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# Natural Paracontact Magnetic Trajectories on Unit Tangent Bundles

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**Abstract:** In this paper, we study natural paracontact magnetic trajectories in the unit tangent bundle, i.e., those that are associated to  $g$ -natural paracontact metric structures. We characterize slant natural paracontact magnetic trajectories as those satisfying a certain conservation law. Restricting to two-dimensional base manifolds of constant Gaussian curvature and to Kaluza–Klein type metrics on their unit tangent bundles, we give a full classification of natural paracontact slant magnetic trajectories (and geodesics).

**Keywords:** magnetic curve; unit tangent bundle;  $g$ -natural metric; paracontact metric structure.

## 1. Introduction and Main Results

Magnetic curves represent, in physics, the trajectories of charged particles moving on a Riemannian manifold under the action of magnetic fields. A magnetic field  $F$  on a Riemannian manifold  $(M, g)$  is a closed 2-form  $F$  and the Lorentz force associated to  $F$  is an endomorphism field  $\phi$ , such that

$$F(X, Y) = g(\phi(X), Y), \quad (1)$$

for all  $X, Y \in \mathfrak{X}(M)$ .

The magnetic trajectories of  $F$  are curves  $\gamma$  in  $M$  that satisfy the Lorentz equation (called also the Newton equation)

$$\nabla_{\dot{\gamma}} \dot{\gamma} = \phi(\dot{\gamma}), \quad (2)$$

which generalizes the equation of geodesics under arc length parametrization, namely  $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$ . Here  $\nabla$  denotes the Levi–Civita connection associated to the metric  $g$ .

Usually, the investigation is restricted to a single energy level and only unit speed magnetic curves are considered together with a strength  $q \in \mathbb{R}$ . Therefore, the study focuses on normal magnetic curves satisfying the Lorentz equation

$$\nabla_{\dot{\gamma}} \dot{\gamma} = q\phi(\dot{\gamma}),$$

where by dot we denote the derivative with respect to the arc-length parameter  $s$ .

In some settings, magnetic fields arise in a natural way. For instance, in contact (resp. paracontact) metric geometry, there is a naturally given closed two-form, which can be considered as a magnetic field, that we call the contact (resp. paracontact) magnetic field. In [1], the authors considered contact magnetic fields that are associated to the family of  $g$ -natural contact

metric structures on the unit tangent bundle of a Riemannian manifold (cf. [2]) and studied the corresponding contact magnetic trajectories (we also refer to [3,4] for the Sasaki metric case).

In this paper, we consider the paracontact setting on unit tangent bundles. More precisely, we consider the unit tangent bundle  $T_1M$  of a Riemannian manifold  $(M, g)$  endowed with an arbitrary pseudo-Riemannian  $g$ -natural metric, i.e., a metric determined by four fixed constants  $a, b, c, d, a \neq 0, a(a+c) - b^2 \neq 0, a+c+d \neq 0$ , as follows

$$\begin{cases} \tilde{G}_{(x,u)}(X^h, Y^h) = (a+c)g_x(X, Y) + dg_x(X, u)g(Y, u), \\ \tilde{G}_{(x,u)}(X^h, Z^v) = bg_x(X, Z), \\ \tilde{G}_{(x,u)}(Z^v, W^v) = ag_x(Z, W), \end{cases} \tag{3}$$

for all  $(x, u) \in T_1M, X, Y \in M_x$  and  $Z, W \in \{u\}^\perp \subset M_x$ , where  $X^h$  and  $Y^h$  (resp.  $Z^v$  and  $W^v$ ) are the horizontal (resp. vertical) lifts to  $T_1M$  of  $X$  and  $Y$  (resp.  $Z$  and  $W$ ). When  $b = d = 0$ , then  $\tilde{G}$  is said to be a Kaluza–Klein metric, and when  $b = 0$  it is said to be a Kaluza–Klein type metric. G. Calvaruso and V. Martin-Molina proved that paracontact metric structures on the unit tangent bundle associated to pseudo-Riemannian  $g$ -natural metrics constitute a three-parameter family, and they called such structures  $g$ -natural paracontact metric structures (cf. [5]).

Given a  $g$ -natural paracontact metric structure  $(\tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$  on the unit tangent bundle  $T_1M$  of  $M$ , the two-form  $d\tilde{\eta}$  associated to  $\tilde{\varphi}$  is clearly closed, giving rise to a magnetic field. We call its associated magnetic trajectories *natural paracontact magnetic trajectories*, which are characterized by the Lorentz equation

$$\tau(\gamma) = \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} = q\tilde{\varphi}(\dot{\gamma}),$$

where  $q$  is a real constant and  $\tilde{\nabla}$  is the Levi–Civita connection of  $(T_1M, \tilde{G})$ . In this paper, we shall investigate natural paracontact magnetic trajectories.

At first, we give a characterization of natural paracontact magnetic trajectories as solutions of a system of differential equations, which turns out to be a highly nontrivial relationship that involves the curvature tensor, and whose solution in the full generality is very difficult to find. For the particular case of the velocity vector field  $\dot{c}$  of a unit-speed curve  $c$  of  $M$  (which is a curve of  $T_1M$ ), we prove that  $\dot{C}$  is not a natural paracontact magnetic trajectory unless  $c$  is a Riemannian circle and the metric  $\tilde{G}$  on  $T_1M$  is of Kaluza–Klein type (Theorem 5).

In the sequel, we restrict to manifolds  $M$  of constant sectional curvature  $k$  and to pseudo-Riemannian  $g$ -natural metrics of Kaluza–Klein type on  $T_1M$ , and we characterize natural paracontact magnetic trajectories, which are slant, i.e., of constant contact angle. Recall that the contact angle of a curve  $\gamma$  in an almost paracontact metric manifold is defined as the angle between its tangent vector field and the Reeb vector field in the corresponding point. We shall prove the following.

**Theorem 1.** *Let  $(M, g)$  be a Riemannian manifold of constant sectional curvature  $k, \tilde{G}$  be a Kaluza–Klein type metric on  $T_1M$  given by (3) (with  $b = 0$ ) and  $(\tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$  be a  $g$ -natural paracontact metric structure over  $T_1M$ .*

1. *If  $k = \frac{a+c}{a} < 0$ , then every paracontact normal magnetic curve in  $(T_1M, \tilde{G})$  is slant.*
2. *If  $k \neq \frac{a+c}{a}$ , then a paracontact normal magnetic curve  $\gamma(s) = (x(s), V(s))$  is slant if and only if both  $\|\dot{x}\|$  and  $\|\dot{V}\|$  are constant.*

Note that the condition  $k < 0$  in the theorem above is necessary (cf. Remark 2).

Using the preceding theorem, we will give a complete classification of slant magnetic curves on  $T_1M^2(k)$ , when  $M^2(k)$  is a two-dimensional Riemannian manifold of constant Gaussian curvature  $k$ . In particular, for  $k \neq \frac{a+c}{a}$ , we will prove the following

**Theorem 2.** Let  $(M^2(k), g)$  be a Riemannian surface of constant Gaussian curvature  $k$ ,  $\tilde{G}$  be a Kaluza–Klein type metric on  $T_1M^2(k)$  given by (3) (with  $b = 0$ ), such that  $k \neq \frac{a+c}{a}$ , and  $(\tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$  be a  $g$ -natural paracontact metric structure over  $T_1M^2(k)$ . Then a slant paracontact normal magnetic curve of  $T_1M^2(k)$  is either a parallel vector field along a geodesic in  $M^2(k)$  or the velocity vector field along a non-geodesic Riemannian circle in  $M^2(k)$ .

