A New Common Fixed Point Theorem for Three Commuting Mappings

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Abstract: In the present paper, we propose a common fixed point theorem for three commuting mappings via a new contractive condition which generalizes fixed point theorems of Darbo, Hajji and Aghajani et al. An application is also given to illustrate our main result. Moreover, several consequences are derived, which are generalizations of Darbo's fixed point theorem and a Hajji's result.

1. Introduction and Preliminaries

Schauder's fixed point theorem [1] plays a crucial role in nonlinear analysis. Namely, Schauder [1] has proved that if a self-mapping T is continuous on compact and convex subset of Banach spaces, then T has at least one fixed point. In 1955, Darbo [2] has generalized the classical Schauder's fixed point theorem for α-set contraction that is, such that

$$\alpha(T(A)) \leq k\alpha(A), \ \text{with} \ k \in [0,1),$$

on a closed, bounded and convex subsets of Banach spaces. Since then, many interesting works have appeared. For example, in 1967, Sadovskii [3] proved the fixed point property for condensing functions on a closed, bounded and convex subset of Banach spaces, that is, those satisfying

$$\alpha(T(A)) < \alpha(A), \ \text{with} \ \alpha(A) \neq 0.$$

It should be noted that any α-set contraction is a condensing function, but the converse is not true in general (see Reference [4]). In 2007, Hajji and Hanebaly [5] have extended the above contractive conditions and show the existence of a common fixed point for commuting mappings satisfying

$$\alpha(T(A)) \leq k \sup_{i \in I}(\alpha(S_i(A))), \ \alpha(T(A)) < k \sup_{i \in I}(\alpha(S_i(A)), \alpha(A)),$$

on a closed, bounded and convex subset Ω of a locally convex space. Here, S_i and T are continuous functions from Ω into itself, with S_i are affine or linear. In 2013, Hajji [6] established a common fixed point theorems for commuting mappings verifying

$$\alpha(ST(A)) \leq k\alpha(A), \ \alpha(ST(A)) < \alpha(A),$$

which generalize Darbo's and Sadovskii's fixed point theorems. Furthermore, as examples and applications, he studied the existence of common solutions of equations in Banach spaces using the

measure of noncompactness. Recently, in Reference [7], we made use of some axioms of measure of noncompactness to establish the following contractive condition

$$\sigma(H(A)) \leq \varphi(S(A)) - \varphi(S(\overline{\text{conv}}(T(A)))),$$

giving rise to common fixed point theorem for three commuting and continuous mappings H, S and T on a closed, bounded and convex subset of Banach spaces, with H and S are affine. Here, σ satisfies some properties of the measure of noncompactness while the conditions on φ are not needed. For particular choices of φ, σ, H and S Darbo's fixed point theorem can be obtained. As illustration, we have provided a concrete example for which both the classical Darbo's theorem and its generalization due to Hajji [6] are not applicable.

The aim of this paper is to prove the existence of a common fixed point for three mappings H, S and T satisfying the following new contraction

$$\xi(\sigma(H(A))) \leq \varphi(A) - \varphi(\overline{\text{conv}}(ST(A))).$$

Our result generalizes the theorems of Darbo [2], Hajji [6], and Aghajani et al. [8]. As an application, we study the existence of common solutions of the following equations

(1) $u(t) = f(t, Su(t))$,
(2) $u(t) = f(t, u(t))$,
(3) $u(t) = Hu(t)$,
(4) $u(t) = \lambda Hu(t) + (1 - \lambda)f(t, Su(t))$, $\lambda \in [0, 1]$,

under appropriate assumptions on functions S, H, f and ξ. Motivated by contractive conditions investigated in b-metric spaces [9–11] and using a measure of noncompactness, we derive from our main theorem some consequences, which are generalizations of Darbo's fixed point theorem [2] and a Hajji's result [6].

The paper is outlined as follows. Section 2 presents the main result with its proof. An application is provided in Section 3. Finally, several consequences on fixed point results are given in Section 4.

We conclude this introductory section by fixing some notations and recalling basic definitions that will be needed in the sequel. Denote by $\mathbb{N}$ the set of nonnegative integers and put $\mathbb{R}_+ = [0, +\infty)$. Let $(X, \|.\|)$ be a given Banach space. The symbols $\overline{A}$ and $\text{conv}(A)$ stand for the closure and the convex hull of A, respectively. Moreover, we denote by $\mathfrak{M}_X$ the family of all nonempty and bounded subsets of X and by $\mathfrak{N}_X$ its subfamily consisting of all relatively compact sets.

