta p_{\mathrm{BC}}) \in S_{\mathrm{A}}$ it holds that:

$$\Delta p_{\mathrm{AB}} \leq 0 \quad \text{and} \quad \Delta p_{\mathrm{AB}} + \Delta p_{\mathrm{BC}} \leq 0, \tag{12}$$

hence

$$\bar{p} \geq \bar{p} + \Delta p_{\mathrm{AB}} \quad \text{and} \quad \bar{p} \geq \bar{p} + \Delta p_{\mathrm{AB}} + \Delta p_{\mathrm{BC}}. \tag{13}$$

For $(\Delta p_{\mathrm{AB}}, \Delta p_{\mathrm{BC}}) \in S_{\mathrm{B}}$ it holds that:

$$\Delta p_{\mathrm{AB}} \geq 0 \quad \text{and} \quad \Delta p_{\mathrm{BC}} \leq 0, \tag{14}$$

hence

$$\bar{p} + \Delta p_{\mathrm{AB}} \geq \bar{p} \quad \text{and} \quad \bar{p} + \Delta p_{\mathrm{AB}} \geq \bar{p} + \Delta p_{\mathrm{AB}} + \Delta p_{\mathrm{BC}}. \tag{15}$$

For $(\Delta p_{\mathrm{AB}}, \Delta p_{\mathrm{BC}}) \in S_{\mathrm{C}}$ it holds that:

$$\Delta p_{\mathrm{AB}} + \Delta p_{\mathrm{BC}} \geq 0 \quad \text{and} \quad \Delta p_{\mathrm{BC}} \geq 0, \tag{16}$$

hence

$$\bar{p} + \Delta p_{\mathrm{AB}} + \Delta p_{\mathrm{BC}} \geq \bar{p} \quad \text{and} \quad \bar{p} + \Delta p_{\mathrm{AB}} + \Delta p_{\mathrm{BC}} \geq \bar{p} + \Delta p_{\mathrm{AB}}. \tag{17}$$

The same reasoning can be applied for p_{B} and p_{C} completing the proof. $\square$

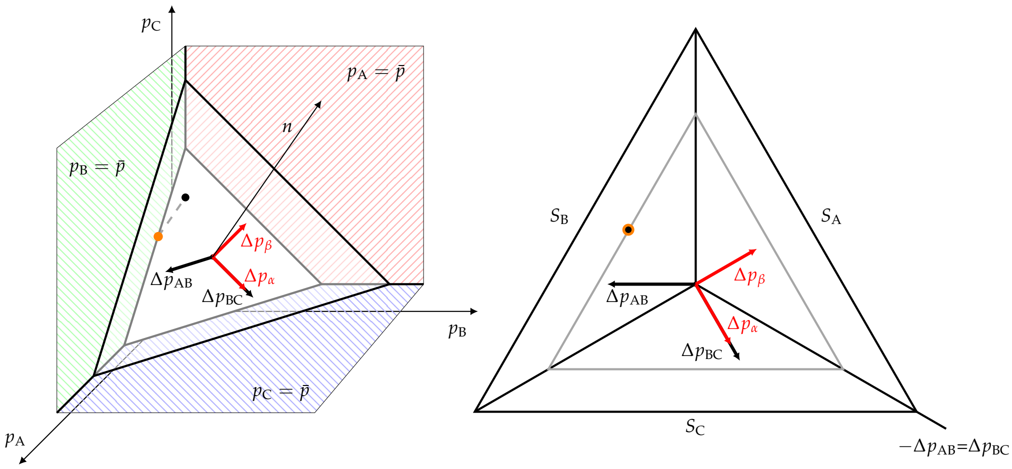

Figure S1. The left plot shows the (positive octant of the) absolute pressure space with p_A, p_B and p_C forming the standard basis. The three planes in red, green and blue correspond to the points where either p_A, p_B or p_C is equal to a certain value of $\bar{p}$. Consequently, the three planes indicate the points where $\min\{p_A, p_B, p_C\} = \bar{p}$ is fulfilled. The plane in white is spanned by Δp_{AB} and Δp_{BC} (or equivalently by Δp_α and Δp_β) with the normal direction indicated by n. The intersection of the white plane with the three colored planes forms an equilateral triangle with its top view shown in the right plot. The black dot indicates a certain value of $(\Delta p_{AB}, \Delta p_{BC})$. Note that the point does not lie on the boundary of the triangle in the right plot and correspondingly does not lie on one of the colored planes in the left plot. Projecting the black dot in the normal direction, n, onto the planes defined by the constraint, $\min\{p_A, p_B, p_C\} = \bar{p}$, results in the orange dot that lies on the boundary of a smaller equilateral triangle (indicated by the gray triangle in the right plot). Therefore, the point is uniquely defined in the absolute pressure space. The triangle in the Δp_α-Δp_β-plane is split by its median lines into three sets S_A, S_B and S_C, where S_A and S_C are split by the line where $-\Delta p_{AB} = \Delta p_{BC}$ holds. If a point $(\Delta p_{AB}, \Delta p_{BC})$ lies in S_B, it is projected onto the green plane that satisfies $p_B = \bar{p}$. The same holds for a point in S_A and the red plane satisfying $p_A = \bar{p}$ and a point in S_C and the blue plane satisfying $p_C = \bar{p}$.

References

1. Zughaibi, J.; Hofer, M.; D'Andrea, R. A Fast and Reliable Pick-and-Place Application with a Spherical Soft Robotic Arm. 2021 IEEE 4th International Conference on Soft Robotics (RoboSoft), 2021, pp. 599–606. doi:10.1109/RoboSoft51838.2021.9479227.
2. Zughaibi, J.; Hofer, M.; D'Andrea, R. A Fast and Reliable Pick-and-Place Application with a Spherical Soft Robotic Arm. Online appendix, 2021. doi:10.3929/ethz-b-000470203.