axioms
MDPI

*Article*

# On the Solvability of Nonlinear Third-Order Two-Point Boundary Value Problems

**Ravi P. Agarwal [1,*], Petio S. Kelevedjiev [2] and Todor Z. Todorov [2]**

1 Department of Mathematics, Texas A and M University-Kingsville, Kingsville, TX 78363-8202, USA
2 Department of Mathematics, Technical University of Sliven, 8800 Sliven, Bulgaria; pskeleved@abv.bg (P.S.K.); tjtodorov@abv.bg (T.Z.T.)
* Correspondence: agarwal@tamuk.edu

Received: 21 April 2020; Accepted: 23 May 2020; Published: 31 May 2020

**Abstract:** Under barrier strips type assumptions we study the existence of $C^3[0,1]$—solutions to various two-point boundary value problems for the equation $x''' = f(t, x, x', x'')$. We give also some results guaranteeing positive or non-negative, monotone, convex or concave solutions.

**Keywords:** third-order differential equation; boundary value problem; existence; sign conditions

**MSC:** 34B15; 34B18

## 1. Introduction

In this paper, we study the solvability of boundary value problems (BVPs) for the differential equation

$$x''' = f(t, x, x', x''), t \in (0, 1), \tag{1}$$

with some of the boundary conditions

$$x(0) = A, x'(1) = B, x''(1) = C, \tag{2}$$
$$x(0) = A, x'(0) = B, x''(1) = C, \tag{3}$$
$$x(0) = A, x(1) = B, x''(1) = C, \tag{4}$$
$$x(0) = A, x'(0) = B, x'(1) = C, \tag{5}$$
$$x(1) = A, x'(0) = B, x'(1) = C, \tag{6}$$

where $f : [0, 1] \times D_x \times D_p \times D_q \to \mathbb{R}$, $D_x, D_p, D_q \subseteq \mathbb{R}$, and $A, B, C \in \mathbb{R}$.

The solvability of BVPs for third-order differential equations has been investigated by many authors. Here, we will cite papers devoted to two-point BVPs which are mostly with some of the above boundary conditions; in each of these works $A, B, C = 0$. Such problems for equations of the form

$$x''' = f(t, x), t \in (0, 1),$$

have been studied by H. Li et al. [1], S. Li [2] (the problem may be singular at $t = 0$ and/or $t = 1$), Z. Liu et al. [3,4], X. Lin and Z. Zhao [5], S. Smirnov [6], Q. Yao and Y. Feng [7]. Moreover, the boundary conditions in References [2,3] are (3), in Reference [4] they are (4), in References [1,5,7] they are (5), and in Reference [6] are

$$x(0) = x(1) = 0, x'(0) = C.$$

Y. Feng [8] and Y. Feng and S. Liu [9] have considered the equation

$$x''' = f(t, x, x'), t \in (0,1),$$

with (6) and (5), respectively. Y. Feng [10] and R. Ma and Y. Lu [11] have studied the equations

$$f(t, x, x', x''') = 0 \text{ and } x''' + Mx'' + f(t, x) = 0, t \in (0,1).$$

with (5). BVPs for the equation

$$x''' = f(t, x, x', x''), t \in (0,1),$$

have been investigated by A. Granas et al. [12], B. Hopkins and N. Kosmatov [13], Y. Li and Y. Li [14]; the boundary conditions in [12] are (5), these in Reference [13] are (2) and (3), and in Reference [14]—(2).

Results guaranteeing positive or non-negative solutions can be found in References [2–4,7–11,13,14], and results that guarantee negative or non-positive ones in References [7,9,10]. The existence of monotone solutions has been studied in References [3,7,9].

As a rule, the main nonlinearity is defined and continuous on a set such that each dependent variable changes in a left- and/or a right-unbounded set; in Reference [13] it is a Carathéodory function on an unbounded set. Besides, the main nonlinearity is monotone with respect to some of the variables in References [1,5], does not change its sign in References [2–4,14] and satisfies Nagumo type growth conditions in Reference [14]. Maximum principles have been used in References [8,10], Green's functions in References [1,2,4,5], and upper and lower solutions in References [1,7–11].

