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Gently Paraconsistent Calculi

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Abstract: In this paper, we consider some paraconsistent calculi in a Hilbert-style formulation with the rule of detachment as the sole rule of interference. Each calculus will be expected to contain all axiom schemas of the positive fragment of classical propositional calculus and respect the principle of gentle explosion.

Keywords: paraconsistent logic; paraconsistency; the principle of explosion; Sette's calculus; paracomplete calculi; paranormal logics

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1. Introduction

The principle of explosion states that from any set $\{\alpha, \neg\alpha\}$ of contradictory formulas any other formula β follows. Paraconsistent logic can be described as a logic in which the principle does not hold. The 'definition' is very simple, but it is also very broad. This may lead to some ambiguity and cause interpretive problems, especially if we aim to draw a sharp distinction between paraconsistent and some other nonclassical logics. In this paper, we discuss some possible consequences of the definition. We examine several paraconsistent calculi that respect the so-called principle of gentle explosion, according to which from any set $\{\alpha, \neg\alpha, \neg\neg\alpha\}$ of formulas any other formula β follows. The calculi (of paraconsistent logic) that admit the principle will be called *gently* paraconsistent.

Let *var* denote a denumerable set of all propositional variables: p_1, p_2, p_3, etc. The set $\mathcal{F}$ of formulas is defined in the standard way using propositional variables from *var* and the symbols $\neg$, $\vee$, $\wedge$ and $\rightarrow$ for negation, disjunction, conjunction and implication, respectively. The connective of equivalence, $\alpha \leftrightarrow \beta$, is treated as an abbreviation for $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$, and hence it will be omitted. We say that a formula α is *atomic*, if $\alpha \in var$; otherwise α will be called *complex*. By *literals* we mean the set $\mathcal{LI}$ of all formulas of the form $\neg_k p_i$, where $i \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$ and $p_i \in var$ (if $k = 0$, then $\neg_0 p_i = p_i$; if $k = 1$, then $\neg_1 p_i = \neg p_i$; etc.). We use lowercase Greek letters for formulas and uppercase Greek letters for subsets of $\mathcal{F}$.

In $\mathcal{F}$, we will consider axiomatic propositional calculi in a Hilbert-style formulation with the rule of detachment, (MP) $\alpha \rightarrow \beta, \alpha \;/\; \beta$, as the sole rule of interference. Each calculus $\mathcal{C}$ discussed in this paper is expected to have all axiom schemas of the positive fragment of classical propositional calculus (CPC^+, for short), that is, all instances of the following schemas:

$(A1)$ $\alpha \rightarrow (\beta \rightarrow \alpha)$
$(A2)$ $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$
$(A3)$ $((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha$
$(A4)$ $(\alpha \wedge \beta) \rightarrow \alpha$
$(A5)$ $(\alpha \wedge \beta) \rightarrow \beta$
$(A6)$ $\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$
$(A7)$ $\alpha \rightarrow (\alpha \vee \beta)$
$(A8)$ $\beta \rightarrow (\alpha \vee \beta)$
$(A9)$ $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \vee \beta \rightarrow \gamma))$,

and include the law of gentle explosion: (DS^2) $\alpha \to (\neg\alpha \to (\neg\neg\alpha \to \beta))$.

Definition 1. *For $\mathcal{C}$, any $\alpha \in \mathcal{F}$ and any $\Gamma \subseteq \mathcal{F}$, we say that α is provable from Γ within $\mathcal{C}$ (in symbols: $\Gamma \vdash_{\mathcal{C}} \alpha$) if, and only if there is a finite sequence of formulas, $\beta_1, \beta_2, \ldots, \beta_n$ such that $\beta_n = \alpha$ and for each $i \leqslant n$, either $\beta_i \in \Gamma$, or β_i is an axiom of $\mathcal{C}$, or for some $j,k \leqslant i$, we have $\beta_k = \beta_j \to \beta_i$. A formula α is a thesis of $\mathcal{C}$ (in symbols: $\emptyset \vdash_{\mathcal{C}} \alpha$) iff α is provable from $\emptyset$ within $\mathcal{C}$ (henceforth, we will use iff as shorthand for if, and only if).*

Notice that each calculus $\mathcal{C}$ may be identified with a triple $\langle \mathcal{F}, Ax_{\mathcal{C}}, \vdash_{\mathcal{C}} \rangle$, but it is determined by its set of axioms $Ax_{\mathcal{C}}$ which is included in $\mathcal{F}$. Moreover, it can be verified that $\vdash_{\mathcal{C}}$ is a finitary consequence relation satisfying the so-called *Tarskian properties* (viz. reflexivity, monotonicity and transitivity).

