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Classes of Entire Analytic Functions of Unbounded Type on Banach Spaces

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Abstract: In this paper we investigate analytic functions of unbounded type on a complex infinite dimensional Banach space X. The main question is: under which conditions is there an analytic function of unbounded type on X such that its Taylor polynomials are in prescribed subspaces of polynomials? We obtain some sufficient conditions for a function f to be of unbounded type and show that there are various subalgebras of polynomials that support analytic functions of unbounded type. In particular, some examples of symmetric analytic functions of unbounded type are constructed.

Keywords: analytic functions on Banach spaces; functions of unbounded type; symmetric polynomials on Banach spaces

MSC: 46G20; 46E25; 46J20

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1. Introduction and Preliminaries

Let X be an infinite dimensional complex Banach space. A function $P\colon X \to \mathbb{C}$ is an n-homogeneous polynomial if there exists a symmetric n-linear map B_P defined on the Cartesian power X^n to $\mathbb{C}$ such that $P(x) = B_P(x, \dots, x)$. The mapping B_P is called *n-linear form associated with* P and is necessary and unique because of the well-known polarization formula (see, e.g., p. 4 [1]). The Banach space of all continuous n-homogeneous polynomials on X with respect to the norm

$$\|P\| = \sup_{\|x\| \le 1} |P(x)|$$

is denoted by $\mathcal{P}(^nX)$. For us the following version of the *polarization inequality* is important (p. 8 [1]).

$$|B_P(x_1^{n_1}, \dots, x_k^{n_k})| \le \frac{n_1! \cdots n_k! m^m}{n_1^{n_1} \cdots n_k^{n_k} m!} \|P\| \|x_1\|^{n_1} \cdots \|x_k\|^{n_k}, \tag{1}$$

where $n_1, \dots, n_k$ are positive integers with $n_1 + \cdots + n_k = m$.

A continuous function $f\colon X \to \mathbb{C}$ is said to be an *entire analytic function* (or just *entire function*) if its restriction on any finite dimensional subspace is analytic. The space of all entire analytic functions on X is denoted by $H(X)$. For every entire function f there exists a sequence of continuous n-homogeneous polynomials (so-called *Taylor polynomials*) such that

$$f(x) = \sum_{n=0}^{\infty} f_n(x) \tag{2}$$

and the series converges for every $x \in X$. Here $f_0 = f(0)$ is a constant. The Taylor series expansion (2) uniformly converges on the open ball rB centered at zero with radius $r = \varrho_0(f)$, where

$$\varrho_0(f) = \frac{1}{\limsup_{n\to\infty} \|f_n\|^{1/n}}.$$

The radius $r = \varrho_0(f)$ is called the *radius of uniform convergence* of f or the *radius of boundedness* of f because the ball rB is the largest open ball at zero such that f is bounded on it. If $\varrho_0(f) = \infty$, then f is bounded on all bounded subsets of X and is called a function of *bounded type*. The set of all functions of bounded type on X is denoted by $H_b(X)$.

It is known that $H_b(X)$ is a Fréchet algebra with respect to the topology of uniform convergence on bounded sets. Algebras $H_b(X)$ were considered first in [2,3] and studied by many authors for various Banach spaces X.

A polynomial P on X is called a polynomial of *finite type* if it is a finite sum of finite products of linear functionals and constants. We denote by $\mathcal{A}_n(X)$ the smallest closed subalgebra of $H_b(X)$ containing the space of all n-homogeneous polynomials $\mathcal{P}_n(X)$. In particular, $\mathcal{A}_1(X)$ is the closure of the space of finite type polynomials in $H_b(X)$. In the general case, $\mathcal{A}_n(X) \neq \mathcal{A}_k(X)$ if $n \neq k$. For example, it is so for $X = \ell_1$ (see [4] for details).

We say that a function f is an entire function of *unbounded type* if $f \in H(X) \setminus H_b(X)$. It is known that for given weak*-null sequences $\phi_n \in X^*$, $\|\phi\| = 1$, which always exists (see p. 157 [5]) the function

$$f(x) = \sum_{n=1}^{\infty} \phi_n^n(x) \tag{3}$$

is an entire function of unbounded type on X. It is easy to see that the Taylor polynomials $f_n = \phi_n^n$ of f are polynomials of finite type. In this paper we consider the following natural question.