Definition 1 ([12]). *A mapping $\mu : \mathfrak{M}_X \to \mathbb{R}_+$ is called a measure of noncompactness in X if it satisfies the following conditions:*
(i) The family $\ker \mu = \{A \in \mathfrak{M}_X : \mu(A) = 0\}$ is nonempty and $\ker \mu \subseteq \mathfrak{N}_X$.
(ii) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$.
(iii) $\mu(\overline{A}) = \mu(A)$.
(iv) $\mu(\text{conv}(A)) = \mu(A)$.
(v) $\mu(\lambda A + (1 - \lambda)B) \leq \lambda\mu(A) + (1 - \lambda)\mu(B)$, for any $\lambda \in [0, 1]$.
(vi) If $(A_n)_n$ is a sequence of closed sets from $\mathfrak{M}_X$ such that $A_{n+1} \subseteq A_n$ for $n = 1, 2, \cdots$, and if $\lim_{n \to +\infty} \mu(A_n) = 0$, then the set $A_\infty = \cap_{n=1}^{+\infty} A_n$ is nonempty.
The family $\ker \mu$ defined in axiom (i) is called the kernel of the measure of noncompactness.

Definition 2 ([13]). *An operator S on a convex set A is said to be affine if it satisfies the identity*

$$S(kx + (1 - k)y) = kSx + (1 - k)Sy,$$

whenever $0 < k < 1$, *and* $x, y \in A$.

2. Main Result

In this section, we present and prove our main result on a common fixed point for three commuting operators. We also deduce from the obtained result a corollary which belongs to the classical metric fixed point theory.

Theorem 1. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and let T, S and H be three continuous and commuting mappings from Ω into itself. Assume that the following conditions are satisfied*
(a) H is affine.
(b) For any nonempty subset A of Ω, we have

$$\xi(\sigma(H(A))) \leq \varphi(A) - \varphi(\overline{\text{conv}}(ST(A))), \tag{1}$$

where $\sigma, \varphi \colon P(\Omega) \to \mathbb{R}_+$ are mappings such that σ satisfies properties (i), (ii), (iii) and (vi) of Definition 1 and $\xi \colon [0, +\infty) \to [0, +\infty)$ is lower semicontinuous function, $\xi(0) = 0$ and $\xi(t) > 0$ for all $t > 0$. Then
(1) If for any nonempty subset A of Ω, we have $S(\overline{\text{conv}}(A)) \subseteq \overline{\text{conv}}(S(A))$, then ST, S and H have a fixed point in Ω.
(2) If S is affine, then T, S and H have a common fixed point in Ω.

Proof. (1) Consider the sequence $\{\Omega_n\}$ defined as

$$\begin{cases} \Omega_0 = \Omega, \\ \Omega_n = \overline{\text{conv}}(ST(\Omega_{n-1})), \quad n = 1, 2, \cdots. \end{cases}$$

Define $\omega_n = \varphi(\Omega_n)$. From inequality (1), we have

$$\varphi(\Omega_n) - \varphi(\overline{\text{conv}}(ST(\Omega_n))) \geq 0, \quad \text{for all } n \in \mathbb{N}.$$

It implies that

$$\omega_{n+1} = \varphi(\Omega_{n+1}) = \varphi(\overline{\text{conv}}(ST(\Omega_n))) \leq \varphi(\Omega_n) = \omega_n, \quad \text{for all } n \in \mathbb{N}.$$

Hence, $\{\omega_n\}$ is a non-increasing sequence of positive real numbers, so it converges to some $\omega \geq 0$ as n tends to +infinity. Using inequality (1) again, we get

$$\xi(\sigma(H(\Omega_n))) \leq \varphi(\Omega_n) - \varphi(\overline{\text{conv}}(ST(\Omega_n))) = \omega_n - \omega_{n+1}.$$

This yields

$$\limsup_{n \to +\infty} \omega_{n+1} \leq \limsup_{n \to +\infty} \omega_n - \liminf_{n \to +\infty} \xi(\sigma(H(\Omega_n))). \tag{2}$$