Lemma 1. *Let $\Gamma, \Delta \subseteq \mathcal{F}$ and $\alpha, \beta \in \mathcal{F}$, then we have*

(1) $\Gamma \vdash_{\mathcal{C}} \alpha$ *iff for some finite* $\Delta \subseteq \Gamma$, $\Delta \vdash_{\mathcal{C}} \alpha$
(2) *If* $\alpha \in \Gamma$, *then* $\Gamma \vdash_{\mathcal{C}} \alpha$
(3) *If* $\Gamma \subseteq \Delta$ *and* $\Gamma \vdash_{\mathcal{C}} \alpha$, *then* $\Delta \vdash_{\mathcal{C}} \alpha$
(4) *If* $\Delta \vdash_{\mathcal{C}} \alpha$, *and for every* $\beta \in \Delta$ *it is true that* $\Gamma \vdash_{\mathcal{C}} \beta$, *then* $\Gamma \vdash_{\mathcal{C}} \alpha$
(5) *If* $\Gamma \cup \{\alpha\} \vdash_{\mathcal{C}} \beta$ *and* $\Delta \vdash_{\mathcal{C}} \alpha$, *then* $\Gamma \cup \Delta \vdash_{\mathcal{C}} \beta$; *in particular,*

if $\Gamma \cup \{\alpha\} \vdash_{\mathcal{C}} \beta$ *and* $\emptyset \vdash_{\mathcal{C}} \alpha$, *then* $\Gamma \vdash_{\mathcal{C}} \beta$.

Proof. We refer the reader to [1] and [2] for details. □

The deduction theorem holds for any calculus having (MP) as the sole rule of inference, and $(A1)$, $(A2)$ as its axiom schemas. Thus we have

Theorem 1. *For any $\Gamma \subseteq \mathcal{F}$ and $\alpha, \beta \in \mathcal{F}$: $\Gamma \cup \{\alpha\} \vdash_{\mathcal{C}} \beta$ iff $\Gamma \vdash_{\mathcal{C}} \alpha \to \beta$.*

It follows from $(A9)$, the deduction theorem and (MP) that the following lemma holds as well:

Lemma 2. *For any $\Gamma, \Delta \subseteq \mathcal{F}$ and $\alpha, \beta, \gamma \in \mathcal{F}$: if $\Gamma \cup \{\alpha\} \vdash_{\mathcal{C}} \gamma$ and $\Gamma \cup \{\beta\} \vdash_{\mathcal{C}} \gamma$, then $\Gamma \cup \{\alpha \vee \beta\} \vdash_{\mathcal{C}} \gamma$.*

Remark 1. *The following formulas are provable in CPC^+:*

(IL) $\alpha \to \alpha$
(LoC) $(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$
(HS) $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$
(C) $(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$
(PoC) $((\alpha \to \beta) \to \gamma) \to ((\alpha \to \gamma) \to \gamma)$.

The formulas will be useful for proving the results presented below.

2. Gently Paraconsistent Calculi

The set of axiom schemas of $\mathcal{C}$ enriched with (ExM) $\alpha \vee \neg\alpha$ and $(NN2)$ $\alpha \to \neg\neg\alpha$ yields the axiom system of classical propositional calculus (in short: CPC). From the viewpoint of paraconsistency, neither (ExM) nor $(NN2)$ seems to be controversial, and therefore they could be generally accepted. There is a problem, however, in admitting (DS^2) and $(NN2)$ simultaneously. This is because the pair of formulas is equivalent, on the grounds of CPC, to (DS) $\alpha \to (\neg\alpha \to \beta)$. The latter, being viewed as a highly contentious logical law, should be rejected. Not surprisingly then, $(NN2)$ cannot be universally accepted either. On the other hand, the formula $(NN1)$ $\neg\neg\alpha \to \alpha$ appears to be more applicable than $(NN2)$, in the sense that its application does not need to be limited, for example, to certain complex formulas (cf. Sections 2.2 and 2.5).

2.1. The Calculus $A1$

The basic *gently* paraconsistent calculus is $A1$. The calculus is defined by the axioms of CPC^+, (DS^2) and (MP) as the sole rule of inference. It is a proper subsystem of the paracomplete logic $CLaN$. The $CLaN$, as proposed in [3], is axiomatized by CPC^+, (DS) and (MP). The calculus $A1$ may be seen as an example of a paranormal calculus (in Miró Quesada's terminology), that is, a calculus which is both paraconsistent and paracomplete. Some other examples of the paranormal calculi are given in [4,5].

Definition 2. *An $A1$-valuation is any function $v : \mathcal{F} \longrightarrow \{1, 0\}$ that satisfies, for any $\alpha, \beta \in \mathcal{F}$, the following conditions:*

$(\wedge)$ $v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = 1$ and $v(\beta) = 1$
$(\vee)$ $v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$
$(\rightarrow)$ $v(\alpha \rightarrow \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$
$(\neg 1)$ if $v(\neg\alpha) = v(\neg\neg\alpha) = 1$, then $v(\alpha) = 0$.