Question 1. *Let $\mathcal{P}_0(^nX)$ be subspaces of $\mathcal{P}(^nX)$, $n \in \mathbb{N}$. Under which conditions is there a function $f = \sum_{n=0}^{\infty} f_n \in H(X) \setminus H_b(X)$ such that $f_n \in \mathcal{P}_0(^nX)$?*

In Section 2 we obtain some general results on entire functions of unbounded type. In particular, we show that if X is such that $\mathcal{A}_m(X) \neq \mathcal{A}_{m-1}(X)$ for some $m > 1$, then there exists a function $f \in H(X) \setminus H_b(X)$ such that all its Taylor polynomials are in $\mathcal{A}_m(X)$. Also, we establish some sufficient conditions for a function f to be in $H(X) \setminus H_b(X)$. In Section 3 the obtained results are applied to construct examples of symmetric analytic functions of unbounded type on ℓ_p, $1 \leq p \leq \infty$. These examples can be considered as generalizations of the example, constructed in [6] on ℓ_1. In addition, the paper contains some discussions and open questions.

For details on analytic functions of bounded type we refer the reader to [1,5,7]. Entire analytic functions of unbounded type were investigated in [8–10]. Symmetric analytic functions on Banach spaces were studied in [11–18].

2. General Results

Evidently, the set of entire functions of unbounded type is not a linear space. Moreover, the following example shows that the product of two functions of unbounded type is not necessarily a function of unbounded type because there are invertible analytic functions of unbounded type.

Example 1. *Let $X = \ell_p$ or c_0 for $1 \leq p < \infty$ and ϕ_n be the coordinate functionals multiplied by -1*

$$\phi_n \colon x = (x_1, x_2, \dots, x_n, \dots) \mapsto -x_n.$$

So the function f of the form (3) is of unbounded type. We set

$$g(x) = e^{f(x)}.$$

Clearly, $g \in H(X) \setminus H_b(X)$ because it is unbounded on the bounded set

$$z_j = (\underbrace{0, \ldots, 0, 2}_{j}, 0, \ldots), \quad \textit{since} \quad g(z_{2j}) = e^{2^{2j}}.$$

On the other hand,

$$\frac{1}{g(z_{2j+1})} = e^{-f(z_{2j+1})} = e^{2^{2j+1}}$$

and so $1/g \in H(X) \setminus H_b(X)$. However, the product of g and $1/g$ equals 1, which is bounded.

Note that $H(X) \setminus H_b(X)$ surprisedly contains infinite dimension linear subspaces and subalgebras without the zero vector [19].

The following proposition shows that the set of entire functions of unbounded type has some kind of ideal property.

Proposition 1. *Let*

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \in H(X) \setminus H_b(X)$$

and $P_n \in \mathcal{P}(^n X)$, $0 < c \le \|P_n\| \le C < \infty$ for some constants $c, C > 0$, and $n \in \mathbb{N}$. Then

$$g(x) := \sum_{n=1}^{\infty} P_n(x) f_n(x) \in H(X) \setminus H_b(X).$$

Proof. Let us show that g is well-defined on X. Note first that since $f(x)$ is well-defined on X, the series

$$\sum_{n=1}^{\infty} |f_n(x)|$$

is convergent on X. Indeed, for every fixed $x \in X$ the function $\gamma(t) := f(tx)$, $t \in \mathbb{C}$ is in $H(\mathbb{C})$ and so the power series

$$\sum_{n=1}^{\infty} t^n f_n(x)$$

is absolutely convergent. It is true, in particular, for $t = 1$.