The rest of the proof needs to show that the sequence $\{\Omega_n\}$ is nested. Indeed,
for $n = 1$, we have $\Omega_1 \subseteq \Omega_0$. Suppose that $\Omega_n \subseteq \Omega_{n-1}$ is true for some $n \geq 1$. Then,

$$\Omega_{n+1} = \overline{\text{conv}}(ST(\Omega_n)) \subseteq \overline{\text{conv}}(ST(\Omega_{n-1})) = \Omega_n.$$

By induction, we get $\Omega_n \subseteq \Omega_{n-1}$ for every $n \geq 1$. It follows that $H(\Omega_n) \subseteq H(\Omega_{n-1})$ for every $n \geq 1$. In view of (ii) in Definition 1, $\{\sigma(H(\Omega_n))\}$ is a positive non-increasing sequence of real numbers,

we deduce that $\sigma(H(\Omega_n)) \to r$ when n tends to +infinity, where $r \geq 0$. Then, from inequality (2), we get

$$\omega \leq \omega - \xi(r).$$

Therefore $\xi(r) = 0$ and so $\lim\limits_{n \to +\infty} \sigma(H(\Omega_n)) = r = 0$. Now, if we set $\Omega'_n = \overline{H(\Omega_n)}$, we can make use of (iii) of Definition 1, to show

$$\sigma(\Omega'_n) = \sigma(\overline{H(\Omega_n)}) = \sigma(H(\Omega_n)),$$

which implies that $\lim\limits_{n \to +\infty} \sigma(\Omega'_n) = 0$. Since the sequence $\{\Omega_n\}$ is nested, we have $\Omega'_{n+1} \subseteq \Omega'_n$ for all $n \in \mathbb{N}$. Consequently, by the axiom (vi) of Definition 1 $\Omega'_\infty = \cap_{n=1}^{+\infty} \Omega'_n$ is nonempty. In addition, from (ii) of Definition 1, we obtain

$$\sigma(\Omega'_\infty) \leq \sigma(\Omega'_n), \quad \text{for all } n \in \mathbb{N}.$$

Passing to the limit, we get $\sigma(\Omega'_\infty) = 0$, which together with property (i) of Definition 1 imply that $\overline{\Omega'_\infty} = \Omega'_\infty$ is compact and convex since H is affine. Note also that $ST(\Omega_n) \subseteq \Omega_n$. Indeed,

$$ST(\Omega_n) \subseteq \overline{\text{conv}}(ST(\Omega_n)) \subseteq \overline{\text{conv}}(ST(\Omega_{n-1})) = \Omega_n, \quad n = 1, 2, \cdots.$$

For $n = 1$, we have

$$S(\Omega_1) = S(\overline{\text{conv}}(ST(\Omega_0))) \subseteq \overline{\text{conv}}(ST(S(\Omega_0))) \subseteq \overline{\text{conv}}(ST(\Omega_0)) = \Omega_1.$$

Assuming now that $S(\Omega_n) \subseteq \Omega_n$ is true for some $n \geq 1$. Then

$$S(\Omega_{n+1}) = S(\overline{\text{conv}}(ST(\Omega_n))) \subseteq \overline{\text{conv}}(ST(S(\Omega_n))) \subseteq \overline{\text{conv}}(ST(\Omega_n)) = \Omega_{n+1}.$$

By induction, we obtain $S(\Omega_n) \subseteq \Omega_n$. Similarly as for S, we can prove $H(\Omega_n) \subseteq \Omega_n$. So we get

$$ST(\Omega'_n) = ST(\overline{H(\Omega_n)}) \subseteq \overline{ST(H(\Omega_n))} \subseteq \overline{H(ST(\Omega_n))} \subseteq \overline{H(\Omega_n)} = \Omega'_n,$$

$$S(\Omega'_n) = S(\overline{H(\Omega_n)}) \subseteq \overline{H(S(\Omega_n))} \subseteq \overline{H(\Omega_n)} = \Omega'_n,$$

and

$$H(\Omega'_n) = H(\overline{H(\Omega_n)}) \subseteq \overline{H(H(\Omega_n))} \subseteq \overline{H(\Omega_n)} = \Omega'_n, \quad \text{for all } n \in \mathbb{N}.$$

Therefore, $ST(\Omega'_\infty) \subseteq \Omega'_\infty$, $S(\Omega'_\infty) \subseteq \Omega'_\infty$ and $H(\Omega'_\infty) \subseteq \Omega'_\infty$. Thus, applying Schauder's fixed point theorem leads us to conclude that ST, S and H have a fixed point.

