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Axiomatic Approach in the Analytic Theory of Singular Perturbations

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Received: 18 November 2019; Accepted: 25 December 2019; Published: 16 January 2020

Abstract: Introduced by S.A. Lomov, the concept of a pseudoanalytic (pseudoholomorphic) solution laid the foundation for the development of the singular perturbation analytical theory. In order for this concept to work in case of linear problems, an apparatus for the theory of exponential type vector spaces was developed. When considering nonlinear singularly perturbed problems, an algebraic approach is currently used. This approval is based on the properties of algebra homomorphisms for holomorphic functions with various numbers of variables, as a result of which it is possible to obtain pseudoholomorphic solutions. In this paper, formally singularly perturbed equations are considered in topological algebras, which allows the authors to formulate the main concepts of the singular perturbation analytical theory from the standpoint of maximal generality.

Keywords: ε-regular function; invariants of equations and systems; ε-pseudoregular solution; essentially singular manifold

MSC: 34E15

1. Introduction

The basic concept of the singular perturbation analytic theory is the concept pseudoholomorphic solution, i.e., such a solution, which can be presented as a series in powers of a small parameter that converges in the usual sense (and not asymptotically). The nature of this convergence is determined by the topology of the spaces in which the investigated problems are considered. As a rule, spaces of holomorphic functions (of one or several variables) are used. In this regard, it was possible to formulate the main principles for the theory of singularly perturbed differential equations and systems—under fairly general assumptions that they possess holomorphics in small parameter first integrals [1,2]. Moreover, a connection between the first integrals and homomorphisms of algebras of holomorphic functions with various numbers of variables was established. The pseudoholomorphic solutions themselves are obtained as a result of applying the implicit function theorem. In the presented paper, all of these constructions will be carried out in topological algebras for formally singularly perturbed equations.

2. Algebraic and Analytic Aspects of the Theory of Singular Perturbations

Let $\mathcal{J}_a$ be a complete topological commutative algebra with unit e and let $X, Y_1, \dots, Y_k, \dots$ be a sequence of open sets $\mathcal{J}_a$. Let us denote by $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_k, \dots$ the spaces of functions continuous on the sets $X, X \times Y_1, \dots, X \times Y_1 \times \dots \times Y_k, \dots$ respectively with their values in $\mathcal{J}_a$. Let us formulate the block I of necessary conditions:

(1°) If the sequence $\{x_i\}_{i=0}^{\infty}$ is a bounded set [3] in $\mathcal{J}_a$, then the series $x_0 + \varepsilon x_1 + \ldots + \varepsilon^i x_i + \ldots$ converges at $|\varepsilon| < 1$.

(2°) If the sequence $\{h_{i,k}\}_{i=1}^{\infty} \subset \mathcal{A}_k$ is such that the series

$$\sum_{i=0}^{\infty} \varepsilon^i h_{i,k}(x, y_1, \ldots, y_k) \tag{1}$$

converges on each set $T \times T_1 \times \ldots \times T_k$, where T is an arbitrary compact from X; T_1 is an arbitrary compact from $Y_1, \cdots$; T_k is an arbitrary compact set from Y_k in some neighborhood of the value $\varepsilon = 0$, the function $\Phi \in \mathcal{A}_0$ and it can be extended to all $\mathcal{J}_a$, then we have

$$\Phi\left(\sum_{i=0}^{\infty} \varepsilon^i h_{i,k}\right) = \Phi(h_{0,k}) + \sum_{i=1}^{\infty} \varepsilon^i g_{i,k}$$

and the last row with coefficients from $\mathcal{A}_k$ is convergent.

Definition 1. *The function $f(x, y_1, \ldots, y_k, \varepsilon) \in \mathcal{A}_k$ represented by (1), is called ε-regular.*

(3°) If the system

$$\begin{cases} F_1(x, y_1, \ldots, y_k, \varepsilon) = q_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_k(x, y_1, \ldots, y_k, \varepsilon) = q_k \end{cases}$$

with ε-regular left-hand sides is uniquely solvable with respect to $\{y_1, \ldots, y_k\}$ for $\varepsilon = 0$ in some neighborhood of the point $x_0 \in X$, then it is also uniquely solvable in some neighborhood of the same point and thus functions $y_m(x, \varepsilon) \in \mathcal{A}_0$ $(m = \overline{1,k})$ are ε-regular.

Remark 1. *The conditions of the block I are satisfied if $\mathcal{J}_a = \mathbb{C}$, $X, Y_1, Y_2, \ldots$ are simply connected regions, $\mathcal{A}_0, \mathcal{A}_1, \ldots$ are spaces of holomorphic functions on $X, X \times Y_1, X \times Y_1 \times Y_2, \ldots$ respectively.*

In order to formulate the conditions of block II, we give some definitions.

Definition 2. *s-product of tuples $\boldsymbol{\varphi} = \{\varphi_1, \ldots, \varphi_k\}$ and $\boldsymbol{\psi} = \{\psi_1, \ldots, \psi_k\}$ is a function $\boldsymbol{\varphi} \circledS \boldsymbol{\psi} = \varphi_1\psi_1 + \ldots + \varphi_k\psi_k$.*

Definition 3. *Let $f \in \mathcal{A}_k$, $\varphi_i(x) \in \mathcal{A}_0$, $i = \overline{1,k}$. The composition f and $\boldsymbol{\varphi} = \{\varphi_1, \ldots, \varphi_k\}$ is determined by the formula $f \circ \boldsymbol{\varphi} = f(x, \varphi_1(x), \ldots, \varphi_k(x))$ as usual.*

Block of conditions II:

(1°) All algebras $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_k, \ldots$ contain constant functions and linear functions. We consider the embeddings $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \ldots \subset \mathcal{A}_k \subset \ldots$ together with topologies to be obvious.