For every $x \in X$

$$|g(x)| \le \sum_{n=1}^{\infty} |g_n(x)| = \sum_{n=1}^{\infty} |P_n(x) f_n(x)| \le C \sum_{n=1}^{\infty} |f_n(x)| < \infty.$$

To prove that g is analytic but not of bounded type we need to show that the radius of boundedness $\varrho_0(g)$ of g at zero satisfies $0 < \varrho_0(g) < \infty$. One can check that

$$\frac{1}{\varrho_0(g)} = \limsup_{n\to\infty} \|g_n\|^{1/n} = \limsup_{n\to\infty} \|P_n f_n\|^{1/2n}.$$

So

$$\limsup_{n\to\infty} c^{1/2n} \|f_n\|^{1/2n} \le \frac{1}{\varrho_0(g)} \le \limsup_{n\to\infty} C^{1/2n} \|f_n\|^{1/2n}.$$

That is, $\varrho_0(g) = \sqrt{\varrho_0(f)}$ and so $g \in H(X) \setminus H_b(X)$. $\square$

Corollary 1. *Let X be such that* $\mathcal{A}_m(X) \neq \mathcal{A}_{m-1}(X)$ *for some* $m > 1$. *Then there exists a function* $f \in H(X) \setminus H_b(X)$ *such that all* f_n, $n \geq m$ *are in* $\mathcal{A}_m(X)$.

Theorem 1. *Let us suppose that there is a dense subset* $\Omega \subset X$ *and a sequence of polynomials* $P_n \in \mathcal{P}(^n X)$, $\limsup_{n\to\infty} \|P_n\|^{1/n} = c$, $0 < c < \infty$ *such that for every* $z \in \Omega$ *there exists* $m \in \mathbb{N}$ *such that for every* $y \in X$,

$$B_{P_n}(\underbrace{z,\ldots,z}_{k},\underbrace{y,\ldots,y}_{n-k}) = 0$$

for all $k > m$ *and* $n > k$. *Then*

$$g(x) = \sum_{n=1}^{\infty} P_n(x) \in H(X) \setminus H_b(X).$$

Proof. Note first that

$$\varrho_0(g) = \frac{1}{\limsup_{n\to\infty} \|P_n\|^{1/n}} = \frac{1}{c}$$

and so $0 < \varrho_0(g) < \infty$. Thus g is locally bounded and if it is well-defined on X, then it belongs to $H(X) \setminus H_b(X)$.

Let us show that g is well-defined on X. Let $x \in X$ and $z \in \Omega$ such that $\|x - z\| < 1/2c$. Let $y = x - z$. Then

$$g(x) = g(z+y) = \sum_{n=1}^{\infty} \sum_{k\leq m} B_{P_n}(\underbrace{z,\ldots,z}_{k},\underbrace{y,\ldots,y}_{n-k}).$$

For every $0 \leq k \leq m$ we set

$$g^{[k]}(y) = \sum_{n=1}^{\infty} B_{P_n}(\underbrace{z,\ldots,z}_{k},\underbrace{y,\ldots,y}_{n-k}).$$

For the fixed z, $g^{[k]}(y)$ is an entire function of y and its j-homogeneous Taylor polynomial is

$$g_j^{[k]}(y) = B_{P_{k+j}}(\underbrace{z,\ldots,z}_{k},\underbrace{y,\ldots,y}_{j}).$$

Taking into account inequality (1) we have

$$\|g_j^{[k]}\| = \sup_{\|y\|\leq 1} \left|B_{P_{k+j}}(\underbrace{z,\ldots,z}_{k},\underbrace{y,\ldots,y}_{j})\right| \leq \frac{k!j!\|z\|^k (k+j)^{(k+j)}}{k^k j^j (k+j)!} \|P_{k+j}\|.$$

So, using the Stirling asymptotic formula for $j!/j^j$, we have

$$\begin{aligned}\limsup_{j\to\infty} \|g_j^{[k]}\|^{1/j} &= \limsup_{j\to\infty} \left(\frac{k!j!\|z\|^k (k+j)^{(k+j)}}{k^k j^j (k+j)!} \|P_{k+j}\|\right)^{1/j} \\ &= \limsup_{j\to\infty} \left(\frac{e^{(j+k)}\sqrt{2\pi j}}{e^j\sqrt{2\pi(j+k)}} \|P_{k+j}\|\right)^{1/j} = \limsup_{j\to\infty} (\|P_{k+j}\|)^{1/j}.\end{aligned}$$