Conversely, we will give explicitly sufficient conditions for the existence of such slant paracontact normal magnetic curves (Proposition 5).

For  $k = \frac{a+c}{a}$ , we find a third type of slant paracontact normal magnetic curves on  $T_1M^2(k)$ , i.e., those along curves that are not necessarily Riemannian circles. More precisely, we have

**Theorem 3.** Let  $(M^2(k), g)$  be a Riemannian surface of constant Gaussian curvature  $k < 0$ ,  $\tilde{G}$  be a Kaluza–Klein type metric on  $T_1M^2(k)$  given by (3) ( $b = 0$ ), with  $a \neq 0$ ,  $c = a(k - 1)$  (i.e.,  $k = \frac{a+c}{a}$ ) and  $d \neq ak$ , and  $(\tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$  be a  $g$ -natural paracontact metric structure over  $T_1M^2(k)$ . Subsequently, a curve  $\gamma(s) = (x(s); V(s))$  is a paracontact normal magnetic trajectory with strength  $q$  and a contact angle  $\theta$  in  $(T_1M^2(k), \tilde{G})$  if and only if one of the following cases occurs:

1.  $q = -\frac{2d}{\varphi} \cos \theta$ ,  $x$  is a geodesic on  $M^2(k)$  and  $V$  is parallel along  $x$ ;
2.  $q = -\frac{2d}{\varphi} \cos \theta$ ,  $\theta$  is constant,  $x$  is a Riemannian circle in  $M^2(k)$  with constant speed and  $\|\dot{V}\|$  (which is constant) is non zero;
3.  $q \neq -\frac{2d}{\varphi} \cos \theta$ ,  $\theta$  is constant; and,

$$\begin{cases} \dot{x} = \frac{1}{\sqrt{\varphi}} \cos \theta V + (A_1 \exp(\lambda s) + A_2 \exp(-\lambda s)) R_{\frac{\pi}{2}} V, \\ \nabla_x V = \sqrt{-k} (A_2 \exp(-\lambda s) - A_1 \exp(\lambda s)) R_{\frac{\pi}{2}} V, \end{cases} \tag{4}$$

where  $A_1, A_2$  are constants satisfying  $A_1 A_2 = \frac{\sin^2 \theta}{4ak}$  and  $\lambda := q + \frac{2d}{\varphi} \cos \theta$ .

Note that, as before, the condition  $k < 0$  is necessary in the theorem above, and that, by Theorem 1,  $\theta$  is constant.

In [1], the authors gave the classification of geodesics on unit tangent bundles of constant Gaussian curvature surfaces endowed with pseudo-Riemannian Kaluza-Klein type metrics, except in the case when the Gaussian curvature is negative equal to  $\frac{a+c}{a}$ . As a consequence of Theorem 3, we have the following corollaries that extend the classification of geodesics to the case  $k = \frac{a+c}{a} < 0$ :

**Corollary 1.** Let  $(M^2(k), g)$  be a Riemannian surface of constant Gaussian curvature  $k < 0$ ,  $\tilde{G}$  be a Kaluza–Klein type metric on  $T_1M^2(k)$  given by (3) ( $b = 0$ ), with  $a = -\frac{1}{4}$ ,  $c = \frac{1}{4}(1 - k)$  (i.e.,  $k = 1 - 4c$ ) and  $d = 0$ , and  $(\tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$  be a  $g$ -natural paracontact metric structure over  $T_1M^2(k)$ . Subsequently, a curve  $\gamma(s) = (x(s); V(s))$  is a geodesic in  $(T_1M^2(k), \tilde{G})$  if and only if one of the following cases occurs:

1.  $x$  is a geodesic on  $M^2(k)$  and  $V$  is parallel along  $x$ ;
2.  $x$  is a Riemannian circle in  $M^2(k)$  of constant speed making a constant angle with  $V$  and  $\|\dot{V}\|$  (which is constant) is non zero.

**Corollary 2.** Let  $(M^2(k), g)$  be a Riemannian surface of constant Gaussian curvature  $k < 0$ ,  $\tilde{G}$  be a Kaluza–Klein type metric on  $T_1M^2(k)$  given by (3) ( $b = 0$ ), with  $a \neq 0$ ,  $a \neq -\frac{1}{4}$ ,  $c = a(k - 1)$  and  $d = ak(4a + 1)$ , and  $(\tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$  be a  $g$ -natural paracontact metric structure over  $T_1M^2(k)$ . Subsequently, a curve  $\gamma(s) = (x(s); V(s))$  is a geodesic in  $(T_1M^2(k), \tilde{G})$  if and only if one of one of the following cases occurs:

1.  $x$  is a geodesic or a Riemannian circle of constant speed in  $M^2(k)$ , and  $V$  is orthogonal to  $x$ ;
2. the system (4) holds, where  $\theta \in (0, \pi) \setminus \{\frac{\pi}{2}\}$ ,  $A_1, A_2$  are constants satisfying  $A_1A_2 = \frac{\sin^2 \theta}{4ak}$  and  $\lambda := \frac{2d}{\varphi} \cos \theta$ .

Finally, to give a geometric insight to the second type of paracontact normal magnetic trajectories in Theorem 3, we will draw some pictures of slant magnetic curves along Riemannian circles on the unit tangent bundle of the hyperbolic plane of constant Gaussian curvature  $-4$  endowed with a pseudo-Riemannian Kaluza–Klein type metric.

## 2. $g$ -Natural Metrics on Tangent and Unit Tangent Bundles

Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold and  $\nabla$  the Levi–Civita connection of  $g$ . We shall denote by  $M_x$  the tangent space of  $M$  at a point  $x \in M$  and by  $p : TM \rightarrow M$  the bundle projection. For  $(x, u) \in TM$  and  $X \in M_x$ , there exists a unique vector  $X^h \in H_{(x,u)}$ , such that  $p_*X^h = X$ , where  $p : TM \rightarrow M$  is the natural projection. We call  $X^h$  the horizontal lift of  $X$  to the point  $(x, u) \in TM$ . The vertical lift of a vector  $X \in M_x$  to  $(x, u) \in TM$  is a vector  $X^v \in V_{(x,u)}$  such that  $X^v(df) = Xf$ , for all functions  $f$  on  $M$ . Here, we consider 1-forms  $df$  on  $M$  as functions on  $TM$  (i.e.,  $(df)(x, u) = uf$ ).

Observe that the map  $X \rightarrow X^h$  is an isomorphism between the vector spaces  $M_x$  and  $H_{(x,u)}$ . Similarly, the map  $X \rightarrow X^v$  is an isomorphism between the vector spaces  $M_x$  and  $V_{(x,u)}$ . Obviously, each tangent vector  $\tilde{Z} \in (TM)_{(x,u)}$  can be written in the form  $\tilde{Z} = X^h + Y^v$ , where  $X, Y \in M_x$  are uniquely determined vectors. Hence, the tangent space of  $TM$  at any point  $(x, u) \in TM$  splits into the horizontal and vertical subspaces with respect to  $\nabla$ :

$$(TM)_{(x,u)} = H_{(x,u)} \oplus V_{(x,u)}.$$

Horizontal and vertical lifts of vector fields on  $M$  are defined in a corresponding way.