(2°) On all spaces $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_k, \ldots$, a linear operation ∂_0 is defined such that $\partial_0 p = 0$, where p is a constant function, $\partial_0 x = e$ and $\partial_0 f = 0$ if $f \in \mathcal{A}_k$ and does not depend on x. On each space $\mathcal{A}_k$ $(k = 1, 2, \ldots)$, linear operations $\{\partial_i\}_{i=1}^{k}$ are defined and they comply with the following laws:

(a) $\partial_i p = 0$, $i = \overline{1,k}$, where $p \in \mathcal{A}_k$ is a constant function;
(b) $\partial_i y_i = e$, $i = \overline{1,k}$;
(c) if the function $f \in \mathcal{A}_k$ does not depend on y_m, then $\partial_i f = 0$ for $i \neq m$.

(3°) The operations $\{\partial_i\}_{i=0}^{\infty}$ form a commutative ring.

(4°) An operation d is introduced, and it satisfies the following rules:

(a) $d \equiv \partial_0$ on $\mathcal{A}_0$;
(b) $d(f \circ g) = \partial_0 f \cdot \partial_0 g \;\; \forall f, g \in \mathcal{A}_0$;

(c) if $f \in \mathcal{A}_k$, $\varphi_i(x) \in \mathcal{A}_0$ $(i = \overline{1,k})$, $\boldsymbol{\varphi} = \{\varphi_1, \dots, \varphi_k\}$, then $d(f \circ \boldsymbol{\varphi}) = \partial f Ⓢ \partial \boldsymbol{\varphi}$, where $\boldsymbol{\partial} f \equiv \{\partial_0 f, \partial_1 f, \dots, \partial_k f\}$, $\partial_0 \boldsymbol{\varphi} = \{e, \partial_0 \varphi_1, \dots, \partial_0 \varphi_k\}$ are tuples of length $(k+1)$.

(5°) For every natural number k in the algebra $\mathcal{A}_k$, there exist a lot of tuples $\mathbf{f} = \{f_1(x, y_1, \dots, y_k), \dots, f_k(x, y_1, \dots, y_k)\}$ such that the operator $D_k^{\mathbf{f}} = \mathbf{f} Ⓢ \boldsymbol{\partial}$, where $\boldsymbol{\partial} = \{\partial_1, \dots, \partial_k\}$, with a specially defined domain $\mathcal{D}(D_k^{\mathbf{f}})$ is surjective and has the inverse $J_k^{\mathbf{f}}$, which has the following property: for arbitrary compact sets $T \subset X, T_1 \subset Y_1, \dots, T_k \subset Y_k$ there is a number $C > 0$ such that, for an arbitrary function $\varphi(x) \in \mathcal{A}_0$, the set $\Gamma_k^{\varphi} = \{C^{-n}(J_k^{\mathbf{f}} \partial_0)^n \varphi(x, y_1, \dots, y_k) \in T \times T_1 \times \dots \times T_k\}_{n=1}^{\infty}$ is bounded in $\mathcal{J}_a$.

Let us consider the case $k = 1$. We investigate the following equation:

$$\varepsilon d y_1 = F(x, y_1), \tag{2}$$

in which $F \in \mathcal{A}_1$ and ε is a small complex parameter. The function $y_1(x) \in \mathcal{A}_0$ satisfying the initial condition

$$y_1(x^0, \varepsilon) = y_1^0, \tag{3}$$

where $x^0 \in X$, $y_1^0 \in Y_1$, is required to be found.

Definition 4. *The invariant of Equation (2) is the function $U(x, y_1, \varepsilon) \in \mathcal{A}_1$, which turns into a constant on the solution $y_1(x, \varepsilon)$ of this equation.*

Theorem 1. *When the blocks of conditions I and II are satisfied, then Equation (2) has ε-regular invariants.*

Proof of Theorem 1. If $U(x, y, \varepsilon)$ is an invariant of the Equation (2), then, as it follows from Definition 4, we have

$$\varepsilon \partial_0 U + D_1^F U = 0, \tag{4}$$

where $D_1^F = F \partial_1$.

We seek a solution of Equation (4) in the form of a series in powers of ε:

$$U(x, y_1, \varepsilon) = U_0(x, y_1) + \varepsilon U_1(x, y_1) + \dots + \varepsilon^n U_n(x, y_1) + \dots \tag{5}$$

for the coefficients of the equation above the following series of equations holds:

$$\begin{aligned} &D_1^F U_0 = 0, \\ &D_1^F U_1 = -\partial_0 U_0, \\ &\dots\dots\dots\dots\dots \\ &D_1^F U_n = -\partial_0 U_{n-1}. \\ &\dots\dots\dots\dots\dots \end{aligned} \tag{6}$$

As a solution to the first equation of this series, we take an arbitrary function $\varphi(x) \in \mathcal{A}_0$. To satisfy the condition (5°) of block II, we assume that the domain of the surjective operator D_1^F consists of functions from $\mathcal{A}_1$ that vanish when $y_1 = y_1^0$ $\forall x \in X$, and the inverse operator J_1^F is such that, for any compact sets $T \subset X$, $T_1 \subset Y_1$, there exists a number $C > 0$ such that, for an arbitrary function $\varphi(x) \in \mathcal{A}_0$ set $\Gamma_1^{\varphi} = \{C^{-n}(J_1^F \partial_0)^n \varphi,\ (x, y_1) \in T \times T_1\}_{n=1}^{\infty}$ is limited in $\mathcal{J}_a$.