Here, we use a different tool—barrier strips which allow the right side of the equation to be defined and continuous on a bounded subset of its domain and to change its sign.

To prove our existence results we apply a basic existence theorem whose formulation requires the introduction of the BVP

$$x''' + a(t)x'' + b(t)x' + c(t)x = f(t, x, x', x''), t \in (0,1), \tag{7}$$

$$V_i(x) = r_i, i = 1,2,3 (i = \overline{1,3} \text{ for short}), \tag{8}$$

where $a, b, c \in C([0,1], \mathbb{R}), f : [0,1] \times D_x \times D_p \times D_q \to \mathbb{R}$,

$$V_i(x) = \sum_{j=0}^{2} [a_{ij} x^{(j)}(0) + b_{ij} x^{(j)}(1)], i = \overline{1,3},$$

with constants $a_{ij}$ and $b_{ij}$ such that $\sum_{j=0}^{2}(a_{ij}^2 + b_{ij}^2) > 0, i = \overline{1,3}$, and $r_i \in \mathbb{R}$, $i = \overline{1,3}$. Next, consider the family of BVPs for

$$x''' + a(t)x'' + b(t)x' + c(t)x = g(t, x, x', x'', \lambda), t \in (0,1), \lambda \in [0,1] \tag*{$(7)_\lambda$}$$

with boundary conditions (8), where $g$ is a scalar function defined $[0,1] \times D_x \times D_p \times D_q \times [0,1]$, and $a, b, c$ are as above. Finally, $BC$ denotes the set of functions satisfying boundary conditions (8), and $BC_0$ denotes the set of functions satisfying the homogeneous boundary conditions $V_i(x) = 0, i = \overline{1,3}$. Besides, let $C^3_{BC}[0,1] = C^3[0,1] \cap BC$ and $C^3_{BC_0}[0,1] = C^3[0,1] \cap BC_0$.

The proofs of our existence results are based on the following theorem. It is a variant of Reference [12] (Chapter I, Theorem 5.1 and Chapter V, Theorem 1.2). Its proof can be found in Reference [15]; see also the similar result in Reference [16] (Theorem 4).

**Lemma 1.** *Suppose:*

*(i)* *Problem* $(7)_0$, (8) *has a unique solution* $x_0 \in C^3[0,1]$.

*(ii)* *Problems* (7), (8) *and* $(7)_1$, (8) *are equivalent.*

*(iii)* *The map* $\mathbf{L}_h : C^3_{BC_0}[0,1] \to C[0,1]$ *is one-to-one: here,*

$$\mathbf{L}_h x = x''' + a(t)x'' + b(t)x' + c(t)x.$$

*(iv)* *Each solution* $x \in C^3[0,1]$ *to family* $(7)_\lambda$, (8) *satisfies the bounds*

$$m_i \le x^{(i)} \le M_i \text{ for } t \in [0,1], i = \overline{0,3},$$

*where the constants* $-\infty < m_i, M_i < \infty, i = \overline{0,3}$, *are independent of* $\lambda$ *and* $x$.
*(v)* *There is a sufficiently small* $\sigma > 0$ *such that*

$$[m_0 - \sigma, M_0 + \sigma] \subseteq D_x, [m_1 - \sigma, M_1 + \sigma] \subseteq D_p, [m_2 - \sigma, M_2 + \sigma] \subseteq D_q,$$

*and* $g(t,x,p,q,\lambda)$ *is continuous for* $(t,x,p,q,\lambda) \in [0,1] \times J \times [0,1]$ *where* $J = [m_0 - \sigma, M_0 + \sigma] \times [m_1 - \sigma, M_1 + \sigma] \times [m_2 - \sigma, M_2 + \sigma]$; $m_i, M_i, i = \overline{0,3}$, *are as in (iv).*

*Then boundary value problem* (7), (8) *has at least one solution in* $C^3[0,1]$.