Now, starting from a Riemannian manifold  $(M, g)$ , a natural construction leads to introduce a wide class of metrics, called  $g$ -natural, on the tangent bundle  $TM$  ([6,7]). Such metrics are characterized by the following (cf. [8]):

**Proposition 1.** Given an arbitrary  $g$ -natural metric  $G$  on the tangent bundle  $TM$  of a Riemannian manifold  $(M, g)$ , there exist six smooth functions  $\alpha_i, \beta_i : \mathbb{R}^+ \rightarrow \mathbb{R}, i = 1, 2, 3$ , such that

$$\begin{cases} G_{(x,u)}(X^h, Y^h) = (\alpha_1 + \alpha_3)(r^2)g_x(X, Y) + (\beta_1 + \beta_3)(r^2)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^h, Y^v) = G_{(x,u)}(X^v, Y^h) \\ \qquad \qquad \qquad = \alpha_2(r^2)g_x(X, Y) + \beta_2(r^2)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^v, Y^v) = \alpha_1(r^2)g_x(X, Y) + \beta_1(r^2)g_x(X, u)g_x(Y, u), \end{cases}$$

for every  $u, X, Y \in M_x$ , where  $r^2 = g_x(u, u)$ .

Putting  $\phi_i(t) = \alpha_i(t) + t\beta_i(t)$ ,  $\alpha(t) = \alpha_1(t)(\alpha_1 + \alpha_3)(t) - \alpha_2^2(t)$  and  $\phi(t) = \phi_1(t)(\phi_1 + \phi_3)(t) - \phi_2^2(t)$ , for all  $t \in \mathbb{R}^+$ , it is known (cf. [9]) that  $G$  is

- non-degenerate if and only if

$$\alpha(t) \neq 0, \quad \phi(t) \neq 0 \quad \text{for all } t \in \mathbb{R}^+;$$

- Riemannian if and only if

$$\alpha_1(t) > 0, \quad \phi_1(t) > 0, \quad \alpha(t) > 0, \quad \phi(t) > 0 \quad \text{for all } t \in \mathbb{R}^+.$$

The wide class of  $g$ -natural metrics includes several well known metrics (Riemannian and not) on  $TM$ . In particular:

- the *Sasaki metric*  $g_S$  is obtained for  $\alpha_1 = 1$  and  $\alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0$ .
- *Kaluza–Klein metrics*, as commonly defined on principal bundles (see for example [10]), are obtained for  $\alpha_2 = \beta_2 = \beta_1 + \beta_3 = 0$ .
- Metrics of Kaluza–Klein type are defined by the geometric condition of orthogonality between horizontal and vertical distributions [11]. Thus, a  $g$ -natural metric  $G$  is of Kaluza–Klein type if  $\alpha_2 = \beta_2 = 0$ .

The set  $T_1M$  of unit tangent vectors to  $M$  is a hypersurface of  $TM$  called the unit tangent bundle of  $M$ . The tangent space of  $T_1M$  at a point  $(x, u) \in T_1M$  is given by

$$(T_1M)_{(x,u)} = \{X^h + Y^v \mid X \in M_x, Y \in \{u\}^\perp \subset M_x\}. \tag{5}$$

By definition,  $g$ -natural metrics on the unit tangent bundle  $T_1M$  are the metrics induced by  $g$ -natural metrics on  $TM$ . As proved in [12] for the Riemannian case, and extended to pseudo-Riemannian settings in [5], they are completely determined by the values of the four real constants

$$a := \alpha_1(1), \quad b := \alpha_2(1), \quad c := \alpha_3(1), \quad d := (\beta_1 + \beta_3)(1),$$

giving the explicit expression (3).

By a simple calculation, using the Schmidt’s orthonormalization process, it is easy to check that the vector field on  $TM$  defined by

$$N_{(x,u)} = \frac{1}{\sqrt{(a+c+d)\phi}}[-bu^h + (a+c+d)u^v], \tag{6}$$

for all  $(x; u) \in TM$ , is normal to  $T_1M$  and unitary at any point of  $T_1M$ . We define the “tangential lift”  $X^t$  with respect to the metric  $G$  on  $TM$  of a vector  $X \in M_x$  to  $(x, u) \in T_1M$  as the tangential projection of the vertical lift of  $X$  to  $(x, u)$  with respect to  $N$ , which is

$$\begin{aligned} X^t &= X^v - G_{(x,u)}(X^v, N_{(x,u)})N_{(x,u)} \\ &= [X - g_x(u, X)u]^v + \frac{b}{a+c+d}g_x(u, X)u^h. \end{aligned} \tag{7}$$

If  $X \in M_x$  is orthogonal to  $u$ , then  $X^t = X^v$ . The tangent space  $(T_1M)_{(x,u)}$  of  $T_1M$  at  $(x, u)$  is spanned by vectors of the form  $X^h$  and  $Y^t$ , where  $X, Y \in M_x$ .

Using tangential lifts and (3), it is easy to see that  $g$ -natural metrics on  $T_1M$  admit the following explicit description (cf. [12]):

**Proposition 2.** Let  $(M, g)$  be a Riemannian manifold. For every pseudo-Riemannian metric  $\tilde{G}$  on  $T_1M$  induced from a  $g$ -natural  $G$  on  $TM$ , there exist four constants  $a, b, c$  and  $d$ , satisfying the inequalities

$$a \neq 0, \quad \alpha := a(a + c) - b^2 \neq 0, \quad \varphi := a + c + d \neq 0$$

(in particular, they are Riemannian if and only if  $a, \alpha, \varphi > 0$ ), such that

$$\begin{cases} \tilde{G}_{(x,u)}(X^h, Y^h) = (a + c)g_x(X, Y) + dg_x(X, u)g(Y, u) \\ \tilde{G}_{(x,u)}(X^h, Y^t) = bg_x(X, Y) \\ \tilde{G}_{(x,u)}(X^t, Y^t) = ag_x(X, Y) - \frac{\phi}{a+c+d}g_x(X, u)g_x(Y, u) \end{cases} \quad (8)$$

for all  $(x, u) \in T_1M$ , and  $X, Y \in M_x$ , where  $\phi := \alpha + ad$ .

In particular, the Sasaki metric on  $T_1M$  corresponds to the case where  $a = 1$  and  $b = c = d = 0$ ; Kaluza–Klein metrics are obtained when  $b = d = 0$ ; metrics of Kaluza–Klein type are given by the case  $b = 0$ .

### 3. Natural Paracontact Metric Structures on Unit Tangent Bundles

The study of paracontact geometry was initiated by Kaneyuki and Williams [13]. A systematic study of paracontact metric manifolds and their subclasses was started out by Zamkovay [14]. Since then, several geometers studied paracontact metric manifolds and obtained various important properties of them.