As a result, all equations of the series (6), starting from the second, are uniquely solvable:

$$U(x, y_1, \varepsilon) = \varphi - \varepsilon(J_1^F \partial_0)\varphi + \dots + (-1)^n \varepsilon^n (J_1^F \partial_0)^n \varphi + \dots \tag{7}$$

and this series converges in some neighborhood of the value $\varepsilon = 0$ on the set $T \times T_1$. Theorem 1 is proved. □

Remark 2. *As it comes out from the form of series (7), we can consider $U(x, y_1, \varepsilon)$ for each fixed ε as the image of the linear operator $H_\varepsilon : \mathcal{A}_0 \to \mathcal{A}_1$ given by the formula*

$$H_\varepsilon = I - \varepsilon(J_1^F \partial_0) + \ldots + (-1)^n \varepsilon^n (J_1^F \partial_0)^n + \ldots,$$

where I is the identity operator. Thus, $U = H_\varepsilon[\varphi]$.

Theorem 2. *$\{H_\varepsilon\}$ forms a ε-regular family for homomorphisms of the algebra $\mathcal{A}_0$ into the algebra $\mathcal{A}_1$.*

Proof of Theorem 2. Let U and V be invariants of the Equation (2). Obviously, then there exists a function Φ such that $V = \Phi(U)$, and therefore $H_\varepsilon[\varphi(x)] = \Phi(H_\varepsilon[x])$. If in this equality we put $y_1 = y_1^0$, then $\varphi(x) = \Phi(x)\ \forall x \in X$, therefore

$$H_\varepsilon[\varphi(x)] = \varphi(H_\varepsilon[x]). \tag{8}$$

The equality (8) is called the commutation relation.

Now, let $\varphi_1(x), \varphi_2(x) \in \mathcal{A}_0$; then,

$$H_\varepsilon[\varphi_1\varphi_2] = (\varphi_1\varphi_2)(H_\varepsilon[x]) = \varphi_1(H_\varepsilon[x])\varphi_2(H_\varepsilon[x]) = H_\varepsilon[\varphi_1]H_\varepsilon[\varphi_2],$$

where $H_\varepsilon : \mathcal{A}_0 \to \mathcal{A}_1$ is a homomorphism. Theorem 2 is proved. □

For the concepts given below, we need a definition introduced by S.A. Lomov for the notion of the essentially singular manifold [4].

Definition 5. *Let $\varphi(x) \in \mathcal{A}_0$, $\varphi(x_0) = 0$, $\Phi \in \mathcal{A}_0$, let it allow continuation to all $\mathcal{J}_a$, and let T_0 be some compact from X containing the point x_0. The set $Q^+(\varphi, \Phi, T_0) = \{q : \Phi(\varphi(x)/\varepsilon),\ x \in T_0,\ \varepsilon > 0\}$ is called an essentially singular variety generated by the point $\varepsilon = 0$. Moreover, we say that it has the correct structure if*

$$Q^+ = \bigcup_{m=1}^{\infty} \Pi_m,$$

where $\Pi_1 \subset \Pi_2 \subset \ldots$ is an increasing compact system.

We introduce the concept of ε-pseudoregularity necessary for studying the analytic properties of a solution of $y(x, \varepsilon)$.

Definition 6. *The solution to the problems (2), (3) is called ε-pseudoregular if $y_1(x, \varepsilon) = \widetilde{Y}(x, \varphi(x)/\varepsilon, \varepsilon)$, in which $\varphi(x) \in \mathcal{A}_0$; the function $\widetilde{Y}(x, \eta, \varepsilon)$ is ε-regular for all $(x, \eta) \in T_0 \times G$ where T_0 is some compact set containing the point x_0, G is some unlimited set from $\mathcal{J}_a$.*

Theorem 3. *If the essentially singular manifold $Q^+(\varphi, \Phi, T_0)$ is a bounded set in $\mathcal{J}_a$ and the equation*

$$(J_1^F \partial_0)\varphi = \varphi(x)/\varepsilon \tag{9}$$

has a unique solution of the form $y_1 = Y_{1,0}(x, q)\big|_{q=\Phi(\varphi(x)/\varepsilon)}$ such that the function $Y_{1,0}(x, q)$ coincides with the contraction to the set $T_0 \times Q^+$ of some function from $\mathcal{A}_1$, then problems (2), (3) have a ε-pseudoregular solution.

Proof of Theorem 3. For the invariant represented by the Formula (7), we compose the equality

$$(J_1^F \partial_0)\varphi - \varepsilon(J_1^F \partial_0)^2\varphi + \ldots + (-1)^{n-1}\varepsilon^{n-1}(J_1^F \partial_0)^n\varphi + \ldots = \varphi(x)/\varepsilon,$$

which defines the solution to the problems (2), (3). We apply the function Φ to its left-hand and right-hand sides and, using the condition (2°) of block I, we obtain the following equality:

$$\Phi((J_1^F\partial_0)\varphi) + \varepsilon\Psi(x, y_1, \varepsilon) = q, \tag{10}$$

where $\Psi(x, y_1, \varepsilon)$ is some ε-regular function.

