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# Time in Quantum Cosmology of FRW $f(R)$ Theories

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**Abstract:** The time problem is a problem of canonical quantum gravity that has long been known about; it is related to the relativistic invariance and the consequent absence of an explicit time variable in the quantum equations. This fact complicates the interpretation of the wave function of the universe. Following proposals to assign the clock function to a scalar field, we look at the scalar degree of freedom contained in $f(R)$ theories. For this purpose we consider a quadratic $f(R)$ theory in an equivalent formulation with a scalar field, with a FRW metric, and consider its Wheeler-DeWitt equation. The wave function is obtained numerically and is consistent with the interpretation of the scalar field as time by means of a conditional probability, from which an effective time-dependent wave function follows. The evolution the scale factor is obtained by its mean value, and the quantum fluctuations are consistent with the Heisenberg relations and a classical universe today.

**Keywords:** quantum cosmology; modified gravity; time problem

## 1. Introduction

Since its formulation, general relativity has been a successful theory, verified in many ways and at any scale. However, there are instances where it does not reproduce in a precise way the results of observations, in particular the origin of the universe and the early and present inflationary phases, as well as the present matter distribution. The other issues are quantum gravity, for which there are well known approaches, string theory, loop quantum gravity, and supergravity, which has been attracting attention recently. In all these approaches, the problem of time is present and there are diverse proposals for its solution, see [1–3]. In the proposal of Page and Wooters [4], it is argued that all the observables are stationary, but dynamics arises by the behavior of the observables relative to one which plays the role of clock.

One way to approach these issues is by $f(R)$ modified gravity theories, proposed by Starobisnky [5] as an effective action of gravity obtained by coupling it to quantum matter fields, which explains inflation and reheating, and recently it has being used to explain the effects of dark matter and dark energy [6,7], see also [8–12]. Even if these theories appear as effective theories, one appealing feature of them is that they are pure gravity theories, with additional structure as the action is higher order. In fact it is possible to give equivalent actions [13,14], with scalar degrees of freedom.

Quantum cosmology of $f(R)$ theory has been considered in [15], as a proposal for the origin of the universe from a tunneling from "nothing" to the de Sitter phase of the Starobinski model. In this work were also computed, in the WKB approximation, the subsequent curvature fluctuations and the duration of the inflationary phase. Quantum cosmology of $f(R)$ theories has been studied also in [16–19].

In this work we consider the FRW quantum cosmology of $f(R) = R + \alpha R^2$ theory in the form of a second order theory with a scalar field given by O'Hanlon [13,20]. In this approach, the scalar field is auxiliary and has an apparent dynamics which corresponds to its expression in terms of the Ricci scalar. As we do not implement conformal transformations, we do not expect that there are equivalence problems as in the cases discussed in [21].

We solve numerically the Wheeler-deWitt equation, and consider the scalar field as a clock. Considering the positivity of the scale factor $a$, for the hermiticity of the conjugate momentum of $a$, we take a vanishing wave function of the universe at $a = 0$. For the numerical computation, we consider a compact domain in $a$ and $\phi$. The solution is consistent to the vanishing condition on all boundaries, hence it is expected to be normalizable. Moreover, it ensures that the conjugate momentum of $a$ is hermitian [22]. For the time interpretation fo the scalar fields, we follow [22], where an effective wave function corresponding to a conditional probability is proposed. In the second section we make a short analysis of the classical solutions and in the third section we show the numerical solution of the WdW equation considering values of the parameter $\alpha$ for which classically the solutions are qualitatively different. In the last section we draw some conclusions.

## 2. Lagrangian Analysis

Let us consider the $f(R)$ model for gravity without matter, with action

$$A = \frac{1}{2\kappa^2}\int dt\sqrt{-g}f(R). \tag{1}$$

The variation of this action leads to the equations of motion [11] $F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F(R) + g_{\mu\nu}\Box F(R) = 0$, where $F(R) = f'(R)$, see e.g., [8,10]. For a FRW geometry $g_{\mu\nu} = \mathrm{diag}\left(-N^2, \frac{a^2}{1-kr^2}, a^2r^2, a^2r^2\sin^2\theta\right)$, the scalar curvature is $R = \frac{6}{a^2}\left(\frac{a\ddot{a}}{N^2} + \frac{\dot{a}^2}{N^2} - \frac{a\dot{a}\dot{N}}{N^3} + k\right)$. In reference [5], for the quadratic action

$$f(R) = R + \alpha R^2 \tag{2}$$

and for $k = 0$, $F(R)$ is regarded as a scalar degree of freedom, the "scalaron" $\varphi$. In this case the equations of motion are second order, and the dynamics of the scale factor is simply given by $a = \exp\int Hdt$.