A contact manifold is an odd-dimensional manifold  $M^{2n+1}$  equipped with a global 1-form  $\eta$  such that  $\eta \neq 0$  everywhere. Given such a form  $\eta \wedge (d\eta)^{n-1} \neq 0$ , there exists a unique vector field  $\xi$ , called the characteristic vector field or the Reeb vector field of  $\eta$ , satisfying  $\eta(\xi) = 1$  and  $d\eta(X, \xi) = 0$ , for any vector field  $X$  on  $M^{2n+1}$ . A pseudo-Riemannian metric  $g$  is said to be an associated metric if there exists a tensor field  $\phi$  of type  $(1, 1)$ , such that

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y) \quad \text{and} \quad \phi^2(X) = X - \eta(X)\xi, \quad (9)$$

for all vector fields  $X, Y$  on  $M^{2n+1}$ . In this case, the structure  $(\phi, \xi, \eta, g)$  on  $M^{2n+1}$  is called a paracontact metric structure and the manifold  $M^{2n+1}$  equipped with such a structure is said to be a paracontact metric manifold. It can be easily seen that, in a paracontact metric manifold, the following relations hold:

$$\phi(\xi) = 0, \quad \eta\phi = 0, \quad g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \quad (10)$$

for any vector fields  $X, Y$  on  $M^{2n+1}$ .

We now give necessary and sufficient conditions for a pseudo-Riemannian  $g$ -natural metric on  $T_1M$  to be associated to the very natural contact structure given by

$$\tilde{\xi}(x, u) = ru^h, \quad \tilde{\eta}(X^h) = \frac{1}{r^2}g(X, u), \quad \tilde{\eta}(X^t) = brg(X, u),$$

where  $r$  is a positive constant. Let  $\tilde{G}$  be an arbitrary pseudo-Riemannian  $g$ -natural metric over  $T_1M$ . Subsequently, it is easy to see that, by  $d\eta(X, Y) = g(X, \phi Y)$ ,  $\phi$  is completely determined by the relation

$$\begin{cases} \tilde{\varphi}(X^h) = \frac{1}{2r\alpha}[-bX^h + (a+c)X^t + \frac{bd}{\varphi}g(X,u)u^h], \\ \tilde{\varphi}(X^t) = \frac{1}{2r\alpha}[-aX^h + bX^t + \frac{\phi}{\varphi}g(X,u)u^h], \end{cases} \tag{11}$$

We deduce that

$$\begin{aligned} \tilde{\varphi}^2(X^h) &= \frac{1}{4r^2\alpha} \{-X^h + g(X,u)u^h\} \quad \text{and} \\ \tilde{\varphi}^2(X^t) &= \frac{1}{4r^2\alpha} \{-X^t + \frac{b}{\varphi}g(X,u)u^h\}, \end{aligned}$$

so that  $\tilde{\varphi}^2 = I - \tilde{\eta} \otimes \tilde{\xi}$  if and only if  $\frac{1}{r^2} = -4\alpha$  and  $b(\varphi + 4\alpha) = 0$ . But since  $1 = \tilde{\eta}(\tilde{\xi}) = \tilde{G}(\tilde{\xi}, \tilde{\xi}) = r^2\varphi$ , then  $\varphi = \frac{1}{r^2}$ . It follows, on one hand, that  $\varphi > 0$  and, on the other hand, that the relation  $b(\varphi + 4\alpha) = 0$  is always satisfied. We deduce then the following (cf. [5])

**Proposition 3.**  $(\tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$  is a paracontact metric structures over  $T_1M$  if and only if the following holds

$$\frac{1}{r^2} = -4\alpha = \varphi \tag{12}$$

The set of  $(\tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$ , described by Proposition 3, is a three-parameter family of paracontact metric structures on  $T_1M$ , that we call natural paracontact metric structures.

**Remark 1.** The condition  $\frac{1}{r^2} = -4\alpha$  confirms the fact that  $\tilde{G}$  is not Riemannian. It is of signature  $(n, n - 1)$ .

#### 4. Natural Paracontact Magnetic Curves in Unit Tangent Bundles

Let  $(M, g)$  be a Riemannian manifold,  $\nabla$  its Levi-Civita connection and  $R$  its Riemannian curvature. Given a natural paracontact metric structure  $(\tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$  on the unit tangent bundle  $T_1M$  of  $M$ , the two-form  $d\tilde{\eta}$  associated to  $\tilde{\varphi}$  is clearly closed, giving rise to a magnetic field. We call its associated magnetic trajectories natural paracontact magnetic trajectories, which are characterized by the Lorentz equation

$$\tau(\gamma) = \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} = q\tilde{\varphi}(\dot{\gamma}), \tag{13}$$

where  $q$  is a real constant and  $\tilde{\nabla}$  is the Levi-Civita connection of  $(T_1M, \tilde{G})$ . We start this section by giving equations characterizing natural paracontact magnetic trajectories on  $(T_1M, \tilde{G})$ .

Let  $\gamma(s) = (x(s); V(s))$  be a curve in  $(T_1M, \tilde{G})$ . Subsequently,  $V(s)$  is a unit vector field along the base curve  $x(s)$  in  $M$ . The velocity vector field  $\dot{\gamma}(s)$  is given by

$$\dot{\gamma}(s) = \dot{x}(s)_{\gamma(s)}^h + (\nabla_{\dot{x}} V)_{\gamma(s)}^t. \tag{14}$$

In [1], we have proved that the tension vector field  $\tau(\gamma) = \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma}$  is given by

$$\begin{aligned} \tau(\gamma) &= \left\{ \nabla_{\dot{x}} \dot{x} - \frac{ab}{\alpha} R(\dot{x}, V)\dot{x} + \frac{bd}{\alpha} g(\dot{x}, V)\dot{x} - \frac{a^2}{\alpha} R(\dot{V}, V)\dot{x} + \frac{ad}{\alpha} g(\dot{x}, V)\dot{V} \right. \\ &\quad + \frac{1}{\alpha\varphi} [a(ad + b^2)g(R(\dot{x}, V)\dot{V}, V) + d\alpha g(\dot{x}, \dot{V}) \\ &\quad + b(ad + b^2)g(R(\dot{x}, V)\dot{x}, V) - db\varphi g^2(\dot{x}, V)] V\}^h \\ &\quad + \left\{ \nabla_{\dot{x}} \dot{V} + \frac{b^2}{\alpha} R(\dot{x}, V)\dot{x} - \frac{(a+c)d}{\alpha} g(\dot{x}, V)\dot{x} + \frac{ab}{\alpha} R(\dot{V}, V)\dot{x} - \frac{bd}{\alpha} g(\dot{x}, V)\dot{V} \right. \\ &\quad \left. + \frac{1}{\alpha} [-b^2g(R(\dot{x}, V)\dot{x}, V) + d(a+c)g^2(\dot{x}, V) - abg(R(\dot{x}, V)\dot{V}, V)] V\}^t. \end{aligned} \tag{15}$$

On the other hand, using (11) and (14), we get

$$\tilde{\varphi}(\dot{\gamma}) = \frac{1}{2r\alpha} \left( \left[ -b\dot{x} - a\dot{V} + \frac{bd}{\varphi}g(\dot{x}, V)V \right]^h + [(a+c)\dot{x} + b\dot{V}]^t \right). \tag{16}$$