Let the small parameter $\varepsilon > 0$ in Equation (2) be such that the following expression holds:

$$\{q : q = \Phi(\varphi(x)/\varepsilon),\ x \in T_0\} = \Pi_m$$

for some natural number m (depending on ε). In accordance with the theorem conditions and the condition (3°) of block I, Equation (10) is solvable in some neighborhood σ_{xq} of each point $(x, q) \in T_0 \times \Pi_m$ and has a solution $y_1 = Y_1(x, q, \varepsilon)$ that is ε-regular in a neighborhood of $|\varepsilon| < \varepsilon_{xq}$, where $\varepsilon_{xq} > 0$ and is determined by this neighborhood. From the cover $\{\sigma_{xq}\}$ of the compact set $T_0 \times \Pi_m$, we choose the finite subcover $\{\sigma_{xq}\}_{i=1}^N$. Then, $y_1 = Y_1(x, q, \varepsilon)$ will be a ε-regular function in the smallest neighborhood of the point $\varepsilon = 0$ defined by a finite subcover; the function $y_1 = Y_1(x, \Phi(\varphi(x)/\varepsilon), \varepsilon)$ will give a ε-pseudoregular solution to the problem (2), (3) on the part $\widetilde{T}_0 \subset T_0$ such that the set $\{(x, q) : x \in \widetilde{T}_0,\ q = \Phi(\varphi(x)/\varepsilon)\} \subset T_0 \times \Pi_m$. The theorem is proved. □

3. Invariants and ε-Pseudoregular Solutions of Systems of Equations

We take into the consideration the system of equations

$$\begin{cases} \varepsilon dy_1 = F_1(x, y_1, \dots, y_k), \\ \dots\dots\dots\dots\dots\dots\dots \\ \varepsilon dy_k = F_k(x, y_1, \dots, y_k), \end{cases} \tag{11}$$

the right-hand sides of which belong to the algebra $\mathcal{A}_k$. It is required to find its solution $\mathbf{y}(x, \varepsilon) = \{y_1(x, \varepsilon), \dots, y_k(x, \varepsilon)\}$ satisfying the initial conditions

$$y_1(x^0, \varepsilon) = y_1^0,\ \dots,\ y_k(x^0, \varepsilon) = y_k^0. \tag{12}$$

We rewrite system (11) by introducing the following denotation:

$$\begin{aligned} &\mathbb{F}(x, \mathbf{y}) = \{F_1(x, \mathbf{y}), \dots, F_k(x, \mathbf{y})\}, \\ &\mathbf{y}^0 = \{y_1^0, \dots, y_k^0\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\varepsilon d\mathbf{y} = \mathbb{F}(x, \mathbf{y}), \\ &\mathbf{y}(x^0, \varepsilon) = \mathbf{y}^0 \end{aligned} \tag{13}$$

to be the initial investigated problem.

Definition 7. *The function $U(x, \mathbf{y}, \varepsilon) \in \mathcal{A}_k$ is called the invariant of the system (11) if it turns into a constant on the solution $\mathbf{y}(x, \varepsilon)$.*

We formulate a theorem similar to Theorem 1.

Theorem 4. *The system (11) has ε-regular invariants.*

Proof of Theorem 4. The proof is carried out according to the same scheme as in the case of a single equation. □

Definition 8. *The solution of the* $\mathbf{y}(x,\varepsilon)$ *problem* (13) *is called* ε*-pseudoregular if* $\mathbf{y}(x,\varepsilon) = \widetilde{\mathbb{Y}}(x,\boldsymbol{\varphi}(x)/\varepsilon,\varepsilon)$, *in which* $\boldsymbol{\varphi}(x) = \{\varphi_1(x),\ldots,\varphi_k(x)\}$, $\varphi_i(x) \in \mathcal{A}_0$ $(i = \overline{1,k})$ *and the function* $\widetilde{\mathbb{Y}}(x,\boldsymbol{\eta},\varepsilon)$ *in which* $\boldsymbol{\eta} = (\eta_1,\ldots,\eta_k)$, *is* ε*-regular for all* $(x,\eta_1,\ldots,\eta_k) \in T_0 \times G_1 \times \ldots \times G_k$ *where* T_0 *is some compact set from* X *containing* x_0 *and* G_i $(i = \overline{1,k})$ *are unbounded sets from* $\mathcal{J}_a$.

Theorem 5. *Let the following conditions be fulfilled:*

(1°) *The functions* Φ_i *and* φ_i *are such that the essentially singular manifolds* $Q_i^+(\varphi_i,\Phi_i,T_0)$ $(i = \overline{1,k})$ *are bounded sets in* $\mathcal{J}_a$.

(2°) *The equation* $D_k^{\mathbb{F}} V = e$ *has the solutions* $\{V_1(x,\mathbf{y}),\ldots,V_k(x,\mathbf{y})\}$ *such that system*

$$\begin{cases} \partial_0\varphi_1 \cdot V_1(x,\mathbf{y})) = \varphi_1(x)/\varepsilon, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \partial_0\varphi_k \cdot V_k(x,\mathbf{y})) = \varphi_k(x)/\varepsilon \end{cases}$$

has the only solution

$$\mathbf{y} = \mathbb{Y}_0(x,\mathbf{q})\Big|_{\substack{\mathbf{q}=(q_1,\ldots,q_k) \\ q_i=\Phi_i(\varphi_i(x)/\varepsilon)}},$$

and each component $Y_{0,i}(x,\mathbf{q})$ $(i = \overline{1,k})$ *of it coincides with the restriction to the set* $T_0 \times Q_1^+ \times \ldots \times Q_k^+$ *of some function from* $\mathcal{A}_k$.