For us, the equation from $(7)_\lambda$ has the form

$$x''' = \lambda f(t,x,x',x''). \tag*{$(1)_\lambda$}$$

Preparing the application of Lemma 1, we impose conditions which ensure the a priori bounds from *(iv)* for the eventual $C^3[0,1]$ - solutions of the families of BVPs for $(7)_\lambda, \lambda \in [0,1]$, with one of the boundary conditions $(k)$, $k = \overline{2,6}$.

So, we will say that for some of the BVPs (1), $(k)$, $k = \overline{2,6}$, the conditions $(\mathbf{H_1})$ and $(\mathbf{H_2})$ hold for a $K \in \mathbb{R}$ (it will be specified later for each problem) if:

$(\mathbf{H_1})$ There are constants $F_i', L_i', i = 1,2$, such that

$$F_2' < F_1' \le K \le L_1' < L_2', [F_2', L_2'] \subseteq D_q,$$

$$f(t,x,p,q) \ge 0 \text{ for } (t,x,p,q) \in [0,1] \times D_x \times D_p \times [L_1', L_2'], \tag{9}$$
$$f(t,x,p,q) \le 0 \text{ for } (t,x,p,q) \in [0,1] \times D_x \times D_p \times [F_2', F_1']. \tag{10}$$

$(\mathbf{H_2})$ There are constants $F_i, L_i, i = 1,2$, such that

$$F_2 < F_1 \le K \le L_1 < L_2, [F_2, L_2] \subseteq D_q,$$

$$f(t,x,p,q) \le 0 \text{ for } (t,x,p,q) \in [0,1] \times D_x \times D_p \times [L_1, L_2],$$

$$f(t,x,p,q) \ge 0 \text{ for } (t,x,p,q) \in [0,1] \times D_x \times D_p \times [F_2, F_1].$$

Besides, we will say that for some of the BVPs (1), $(k)$, $k = \overline{2,6}$, the condition $(\mathbf{H_3})$ holds for constants $m_i \le M_i, i = \overline{0,2}$, (they also will be specified later for each problem) if:

$(\mathbf{H_3})$ $[m_0 - \sigma, M_0 + \sigma] \subseteq D_x$, $[m_1 - \sigma, M_1 + \sigma] \subseteq D_p$, $[m_2 - \sigma, M_2 + \sigma] \subseteq D_q$ and $f(t,x,p,q)$ is continuous on the set $[0,1] \times J$, where $J$ is as in $(v)$ of Lemma 1, and $\sigma > 0$ is sufficiently small.

In fact, the present paper supplements P. Kelevedjiev and T. Todorov [15] where only conditions $(\mathbf{H_2})$ and $(\mathbf{H_3})$ have been used for studying the solvability of various BVPs for (1) with other boundary conditions. Here, $(\mathbf{H_1})$ is also needed. Now, only $(\mathbf{H_1})$ guarantees the a priori bounds for $x''(t), x'(t)$ and $x(t)$, in this order, for each eventual solution $x \in C^3[0,1]$ to the families $(1)_\lambda, (k), k = \overline{2,4}$, and $(\mathbf{H_1})$ and $(\mathbf{H_2})$ together guarantee these bounds for the families $(1)_\lambda$, $(k)$, $k = 5,6$. As in Reference [15], $(\mathbf{H_3})$ gives the bounds for $x'''(t)$.

The auxiliary results which guarantee a priori bounds are given in Section 2, and the existence theorems are in Section 3. The ability to use $\mathbf{(H_1)}$ and $\mathbf{(H_2)}$ for studying the existence of solutions with important properties is shown in Appendix A. Examples are given in Section 4.

## 2. Auxiliary Results

This part ensures a priori bounds for the eventual $C^3[0,1]$-solutions of each family $(1)_\lambda$, $(k), k=\overline{2,6}$, that is, it ensures the constants $m_i, M_i, i=\overline{0,2}$, from *(iv)* of Lemma 1 and $\mathbf{(H_3)}$.