In terms of horizontal and vertical lifts, the previous relation becomes:

$$\tilde{\varphi}(\dot{\gamma}) = \frac{1}{2r\alpha} \{ [-b\dot{x} - a\dot{V} + bg(\dot{x}, V)V]^h + [(a+c)\dot{x} + b\dot{V} - (a+c)g(\dot{x}, V)V]^v \}$$

Taking into account (15) and (16), Lorentz Equation (13) gives the following characterization of natural paracontact magnetic curves on the unit tangent bundle:

**Theorem 4.** A curve  $\gamma(s) = (x(s); V(s))$  in  $(T_1M, \tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$  is a paracontact magnetic trajectory with strength  $q$  if and only if

$$\left\{ \begin{array}{l} \nabla_{\dot{x}}\dot{x} - \frac{ab}{\alpha}R(\dot{x}, V)\dot{x} + [\frac{bd}{\alpha}g(\dot{x}, V) + \frac{bq}{2r\alpha}]\dot{x} - \frac{a^2}{\alpha}R(\dot{V}, V)\dot{x} + [\frac{ad}{\alpha}g(\dot{x}, V) \\ + \frac{aq}{2r\alpha}]\dot{V} + \frac{1}{\alpha\varphi}[a(ad + b^2)g(R(\dot{x}, V)\dot{V}, V) + b(ad + b^2)g(R(\dot{x}, V)\dot{x}, V) \\ + d\alpha g(\dot{x}, \dot{V}) + b\alpha g(\nabla_{\dot{x}}\dot{V}, V) - db\varphi g^2(\dot{x}, V) - \frac{qb\varphi}{2r}g(\dot{x}, V)]V = 0 \\ \nabla_{\dot{x}}\dot{V} + \frac{b^2}{\alpha}R(\dot{x}, V)\dot{x} - [\frac{(a+c)d}{\alpha}g(\dot{x}, V) + \frac{q(a+c)}{2r\alpha}]\dot{x} + \frac{ab}{\alpha}R(\dot{V}, V)\dot{x} \\ - [\frac{bq}{2r\alpha} + \frac{bd}{\alpha}g(\dot{x}, V)]\dot{V} - \frac{1}{\alpha}[abg(R(\dot{V}, V)\dot{x}, V) + b^2g(R(\dot{x}, V)\dot{x}, V) \\ + \alpha g(\nabla_{\dot{x}}\dot{V}, V) - (a+c)d g^2(\dot{x}, V) - \frac{q(a+c)}{2r}g(\dot{x}, V)]V = 0. \end{array} \right. \tag{17}$$

In the special case of velocity vector fields, we have

**Corollary 3.** Let  $x(s)$  be a unit speed curve in  $(M, g)$ . Subsequently, its velocity vector field  $\gamma(s) = (x(s), \dot{x}(s))$  is a paracontact magnetic trajectory with strength  $q$  in  $(\tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$  if and only if

$$\left\{ \begin{array}{l} (\frac{2r\varphi+aq}{2r})\nabla_{\dot{x}}\dot{x} + a^2R(\dot{x}, \nabla_{\dot{x}}\dot{x})\dot{x} - \frac{b\alpha}{\varphi}\|\nabla_{\dot{x}}\dot{x}\|^2\dot{x} = 0, \\ \nabla_{\dot{x}}\nabla_{\dot{x}}\dot{x} - \frac{ab}{\alpha}R(\dot{x}, \nabla_{\dot{x}}\dot{x})\dot{x} - \frac{b(q+2rd)}{2r\alpha}\nabla_{\dot{x}}\dot{x} + \|\nabla_{\dot{x}}\dot{x}\|^2\dot{x} = 0. \end{array} \right. \tag{18}$$

As a consequence of system (18), we find

$$\nabla_{\dot{x}}\nabla_{\dot{x}}\dot{x} + \frac{b}{a}\nabla_{\dot{x}}\dot{x} + [1 - \frac{b^2}{a\varphi}]\|\nabla_{\dot{x}}\dot{x}\|^2\dot{x} = 0. \tag{19}$$

We distinguish two situations:

- If  $x$  is a geodesic, then Equation (18) is automatically satisfied.
- If  $x$  is not a geodesic, then let  $\nu_1$  be the (first) normal, that is  $\nabla_{\dot{x}}\dot{x} = \kappa\nu_1$ , where  $\kappa = \|\nabla_{\dot{x}}\dot{x}\| \neq 0$ . Then we have

$$\nabla_{\dot{x}}\nabla_{\dot{x}}\dot{x} = \kappa\nabla_{\dot{x}}\nu_1 + \kappa'\nu_1. \tag{20}$$

While using (20) in (19), we obtain

$$\left\{ \begin{array}{l} \kappa' + \frac{b}{a}\kappa = 0 \\ b\kappa = 0. \end{array} \right. \tag{21}$$

Hence, one gets  $b = 0$ , i.e.,  $\tilde{G}$  is a Kaluza-Klein type metric on  $T_1M$ .

We conclude with the following result:

**Theorem 5.** *Let  $x(s)$  be a non-geodesic unit speed curve in  $(M, g)$ . Subsequently, its velocity vector field  $\gamma(s) = (x(s), \dot{x}(s))$  is a paracontact magnetic trajectory with strength  $q$  in  $(\tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$  if and only if*

1.  $\tilde{G}$  is a Kaluza-Klein type metric on  $T_1M$ ,
2.  $R(\nabla_{\dot{x}}\dot{x}, \dot{x})\dot{x} = (\frac{2r\varphi+aq}{2ra^2})\nabla_{\dot{x}}\dot{x}$ ,
3.  $x$  is a Riemannian circle.

**Corollary 4.** *Let  $(M, g)$  be a Riemannian manifold of constant curvature  $k$  and  $x(s)$  be a non-geodesic unit speed curve in  $(M, g)$ . Then its velocity vector field  $\gamma(s) = (x(s), \dot{x}(s))$  is a paracontact magnetic trajectory with strength  $q$  in  $(\tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$  if and only if*

1.  $\tilde{G}$  is a Kaluza-Klein type metric on  $T_1M$ ,
2.  $x$  is a Riemannian circle,
3.  $q = 2[ka - \varphi]r$ .

### 5. Contact Angle

Recall that the contact angle of a curve  $\gamma$  in an almost paracontact metric manifold is defined as the angle between its tangent vector field and the Reeb vector field in the corresponding point. For a unit speed curve  $\gamma(s)$  in  $(T_1M, \tilde{G})$ , we have  $\dot{\gamma}(s) = \dot{x}_{\gamma(s)}^h + (\nabla_{\dot{x}}V)_{\gamma(s)}^t$  and so the contact angle  $\theta$  of  $\gamma$  is given by

$$\cos \theta(s) := \frac{\tilde{G}(\dot{\gamma}, \tilde{\xi})}{\|\dot{\gamma}\| \cdot \|\tilde{\xi}\|} = \frac{1}{r\sqrt{\varphi}} \tilde{\eta}_{\gamma(s)}(\dot{\gamma}(s)) = \sqrt{\varphi} g_{x(s)}(\dot{x}, V). \tag{22}$$

Hence, a unit speed curve is slant, which is the contact angle is constant, if and only if  $g(\dot{x}, V)$  is constant.