Then, the solution $\mathbf{y}(x,\varepsilon)$ *of the problem* (13) *is* ε*-pseudoregular.*

Proof of Theorem 5. We write the equalities for the invariants of the system (11) in the following form:

$$\begin{cases} \partial_0\varphi_1 \cdot V_1(x,\mathbf{y}) - \varepsilon(J_k^{\mathbb{F}}\partial_0)^2\varphi_1 + \varepsilon^2(J_k^{\mathbb{F}}\partial_0)^3\varphi_1 - \ldots = \varphi_1(x)/\varepsilon, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \partial_0\varphi_k \cdot V_k(x,\mathbf{y}) - \varepsilon(J_k^{\mathbb{F}}\partial_0)^2\varphi_k + \varepsilon^2(J_k^{\mathbb{F}}\partial_0)^3\varphi_k - \ldots = \varphi_k(x)/\varepsilon. \end{cases} \tag{14}$$

In order for this system to determine the solution of the problem (11), (12) (or (13)), we assume (see condition (5°) of block II) that $\mathcal{D}(D_k^{\mathbb{F}})$ consists of functions that vanish when $\mathbf{y} = \mathbf{y}^0$ for any $x \in X$ and $V_i(x,\mathbf{y}^0) = 0$, $i = \overline{1,k}$.

We apply the functions $\Phi_1,\ldots,\Phi_k$ to the equations of the system (14), respectively. Then, in accordance with the condition (2°) of block I, we obtain the system

$$\begin{cases} \Phi_1(\partial_0\varphi_1 \cdot V_1(x,\mathbf{y})) + \varepsilon\Psi_1(x,\mathbf{y},\varepsilon) = q_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \Phi_k(\partial_0\varphi_k \cdot V_k(x,\mathbf{y})) + \varepsilon\Psi_k(x,\mathbf{y},\varepsilon) = q_k. \end{cases} \tag{15}$$

Let the small parameter $\varepsilon > 0$ in the system (13) be such that

$$\{q_i : q_i = \Phi_i(\varphi(x)/\varepsilon),\ x \in T_0\} = \Pi_{m_i},\ \ i = \overline{1,k}$$

for natural $m_1,\ldots,m_k$. By the condition of the (2°) Theorem 5, the system (15) for $\varepsilon = 0$ has a unique solution $\mathbf{y} = \mathbb{Y}_0(x,\mathbf{q})$ and, therefore, in accordance with the condition (3°) of block I, this system is solvable in some neighborhood $\sigma_{x\mathbf{q}}$ of each point $(x,\mathbf{q}) \in T_0 \times \Pi_{m_1} \times \ldots \times \Pi_{m_k}$, and its solution $\mathbb{Y}(x,\mathbf{q},\varepsilon)$ is ε-regular there for $|\varepsilon| < \varepsilon_{x\mathbf{q}}$. After that, from the cover $\{\sigma_{x\mathbf{q}}\}$ of the compact set $T_0 \times \Pi_{m_1} \times \ldots \times \Pi_{m_k}$, we choose a finite subcover and $\mathbb{Y}(x,\mathbf{q},\varepsilon)$ will be ε-regular in the minimal neighborhood from the neighborhood of $\varepsilon = 0$ corresponding to a finite subcover. As in the proof of Theorem 3, we choose $\widetilde{T}_0 \subset T_0$, a compact set on which there exists a ε-regular solution

$$\mathbf{y}(x,\varepsilon) = \mathbb{Y}(x,\mathbf{q},\varepsilon)\Big|_{\substack{q_i=\Phi_i(\varphi_i(x)/\varepsilon)\\(i=\overline{1,k})}}.$$

The theorem is proved. □

4. Concrete Implementations of the Theory

In this section of the article, we assume that $\mathcal{J}_a = \mathbb{C}$, $X = P_0 \equiv \{z \in \mathbb{C} : |z - z^0| < r_0\}$, $Y_i = P_i \equiv \{w_i \in \mathbb{C} : |w_i - w_i^0| < r_i\}$, $i = \overline{1,k}$. We shall use the following denotations $\mathbf{w} = \{w_1, \ldots, w_k\}$, $\mathbf{w}^0 = \{w_1^0, \ldots, w_k^0\}$, $\mathbb{P}^k = P_1 \times \ldots \times P_k$ a polycircle of $\mathbb{C}^k$.

Let $\mathcal{A}_0$ be the algebra of holomorphic functions in the P_0 circle of the variable z; let $\mathcal{A}_1$ be the algebra of holomorphic functions in the $P_0 \times P_1$ bicircle of the variables (z, w_1), …; let $\mathcal{A}_k$ be the algebra of holomorphic functions of the variables $(z, w_1, \ldots, w_k)$ in the polycircle $P_0 \times \mathbb{P}^k$. It is clear that, if $\partial_0 = \partial_z$, $\partial_1 = \partial_{w_1}$, …, $\partial_k = \partial_{w_k}$, then all the conditions of block I and the conditions (1°)—(4°) of block II are satisfied. In the concepts given below, we show that the condition (5°) also holds under fairly general assumptions.