Definition 3. *A formula α is an $A1$-tautology iff for every $A1$-valuation v, $v(\alpha) = 1$. For any $\alpha \in \mathcal{F}$ and $\Gamma \subseteq \mathcal{F}$, α is a semantic consequence of Γ (in symbols: $\Gamma \models_{A1} \alpha$) iff for any $A1$-valuation v: if $v(\beta) = 1$ for any $\beta \in \Gamma$, then $v(\alpha) = 1$.*

Theorem 2. *For every $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$: if $\Gamma \vdash_{A1} \alpha$ then $\Gamma \models_{A1} \alpha$.*

The proof of soundness proceeds by induction on the length of a derivation in $A1$. To prove the completeness, we apply the method which is based on the notion of maximal nontrivial sets of formulas (see [6,7]). To begin with, let us recall some important definitions and results.

Definition 4. *Let $\mathcal{C} = \langle \mathcal{F}, Ax_{\mathcal{C}}, \vdash_{\mathcal{C}} \rangle$ be a calculus (satisfying Tarskian properties) and $\Delta \subseteq \mathcal{F}$. We say that Δ is a closed theory of $\mathcal{C}$ iff for any $\beta \in \mathcal{F}$, we have $\Delta \vdash_{\mathcal{C}} \beta$ iff $\beta \in \Delta$. We say that Δ is maximal nontrivial with respect to $\alpha \in \mathcal{F}$ in $\mathcal{C}$ iff (i) $\Delta \nvdash_{\mathcal{C}} \alpha$ and (ii) for every $\beta \in \mathcal{F}$, if $\beta \notin \Delta$ then $\Delta \cup \{\beta\} \vdash_{\mathcal{C}} \alpha$.*

Lemma 3 ([6], Lemma 2.2.5.)**.** *Every maximal nontrivial set with respect to some formula is a closed theory.*

Observe that the lemma holds for $A1$. Additionally, we have

Lemma 4. *For any maximal nontrivial set Δ with respect to α in $A1$, any $\delta \in \mathcal{F}$, the mapping $v : \mathcal{F} \longrightarrow \{1, 0\}$ defined as $(\star)$: $v(\delta) = 1$ iff $\delta \in \Delta$, is an $A1$-valuation.*

Proof. We need to prove that the mapping v is an $A1$-valuation. The proof splits into a number of cases. The case $(\wedge)$ follows directly from the definition of $(\star)$, the axioms $(A4)$–$(A6)$ and Lemma 1; the case $(\vee)$ from $(\star)$, $(A7)$–$(A9)$ and Lemma 1.

Case $(\rightarrow)$: (*if–then*) Assume, for a contradiction, that $v(\beta \rightarrow \gamma) = 1$, $v(\beta) = 1$ and $v(\gamma) = 0$. Then, by $(\star)$, we have that $\beta \rightarrow \gamma \in \Delta$, $\beta \in \Delta$ and $\gamma \notin \Delta$. Now, by Lemma 1, we get $\Delta \vdash_{A1} \beta \rightarrow \gamma$, $\Delta \vdash_{A1} \beta$, that is, $\Delta \vdash_{A1} \{\beta \rightarrow \gamma, \beta\}$. The formula (IL) is a thesis of $A1$ and the deduction theorem holds, so $\{\beta \rightarrow \gamma, \beta\} \vdash_{A1} \gamma$. Since the relation $\vdash_{A1}$ is transitive (see Lemma 1), then $\Delta \vdash_{A1} \gamma$, which means that $\gamma \in \Delta$. However, $\gamma \notin \Delta$. This entails a contradiction.

(*then–if*) There are two subcases to consider. Subcase (i): Suppose, for a contradiction, that $v(\beta) = 0$ and $v(\beta \rightarrow \gamma) = 0$. This implies that $\beta \notin \Delta$ and $\beta \rightarrow \gamma \notin \Delta$, by $(\star)$. Since Δ is a maximal nontrivial set with respect to α, then $\Delta \cup \{\beta\} \vdash_{A1} \alpha$ and $\Delta \cup \{\beta \rightarrow \gamma\} \vdash_{A1} \alpha$. Hence, $\Delta \vdash_{A1} \beta \rightarrow \alpha$, $\Delta \vdash_{A1} (\beta \rightarrow \gamma) \rightarrow \alpha$, by the deduction theorem, and consequently, $\Delta \vdash_{A1} \{\beta \rightarrow \alpha, (\beta \rightarrow \gamma) \rightarrow \alpha\}$. Observe that (PoC) is a thesis of $A1$, so $\{\beta \rightarrow \alpha, (\beta \rightarrow \gamma) \rightarrow \alpha\} \vdash_{A1} \alpha$, by the deduction theorem. The relation $\vdash_{A1}$ is transitive, and therefore $\Delta \vdash_{A1} \alpha$. Since Δ is deductively closed, then $\alpha \in \Delta$. However, $\alpha \notin \Delta$, by the main assumption. This yields a contradiction. Subcase (ii): Suppose that

$v(\gamma) = 1$. Then, by $(\star)$, we get $\gamma \in \Delta$. This implies, by Lemma 1, that $\Delta \vdash_{A1} \gamma$. Since $(A1)$ is an axiom schema of $A1$, then, by the deduction theorem, we have $\{\gamma\} \vdash_{A1} \beta \to \gamma$. The relation $\vdash_{A1}$ is transitive, and hence $\Delta \vdash_{A1} \beta \to \gamma$. If $\Delta \vdash_{A1} \beta \to \gamma$, then $\beta \to \gamma \in \Delta$, which means that $v(\beta \to \gamma) = 1$.