It is easy to check (c.f. Lemma 2 [6]) that

$$\limsup_{j\to\infty} \|P_{k+j}\|^{1/j} = \limsup_{j\to\infty} \|P_j\|^{1/j}.$$

So the radius of boundedness of $g^{[k]}$ is equal to $1/c$. Thus $g^{[k]}$ is defined at y because $\|y\| < 1/2c$. Since

$$g(x) = g(z+y) = \sum_{k=0}^{m} g^{[k]}(y),$$

g is well-defined at x. $\square$

3. Symmetric Analytic Functions of Unbounded Type

Let X be a complex Banach space and $\mathcal{S}$ be a semigroup of isometric operators on X. A polynomial $P \in \mathcal{P}(X)$ is said to be *$\mathcal{S}$-symmetric* if it is invariant with respect to the action of $\mathcal{S}$, that is, $P(\sigma(x)) = P(x)$ for every $\sigma \in \mathcal{S}$. $\mathcal{S}$-symmetric polynomials from the general point of view were considered in [12,13].

Symmetric polynomials on ℓ_p, $1 \le p < \infty$ can be defined as $\mathcal{S}$-symmetric polynomials if $\mathcal{S}$ is the group of permutation of the standard basis vectors in ℓ_p. Due to [20] we know that polynomials

$$F_k(x) = \sum_{i=1}^{\infty} x_i^k, \qquad k = \lceil p \rceil, \lceil p \rceil + 1 \ldots$$

form an *algebraic basis* in the algebra of all symmetric polynomials on ℓ_p (here $\lceil p \rceil$ is the smallest integer that is greater than or equal to p), $x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell_p$. For the case ℓ_1 we can use, also, another algebraic basis

$$G_k(x) = \sum_{n_1 < n_2 < \cdots < n_k} x_{n_1} \ldots x_{n_k}.$$

In [15] it is proved that $\|G_n\| = 1/n!$. Let $P_n = n!G_n$. It is easy to check that polynomials P_n satisfy the condition of Theorem 1 if Ω is the subspace c_{00} of all finite sequences in ℓ_1. In [6] there is a direct proof that the following function

$$g(x) = \sum_{n=1}^{\infty} P_n(x)$$

belongs to $H(\ell_1) \setminus H_b(\ell_1)$. We will prove it for more general situation. For a given positive integer number s we denote

$$G_k^{(s)}(x) = \sum_{n_1 < n_2 < \cdots < n_k} x_{n_1}^s \ldots x_{n_k}^s.$$

Theorem 2. *If $s \ge p$, then $G_k^{(s)} \in \mathcal{P}(^{k+s}\ell_p)$ and polynomials*

$$P_n^{(s)}(x) = \frac{G_n^{(s)}(x)}{\|G_n^{(s)}\|}$$

satisfy the condition of Theorem 1 for $\Omega = c_{00}$. In particular,

$$g^{(s)}(x) := \sum_{n=1}^{\infty} P_n^{(s)}(x) \in H(\ell_p) \setminus H_b(\ell_p).$$

Proof. If $s \ge p$, then for every $x \in \ell_p$, $\|x\| \le 1$ we have

$$|G_k^{(s)}(x)| \le \left(\sum_{n=1}^{\infty} \|x_n\|^s \right)^k \le \|x\|^{ps} \le 1.$$

So $\|G_k^{(s)}\| \le 1$ for every k. Thus $G_k^{(s)}$ is continuous and well-defined on ℓ_p.

Let $z \in c_{00}$, $z = (z_1, \ldots, z_m, 0, \ldots, 0)$. Then, for $n > k > sm$

$$B_{P_n^{(s)}}(\underbrace{z, \ldots, z}_{k}, \underbrace{y, \ldots, y}_{n-k}) = 0$$

because it is a linear span of elements

$$\sum_{j_1 < \cdots < j_{n-k}} z_1^{s_1} \cdots z_m^{s_m} \underbrace{0 \cdots 0}_{k-s_1-\cdots-s_m} y_{j_1} \cdots y_{j_{n-k}} = 0,$$

where $0 \le s_1 + \cdots + s_n \le s$. So $P_n^{(s)}$ satisfies the condition of Theorem 1 for $\Omega = c_{00}$. $\square$

From Theorem 3 it follows that there exists an entire function of unbounded type $g(x)$ on ℓ_p such that its Taylor polynomials $P_n^{(s)}$ are symmetric. From [4] it follows, also that $P_n^{(s)} \in \mathcal{A}_{sn}(X)$ for $s \ge p$. Using similar arguments like in Theorem 3, it is possible to construct examples of separately symmetric and entire block-symmetric functions on a finite Cartesian power of ℓ_p, $1 \le p < \infty$. For the definition and properties of *separately symmetric* analytic functions we refer the reader to [21]. *Block symmetric* polynomials on a Cartesian power of ℓ_p were studied in [22,23]. Here we consider the case of ℓ_∞.