We now investigate natural paracontact magnetic curves that are slant in the unit tangent bundle  $(T_1M, \tilde{G})$  of a space form  $M(k)$  with  $\tilde{G}$  is a metric of Kaluza–Klein type. We first reformulate the equation of natural paracontact magnetic curves in a space form, in terms of the contact angle.

**Proposition 4.** *Let  $M = M(k)$  be a space form of curvature  $k$ . Subsequently, any paracontact magnetic curve on  $(T_1M, \tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$ , where  $\tilde{G}$  is a metric of Kaluza–Klein type satisfies the following differential equations system:*

$$\begin{cases} \nabla_{\dot{x}}\dot{x} + \frac{1}{a+c}[r(d - ak) \cos \theta + \frac{q}{2r}]\dot{V} + r^2(ak + d)g(\dot{x}, \dot{V})V = 0 \\ \nabla_{\dot{x}}\dot{V} - g(\nabla_{\dot{x}}\dot{V}, V)V - \frac{1}{a}[rd \cos \theta + \frac{q}{2r}][\dot{x} - r \cos \theta V] = 0. \end{cases} \tag{23}$$

We are now in position to prove Theorem 1, which gives a characterization of slant paracontact magnetic curves in  $T_1M$ .

**Proof of Theorem 1.** By normality of  $\gamma$ , we have

$$\begin{aligned} 1 = \|\dot{\gamma}\|^2 &= (a + c)\|\dot{x}\|^2 + dg^2(\dot{x}, V) + a\|\dot{V}\|^2 \\ &= (a + c)\|\dot{x}\|^2 + r^2d \cos^2 \theta + a\|\dot{V}\|^2 \end{aligned} \tag{24}$$

and, by the first equation of (23), we obtain

$$\frac{1}{2} \frac{d}{ds} g(\dot{x}, \dot{x}) = g(\nabla_{\dot{x}} \dot{x}, \dot{x}) = -\frac{1}{a+c} \left[ r^3 d(\varphi + (a+c) - ak) \cos \theta + \frac{q}{2r} \right] g(\dot{x}, \dot{V}). \tag{25}$$

On the other hand, deriving (22) with respect to  $s$  and using the first equation of (23), we obtain the following:

$$-\dot{\theta} \sin \theta = r[(a+c) - ak]g(\dot{x}, \dot{V}). \tag{26}$$

So, we have two possibilities:

1.  $k = \frac{a+c}{a}$ . In this case, we have by (26)  $\dot{\theta} = 0$ , and hence  $\gamma$  is slant.
2.  $k \neq \frac{a+c}{a}$ . Suppose that  $\gamma$  is slant. Subsequently,  $\theta$  is constant and, then, by (26), we have  $g(\dot{x}, \dot{V}) = 0$  since  $k \neq \frac{a+c}{a}$ . Hence, (25) gives  $\frac{d}{ds} g(\dot{x}, \dot{x}) = 0$ , i.e.,  $\|\dot{x}\|$  is constant. Because, in (24)  $\|\dot{x}\|$  and  $\theta$  are constant, then  $\|\dot{V}\|$  is constant.

Conversely, if we suppose that both  $\|\dot{x}\|$  and  $\|\dot{V}\|$  are constant, then we can distinguish two cases:

- $d \neq 0$ . In this case, from (24),  $\theta$  is constant, i.e.,  $\gamma$  is slant.
- $d = 0$  and  $q \neq 0$ . In this case, we have by virtue of (25),  $g(\dot{x}, \dot{V}) = 0$ . Hence (26) implies that  $\theta$  is constant, i.e.,  $\gamma$  is slant.

□

**Remark 2.** The condition  $k < 0$  in Theorem 1 is mandatory. Indeed, since  $\frac{1}{r^2} = -4\alpha$ , then we obtain  $\frac{1}{r^2} = -4a^2k > 0$ , i.e.,  $k < 0$ .

### 6. Slant Magnetic Curves on the Unit Tangent Bundle of $M^2(k)$

We are now interested in what happens when  $n = 2$ , i.e., when  $M$  is a Riemannian surface  $M^2(k)$  of constant Gaussian curvature  $k$  and we shall restrict ourselves to the Kaluza–Klein type metrics on the unit tangent bundle  $T_1M^2(k)$ . The investigation yields to Theorems 2 ( $k \neq \frac{a+c}{a}$ ) and 3 ( $k = \frac{a+c}{a}$ ), whose proofs are given below.

**Proof of Theorem 2.** Suppose that  $\gamma$  is a slant paracontact normal magnetic curve. Subsequently, by Theorem 1, (24) and (26), we have

$$\begin{aligned} g(\dot{x}, V) &= r \cos \theta, & g(\dot{x}, \dot{V}) &= 0, & \|\dot{x}\| &= \sigma, \\ \|\dot{V}\|^2 &= \frac{1}{a} [1 - (a+c)\sigma^2 - r^2 d \cos^2 \theta], \end{aligned} \tag{27}$$

where  $\sigma$  is a positive real constant.

Let us distinguish two situations:

**Case A:** If  $\sigma = \sqrt{\frac{1-r^2d \cos^2 \theta}{a+c}}$ , we deduce that  $\dot{V} = 0$ .

Moreover, from the first Equation (23), we obtain that  $\nabla_{\dot{x}} \dot{x} = 0$ , namely  $x$  is a geodesic on  $M^2(k)$ .

On the other hand, from the second equation of (23), we get

$$\left( rd \cos \theta + \frac{q}{2r} \right) (r \cos \theta V - \dot{x}) = 0. \tag{28}$$

**Subcase A1** If we choose  $q \neq -2r^2d \cos \theta$ , we get  $r \cos \theta V = \dot{x}$ . As  $V$  is unitary and  $\|\dot{x}\| = \sigma$ , we obtain  $\cos^2 \theta = 1$ . We deduce that  $V = \pm r\dot{x} = \pm \frac{1}{\sqrt{\varphi}}\dot{x}$ . Accordingly, we obtain the magnetic curve  $\gamma(s) = (x(s); \pm \frac{1}{\sqrt{\varphi}}\dot{x}(s))$ , with strength  $q \neq \pm \frac{2d}{\varphi}$ , where  $x$  is geodesic on  $M^2(k)$ .