Case $(\neg 1)$: Assume, for a contradiction, that $v(\neg\beta) = 1$, $v(\neg\neg\beta) = 1$ and $v(\beta) = 1$. Then by $(\star)$, we have $\neg\beta \in \Delta$, $\neg\neg\beta \in \Delta$ and $\beta \in \Delta$. By Lemma 1, we obtain $\Delta \vdash_{A1} \neg\beta$, $\Delta \vdash_{A1} \neg\neg\beta$ and $\Delta \vdash_{A1} \beta$. This implies that $\Delta \vdash_{A1} \{\beta, \neg\beta, \neg\neg\beta\}$. Since (DS^2) is an axiom schema of $A1$, then $\{\beta, \neg\beta, \neg\neg\beta\} \vdash_{A1} \alpha$, by the deduction theorem. The relation $\vdash_{A1}$ is transitive, so $\Delta \vdash_{A1} \alpha$. Observe that Δ is deductively closed, then $\alpha \in \Delta$. However, by the main assumption, $\alpha \notin \Delta$. This entails a contradiction. □

Notice that the so-called Lindenbaum–Łoś theorem holds,for any finitary calculus $\mathcal{C} = \langle \mathcal{F}, Ax_{\mathcal{C}}, \vdash_{\mathcal{C}} \rangle$.

Lemma 5 ([2], Theorem 3.31; [6], Theorem 2.2.6)**.** *For any $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$ such that $\Gamma \nvdash_{\mathcal{C}} \alpha$, there is a maximal nontrivial set Δ with respect to α in $\mathcal{C}$ such that $\Gamma \subseteq \Delta$.*

Thus, the completeness of $A1$ follows

Theorem 3. *For all $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$: if $\Gamma \models_{A1} \alpha$, then $\Gamma \vdash_{A1} \alpha$.*

Proof. Assume that $\Gamma \nvdash_{A1} \alpha$ and let Δ be a maximal nontrivial set with respect to α in $A1$ such that $\Gamma \subseteq \Delta$. Then, $\alpha \notin \Delta$. Because Lemma 4 holds, there is an $A1$-valuation v such that $v(\alpha) = 0$ and, for any $\beta \in \Gamma$, $v(\beta) = 1$. Hence, $\Gamma \not\models_{A1} \alpha$. □

Though the calculus $A1$ is very weak and does not provide any adequate grounds for practical inference, it offers a good starting point for further research. In the subsequent paragraphs, we will discuss various *gently* paraconsistent extensions of $A1$.

2.2. The Calculus E1

The calculus $E1$ is defined by CPC^+, (DS^2), $(DS^\ddagger)$ $(\alpha \ddagger \beta) \to (\neg(\alpha \ddagger \beta) \to \gamma)$, where $\ddagger \in \{\wedge, \vee, \to\}$, and (MP). There are only few paraconsistent calculi in which $(DS^\ddagger)$ is provable. One of them is Sette's calculus $P1$. Anticipating what comes next in Section 2.5, Sette's calculus will be the *top* paraconsistent extension of the calculi admitting (DS^2) and $(DS^\ddagger)$, simultaneously.

Definition 5. *An E1-valuation is any function $v : \mathcal{F} \longrightarrow \{1, 0\}$ that, for any $\alpha, \beta \in \mathcal{F}$, satisfies all the conditions of A1-valuation and, additionally: $(\neg\ddagger)$ if $v(\neg(\alpha \ddagger \beta)) = 1$, then $v(\alpha \ddagger \beta) = 0$, where $\ddagger \in \{\wedge, \vee, \to\}$.*

Definition 6. *A formula α is an E1-tautology iff for every E1-valuation v, $v(\alpha) = 1$. For any $\alpha \in \mathcal{F}$ and $\Gamma \subseteq \mathcal{F}$, α is a semantic consequence of Γ (in symbols: $\Gamma \models_{E1} \alpha$) iff for any E1-valuation v: if $v(\beta) = 1$ for any $\beta \in \Gamma$, then $v(\alpha) = 1$.*

Theorem 4. *For every $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$: $\Gamma \vdash_{E1} \alpha$ iff $\Gamma \models_{E1} \alpha$.*

The proof of soundness is by induction on the structure of proofs in $E1$. The completeness proof strategy is exactly the same as that of the proof of Theorem 3. The key point is to show that the following lemma holds:

Lemma 6. *For any maximal nontrivial set Δ with respect to α in E1, any $\delta \in \mathcal{F}$, the mapping $v : \mathcal{F} \longrightarrow \{1, 0\}$ defined as $(\star)$: $v(\delta) = 1$ iff $\delta \in \Delta$, is an E1-valuation.*

Proof. Case $(\wedge)$, $(\vee)$, $(\to)$ and $(\neg 1)$: The proof proceeds analogously to that of Lemma 4. Case $(\neg\ddagger)$: Assume, for a contradiction, that $v(\neg(\beta \ddagger \gamma)) = 1$ and $v(\beta \ddagger \gamma) = 1$, where $\ddagger \in \{\wedge, \vee, \to\}$. Then by $(\star)$, we have $\neg(\beta \ddagger \gamma) \in \Delta$ and $\beta \ddagger \gamma \in \Delta$. It follows from Lemma 1 that $\Delta \vdash_{E1} \neg(\beta \ddagger \gamma)$ and $\Delta \vdash_{E1} \beta \ddagger \gamma$, which results in $\Delta \vdash_{E1} \{\beta \ddagger \gamma, \neg(\beta \ddagger \gamma)\}$. By $(DS^{\ddagger})$ and the deduction theorem, we easily show that $\{\beta \ddagger \gamma, \neg(\beta \ddagger \gamma)\} \vdash_{E1} \alpha$. The relation $\vdash_{E1}$ is transitive, so $\Delta \vdash_{E1} \alpha$. Since Δ is deductively closed, then $\alpha \in \Delta$. However, $\alpha \notin \Delta$ (the main assumption). This entails a contradiction. □

Definition 7. *Let $\mathcal{TH}(\mathcal{C})$ be the set of all theses of $\mathcal{C}$. For any calculi $\mathcal{C}$ and $\mathcal{C}\star$ in $\mathcal{F}$, we say that $\mathcal{C}$ is an extension of $\mathcal{C}\star$ iff $\mathcal{TH}(\mathcal{C}\star) \subseteq \mathcal{TH}(\mathcal{C})$. We say that $\mathcal{C}\star$ is a proper subsystem of $\mathcal{C}$ (in symbols: $\mathcal{C}\star \sqsubset \mathcal{C}$) iff $\mathcal{TH}(\mathcal{C}\star) \subseteq \mathcal{TH}(\mathcal{C})$ and $\mathcal{TH}(\mathcal{C}) \nsubseteq \mathcal{TH}(\mathcal{C}\star)$.*

Remark 2. *$CPC^{+} \sqsubset A1 \sqsubset E1$.*

There is an alternative way to extend $A1$ so that the resulting calculus preserves $(DS^{\ddagger})$ as provable. Let $E1^{\star}$ be the calculus defined by CPC^{+}, (DS^{2}), $(NN2^{\ddagger})$ $(\alpha \ddagger \beta) \to \neg\neg(\alpha \ddagger \beta)$, where $\ddagger \in \{\wedge, \vee, \to\}$, and (MP). It then follows from the deduction theorem, $(NN2^{\ddagger})$, (DS^{2}) and (MP) that $(DS^{\ddagger})$ is a thesis of $E1^{\star}$. Note that $(p_1 \to p_2) \to \neg\neg(p_1 \to p_2)$ of the form $(NN2^{\ddagger})$ is not an $E1$-tautology. So, by completeness, it is not provable in $E1$, either. This suggests that the new calculus is strictly stronger than $E1$, i.e., $E1 \sqsubset E1^{\star}$.

Another example is the paranormal logic I^1P^1. The logic was considered in [4,8–10]. It is characterized by the four-valued matrix

$$\mathcal{M}_{I1P1} = \langle \{1,2,3,0\}, \{1,2\}, \neg, \wedge, \vee, \to \rangle,$$

where $\{1,2,3,0\}$ and $\{1,2\}$ are the sets of logical values and designated values, respectively; the connectives $\neg, \wedge, \vee, \to$ are defined in the following way:

	$\neg$
1	0
2	1
3	0
0	1

$\to$	1	2	3	0
1	1	1	0	0
2	1	1	0	0
3	1	1	1	1
0	1	1	1	1

$\wedge$	1	2	3	0
1	1	1	0	0
2	1	1	0	0
3	0	0	0	0
0	0	0	0	0

$\vee$	1	2	3	0
1	1	1	1	1
2	1	1	1	1
3	1	1	0	0
0	1	1	0	0

An I^1P^1-valuation is any function $v : \mathcal{F} \longrightarrow \{1,2,3,0\}$ compatible with the above truth tables. An I^1P^1-tautology is a formula which under every valuation v takes on the designated values $\{1,2\}$.

All axioms of $E1$ are I^1P^1-tautologies, and (MP) preserves tautologicality. However, the formula $\neg p_1 \to (\neg\neg p_1 \to p_2)$, being an I^1P^1-tautology, is unprovable in $E1$. This yields that $E1 \sqsubset I^1P^1$.