Thus, we investigate the Cauchy problem for $\varepsilon > 0$:

$$\begin{aligned} &\varepsilon\frac{d\mathbf{w}}{dz} = \mathbb{F}(z,\mathbf{w}),\ \ z \in \widetilde{P}_0 = \{z \in \mathbb{C} : |z - z^0| < \widetilde{r}_0,\ 0 < \widetilde{r}_0 < r_0\},\\ &\mathbf{w}(z^0,\varepsilon) = \mathbf{w}^0, \end{aligned} \tag{16}$$

where $\mathbb{F}(z,\mathbf{w}) = \{F_1(z,\mathbf{w}), \ldots, F_k(z,\mathbf{w})\}$, $F_i(z,\mathbf{w}) \in \mathcal{A}_k$ for $i = \overline{1,k}$.

From the nonlinear system (16), we come to the linear equation of its integrals (invariants):

$$\varepsilon\partial_z\mathbb{U} + D_k^{\mathbb{F}}\mathbb{U} = 0. \tag{17}$$

Here, $D_k^{\mathbb{F}} = F_1\partial_{w_1} + \ldots + F_k\partial_{w_k}$ is the linear partial differential operator of the first order in partial derivatives: $\mathbb{U} = \{U^{[1]}, \ldots, U^{[k]}\}$, where $\{U^{[i]}\}_{i=1}^k$ is the system of independent integrals.

First of all, we present an integral method for solving inhomogeneous linear differential equations of the first order with partial derivatives [5].

Let Λ be a holomorphically smooth surface in $\mathbb{C}^k$ and we need to solve the initial problem

$$\begin{aligned} &D_k^{\mathbb{F}}V = f,\ \ f \in \mathcal{A}_k,\\ &V\big|_{\mathbf{w}\in\Lambda} = 0. \end{aligned} \tag{18}$$

Let us suppose that the surface Λ is given by the coordinates $\widetilde{\mathbf{w}} = \{\widetilde{w}_1, \ldots, \widetilde{w}_{k-1}\}$ and, namely, $\Lambda = \{\mathbf{w} \in \mathbb{C}^k : w_i = \lambda(\widetilde{\mathbf{w}}),\ i = \overline{1,k}\}$, where $\lambda_i(\widetilde{\mathbf{w}})$ are functions holomorphic in some region $\mathbb{C}^{k-1}$. Next, we compose the equation system for the characteristic equation

$$\frac{d\mathbf{w}}{ds} = \mathbb{F}(z,\mathbf{w}), \tag{19}$$

in which $s \in \mathbb{C}$ is an independent variable, and z acts as a parameter. Let $\mathbf{w} = \mathbf{g}(z,\widetilde{\mathbf{w}},s)$ be a solution to the system (19) with the initial condition

$$\mathbf{w}\big|_{s=0} = \boldsymbol{\lambda}(\widetilde{\mathbf{w}}),$$

where $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_k\}$.

The existence and uniqueness theorem guarantees the unique solvability of the system $\mathbf{g}(z,\widetilde{\mathbf{w}},s) = \mathbf{w}$ relative to $\widetilde{\mathbf{w}}$ and s: $s = S(z,\mathbf{w})$, $\widetilde{\mathbf{w}} = \widetilde{\mathbb{W}}(z,\mathbf{w})$. We denote the operator of replacing variables $(s,\widetilde{\mathbf{w}})$ by the variable $\mathbf{w}$ by $R(z)$ and the backward replacement operator is denoted by $R^{-1}(z,s)$:

$$R(z)[\chi(z,s,\widetilde{\mathbf{w}})] = \chi(z, S(z,\mathbf{w}), \widetilde{\mathbb{W}}(z,\mathbf{w})),$$
$$R^{-1}(z,s)[\theta(z,\mathbf{w})] = \theta(z, \mathbf{g}(z,\widetilde{\mathbf{w}},s)).$$

Then, as you know, if the phase trajectories in the system of characteristics are transversal (not tangent) to the surface Λ, then a solution to the Cauchy problem (17) exists, is unique, and is expressed by the following formula:

$$V(z,\mathbf{w}) = \int_0^s f(z, \mathbf{g}(z,\widetilde{\mathbf{w}},s_1))ds_1 \Big|_{\substack{s=S(z,\mathbf{w}),\\ \widetilde{\mathbf{w}}=\widetilde{\mathbb{W}}(z,\mathbf{w})}}. \tag{20}$$

We return to Equation (18). We have

$$\mathbb{U}(z,\mathbf{w},\varepsilon) = \mathbb{U}_0(z,\mathbf{w}) + \varepsilon\mathbb{U}_1(z,\mathbf{w}) + \ldots + \varepsilon^n\mathbb{U}_n(z,\mathbf{w}) + \ldots, \tag{21}$$

and, as this takes place

$$\begin{aligned}
&D_k^{\mathbb{F}}\mathbb{U}_0(z,\mathbf{w}) = 0,\\
&D_k^{\mathbb{F}}\mathbb{U}_1(z,\mathbf{w}) = -\partial_z\mathbb{U}_0(z,\mathbf{w}),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&D_k^{\mathbb{F}}\mathbb{U}_n(z,\mathbf{w}) = -\partial_z\mathbb{U}_{n-1}(z,\mathbf{w}).\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{22}$$