Each vector x in ℓ_∞ is a bounded number sequence $x = (x_1, x_2, \ldots, x_n, \ldots)$. So we can consider symmetric polynomials on ℓ_∞, that is, invariant polynomials with respect to all permutations $\sigma\colon \mathbb{N} \to \mathbb{N}$

$$(x_1, x_2, \ldots, x_n, \ldots) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}, \ldots).$$

In [17] it is proved that only constants are symmetric polynomials on ℓ_∞. However, ℓ_∞ admits polynomials that are invariant with respect to the subgroup of finite permutations. Such polynomials are called *finitely symmetric*. A permutation $\sigma\colon \mathbb{N} \to \mathbb{N}$ is *finite* if there is $m \in \mathbb{N}$ such that $\sigma(n) = n$ for every $n > m$. The following example shows that there exists a finitely symmetric entire function of unbounded type on ℓ_∞.

Example 2. *Let us fix a presentation of the set of positive integers as a disjoint union*

$$\mathbb{N} = \coprod_{i=1}^{\infty} N_i, \quad |N_i| = \infty.$$

Let $\mathcal{U}_i$ be a free ultrafilter on N_i for every $i \in \mathbb{N}$. We denote by

$$\phi_i(x) := \lim_{\mathcal{U}_i} x_n, \quad x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell_\infty.$$

Set

$$Q_1(x) = \phi_1(x) - \phi_2(x), \ldots, Q_k(x) = \prod_{i<j\le k} (\phi_i(x) - \phi_j(x)), \ldots.$$

It is clear that Q_k are nontrivial continuous n-homogeneous polynomials for

$$n = \binom{k}{2} = \frac{k(k-1)}{2}, \quad k \in \mathbb{N}.$$

Let us define

$$P_n(x) = \begin{cases} \frac{Q_k(x)}{\|Q_k\|} & \text{if} \quad n = \binom{k}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\Omega_m = \{x \in \ell_\infty \colon x_j \in \{c_1, c_2, \ldots, c_m\}\}$$

for some fixed constants $c_1, c_2, \ldots, c_m$ and

$$\Omega = \bigcup_{m \in \mathbb{N}} \Omega_m.$$

In other words, if $x \in \Omega$, then the function $x(n) = x_n$ has just a finite number of different values. By the definition of P_n one can check that if $x \in \Omega_m$, then

$$P_n(x) = 0 \quad \text{for} \quad n > \binom{m}{2} = \frac{m(m-1)}{2}.$$

Since Ω is dense in ℓ_∞, polynomials P_n satisfy the condition of Theorem 1. In addition, polynomials P_n, $n \in \mathbb{N}$ are finitely symmetric because all functionals are evidently so.

4. Discussion, Conclusions and Open Questions

As a result, we can say that various classes of polynomials on Banach spaces support entire analytic functions of unbounded type. However, does an entire symmetric analytic function of unbounded type exist on $L_\infty[0;1]$? A function on $L_\infty[0;1]$ is *symmetric* if it is invariant with respect to measuring and preserving automorphisms of the interval $[0;1]$. Polynomials

$$R_n(x) = \int_{[0;1]} (x(t))^n \, dt, \quad x(t) \in L_\infty[0;1], \quad n \in \mathbb{N}$$

form an algebraic basis in the space of all symmetric polynomials on $L_\infty[0;1]$ [18]. It would be interesting, also, to construct an entire supersymmetric function of unbounded type on $\ell_1(\mathbb{Z}_0)$, where $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$. Let us recall that a polynomial on $\ell_1(\mathbb{Z}_0)$ is *supersymmetric* if it is an algebraic combination of polynomials

$$T_k(x) = \sum_{n=1}^{+\infty} x_n^k - \sum_{n=-1}^{-\infty} x_n^k, \quad x = (\ldots, x_{-n}, \ldots, x_{-1}, x_1, \ldots, x_n, \ldots) \in \ell_1(\mathbb{Z}_0)$$

(see [24]).