**Lemma 2.** *Let $x \in C^3[a,b]$ be a solution to $(1)_\lambda$. Suppose* $\mathbf{(H_1)}$ *holds with $[0,1]$ replaced by $[a,b]$ and $K = x''(b)$. Then*

$$F_1' \le x''(t) \le L_1' \text{ on } [a,b].$$

**Proof.** By contradiction, assume that $x''(t) > L_1'$ for some $t \in [a,b)$. This means that the set

$$S_+ = \{t \in [a,b] : L_1' < x''(t) \le L_2'\}$$

is not empty because $x''(t)$ is continuous on $[a,b]$ and $x''(b) \le L_1'$. Besides, there is a $\gamma \in S_+$ such that

$$x'''(\gamma) < 0.$$

As $x(t)$ is a $C^3[a,b]$—solution to $(1)_\lambda$,

$$x'''(\gamma) = \lambda f(\gamma, x(\gamma), x'(\gamma), x''(\gamma)).$$

But, $(\gamma, x(\gamma), x'(\gamma), x''(\gamma)) \in S_+ \times D_x \times D_p \times (L_1', L_2']$ and (9) imply

$$x'''(\gamma) \ge 0,$$

a contradiction. Consequently,

$$x''(t) \le L_1' \text{ for } t \in [a,b].$$

Along similar lines, assuming on the contrary that the set

$$S_- = \{t \in [a,b] : F_2' \le x''(t) < F_1'\}$$

is not empty and using (10), we achieve a contradiction which implies that

$$F_1' \le x''(t) \text{ for } t \in [a,b].$$

□

The proof of the next assertion is virtually the same as that of Lemma 2 and is omitted; it can be found in [15].

**Lemma 3.** *Let $x \in C^3[a,b]$ be a solution to $(1)_\lambda$. Suppose* $\mathbf{(H_2)}$ *holds with $[0,1]$ replaced by $[a,b]$ and $K = x''(a)$. Then*

$$F_1 \le x''(t) \le L_1 \text{ on } [a,b].$$

Let us recall, conditions of type $\mathbf{(H_1)}$ and $\mathbf{(H_2)}$ are called barrier strips, see P. Kelevedjiev [17]. As can we see from Lemmas 2 and 3 they control the behavior of $x''(t)$ on $[a,b]$, depending on the sign of $f(t,x,x',x'')$ the curve of $x''(t)$ on $[a,b]$ crosses the strips $[a,b] \times [L_1', L_2']$, $[a,b] \times [L_1, L_2]$, $[a,b] \times [F_2', F_1']$ and $[a,b] \times [F_2, F_1]$ not more than once. This property ensures the a priori bounds for $x''(t)$.

**Lemma 4.** *Let* **(H$_1$)** *hold for* $K = C$. *Then every solution* $x \in C^3[0,1]$ *to* (1)$_\lambda$, (2) *or* (1)$_\lambda$, (3) *satisfies the bounds*

$$|x(t)| \le |A| + |B| + \max\{|F_1'|, |L_1'|\}, t \in [0,1],$$

$$|x'(t)| \le |B| + \max\{|F_1'|, |L_1'|\}, t \in [0,1],$$

$$F_1' \le x''(t) \le L_1', t \in [0,1]. \tag{11}$$

**Proof.** Let first $x(t)$ be a solution to (1)$_\lambda$, (2). Using Lemma 2 we conclude that (11) is true. Then, according to the mean value theorem, for each $t \in [0,1)$ there is a $\xi \in (t,1)$ such that

$$x'(1) - x'(t) = x''(\xi)(1-t),$$

which together with (11) gives the bound for $|x'(t)|$. Again from the mean value theorem for each $t \in (0,1]$ there is an $\eta \in (0,t)$ with the property

$$x(t) - x(0) = x'(\eta)t,$$

which yields the bound for $|x(t)|$. The assertion follows similarly for (1)$_\lambda$, (3). □