(2) By the same argument as in part (3) of the proof of Theorem 2.1 [7], we see that $E = \{x \in \Omega : H(x) = x\}$ is convex, closed and bounded subset of Ω, $S(E) \subseteq E$ and $H(E) \subseteq E$. Furthermore, from inequality (1), we have

$$\xi(\sigma(H(A))) \leq \varphi(A) - \varphi(\overline{\text{conv}}(ST(A))), \quad \text{for every } A \subseteq E.$$

Then by part (1), the mapping S has a fixed point in E and therefore S and H have a common fixed point. In a similar way, we can show that T has a fixed point in $F = \{x \in \Omega : S(x) = H(x) = x\}$. Thus, S, H and T have a common fixed point. $\square$

Remark 1. *By letting H and ξ be the identity mappings, and taking $\sigma = \mu$ and $\varphi = (\frac{1}{1-k})\mu$, where μ is a measure of noncompactness and $k \in [0, 1)$, one can deduce Hajji's fixed point theorem [6] and when we take furthermore S the identity mapping, we obtain Darbo's fixed point theorem [2].*

Taking H and S the identity mappings, $\sigma = \mu$ and $\varphi = \psi \circ \mu$, in Theorem 1, we obtain the following result due to Aghajani et al. [8].

Theorem 2. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and let $T : \Omega \to \Omega$ be a continuous mapping such that*

$$\psi(\mu(T(A))) \leq \psi(\mu(A)) - \xi(\mu(A)),$$

for any nonempty subset A of Ω, where μ is an arbitrary measure of noncompactness and $\xi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ are given functions such that ξ is lower semicontinuous and ψ is continuous. Moreover, $\xi(0) = 0$ and $\xi(t) > 0$ for $t > 0$. Then, T has at least one fixed point in Ω.

Now, let us pay attention to the following corollary from the Theorem 1.

Corollary 1. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and let T, S and H be commuting and continuous mappings from Ω into itself such that*
(a) H and S are affine.
(b) For all $u, v \in \Omega$, we have

$$\xi(\|H(u) - H(v)\|) \leq \alpha(u,v) - \alpha(ST(u), ST(v)), \tag{3}$$

where $\alpha : \Omega \times \Omega \to \mathbb{R}_+$ is a mapping and $\xi : [0, +\infty) \to [0, +\infty)$ is lower semicontinuou and bounded function such that $\xi(0) = 0$ and $\xi(t) > 0$ for all $t > 0$. Then, the set $\{u \in \Omega : T(u) = S(u) = H(u) = u\}$ is nonempty.

Proof. Let $\sigma : \mathfrak{M}_X \to [0, +\infty)$ be a set quantity defined by the formula

$$\sigma(A) = \mathrm{diam}(A),$$

where $\mathrm{diam}(A) = \sup\{\|u - v\| : u, v \in A\}$ stands for the diameter of A. It is easily seen that σ is a measure of noncompactness in X. Thus, in view of (3), we have

$$\inf_{u,v \in A} \alpha(ST(u), ST(v)) \leq \inf_{u,v \in A} \alpha(u,v) - \sup_{u,v \in A} \xi(\|H(u) - H(v)\|).$$

This yields that

$$\xi(\sup_{u,v \in A} \|H(u) - H(v)\|) \leq \inf_{u,v \in A} \alpha(u,v) - \inf_{u,v \in A} \alpha(ST(u), ST(v)).$$

Set $\varphi(A) = \inf_{u,v \in A} \alpha(u,v)$. Hence,

$$\xi(\mathrm{diam}(H(A))) \leq \varphi(A) - \varphi(ST(A))$$

and therefore

$$\xi(\sigma(H(A))) \leq \varphi(A) - \varphi(ST(A)).$$

Thus, the desired result is obtained by Theorem 1. $\quad\square$

3. Application

This section is concerned with the existence problem of common solutions for the following equations:

(1) $u(t) = f(t, Su(t))$,
(2) $u(t) = f(t, u(t))$,
(3) $u(t) = Hu(t)$,

(4) $u(t) = \lambda H u(t) + (1 - \lambda) f(t, Su(t))$, $\lambda \in [0, 1]$,

under some appropriate assumptions on the functions f, S and H. Let $(E, \|.\|)$ be a Banach space and B be a convex, closed and bounded subset of E. Denote by $C([0, b], B)$ the space of all continuous functions from $[0, b]$; $b > 0$, into B endowed with the norm $\|u\|_\infty = \sup_{t \in [0, b]} \|u(t)\|$.
Assume that
(a) $S, H \colon B \to B$ are linear continuous functions.
(b) $f \colon [0, b] \times B \to B$ is continuous function such that

$$\|Ha - Hb\| \leq \gamma(a, b) - \gamma(S(f(t, a)), S(f(t, b))), \tag{4}$$

for all $a, b \in B$ and $t \in [0, b]$, where $\gamma \colon B \times B \to \mathbb{R}_+$ is a mapping.
(c) For any $(t, a) \in [0, b] \times B$

$$S(f(t, a)) = f(t, S(a)) \quad \text{and} \quad H(f(t, a)) = f(t, H(a)).$$

Theorem 3. *Under hypotheses (a), (b), and (c), equations (1), (2), (3), and (4) have at least one common solution in $C([0, b], B)$.*

Proof. It is clear that $C([0, b], B)$ is a closed, bounded and convex subset of $C([0, b], X)$. On the other hand, by considering $Tu(t) = f(t, u(t))$, for $u \in C([0, b], B)$, we obtain that

$$\|Hu(t) - Hv(t)\| \leq \gamma(u(t), v(t)) - \gamma(STu(t), STv(t)).$$

It follows that

$$\inf_{t \in [0, b]} \gamma(STu(t), STv(t)) \leq \inf_{t \in [0, b]} \gamma(u(t), v(t)) - \sup_{t \in [0, b]} \|Hu(t) - Hv(t)\|.$$

Define $\alpha(u, v) = \inf_{t \in [0, b]} \gamma(u(t), v(t))$. Then,

$$\|H(u) - H(v)\|_\infty \leq \alpha(u, v) - \alpha(ST(u), ST(v)).$$

So by taking ξ the identity function, we get

$$\xi(\|H(u) - H(v)\|_\infty) \leq \alpha(u, v) - \alpha(ST(u), ST(v)),$$

for any $u, v \in C([0, b], B)$. Finally, since T, S and H commute, we conclude from Corollary 1 that T, S and H have a common fixed point in $C([0, b], B)$. Therefore, equations (1), (2), (3), and (4) have at least one common solution in $C([0, b], B)$. □

4. Consequences

In this section, we establish several consequences of our main result.

Corollary 2. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and $T, S \colon \Omega \to \Omega$ are continuous mappings satisfying:*
(a) S is affine.
(b) $TS = ST$.
(c) For any nonempty subset A of Ω, we have

$$\mu(ST(A)) \leq \eta(\mu(A)), \tag{5}$$

where μ is a measure of noncompactness defined in X and $\eta \colon [0, +\infty) \to [0, +\infty)$ is function such that $\eta(t) < t$ for each $t > 0$ and $\frac{\eta(t)}{t}$ is non-decreasing. Then, the set $\{u \in \Omega : T(u) = S(u) = u\}$ is nonempty.

Proof. Taking H and ξ as the identity functions and $\varphi(A) = \frac{\mu(A)}{1 - \frac{\eta(\mu(A))}{\mu(A)}}$, if $\mu(A) \neq 0$ and $\varphi(A) = 0$, otherwise. Then (5) schows that

First case, if $\mu(A) \neq 0$, we have

$$\mu(ST(A)) \leq \mu(A) - \left[1 - \frac{\eta(\mu(A))}{\mu(A)})\right]\mu(A).$$

It implies that

$$\frac{\mu(ST(A))}{1 - \frac{\eta(\mu(A))}{\mu(A)}} \leq \frac{\mu(A)}{1 - \frac{\eta(\mu(A))}{\mu(A)}} - \mu(A).$$