In a more general case, let $\mathbf{P} = (P_n)$, $n \in \mathbb{N}$ be a sequence of algebraically independent polynomials on a Banach space X. Let us denote by $H_{b\mathbf{P}}(X)$ the minimal subalgebra of $H_b(X)$ containing all polynomials in $\mathbf{P}$. Algebras $H_{b\mathbf{P}}(X)$ were studied in [16,25]. It is natural to ask:

Question 2. *Under which conditions on $\mathbf{P}$ does there exist an entire functions f of unbounded type on X such that its Taylor polynomials are in $H_{b\mathbf{P}}(X)$?*

Note that algebras of symmetric (block-symmetric, supersymmetric) analytic functions of bounded types are partial cases of $H_{b\mathbf{P}}(X)$. Let us consider the following example.

Example 3. *Let $X = \ell_p$ for $1 \le p \le \infty$ and*

$$P_n(x) = x_n^n, \quad x = (x_1, \ldots, x_n, \ldots) \in \ell_p.$$

If $p < \infty$, then polynomials in $H_{b\mathbf{P}}(\ell_p)$ support a function of unbounded type, for example, $f(x) = \sum_{n=1}^{\infty} P_n(x)$. However, if $p = \infty$, the function f is no longer defined on the whole space.

Question 3. *Does there exist an entire function of unbounded type on ℓ_∞ with Taylor polynomials in $H_{b\mathbf{P}}(\ell_p)$, where $\mathbf{P}$ is as in Example 3?*

Let us make a note about products of functions of unbounded type. In [19] it is proved that if f is a function of unbounded type on X of the form (3) and P is a nonzero continuous polynomial on X, then $Pf \in H(X) \setminus H_b(X)$. Example 1 shows that the product of two functions of unbounded type is not necessarily of unbounded type.

Question 4. *Does a nonzero function $f \in H_b(X)$ and $g \in H(X) \setminus H_b(X)$ exist such that $fg \in H_b(X)$?*

By the following example we can see that the answer to this question is positive for the *real* case but we do not know the answer for the complex case. Analytic functions of unbounded type on real Banach spaces were studied in [26].

Example 4. *Let f be a function of unbounded type defined by (3) on a real Banach space X. We set*

$$g(x) = e^{(f(x))^2}.$$

Clearly $g \in H(X) \setminus H_b(X)$. However,

$$\frac{1}{g} = e^{-(f(x))^2}$$

has a bounded range and so is of bounded type.

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References

1. Dineen, S. *Complex Analysis in Locally Convex Spaces*; Mathematics Studies; North-Holland: Amsterdam, The Netherlands; New York, NY, USA; Oxford, UK, 1981.
2. Aron, R.M.; Cole, B.J.; Gamelin, T.W. Spectra of algebras of analytic functions on a Banach space. *J. Reine Angew. Math.* **1991**, 51–94. [CrossRef]
3. Aron, R.M.; Cole, B.J.; Gamelin, T.W. Weak-star continuous analytic functions. *Can. J. Math.* **1995**, *47*, 673–683. [CrossRef]
4. Dimant, V.; Gonzalo, R. Block diagonal polynomials. *Trans. Am. Math. Soc.* **2001**, *353*, 733–747. [CrossRef]
5. Dineen, S. *Complex Analysis on Infinite Dimensional Spaces*; Springer: Berlin/Heidelberg, Germany, 1999; p. 543. [CrossRef]
6. Chernega, I.; Zagorodnyuk, A. Unbounded symmetric analytic functions on ℓ_1. *Math. Scand.* **2018**, *122*, 84–90. [CrossRef]
7. Mujica, J. *Complex Analysis in Banach Spaces*; North-Holland: Amsterdam, The Netherlands; New York, NY, USA; Oxford, UK, 1986.
8. Ansemil, J.M.; Aron, R.M.; Ponte, S. Behavior of entire functions on balls in a Banach space. *Indag. Math.* **2009**, *20*, 483–489. [CrossRef]
9. Ansemil, J.M.; Aron, R.M.; Ponte, S. Representation of spaces of entire functions on Banach spaces. *Publ. Res. Inst. Math. Sci.* **2009**, *45*, 383–391. [CrossRef]
10. Aron, R.M. Entire functions of unbounded type on a Banach space. *Boll. dell'Unione Mat. Ital.* **1974**, *9*, 28–31.
11. Alencar, R.; Aron, R.; Galindo, P.; Zagorodnyuk, A. Algebras of symmetric holomorphic functions on ℓ_p. *Bull. Lond. Math. Soc.* **2003**, *35*, 55–64. [CrossRef]
12. Aron, R.; Galindo, P.; Pinasco, D.; Zalduendo, I. Group-symmetric holomorphic functions on a Banach space. *Bull. Lond. Math. Soc.* **2016**, *48*, 779–796. [CrossRef]
13. Aron, R.M.; Falcó, J.; Maestre, M. Separation theorems for group invariant polynomials. *J. Geom. Anal.* **2018**, *28*, 393–404. [CrossRef]
14. Chernega, I.; Galindo, P.; Zagorodnyuk, A. Some algebras of symmetric analytic functions and their spectra. *Proc. Edinb. Math. Soc.* **2012**, *55*, 125–142. [CrossRef]