**Lemma 5.** *Let* **(H$_1$)** *hold for* $K = C$. *Then every solution* $x \in C^3[0,1]$ *to* (1)$_\lambda$, (4) *satisfies the bounds*

$$|x(t)| \le |A| + |B - A| + \max\{|F_1'|, |L_1'|\},\ t \in [0,1],$$

$$|x'(t)| \le |B - A| + \max\{|F_1'|, |L_1'|\},\ t \in [0,1],$$

$$F_1' \le x''(t) \le L_1', t \in [0,1].$$

**Proof.** By Lemma 2, $F_1' \le x''(t) \le L_1'$ on $[0,1]$. Clearly, there is a $\mu \in (0,1)$ for which $x'(\mu) = B - A$. Further, for each $t \in [0,\mu)$ there is a $\xi \in (t,\mu)$ such that

$$x'(\mu) - x'(t) = x''(\xi)(\mu - t),$$

from where, using the obtained bounds for $x''(t)$, we get

$$|x'(t)| \le |B - A| + \max\{|F_1'|, |L_1'|\},\ t \in [0,\mu].$$

We can proceed analogously to see that the same bound is valid for $t \in [\mu,1]$. Finally, for each $t \in (0,1]$ there is an $\eta \in (0,t)$ such that

$$x(t) - x(0) = x'(\eta)t,$$

which together with the obtained bound for$|x'(t)|$ yields the bound for $|x(t)|$. □

**Lemma 6.** *Let* **(H$_1$)** *and* **(H$_2$)** *hold for* $K = C - B$. *Then every solution* $x \in C^3[0,1]$ *to* (1)$_\lambda$, (5) *or* (1)$_\lambda$, (6) *satisfies the bounds*

$$|x(t)| \le |A| + |B| + \max\{|F_1|, |L_1|, |F_1'|, |L_1'|\}, t \in [0,1],$$

$$|x'(t)| \le |B| + \max\{|F_1|, |L_1|, |F_1'|, |L_1'|\}, t \in [0,1],$$

$$\min\{F_1, F_1'\} \le x''(t) \le \max\{L_1, L_1'\}, t \in [0,1].$$

**Proof.** Let $x(t)$ be a solution to $(1)_\lambda$, (5); the proof is similar for $(1)_\lambda$, (6). We know there is a $\nu \in (0,1)$ for which $x''(\nu) = C - B$. Then, applying Lemmas 2 and 3 on the intervals $[0,\nu]$ and $[\nu,1]$, respectively, we get

$$F_1' \leq x''(t) \leq L_1' \text{ on } [0,\nu] \text{ and } F_1 \leq x''(t) \leq L_1 \text{ on } [\nu,1]$$

and so the bounds for $x''(t)$ follow. Further, as in the proof of Lemma 4 we establish consecutively the bounds for $|x'(t)|$ and $|x(t)|$. □

## 3. Existence Results

**Theorem 1.** *Let* **(H$_1$)** *hold for* $K = C$ *and* **(H$_3$)** *hold for*

$$M_0 = |A| + |B| + \max\{|F_1'|, |'L_1|\}, m_0 = -M_0,$$

$$M_1 = |B| + \max\{|F_1'|, |L_1'|\}, m_1 = -M_1, m_2 = F_1', M_2 = L_1'.$$

*Then each of BVPs* (1), (2) *and* (1), (3) *has at least one solution in* $C^3[0,1]$.

**Proof.** We will establish that the assertion is true for problem (1), (2) after checking that the hypotheses of Lemma 1 are fulfilled; it follows similarly and for (1), (3). We easily check that *(i)* holds for $(1)_0$, (2). Clearly, BVP (1), (2) is equivalent to BVP $(1)_1$, (2) and so *(ii)* is satisfied. Since now $\mathbf{L}_h = x'''$, *(iii)* also holds. Next, according to Lemma 4, for each solution $x \in C^3[0,1]$ to $(1)_\lambda$, (2) we have

$$m_i \leq x^{(i)}(t) \leq M_i, t \in [0,1], i = 0,1,2.$$

Now use that $f$ is continuous on $[0,1] \times J$ to conclude that there are constants $m_3$ and $M_3$ such that

$$m_3 \leq \lambda f(t,x,p,q) \leq M_3 \text{ for } \lambda \in [0,1] \text{ and } (t,x,p,q) \in [0,1] \times J,$$

which together with $(x(t), x'(t), x''(t)) \in J$ for $t \in [0,1]$ and Equation $(1)_\lambda$ implies

$$m_3 \leq x'''(t) \leq M_3, t \in [0,1].$$

These observations imply that *(iv)* holds, too. Finally, the continuity of $f$ on the set $J$ gives *(v)* and so the assertion is true by Lemma 1. □