As a solution to the first equation of this series, we take the vector function $\mathbb{U}_0 = \{\varphi_1(z),\ldots,\varphi_k(z)\}$, $\varphi_i(z) \in \mathcal{A}_0$ for $i = \overline{1,k}$. The solution to the second equation of the series (22) is the vector function $\mathbb{U}_1 = \{-\partial_z\varphi_1 V^{[1]},\ldots,-\partial_z\varphi_k V^{[k]}\}$ where $\{V^{[1]},\ldots,V^{[k]}\}$ are functionally independent solutions of the equation $D_k^{\mathbb{F}}V = 1$ and such that $V^{[i]}(z,\mathbf{w}^0) = 0\ \forall z \in P_0$, $i = \overline{1,k}$. We find solutions to other equations using Formula (20), assuming that $\mathbf{w}^0 \in \Lambda$:

$$\begin{aligned}
&\mathbb{U}_2(z,\mathbf{w}) = -R(z)\int_0^s R^{-1}(z,s_1)\partial_z\mathbb{U}_1(z,\mathbf{w})ds_1,\\
&\mathbb{U}_3(z,\mathbf{w}) = R(z)\int_0^s ds_1 R^{-1}(z,s_1)\partial_z R(z)\int_0^{s_1} R^{-1}(z,s_2)\partial_z\mathbb{U}_1(z,\mathbf{w})ds_2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\mathbb{U}_n(z,\mathbf{w}) = (-1)^{n-1}R(z)\int_0^s ds_1 R^{-1}(z,s_1)\partial_z R(z)\int_0^{s_1} ds_2 R^{-1}(z,s_2)\partial_z\ldots\\
&\qquad\ldots R(z)\int_0^{s_{n-2}} R^{-1}(z,s_{n-1})\partial_z\mathbb{U}_1(z,\mathbf{w})ds_{n-1}.\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{23}$$

Next, to each natural $n \geq 2$, we associate $(n-1)$ concentric circles $C_m = \{z : |z - z^0| = t_m\}$ where

$$t_m = \widetilde{r}_0 + \frac{r - \widetilde{r}_0}{n-1}m, \quad m = \overline{1, n-1}$$

and $\widetilde{r}_0 < r < r_0$.

These circles are situated at the same distance from each other:

$$\rho = t_{n-1} - t_{n-2} = \ldots = t_2 - \widetilde{r}_0 = \frac{r - \widetilde{r}_0}{n-1}.$$

We use the equalities (23) with the Cauchy integral formula:

$$\begin{aligned}
\mathbb{U}_2(z,\mathbf{w}) &= -\tfrac{1}{2\pi i} R(z) \int\limits_0^s R^{-1}(z,s_1) \oint\limits_{C_1} \tfrac{dz_1}{(z_1-z)^2} \mathbb{U}_1(z_1,\mathbf{w}) ds_1, \\
\mathbb{U}_3(z,\mathbf{w}) &= \tfrac{1}{(2\pi i)^2} R(z) \int\limits_0^s ds_1 R^{-1}(z,s_1) \oint\limits_{C_1} \tfrac{dz_1}{(z_1-z)^2} R(z_1) \int\limits_0^{s_1} R^{-1}(z_1,s_2) \oint\limits_{C_2} \tfrac{dz_2}{(z_2-z_1)^2} \mathbb{U}_1(z_2,\mathbf{w}) ds_2, \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\mathbb{U}_n(z,\mathbf{w}) &= \tfrac{(-1)^{n-1}}{(2\pi i)^{n-1}} R(z) \int\limits_0^s ds_1 R^{-1}(z,s_1) \oint\limits_{C_1} \tfrac{dz_1}{(z_1-z)^2} R(z_1) \int\limits_0^{s_1} ds_2 R^{-1}(z_1,s_2) \oint\limits_{C_2} \tfrac{dz_2}{(z_2-z_1)^2} \cdots \\
& \quad \ldots R(z_{n-2}) \int\limits_0^{s_{n-2}} R^{-1}(z_{n-2},s_{n-1}) \oint\limits_{C_{n-1}} \tfrac{dz_{n-1}}{(z_{n-1}-z_{n-2})^2} \mathbb{U}_1(z_{n-1},\mathbf{w}) ds_{n-1}. \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{24}$$

We represent $\mathbb{U}_n(z,\mathbf{w})$ in the following form:

$$\mathbb{U}_n(z,\mathbf{w}) = \frac{(-1)^{n-1}}{(2\pi i)^{n-1}} \int\limits_0^s ds_1 \int\limits_0^{s_1} ds_2 \ldots \int\limits_0^{s_{n-2}} ds_{n-1} \oint\limits_{C_1} \frac{dz_1}{(z-z_1)^2} \cdots \oint\limits_{C_{n-2}} \frac{dz_{n-2}}{(z_{n-3}-z_{n-2})^2} \cdot$$
$$\cdot \oint\limits_{C_{n-1}} \frac{R(z)R^{-1}(z,s_1)R(z_1)R^{-1}(z_1,s_2)\ldots R(z_{n-2})R^{-1}(z_{n-2},s_{n-1})\mathbb{U}_1(z_{n-1},\mathbf{w})dz_{n-1}}{(z_{n-2}-z_{n-1})^2}.$$