A slightly different example is with $LPPL$. The logic, as proposed in [5,11], is axiomatizable by $(A1)$, $(A2)$, $(A4)$–$(A9)$, (Con^{L}) $(\neg\phi \to \neg\psi) \to (\psi \to \phi)$, where $\phi, \psi \notin \mathcal{LI}$, and (MP). The formula $p_1 \to (\neg p_1 \to (\neg\neg p_1 \to p_2))$ of the form (DS^{2}) is unprovable in $LPPL$, and neither is $(\neg(p_1 \to p_2) \to \neg(p_3 \to p_4)) \to ((p_3 \to p_4) \to (p_1 \to p_2))$ of the form (Con^{L}) provable in $E1$, then we have that $E1 \not\sqsubset LPPL$ and $LPPL \not\sqsubset E1$.

2.3. The Calculus $B1$

The calculus $B1$ is obtained from $A1$ by adding the formula (ExM) as a new axiom schema, which indicates that $B1$ is axiomatizable by CPC^{+}, (DS^{2}), (ExM) and (MP). Since the law of excluded middle is unprovable in $A1$, we obviously have that $A1 \sqsubset B1$. The calculus $B1$ was considered in [12,13] as the strongest in the hierarchy of B^n-calculi ($n \in \mathbb{N}$).

Definition 8. *A $B1$-valuation is any function $v : \mathcal{F} \longrightarrow \{1,0\}$ that satisfies, for any $\alpha, \beta \in \mathcal{F}$, the following conditions:*

$(\wedge)$ $v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = 1$ and $v(\beta) = 1$
$(\vee)$ $v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$
$(\to)$ $v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$

($\neg$0) *if* $v(\neg\alpha) = 0$*, then* $v(\alpha) = 1$
($\neg\neg$1) *if* $v(\neg\neg\alpha) = 1$*, then* $v(\alpha) = 0$ *or* $v(\neg\alpha) = 0$*.*

Definition 9. *A formula* α *is a* $B1$*-tautology iff for every* $B1$*-valuation* v*,* $v(\alpha) = 1$*. For any* $\alpha \in \mathcal{F}$ *and* $\Gamma \subseteq \mathcal{F}$*,* α *is a semantic consequence of* Γ *(in symbols:* $\Gamma \models_{B1} \alpha$*) iff for any* $B1$*-valuation* v*: if* $v(\beta) = 1$ *for any* $\beta \in \Gamma$*, then* $v(\alpha) = 1$*.*

Theorem 5. *For every* $\Gamma \subseteq \mathcal{F}$ *and* $\alpha \in \mathcal{F}$*:* $\Gamma \vdash_{B1} \alpha$ *iff* $\Gamma \models_{B1} \alpha$*.*

The completeness proof is carried out similarly as for the calculus $A1$. In addition to Lemmas 3 and 5 given in Section 2.1, the following lemma is of particular importance:

Lemma 7. *For any maximal nontrivial set* Δ *with respect to* α *in* $B1$*, any* $\delta \in \mathcal{F}$*, the mapping* $v : \mathcal{F} \longrightarrow \{1, 0\}$ *defined as* ($\star$)*:* $v(\delta) = 1$ *iff* $\delta \in \Delta$*, is a* $B1$*-valuation.*

Proof. Case ($\wedge$), ($\vee$), ($\rightarrow$) and ($\neg$1): We proceed analogously to the proof of Lemma 4. Case ($\neg$0): Assume, for a contradiction, that $v(\neg\beta) = 0$ and $v(\beta) = 0$. Then by ($\star$), we obtain $\neg\beta \notin \Delta$ and $\beta \notin \Delta$. Since Δ is a maximal nontrivial set with respect to α, then $\Delta \cup \{\beta\} \vdash_{B1} \alpha$ and $\Delta \cup \{\neg\beta\} \vdash_{B1} \alpha$. By Lemma 2, we get $\Delta \cup \{\beta \vee \neg\beta\} \vdash_{B1} \alpha$. Since ($ExM$) is a thesis of $B1$ and Lemma 1 holds, then $\Delta \vdash_{B1} \alpha$. Recall that Δ is a closed theory, so $\alpha \in \Delta$. However, $\alpha \notin \Delta$. This entails a contradiction.

Case ($\neg\neg$1): Suppose, for a contradiction, that $v(\neg\neg\beta) = 1$, $v(\beta) = 1$ and $v(\neg\beta) = 1$. The remaining part of the proof is similar to the case ($\neg$1) of Lemma 4 and thus omitted. □

As $p_1 \vee \neg p_1$ of the form (ExM) is not a thesis of $E1$ and $(p_1 \rightarrow p_2) \rightarrow (\neg(p_1 \rightarrow p_2) \rightarrow p_3)$ of the form ($DS^{\ddagger}$) is not provable in $B1$, it follows that $E1 \not\sqsubset B1$ and $B1 \not\sqsubset E1$.