**Theorem 2.** *Let* **(H$_1$)** *hold for* $K = C$ *and* **(H$_3$)** *hold for*

$$M_0 = |A| + |B - A| + \max\{|F_1'|, |L_1'|\}, m_0 = -M_0,$$

$$M_1 = |B - A| + \max\{|F_1'|, |L_1'|\}, m_1 = -M_1, m_2 = F_1', M_2 = L_1'.$$

*Then BVP* (1), (4) *has at least one solution in* $C^3[0,1]$.

**Proof.** It follows the lines of the proof of Theorem 1. Now the bounds

$$m_i \leq x^{(i)}(t) \leq M_i, t \in [0,1], i = 0,1,2,$$

for each solution $x \in C^3[0,1]$ to a $(1)_\lambda$, (4) follow from Lemma 5. □

**Theorem 3.** *Let* **(H$_1$)** *and* **(H$_2$)** *hold for* $K = C - B$ *and* **(H$_3$)** *hold for*

$$M_0 = |A| + |B| + \max\{|F_1|, |L_1|, |F_1'|, |L_1'|\}, m_0 = -M_0,$$

$$M_1 = |B| + \max\{|F_1|, |L_1|, |F_1'|, |L_1'|\}, m_1 = -M_1,$$

$$m_2 = \min\{F_1, F_1'\}, M_2 = \max\{L_1, L_1'\}.$$

*Then each of BVPs (1), (5) and (1), (6) has at least one solution in* $C^3[0,1]$.

**Proof.** Arguments similar to those in the proof of Theorem 1 yield the assertion. Now the bounds

$$m_i \le x^{(i)}(t) \le M_i, t \in [0,1], i = 0,1,2,$$

for each solution $x \in C^3[0,1]$ to $(1)_\lambda$, (5) and $(1)_\lambda$, (6) follow from Lemma 6. □

## 4. Examples

Through several examples we will illustrate the application of the obtained results.

**Example 1.** *Consider the BVPs for the equation*

$$x'''(t) = \exp(x'' - 3) + 5x''(x'^2 + 1) - t \sin x, t \in (0,1),$$

*with boundary conditions* (2) *or* (3).

For $F_2' = -|C| - 2$, $F_1' = -|C| - 1$, $L_1' = \max\{|C|, 3\} + 1$, $L_2' = \max\{|C|, 3\} + 2$ and $\sigma = 0.1$, for example, each of these problems has a solution in $C^3[0,1]$ by Theorem 1.

**Example 2.** *Consider the BVP*

$$x'''(t) = \varphi(t, x, x')\Big(\lg\big((x'' + 50)(60 - x'')\big) - 3\Big), t \in (0,1),$$

$$x(0) = 5, x'(0) = 10, x'(1) = 40,$$

*where* $\varphi : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous and does not change its sign.*

If $\varphi(t,x,p) \ge 0$ on $[0,1] \times \mathbb{R}^2$, the assumptions of Theorem 3 are satisfied for $F_2 = -36$, $F_1 = -35$, $F_2' = -46$, $F_1' = -45$, $L_1' = 40$, $L_2' = 41$, $L_1 = 55$, $L_2 = 56$ and $\sigma = 0.01$, for example, and if $\varphi(t,x,p) \le 0$ on $[0,1] \times \mathbb{R}^2$, they are satisfied for $F_2' = -36$, $F_1' = -35$, $F_2 = -46$, $F_1 = -45$, $L_1 = 40$, $L_2 = 41$, $L_1' = 55$, $L_2' = 56$ and $\sigma = 0.01$, for example; it is clear, $K = 30$. Thus, the considered problem has at least one solution in $C^3[0,1]$. Let us note, here $D_q = (-50, 60)$.