Let $\|\cdot\|_k$ be the norm in $\mathbb{C}^k$; then, for all $z \in \widehat{P}_0 = \{z \in \mathbb{C} : |z-z_0| < r_0\}$ and all $\mathbf{w}$ from some subregion $\widehat{\mathbb{P}}_k$ of the polycircle $\mathbb{P}_k$, the following inequality takes place:

$$\|\mathbb{U}_n(z,\mathbf{w})\|_k \le \left| \int\limits_0^s ds_1 \int\limits_0^{s_1} ds_2 \ldots \int\limits_0^{s_{n-2}} ds_{n-1} \right| \frac{1}{(2\pi)^{n-1}} H_{n-1} \|\mathbb{U}_1(z,\mathbf{w})\|_k,$$

where

$$\begin{aligned}
H_{n-1} &= \oint\limits_{C_1} \tfrac{|d\,z_1|}{|z-z_1|^2} \cdots \oint\limits_{C_{n-1}} \tfrac{|d\,z_{n-1}|}{|z_{n-2}-z_{n-1}|^2} = \\
&= \int\limits_0^{2\pi} \tfrac{t_1 d\alpha}{t_1^2+|z|^2-2t_1|z|\cos\alpha} \cdots \int\limits_0^{2\pi} \tfrac{t_{n-1} d\alpha}{t_{n-1}^2+t_{n-2}^2-2t_{n-1}t_{n-2}\cos\alpha} = \\
&= \tfrac{(2\pi)^{n-1} t_1 t_2 \ldots t_{n-1}}{(t_1-|z|^2)(t_2^2-t_1^2)\ldots(t_{n-1}^2-t_{n-2}^2)} \le \tfrac{(2\pi)^{n-1} r_0^{n-1} (n-1)^{n-1}}{2^{n-2} \widetilde{r}_0^{\,n-1} (r-\widetilde{r}_0)^{n-1}}.
\end{aligned}$$

As we have

$$\left| \int\limits_0^s ds_1 \int\limits_0^{s_1} ds_2 \ldots \int\limits_0^{s_{n-2}} ds_{n-1} \right| = \frac{|s|^{n-1}}{(n-1)!},$$

then

$$\|\mathbb{U}_n(z,\mathbf{w})\|_k \le \frac{r_0^{n-1}(n-1)^{n-1}}{2^{n-2}\widetilde{r}_0^{\,n-1}(r-\widetilde{r}_0)^{n-1}(n-1)!} \|\mathbb{U}_1(z,\mathbf{w})\|_k,$$

and from that the convergence of the series (21) on any compact set from the set $\widehat{P}_0 \times \widehat{\mathbb{P}}_k$ follows.

Thus, it is proved that the components of the vector $\mathbb{U}(z,\mathbf{w},\varepsilon)$ form an independent system of integrals (invariants) and are holomorphic (ε-regular) at the point $\varepsilon = 0$. It is also clear that there is a statement similar to Theorem 5 on the existence of a pseudoholomorphic (ε-pseudoregular) solution of the Cauchy problem (16). Without loss of generality, we assume that $z_0 = 0$.

Theorem 6. *Let the entire functions* $\{\Phi_1, \ldots, \Phi_k\}$ *and the functions* $\{\varphi_1(z), \ldots, \varphi_k(z)\}$ *which are holomorphic in the circle* P_0 *be such that the essentially singular manifolds* $\{Q_1^+(\varphi_1, \Phi_1, T_0), \ldots, Q_k^+(\varphi_k, \Phi_k, T_0)\}$ *created by the functions described above where* T_0 *is some segment of the real axis, the left end of which coincides with the origin and belongs to the circle* $\widehat{P}_0$*, are sets bounded in* $\mathbb{C}$*; and the system of equations*

$$\begin{cases} \varphi_1'(z)V_1(z, \mathbf{w}) = \varphi_1(z)/\varepsilon, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \varphi_k'(z)V_k(z, \mathbf{w}) = \varphi_k(z)/\varepsilon, \end{cases}$$

in which $\{V_1(z, \mathbf{w}), \ldots, V_k(z, \mathbf{w})\}$ *are independent solutions of the equation* $D_k^{\mathbb{F}} V = 1$*, has a solution of the form*

$$\mathbf{w} = \mathbb{W}_0(z, \mathbf{q})\Big|_{\mathbf{q}=\{\Phi_1(\varphi_1(z)/\varepsilon), \ldots, \Phi_k(\varphi_k(z)/\varepsilon)\}},$$

each component $W_{0,i}(z, \mathbf{q})$ $(i = \overline{1,k})$ *of it is holomorphic on the set* $T_0 \times Q_1^+ \times \ldots \times Q_k^+$*. Then, the solution* $\mathbf{w}(z, \varepsilon)$ *of the initial problem* (16) *is pseudoholomorphic at the point* $\varepsilon = 0$ *(*ε*-pseudoregular).*

It should be noted [6] that this solution can be continued in a pseudoholomorphic way for a fixed $\varepsilon > 0$ from some segment $[0, \widetilde{T}_0]$ (see the end of the proof of Theorem 5) by segment $[0, T_0]$.

5. Conclusions

Further development of the axiomatic approach in the analytical singular perturbation theory will allow us to consider a more general class of equations with a small parameter, in particular, an analogue of nonlinear differential equations in partial derivatives (for example, equations of the Navier–Stokes type, etc.). This is very urgent since the range of problems leading to singularly perturbed problems is constantly expanding. In this sense, the "Dyson argument" that appeared in theoretical physics is quite indicative—the solutions of the equations arising in astrophysics can depend holomorphically on the gravitational constant only after isolating a revealing the essentially singular manifold [7].

As for the current state of the singular perturbation of theory, the asymptotic approach prevails there. In our opinion, when solving most singularly perturbed equations, the following methods are used: the Vasilieva–Butuzov–Nefedov boundary function method [8,9], the Maslov method [10], the Lomov regularization method [4,11], and the Bogolyubov–Krylov–Mitropolsky average method [12,13]. In the case of more specific situations, these methods are combined and new approaches to the asymptotic integration are proposed [14].

Author Contributions: Conceptualization and writing—review and editing by M.B. and V.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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