15. Chernega, I.; Galindo, P.; Zagorodnyuk, A. The convolution operation on the spectra of algebras of symmetric analytic functions. *J. Math. Anal. Appl.* **2012**, *395*, 569–577. [CrossRef]
16. Chernega, I.; Holubchak, O.; Novosad, Z.; Zagorodnyuk, A. Continuity and hypercyclicity of composition operators on algebras of symmetric analytic functions on Banach spaces. *Eur. J. Math.* **2020**, *6*, 153–163. [CrossRef]
17. Galindo, P.; Vasylyshyn, T.; Zagorodnyuk, A. Symmetric and finitely symmetric polynomials on the spaces ℓ_∞ and $L_\infty[0, +\infty]$. *Math. Nachrichten* **2018**, *291*, 1712–1726. [CrossRef]
18. Galindo, P.; Vasylyshyn, T.; Zagorodnyuk, A. Analytic structure on the spectrum of the algebra of symmetric analytic functions on L_∞. *Rev. Real Acad. Cienc. Exact. Fís. Nat. Ser. A Mat.* **2020**, *114*, 56. [CrossRef]
19. Lopez-Salazar Codes, J. Vector spaces of entire functions of unbounded type. *Proc. Am. Math. Soc.* **2011**, *139*, 1347–1360. [CrossRef]
20. González, M.; Gonzalo, R.; Jaramillo, J.A. Symmetric polynomials on rearrangement-invariant function spaces. *J. Lond. Math. Soc.* **1999**, *59*, 681–697. [CrossRef]
21. Jawad, F. Note on separately symmetric polynomials on the Cartesian product of ℓ_1. *Mat. Stud.* **2018**, *50*, 204–210. [CrossRef]
22. Kravtsiv, V.; Vasylyshyn, T.; Zagorodnyuk, A. On algebraic basis of the algebra of symmetric polynomials on $\ell_p(\mathbb{C}^n)$. *J. Funct. Spaces* **2017**. [CrossRef]
23. Kravtsiv, V.V. Analogues of the Newton formulas for the block-symmetric polynomials on $\ell_p(\mathbb{C}^s)$. *Carpathian Math. Publ.* **2020**, *12*, 17–22. [CrossRef]
24. Jawad, F.; Zagorodnyuk, A. Supersymmetric polynomials on the space of absolutely convergent series. *Symmetry* **2019**, *11*, 1111. [CrossRef]
25. Halushchak, S.I. Spectra of some algebras of entire functions of bounded type, generated by a sequence of polynomials. *Carpathian Math. Publ.* **2019**, *11*, 311–320. [CrossRef]
26. Ansemil, J.M.; López-Salazar, J.; Ponte, S. On real analytic functions of unbounded type. *Rev. Mat. Complut.* **2013**, *26*, 549–560. [CrossRef]