**Example 3.** *Consider the BVP*

$$x'''(t) = \frac{t(x''+8)(x''+3)\sqrt{625 - x'^2}}{\sqrt{900 - x^2}\sqrt{100 - x''^2}}, t \in (0,1),$$

$$x(0) = 9, x(1) = 1, x''(1) = -4.$$

For $F_2' = -6$, $F_1' = -5$, $L_1' = -3$, $L_2' = -2$ and $\sigma = 0.1$, for example, this problem has a positive, decreasing, concave solution in $C^3[0,1]$ by Theorem A1; notice, here $D_x$, $D_p$ and $D_q$ are bounded.

**Author Contributions:** All authors contributed equally. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Conflicts of Interest:** The authors declare no conflict of interest.

## Appendix A

In this part we show how the barrier strips can be used for studying the existence of positive or non-negative, monotone, convex or concave $C^3[0,1]$ - solutions. Here, we demonstrate this on problem (1), (4) but it can be done for the rest of the BVPs considered in this paper. Similar results for various other two-point boundary conditions can be found in R. Agarwal and P. Kelevedjiev [16] and P. Kelevedjiev and T. Todorov [15].

**Lemma A1.** *Let $A, B \geq 0, C \leq 0$. Suppose* **($H_1$)** *holds for $K = C$ with $L_1' \leq 0$. Then each solution $x \in C^3[0,1]$ to (1)$_\lambda$, (4) satisfies the bounds*

$$\min\{A, B\} \leq x(t) \leq A + |B - A| + |F_1'|, \ t \in [0,1],$$

$$B - A + F_1' \leq x'(t) \leq B - A - F_1', \ t \in [0,1].$$

**Proof.** From Lemma 2 we know that $F_1' \leq x''(t) \leq L_1'$ for $t \in [0,1]$. Besides, for some $\mu \in (0,1)$ we have $x'(\mu) = B - A$. Then,

$$\int_t^\mu F_1' ds \leq \int_t^\mu x''(s) ds \leq \int_t^\mu L_1' ds, t \in [0, \mu),$$

gives

$$B - A \leq x'(t) \leq B - A - F_1', t \in [0, \mu],$$

and

$$\int_\mu^t F_1' ds \leq \int_\mu^t x''(s) ds \leq \int_\mu^t L_1' ds, t \in (\mu, 1],$$

implies

$$B - A + F_1' \leq x'(t) \leq B - A, t \in [\mu, 1].$$

As a result,

$$B - A + F_1' \leq x'(t) \leq B - A - F_1', t \in [0,1].$$

Using Lemma 5, conclude

$$|x(t)| \leq A + |B - A| + |F_1'| \text{ for } t \in [0,1].$$

From $x''(t) \leq L_1' \leq 0$ for $t \in [0,1]$ it follows that $x(t)$ is concave on $[0,1]$ and so, in view of $A, B \geq 0$, $x(t) \geq \min\{A, B\}$ on $[0,1]$, which completes the proof. □

**Theorem A1.** *Let $A \geq B \geq 0$ and $C \leq 0$ ($A \geq B > 0$ and $C < 0$). Suppose* **($H_1$)** *holds for $K = C$ with $B - A \leq F_1'$ ($B - A < F_1'$) and $L_1' \leq 0$, and* **($H_3$)** *holds for*

$$m_0 = B, M_0 = 2A - B + |F_1'|,$$

$$m_1 = B - A + F_1', M_1 = B - A - F_1', m_2 = F_1', M_2 = L_1'.$$

*Then BVP (1), (4) has at least one non-negative, non-increasing (positive, decreasing), concave solution in $C^3[0,1]$.*