Instead of it, here we adopt the O'Hanlon action [13]

$$A = \frac{1}{2\kappa^2}\int\sqrt{-g}\left[R + \phi(\beta\phi + R)\right], \tag{3}$$

where $\beta$ is a free parameter. This action resembles the action used in [15], where the definition of the scalar curvature is regarded as a constraint. A variation with respect to $\phi$ gives $\phi = -\frac{1}{2\beta}R$, which leads to $A = \frac{1}{2\kappa^2}\int\sqrt{-g}\left(R - 1/4\,\beta^{-1}R^2\right)$, i.e., $\alpha = -1/4\beta^{-1}$. Thus, with the FRW metric, action (3) can be written as

$$A = \frac{1}{\kappa}\int dt\left[3(1+\phi)(Nak - N^{-1}a\dot{a}^2) - 3N^{-1}a^2\dot{a}\dot{\phi} + \frac{1}{2}\beta Na^3\phi^2\right].$$

From the equation of motion of $\phi$ follows

$$\phi = -\frac{3}{\beta a^2}\left(a\ddot{a} + \dot{a}^2 + k\right), \tag{4}$$

as well as the higher order equation for $a$ [11]

$$3k^2 + \dot{a}^2\left(6a\ddot{a} - 9\dot{a}^2 - 2\beta a^2 - 6k\right) + a^2\left(6\dot{a}\dddot{a} - 3\ddot{a}^2 - 2k\beta\right) = 0. \tag{5}$$

A numerical solution of the last equation with vanishing initial scale factor, at $t = 0$, leads as shown in [20], to an inflationary stage with exit, see Figure 1. See [11] for an analytical solution.

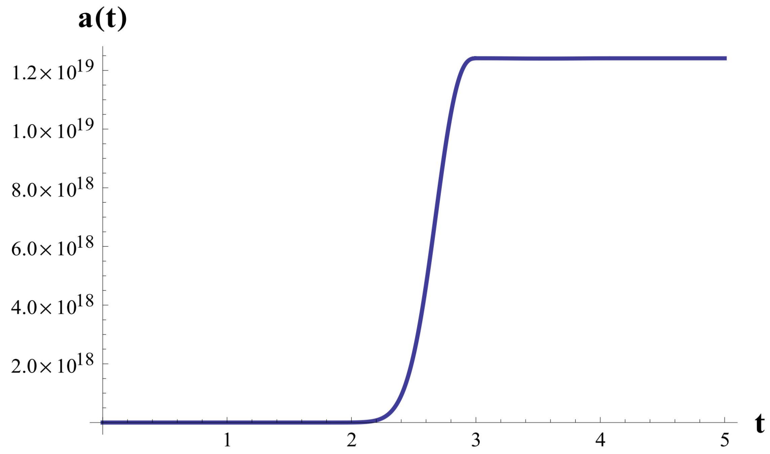


**Figure 1.** Numerical profile of $a(t)$ with initial conditions $a(0) = 10^{-20}$, $\dot{a}(0) = 10^{-20}$, $\ddot{a}(0) = 10^{-17}$, $\beta = -200$ and $k = 0$.

Substituting this solution in the expression for $\phi(t)$ (4), we get Figure 2.

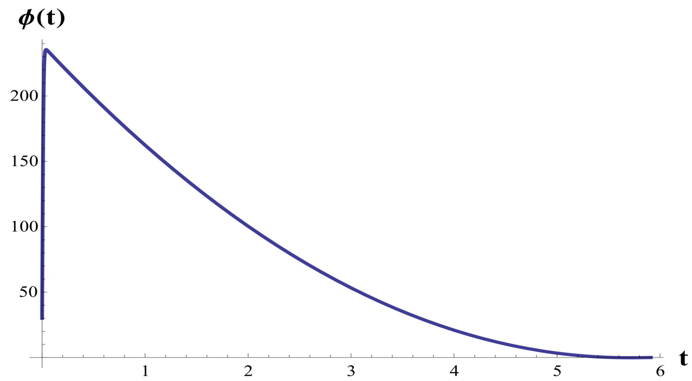


**Figure 2.** Numerical profile of $\phi(t)$ corresponding to $a(t)$ given in 1.

Thus, as can be also deduced from the solution of $\phi$ in terms of the scalar of curvature, the scalar $\phi$ take positive values. From these graphics we see that this scalar could be taken as a clock inverting the time, omitting far times, as shown in Figure 3.