Since $\frac{\eta(t)}{t}$ is non-decreasing and $\mu(ST(A)) < \mu(A)$, we have to distinguish two subcases
(a) If $\mu(ST(A)) \neq 0$, then

$$\frac{\mu(ST(A))}{1 - \frac{\eta(\mu(ST(A)))}{\mu(ST(A))}} \leq \frac{\mu(A)}{1 - \frac{\eta(\mu(A))}{\mu(A)}} - \mu(A),$$

so

$$\xi(\mu(H(A))) \leq \varphi(A) - \varphi(ST(A)).$$

(b) If $\mu(ST(A)) = 0$, we have $\varphi(ST(A)) = \mu(ST(A)) = 0$. On the other hand, we see that

$$1 - \frac{\eta(\mu(A))}{\mu(A)} \leq 1.$$

It means that

$$1 \leq \frac{1}{1 - \frac{\eta(\mu(A))}{\mu(A)}},$$

and so

$$\mu(A) \leq \frac{\mu(A)}{1 - \frac{\eta(\mu(A))}{\mu(A)}},$$

Consequently, we get

$$\xi(\mu(H(A))) \leq \varphi(A) - \varphi(ST(A)).$$

Now, if $\mu(A) = 0$, from assertions (i), (ii) and the fact that T and S are continuous, we have $\mu(ST(A)) = 0$, so

$$\xi(\mu(H(A))) \leq \varphi(A) - \varphi(ST(A)).$$

Then, by Theorem 1, T has a fixed point in Ω. $\square$

Remark 2. *1. Note that taking S the identity function and $\eta(t) = kt$ for all $t \in [0, +\infty)$ with $k \in [0, 1)$ gives Darbo's fixed point theorem.*
2. Taking $\eta(t) = kt$ for all $t \in [0, +\infty)$ with $k \in [0, 1)$, then Corollary 2 is a generalization of the Theorem 3.1 due to Hajji [6].

The above result gives rise to two corollaries, which are also generalizations of the both theorems due to Darbo [2] and Hajji [6].

Corollary 3. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and let T, $S \colon \Omega \to \Omega$ are continuous mappings such that*

(a) S is affine
(b) TS = ST.
(c) For any nonempty subset A of Ω, we have

$$\mu(ST(A)) \leq \mu(A) - \theta(\mu(A)), \tag{6}$$

where μ is a measure of noncompactness defined in X and $\theta\colon (0, +\infty) \to (0, +\infty)$ is function such that $\frac{\theta(t)}{t}$ is non-increasing. Then the set $\{u \in \Omega : T(u) = S(u) = u\}$ is nonempty.

Proof. Let $\eta(t) = t - \theta(t)$, for each $t > 0$. Then $\eta(t) < t$, for each $t > 0$ and $\frac{\eta(t)}{t} = 1 - \frac{\theta(t)}{t}$ is non-decreasing. Thus, the result is obtained by making use of Corollary 2. □

Remark 3. *1. For S being the identity function and $\theta(t) = (1 - k)t$ for all $t > 0$ with $k \in [0, 1)$, a generalization of Darbo's fixed point theorem, is obtained.*
2. By letting $\theta(t) = (1 - k)t$ for all $t > 0$ with $k \in [0, 1)$, we recover the Theorem 3.1 due to Hajji [6].

Corollary 4. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and T, $S\colon \Omega \to \Omega$ are continuous operators such that*
(a) S is affine.
(b) TS = ST.
(c) For any nonempty subset A of Ω, we have

$$\mu(ST(A)) \leq \phi(\mu(A))\mu(A), \tag{7}$$

where μ is a measure of noncompactness defined in X and $\phi\colon [0, +\infty) \to [0, 1)$ is a non-decreasing function. Then, the set $\{u \in \Omega : T(u) = S(u) = u\}$ is nonempty.

Proof. Let $\eta(t) = \phi(t)t$ for all $t > 0$. Then $\eta(t) < t$ for all $t > 0$ and $\frac{\eta(t)}{t} = \phi(t)$ is non-decreasing. Therefore by Corollary 2, T has a fixed point. □

Remark 4. *1. By taking S the identity function and $\phi(t) = k$ for all $t \in [0, +\infty)$ with $k \in [0, 1)$, it not hard to see that the Corollary 4 is a generalization of Darbo's fixed point theorem.*
2. For the specific function $\phi(t) = k$ for all $t \in [0, +\infty)$ with $k \in [0, 1)$, we obtain the Theorem 3.1 due to Hajji [6].

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