**Proof.** By Lemma 5, for every solution $x \in C^3[0,1]$ to (1)$_\lambda$, (4) we have $F_1' \leq x''(t) \leq L_1'$ on $[0,1]$, and Lemma A1 yields

$$B - A + F_1' \leq x'(t) \leq B - A - F_1', \ t \in [0,1]$$

$$\min\{A, B\} \leq x(t) \leq A + |B - A| + |F_1'|, \ t \in [0,1].$$

Because of $A \geq B$, the last inequality gets the form

$$B \leq x(t) \leq 2A - B + |F_1'|, \ t \in [0,1].$$

So, $x(t)$ satusfies the bounds

$$m_0 \leq x^{(i)}(t) \leq M_0, t \in [0,1], i = 0,1,2.$$

Essentially the same reasoning as in the proof of Theorem 1 establishes that (1), (4) has a solution in $C^3[0,1]$. Since $m_0 = B \geq 0 (m_0 > 0), M_1 = B - A - F_1' \leq 0 (M_1 < 0)$ and $M_2 = L_1' \leq 0$, this solution has the desired properties. □

## Reference


1. Li, H.; Feng, Y.; Bu, C. Non-conjugate boundary value problem of a third order differential equation. *Electron. J. Qual. Theory Differ. Equ.* **2015**, *2015*, 1–19. [CrossRef]
2. Li, S. Positive solutions of nonlinear singular third-order two-point boundary value problem. *J. Math. Anal. Appl.* **2006**, *323*, 413–425. [CrossRef]
3. Liu, Z.; Debnath, L.; Kang, S.M. Existence of monotone positive solutions to a third order two-point generalized right focal boundary value problem. *Comput. Math. Appl.* **2008**, *55*, 356–367. [CrossRef]
4. Liu, Z.; Ume, J.S.; Kang, S.M. Positive solutions of a singular nonlinear third order two-point boundary value problem. *J. Math. Anal. Appl.* **2007**, *326*, 589–601. [CrossRef]
5. Lin, X.; Zhao, Z. Sign-changing solution for a third-order boundary-value problem in ordered Banach space with lattice structure. *Bound. Value Probl.* **2014**, *2014*, 132. [CrossRef]
6. Smirnov, S. On the third order boundary value problems with asymmetric nonlinearity. *Nonlinear Anal. Model. Control* **2011**, *16*, 231–241. [CrossRef]
7. Yao, Q.; Feng, Y. The existence of solutions for a third-order two-point boundary value problem. *Appl. Math. Lett.* **2002**, *15*, 227–232. [CrossRef]
8. Feng, Y. Solution and positive solution of a semilinear third-order equation. *J. Appl. Math. Comput.* **2009**, *29*, 153–161. [CrossRef]
9. Feng, Y.; Liu, S. Solvability of a third-order two-point boundary value problem. *Appl. Math. Lett.* **2005**, *18*, 1034–1040. [CrossRef]
10. Feng, Y. Existence and uniqueness results for a third-order implicit differential equation. *Comput. Math. Appl.* **2008**, *56*, 2507–2514. [CrossRef]
11. Ma, R.; Lu, Y. Disconjugacy and extremal solutions of nonlinear third-order equations. *Commun. Pure Appl. Anal.* **2014**, *13*, 1223–1236. [CrossRef]
12. Granas, A.; Guenther, R.B.; Lee, J.W. *Nonlinear Boundary Value Problems for Ordinary Differential Equations*; Dissnes Math.: Warszawa, Poland, 1985.
13. Hopkins, B.; Kosmatov, N. Third-order boundary value problems with sign-changing solutions. *Nonlinear Anal.* **2007**, *67*, 126–137. [CrossRef]
14. Li, Y.; Li, Y. Positive solutions of a third-order boundary value problem with full nonlinearity. *Mediterr. J. Math.* **2017**, *14*, 128. [CrossRef]
15. Kelevedjiev, P.; Todorov, T.Z. Existence of solutions of nonlinear third-order two-point boundary value problems. *Electron. J. Qual. Theory Differ. Equ.* **2019**, *2019*, 1–15. [CrossRef]
16. Agarwal, R.P.; Kelevedjiev, P.S. On the solvability of fourth-order two-point boundary value problems. *Mathematics* **2020**, *8*, 603. [CrossRef]
17. Kelevedjiev, P. Existence of solutions for two-point boundary value problems *Nonlinear Anal.* **1994**, *22*, 217–224. [CrossRef]