**Subcase A2** If  $q = -2r^2d \cos \theta$ , then we can distinguish two possibilities:

- $d = 0$ , then  $q = 0$ ,  $a + c > 0$  and  $\sigma = \frac{1}{\sqrt{a+c}}$ .
- $d \neq 0$ . Then  $\cos^2 \theta = \frac{1-(a+c)\sigma^2}{r^2d} = \frac{\varphi}{d}[1 - (a + c)\sigma^2]$ . Thus we should have

$$|1 - (a + c)\sigma^2| \leq \frac{|d|}{\varphi}. \tag{29}$$

**Case B:**  $\sigma < \sqrt{\frac{1-r^2d \cos^2 \theta}{a+c}}$ , then the vector fields  $\dot{V}$  and  $V$  are linearly independent. Hence, at every point  $x(s)$ , the vector  $\dot{x}$  is a linear combination of  $V$  and  $\dot{V}$ . Using  $g(\dot{x}, \dot{V}) = 0$ , we find  $\dot{x} = g(\dot{x}, V)V = r \cos \theta V$  and  $\sigma = r|\cos \theta|$ . Because  $\sigma \neq \sqrt{\frac{1-r^2d \cos^2 \theta}{a+c}}$ , we have  $\cos^2 \theta \neq 1$ , i.e.,  $\theta \in ]0, \pi[$ . As  $\dot{V}$  does not vanish, we have  $\nabla_{\dot{x}}\dot{x} = r \cos \theta \dot{V} \neq 0$ . From the first equation in (23), it follows that  $q = 2r^2(ak - \varphi) \cos \theta$ . If we put  $\rho := \|\nabla_{\dot{x}}\dot{x}\|$ , then  $\rho = \|\nabla_{\dot{x}}\dot{x}\|^2 = r^2 \cos^2 \theta \|\dot{V}\|^2 = \sigma^2 \|\dot{V}\|^2 = \frac{\sigma^2}{a}[1 - (a + c)\sigma^2 - dr^2 \cos^2 \theta] = \frac{1}{a\sigma^2}[1 - \varphi\sigma^2]$  is a non-zero constant and, hence,  $x$  is a non-geodesic circle in  $M^2(k)$ . It is then easy to see that the second equation of (23) is equivalent to

$$\nabla_{\dot{x}}\nabla_{\dot{x}}\dot{x} + \frac{\sigma^2}{a}[1 - \varphi\sigma^2]\dot{x} = 0.$$

Thus, the magnetic curve  $\gamma$  is obtained as  $\gamma(s) = (x(s), \pm \frac{1}{\sigma}\dot{x}(s))$ , where  $x$  is a non-geodesic circle in  $M^2(k)$ .  $\square$

In the preceding proof, we have proven that, to have slant paracontact normal magnetic curves, some restrictions should be imposed on the  $g$ -natural metrics and the strengths of the curves. This gives the following classification result:

**Proposition 5.** Let  $(M^2(k), g)$  be a Riemannian surface of constant Gaussian curvature  $k$ ,  $\tilde{G}$  be a Kaluza–Klein type metric on  $T_1M^2(k)$  given by (3) (with  $b = 0$ ), such that  $k \neq \frac{a+c}{a}$ ,  $(\tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$  be a  $g$ -natural paracontact metric structure over  $T_1M^2(k)$  and  $q \in \mathbb{R}^*$ . Then a curve  $\gamma = (x; V)$  of  $T_1M^2(k)$  is a slant paracontact normal magnetic curve of strength  $q$  and contact angle  $\theta \in [0, \pi]$  if and only if one of the following assertions holds:

1.  $x$  is a geodesic in  $M^2(k)$  of speed  $\sigma$ ,  $V$  is parallel and one of the following holds:
  - (i)  $q \neq \pm \frac{2d}{\varphi}$ ,  $\sigma = \frac{1}{\sqrt{\varphi}}$  and  $V = \pm \sqrt{\varphi}\dot{x}$ ;
  - (ii)  $d \neq 0$ ,  $q = -\frac{2d}{\varphi} \cos \theta$ ,  $\theta = \arccos \left( \pm \sqrt{\frac{\varphi[1-(a+c)\sigma^2]}{d}} \right)$  and  $\sigma$  satisfies the inequality (29).
2.  $x$  is a non-geodesic Riemannian circle in  $M^2(k)$  of constant speed  $\sigma$ ,  $V = \pm \frac{1}{\sigma}\dot{x}$ ,  $\theta = \arccos(\pm \sigma \sqrt{\varphi}) \in [0, \pi]$  and  $q = \pm \frac{2\sigma}{\sqrt{\varphi}}(ak - \varphi)$ .

**Remark 3.**

1. In the subcase (i) of the case 1. of the previous proposition, the contact angle  $\theta \in \{0, \pi\}$ .
2. Using the identity  $d = -(a + c)(4a + 1)$ , it is easy to see that condition (29) is equivalent to :

- $\frac{1}{\sqrt{a+c}} \leq \sigma \leq \frac{1}{\sqrt{\varphi}}$ , if  $-\frac{1}{4} < a < 0$ ;
- $\frac{1}{\sqrt{\varphi}} \leq \sigma \leq \frac{1}{\sqrt{a+c}}$ , if  $a < -\frac{1}{4}$ ;
- $0 \leq \sigma \leq \frac{1}{\sqrt{\varphi}}$ , if  $a > 0$ .

**Proof of Theorem 3.** Suppose that  $\gamma(s) = (x(s); V(s))$  is a paracontact normal magnetic trajectory with strength  $q$  and a contact angle  $\theta$  in  $(T_1M^2(k), \tilde{G})$ . Subsequently,  $\theta$  is constant, by Theorem 1. For  $s \in \mathbb{R}$ , let  $n(s) = R_{\frac{\pi}{2}}(V(s))$  be the unit vector normal to  $V(s)$ . Subsequently, we have

$$\dot{x}(s) = r \cos \theta V(s) + A(s)n(s), \quad \dot{V}(s) = B(s)n(s),$$

for certain  $C^\infty$ -functions  $A$  and  $B$ . It follows that

$$\nabla_{\dot{x}}n(s) = -B(s)V(s) \quad \text{and} \quad \nabla_{\dot{x}}\dot{x}(s) = -A(s)B(s)V(s) + (\dot{A}(s) + rB(s) \cos \theta) n(s).$$

Using (23),  $\gamma$  is a magnetic curve if and only if

$$\begin{cases} \dot{A}(s) + a\mu B(s) = 0, \\ \dot{B}(s) - ak\mu A(s) = 0, \end{cases} \tag{30}$$

where  $\mu = \frac{1}{a^2k} [dr \cos \theta + \frac{q}{2r}]$ . Hence,

$$\nabla_{\dot{x}}\dot{x}(s) = -B(s) [A(s)V(s) + (a\mu - r \cos \theta)n(s)]. \tag{31}$$

If  $\mu = 0$ , which is  $q = -2r^2d \cos \theta = -\frac{2d}{\varphi} \cos \theta$ , it follows that  $A(s) = A$  and  $B(s) = B$ , where  $A$  and  $B$  are real constants. Moreover, we obtain  $\nabla_{\dot{x}}\nabla_{\dot{x}}\dot{x} + B^2\dot{x} = 0$ .

- If  $B = 0$ , then  $x$  is a geodesic on  $M^2(k)$  and  $\dot{V} = 0$ .
- If  $B \neq 0$ , we obtain  $n(s) = \frac{1}{B}\dot{V}(s) \neq 0$  and  $\dot{x} = r \cos \theta V + \frac{A}{B}\dot{V}$ . In this case,  $\|\dot{V}\| = |B|$ ,  $\|\dot{x}\|^2 = A^2 + r^2 \cos^2 \theta$  and  $\|\nabla_{\dot{x}}\dot{x}\|^2 = B^2 (A^2 + r^2 \cos^2 \theta)$ , i.e.,  $\|\dot{V}\|$ ,  $\|\dot{x}\|$  and  $\|\nabla_{\dot{x}}\dot{x}\|$  are constants. In particular,  $x$  is a Riemannian circle in  $M^2(k)$ .

Notice that  $\dot{x}$  and  $V$  are collinear vectors if and only if the constant  $A$  is zero.