2.4. The Calculi BE1 and CB1

The calculus $BE1$ comprises the axioms of CPC^{+}, (DS^2), (ExM), ($DS^{\ddagger}$) and (MP), which clearly yields that both $E1 \sqsubset BE1$ and $B1 \sqsubset BE1$. The $BE1$ is an example of calculus which is paraconsistent only at the level of literals: a pair of the formulas α and $\neg\alpha$ yields any β *iff* α is not a propositional variable nor is its iterated negation.

Definition 10. *A* $BE1$*-valuation is any function* $v : \mathcal{F} \longrightarrow \{1, 0\}$ *that satisfies, for any* $\alpha, \beta \in \mathcal{F}$*, all the conditions of* $B1$*-valuation and additionally:* ($\neg\ddagger$) *if* $v(\neg(\alpha \ddagger \beta)) = 1$*, then* $v(\alpha \ddagger \beta) = 0$*, where* $\ddagger \in \{\wedge, \vee, \rightarrow\}$*.*

Definition 11. *A formula* α *is a* $BE1$*-tautology iff for every* $BE1$*-valuation* v*,* $v(\alpha) = 1$*. For any* $\alpha \in \mathcal{F}$ *and* $\Gamma \subseteq \mathcal{F}$*,* α *is a semantic consequence of* Γ *(in symbols:* $\Gamma \models_{BE1} \alpha$*) iff for any* $BE1$*-valuation* v*: if* $v(\beta) = 1$ *for any* $\beta \in \Gamma$*, then* $v(\alpha) = 1$*.*

Theorem 6. *For every* $\Gamma \subseteq \mathcal{F}$ *and* $\alpha \in \mathcal{F}$*:* $\Gamma \vdash_{BE1} \alpha$ *iff* $\Gamma \models_{BE1} \alpha$*.*

Proof. The proof proceeds as in Theorems 2–4. □

The calculus $CB1$ was introduced in [14]. It arose as a result of the extension of $B1$ with the law of double negation ($NN1$) $\neg\neg\alpha \rightarrow \alpha$, which suggests that $B1 \sqsubset CB1$. Moreover, we have

Remark 3. *The calculus* $CB1$ *is axiomatizable by* CPC^{+}*, (*ExM*), (*$DS^{\neg}$*) and (MP).*

Proof. See op. cit., p. 227, for details. □

Definition 12. *A* $CB1$*-valuation is any function* $v : \mathcal{F} \longrightarrow \{1, 0\}$ *that satisfies, for any* $\alpha, \beta \in \mathcal{F}$*, the following conditions:*

($\wedge$) *$v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = 1$ and $v(\beta) = 1$*
($\vee$) *$v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$*
($\rightarrow$) *$v(\alpha \rightarrow \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$*
($\neg 0$) *if $v(\neg\alpha) = 0$, then $v(\alpha) = 1$.*
($\neg\neg 1$) *if $v(\neg\neg\alpha) = 1$, then $v(\neg\alpha) = 0$.*

Definition 13. *A formula α is a $CB1$-tautology iff for every $CB1$-valuation v, $v(\alpha) = 1$. For any $\alpha \in \mathcal{F}$ and $\Gamma \subseteq \mathcal{F}$, α is a semantic consequence of Γ (in symbols: $\Gamma \models_{CB1} \alpha$) iff for any $CB1$-valuation v: if $v(\beta) = 1$ for any $\beta \in \Gamma$, then $v(\alpha) = 1$.*

Theorem 7. *For every $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$: $\Gamma \vdash_{CB1} \alpha$ iff $\Gamma \models_{CB1} \alpha$.*

Proof. We refer the reader to op. cit., pp. 230–231, for details. □

Since $\neg p_1 \rightarrow (\neg\neg p_1 \rightarrow p_2)$ of the form $(DS^{\neg})$ is not a thesis of $BE1$ and $(p_1 \rightarrow p_2) \rightarrow (\neg(p_1 \rightarrow p_2) \rightarrow p_3)$ of the form $(DS^{\ddagger})$ is not provable in $CB1$, it follows that $CB1 \not\subseteq BE1$ and $BE1 \not\subseteq CB1$. There exists, however, some paraconsistent calculi in which $(DS^{\neg})$ does not fail (see [15], for discussion on the topic). The example of such a calculus is $P1$.

2.5. Sette's Calculus $P1$

The calculus $P1$, proposed in [16], is defined in the language with negation and implication as primitives by $(A1)$, $(A2)$, $(AN1)$ $(\neg\alpha \rightarrow \neg\beta) \rightarrow ((\neg\alpha \rightarrow \neg\neg\beta) \rightarrow \alpha)$, $(AN2)$ $\neg(\alpha \rightarrow \neg\neg\alpha) \rightarrow \alpha$, $(NN2^{\rightarrow})$ $(\alpha \rightarrow \beta) \rightarrow \neg\neg(\alpha \rightarrow \beta)$. The sole rule of inference is (MP). Some alternative axiomatizations of $P1$ have been developed since then (see e.g., [9,17–22]).