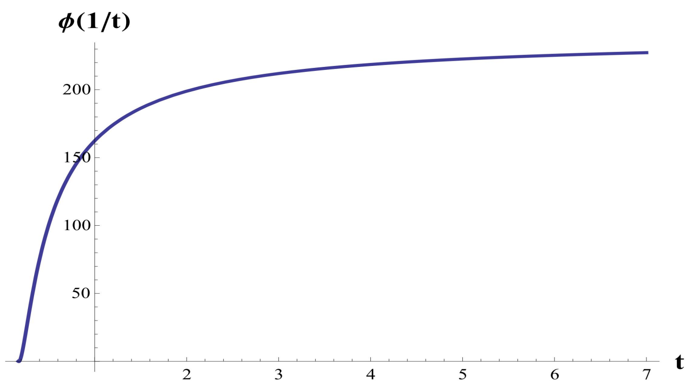


**Figure 3.** Numerical profile of $\phi(1/t)$ corresponding to $a(t)$ given in Figure 2.

## 3. Quantum Cosmology

The canonical momenta of the action (3) are $\pi_a = -\left[\frac{12a\phi\dot{a}}{N} + \frac{12a\dot{a}}{N} + \frac{6a^2\dot{\phi}}{N}\right]$, $\pi_\phi = -\frac{6a^2\dot{a}}{N}$, and $\pi_N = 0$, and the Hamiltonian is $H = NH_0$, where $H_0 = -6ka\phi - 6ka - \beta a^3\phi^2 + \frac{\phi\pi_\phi{}^2}{6a^3} + \frac{\pi_\phi{}^2}{6a^3} - \frac{\pi_a\pi_\phi}{6a^2}$. After choosing Weyl ordering for the ambiguous operator products, the Wheeler-DeWitt equation follows

$$\left[-\hbar^2(\phi+1)\frac{\partial^2}{\partial\phi^2} + a\hbar^2\frac{\partial^2}{\partial\phi\partial a} - 2\hbar^2\frac{\partial}{\partial\phi} - 6\beta a^6\phi^2 - 36ka^4(\phi+1)\right]\psi(a,\phi) = 0. \tag{6}$$

This equation does not admit analytic solutions, unless approximation methods are considered [15]. Further, the scale factor satisfies $a \geq 0$, and as shown in the previous section, the scalar field satisfies the same condition $\phi \geq 0$. Thus, to ensure hermiticity of the momenta $\pi_a$ and $\pi_\phi$, it is required that the wave function vanishes at $a = 0$ [22] and $\phi = 0$. Furthermore, for $a \to \infty$ the wave function must vanish, but if we allow that the universe does not collapse at future, then we do not impose conditions at an upper value of $\phi$.

An analysis in this case can be done with an ansatz with a power series solution of the form

$$\psi(a,\phi) = \sum_{n=0}^{\infty} f_n(\phi)a^n, \tag{7}$$

which entered in (6) leads to a system of differential equations for the coefficients $f_n(\phi)$

$$\begin{aligned}(1+\phi)\,f_0'' + 2f_0' &= 0,\\ (1+\phi)\,f_1'' + f_1' &= 0,\\ f_2'' &= 0,\\ (1+\phi)\,f_3'' + f_3' &= 0,\\ (1+\phi)\,f_4'' - 2f_4' + 36k\,(1+\phi)\,f_0 &= 0,\\ &\vdots\end{aligned}$$

which can be solved iteratively starting from $f_0$

$$\begin{aligned} f_0 &= c_2 - \frac{c_1}{\phi+1},\\ f_1 &= c_3\log(\phi+1) + c_4,\\ f_2 &= c_5\phi + c_4,\\ f_3 &= c_5\left(\frac{\phi^2}{2} + \phi\right) + c_6,\\ f_4 &= \frac{18c_2k\phi^2}{\hbar^2} + \frac{36c_2k\phi}{\hbar^2} - \frac{18c_1k\phi}{\hbar^2} + \frac{c_6\phi^3}{3} + c_6\phi^2 + c_6\phi + c_7,\\ &\vdots\end{aligned}$$

It can be seen that imposing the previous conditions $\psi(0,\phi) = 0$ and $\psi(a,0) = 0$ determines the coefficients in function of $c_1$, $c_3$, and $c_5$. In Figure 4 we illustrate this wave function obtained by numerical analysis of (6) for small values of $a$ and $\phi$.

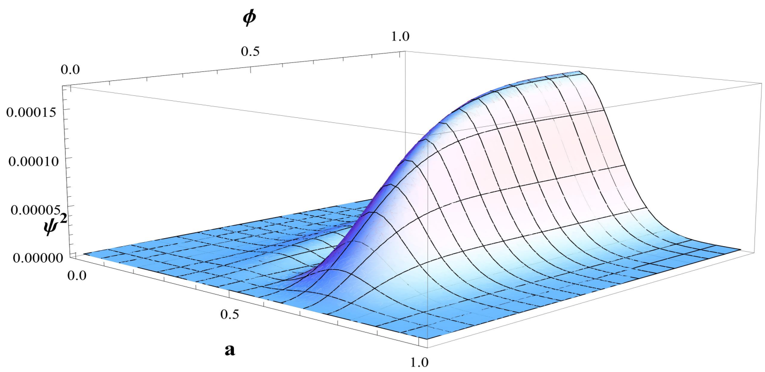


**Figure 4.** Numerical profile of $\psi(a,\phi)^2$ corresponding to (7), with $\beta = -100$, $k = 0$, $c_1 = 1$, $c_3 = c_5 = 0$.