If  $\mu \neq 0$ , then system (30) leads to

$$\ddot{A}(s) + a^2k\mu^2A(s) = 0. \tag{32}$$

Because  $k < 0$ , the general solution of (32) is

$$A(s) = A_1 \exp(\lambda s) + A_2 \exp(-\lambda s), \tag{33}$$

where  $\lambda := \sqrt{-k}|a|\mu = \frac{1}{2\sqrt{-k}|a|r} [2r^2d \cos \theta + q] = q + \frac{2d}{\varphi} \cos \theta$ , and  $A_1, A_2$  are constants. We also obtain

$$B(s) = \sqrt{-k}(A_2 \exp(-\lambda s) - A_1 \exp(\lambda s)). \tag{34}$$

The arc-length condition (24) for  $\gamma$  yields

$$A_1A_2 = \frac{\sin^2 \theta}{4ak}.$$

We obtain then (4).

Conversely, we shall prove that if we have one of the conditions (1), (2), or (3), then (30) is satisfied.

- In the case (1) of the theorem,  $q = -\frac{2d}{\varphi} \cos \theta$  implies that  $\mu = 0$ . Because  $\dot{x}$  and  $V$  are parallel, then  $g(\dot{x}, V)$  is constant, i.e.,  $\theta$  is constant. On the other hand, since  $\|\dot{V}\| = 0$ , then  $B = 0$ , and since  $\dot{x}$  is a geodesic, then its speed is constant, i.e.,  $r^2 \cos^2 \theta + A^2(s)$  is constant, and hence  $A$  is constant. We deduce that (30) is satisfied.
- In the case (2) of the theorem, we also have  $\mu = 0$ . Because  $\|\dot{V}\|$  is constant, then  $B$  is constant. On the other hand,  $x$  is a Riemannian circle, i.e.,  $\|\nabla_{\dot{x}} \dot{x}\|$  is constant. We deduce that  $B^2(r^2 \cos^2 \theta + A^2(s))$  is constant. Because  $\theta$  and  $B$  are constant, then  $A$  is constant, and, consequently, (30) is satisfied.
- In the case (3) of the theorem, it is easy to check that  $A$  and  $B$  given by (33) and (34), respectively, satisfy (30).

□

Examples in  $\mathbb{H}^2(-4)$

We conclude this section drawing some pictures of magnetic curves along Riemannian circles that correspond to different values for the constants that appeared so far. The base manifold is  $M = \mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2; y > 0\}$ , equipped with the metric  $g = \frac{1}{4y^2}(dx^2 + dy^2)$ , and the constant  $\mu$  from the proof of the Theorem 3 is taken as equal to zero.

In [15], the authors give a complete description of Riemannian circles on  $\mathbb{H}^2(-4)$ . Using their results, we consider the normal Riemannian circle on  $\mathbb{H}^2(-4)$  given by the parametrization

$$x(s) = (2R \sin(\mu(s)), 2 - 2 \cos(\mu(s))), \tag{35}$$

where  $\dot{\mu}(s) + 2 \cos(\mu(s)) = 2$ . It is easy to see that  $\dot{x}(s) = (2\dot{\mu}(s) \cos(\mu(s)), 2\dot{\mu}(s) \sin(\mu(s)))$ , so that

$$\nabla_{\dot{x}} \dot{x} = (-4\dot{\mu}(s) \sin(\mu(s)), 4\dot{\mu}(s) \cos(\mu(s))).$$

We deduce that  $B^2 = \|\nabla_{\dot{x}} \dot{x}\|^2 = 4$ .

Moreover, we recall that

$$\begin{cases} \dot{x} = r \cos \theta V + An(s), \\ \nabla_{\dot{x}} \dot{x} = -ABV(s) + Br \cos \theta n(s). \end{cases}$$

Here,  $\|\dot{x}\|^2 = A^2 + r^2 \cos^2 \theta = 1$ . Thus, we obtain

$$V = \frac{1}{A^2 + r^2 \cos^2 \theta} \left( r \cos \theta \dot{x} - \frac{A}{B} \nabla_{\dot{x}} \dot{x} \right) = r \cos \theta \dot{x} - \frac{A}{B} \nabla_{\dot{x}} \dot{x}, \tag{36}$$

that is

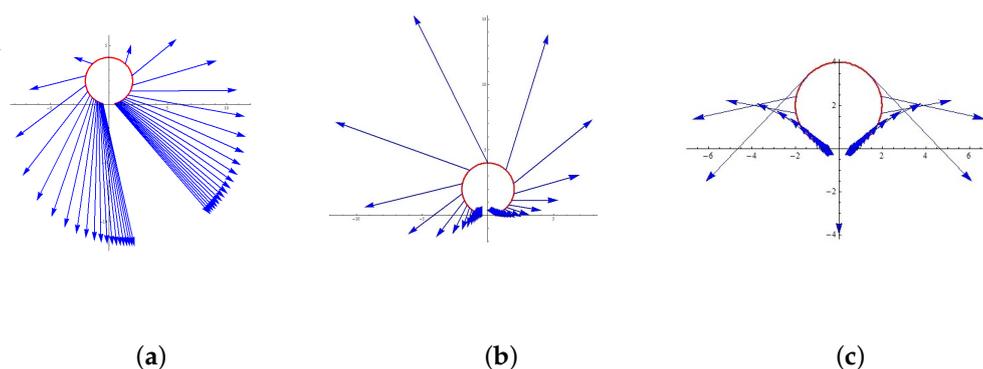
$$V = 2\dot{\mu}(s)(r \cos(\theta) \cos(\mu(s)) + 2\frac{A}{B} \sin(\mu(s))), r \cos(\theta) \sin(\mu(s)) - 2\frac{A}{B} \cos(\mu(s))), \tag{37}$$

where  $a \neq 0$ ,  $c = -5a$ ,  $d = 4a(4a + 1)$ ,  $r = \pm \frac{1}{4a}$ ,  $B = \pm 2$ ,  $q = (\frac{4a+1}{4a}) \cos(\theta)$  and  $\cos^2(\theta) = 16a^2(1 - A^2)$ .

To visualize graphically some slant natural paracontact magnetic trajectories on the unit tangent bundle of  $\mathbb{H}^2(-4)$  along Riemannian circles, Figure 1 below presents the base curve  $x$  on

$\mathbb{H}^2(-4)$ , together with the vector field  $V$  along it, in the three following situations that correspond to different values for the constants that appeared so far:

- (a)  $a = -\frac{1}{4}, c = \frac{5}{4}, d = 0, \alpha = -\frac{1}{4}, r = 1, \theta = \frac{\pi}{4}, B = -2$  and  $q = 0$  (Figure 1a);
- (b)  $a = \frac{1}{4}, c = -\frac{5}{4}, d = 2, \alpha = -\frac{1}{4}, r = 1, \theta = \frac{\pi}{4}$  and  $q = \frac{1}{\sqrt{2}}$  (Figure 1b);
- (c)  $a \neq 0, c = -5a, d = 4a(4a + 1), r = \pm\frac{1}{4a}, B = 2, \theta = \frac{\pi}{2}, A = 1$  and  $q = 0$  (Figure 1c).



**Figure 1.** Examples on  $\mathbb{H}^2(-4)$ . Curves (a–c) are slant geodesics.

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