Sette's calculus is sound and complete with respect to the matrix $\mathcal{M}_{P1} = \langle \{1,2,0\}, \{1,2\}, \neg, \rightarrow \rangle$, where $\{1,2,0\}$ and $\{1,2\}$ are the sets of logical and designated values, respectively. The connectives of $\rightarrow$ and $\neg$ are determined by the following truth tables (cit. per [16], p. 176):

$\rightarrow$	1	2	0
1	1	1	0
2	1	1	0
0	1	1	1

$\neg$	
1	0
2	1
0	1.

A P^1-valuation is any function v from the set of formulas to the set of logical values, i.e., $v : \mathcal{F} \longrightarrow \{1,2,0\}$, compatible with the above truth tables. A P^1-tautology is a formula which under every valuation v takes on the designated values $\{1,2\}$. Conjunction and disjunction are definable connectives: $\alpha \wedge \beta =_{df} \neg(\alpha \rightarrow \neg(\neg\beta \rightarrow \beta))$; $\alpha \vee \beta =_{df} \neg(\neg\alpha \rightarrow \alpha) \rightarrow \beta$ (cit. per [9,20]).

Remark 4. *The calculus $P1$ is axiomatizable by CPC^{+}, (ExM), $(DS^{\neg})$, $(DS^{\ddagger})$, where $\ddagger \in \{\wedge, \vee, \rightarrow\}$, and (MP).*

Proof. The proof splits into two steps. To show that the axioms $(A1)$–$(A9)$, (ExM), $(DS^{\neg})$ and $(DS^{\ddagger})$ are $P1$-tautologies, and (MP) preserves tautologicality, it suffices to apply the three-valued semantics for $P1$ (plus the definitions of 'missing' connectives). Next we need to demonstrate that $(AN1)$, $(AN2)$ and $(NN2^{\rightarrow})$ are provable in the proposed axiomatization. This in turn follows from the results of [17], pp. 270–272, and [19], pp. 1111–1113. □

Sette's calculus is maximal with respect to CPC (see [16], pp. 179–180). Consequently, it is the top extension of all gently paraconsistent calculi discussed in this paper.

3. Final Remarks

We considered several paraconsistent calculi that admitted the principle of gentle explosion, namely

$A1 = CPC^{+} + (DS^{2}) + (\mathrm{MP})$
$E1 = CPC^{+} + (DS^{2}) + (DS^{\ddagger}) + (\mathrm{MP})$
$B1 = CPC^{+} + (DS^{2}) + (ExM) + (\mathrm{MP})$
$BE1 = CPC^{+} + (DS^{2}) + (ExM) + (DS^{\ddagger}) + (\mathrm{MP})$
$CB1 = CPC^{+} + (DS^{\neg}) + (ExM) + (\mathrm{MP})$
$P1 = CPC^{+} + (DS^{\neg}) + (ExM) + (DS^{\ddagger}) + (\mathrm{MP})$.

They all form together the lattice structure shown in Figure 1.

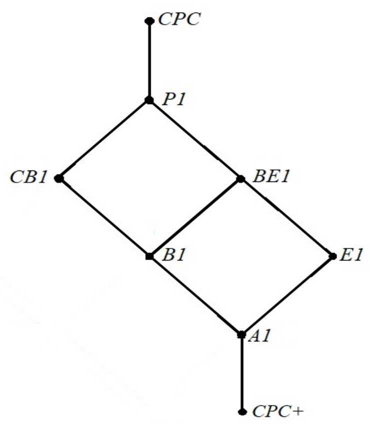

Figure 1. Gently paraconsistent calculi.

It is noteworthy that some well-known (*not-gently*) paraconsistent logics can be obtained by eliminating (DS^{2}) from the axiom schemas. For instance, dropping (DS^{2}) from $B1$ results in obtaining the logic $CLuN$ (see [3], for details); dropping (DS^{2}) from $CB1$ results in obtaining in the logic $Cmin$(see [23]) . The calculi form together the lattice structure shown in Figure 2.

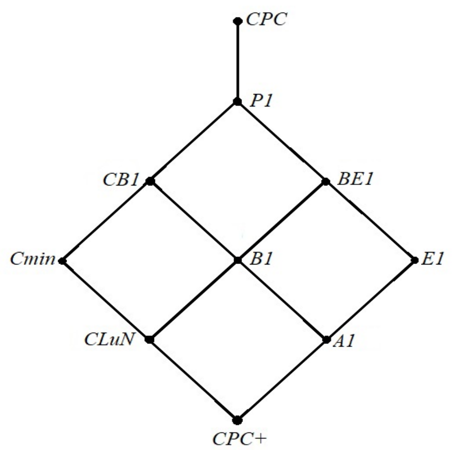

Figure 2. Paraconsistent extensions of CPC^{+}.

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