For a solution around $a = \infty$ we write (6) for $b = 1/a$

$$\left[\hbar^2 b^6 \left(1+\phi\right) \frac{\partial^2}{\partial\phi^2} + \hbar^2 b^7 \frac{\partial^2}{\partial\phi\partial b} + 2b^6\hbar^2 \frac{\partial}{\partial\phi} + 36b^2 k \left(1{+}\phi\right) {+}6\beta\phi^2\right] \psi = 0,$$

it is consistent with $\psi = 0$ at $b = 0$. Regarding the $\phi$-direction, we can see that as $\phi \to \infty$ the $\phi^2$ term dominates in Equation (6), with a vanishing solution in this limit.

## 4. Time

The problem of time in quantum cosmology amounts to the impossibility to implement dynamics by the equation of Wheeler-deWitt, as the Hamiltonian operator vanishes and a time dependent Schrödinger equation implies that the states are time independent [1,2]. As there are no external observators, it has been argued that the universe must contain its own clock. In [2] a scalar field has been proposed as a clock, provided its classical dependence of time is monotonically increasing.

In [22], we proposed an "effective" wave function

$$\Psi(a,\tau) = \left. \frac{1}{\sqrt{\int_0^\infty da \, |\psi(a,\phi)|^2}} \, \psi(a,\phi) \right|_{\phi=\tau} \tag{8}$$

$$|\Psi(a,\tau)|^2 = \left. \frac{|\psi(a,\phi)|^2}{\int_0^\infty da \, |\psi(a,\phi)|^2} \right|_{\phi=\tau} \tag{9}$$

is the conditional probability for the universe [1] to be at $a$, if the universe is at $\phi = \tau$ regardless of $a$. The mean value of the scale factor is

$$a(\tau) = \int_0^\infty a|\Psi(a,\tau)|^2 da \tag{10}$$

In Figure 5 we show the result for this mean value for the wave function given in Figure 4, where the initial values are due to the vanishing boundary condition of the wave function at $\phi = 0$, which requires a closer analysis in the context of the the ansatz (8).

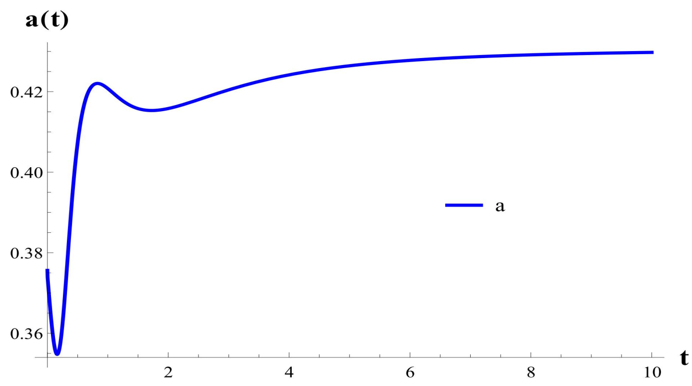


**Figure 5.** Time dependence of the scale factor $a(t)$.

Consistency requires verification of the uncertainty relation $\Delta a(\tau)\Delta\pi_a(\tau) \geq \frac{1}{2}$. Further, the quantum fluctuations of a scale factor measurement must be evaluated considering that the scale factor is determined from its velocity by red shift measurements. Thus, these fluctuations must be computed from $\dot{a} = -\frac{1}{6}a^{-2}\pi_\phi$. As shown in [22], the fluctuations are consistent with a classical universe today.

## 5. Conclusions

We have studied the classical and quantum formulation of a quadratic $f(R)$ modified theory of General Relativity based on the Starobinsky model, with a scalar field in an equivalent action, in a cosmological setting with a FRW metric. The lagrangian and hamiltonian formulations are straightforward. We consider the numerical solutions for the exact equations in both scenarios, classical and quantum, taking a compact domain for the numerical computation in the second case. With suitable boundary conditions, these solutions tend to zero at the boundaries, pointing to normalizability of the wave function, consistently with the probabilistic interpretation. The wave function is interpreted by a conditional probability as in [22], where the scalar field plays the role of time.

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**Author Contributions:** C.R. and V.V.-B. conceived, made the computations and wrote the paper.

**Conflicts of Interest:** The authors declare no conflict of interest.

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