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On a Harmonic Univalent Subclass of Functions Involving a Generalized Linear Operator

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Received: 5 March 2020; Accepted: 21 March 2020; Published: 24 March 2020

Abstract: In this paper, a subclass of complex-valued harmonic univalent functions defined by a generalized linear operator is introduced. Some interesting results such as coefficient bounds, compactness, and other properties of this class are obtained.

Keywords: harmonic univalent functions; generalized linear operator; differential operator; Salagean operator; coefficient bounds

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1. Introduction

Let H represent the continuous harmonic functions which are harmonic in the open unit disk $U = \{z : z \in \mathbb{C}, |z| < 1\}$ and let A be a subclass of H which represents the functions which are analytic in U. A harmonic function in U could be expressed as $f = h + \overline{g}$, where h and g are in A, h is the analytic part of f, g is the co-analytic part of f and $|h'(z)| > |g'(z)|$ is a necessary and sufficient condition for f to be locally univalent and sense-preserving in U (see Clunie and Sheil-Small [1]). Now we write,

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ g(z) = \sum_{n=2}^{\infty} b_n z^n. \tag{1}$$

Let SH represents the functions of the form $f = h + \overline{g}$ which are harmonic and univalent in U, which normalized by the condition $f(0) = f_z(0) - 1 = 0$. The subclass SH^0 of SH consists of all functions in SH which have the additional property $f_{\overline{z}}(0) = 0$. The class SH was investigated by Clunie and Sheil-Smallas [1]. Since then, many researchers have studied the class SH and even investigated some subclasses of it. Also, we observe that the class SH reduces to the class S of normalized analytic univalent functions in U, if the co-analytic part of f is equal to zero. For $f \in S$, the Salagean differential operator $D^n (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ was defined by Salagean [2]. For $f = h + \overline{g}$ given by (1), Jahangiri et al. [3] defined the modified Salagean operator of f as

$$D^m f(z) = D^m h(z) + (-1)^m \overline{D^m g(z)},$$

where

$$D^m h(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n,\ D^m g(z) = \sum_{n=2}^{\infty} n^m b_n z^n.$$

Next, for functions $f \in A$, For $n \in \mathbb{N}_0, \beta \geq \gamma \geq 0$, Yalçın and Altınkaya [4] defined the differential operator of $I^m_{\gamma,\beta} f : SH^0 \to SH^0$. Now we define our differential operator:

$$I^0_{\delta,\mu,\lambda,\eta,\varsigma,\tau} f(z) = h(z) + \overline{g(z)}$$

$$I^1_{\delta,\mu,\lambda,\varsigma,\tau} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\mu+\lambda-(\delta-\varsigma)(\lambda-\tau)D^0 f(z)+(\delta-\varsigma)(\lambda-\tau)D^1 f(z)}{\mu+\lambda} \right) \tag{2}$$

$$= \frac{\mu+\lambda-(\delta-\varsigma)(\lambda-\tau)\left(h(z)+\overline{g(z)}\right)+(\delta-\varsigma)(\lambda-\tau)\left(zh(z)+z\overline{g'(z)}\right)}{\mu+\lambda}$$

$$I^m_{\delta,\mu,\lambda,\varsigma,\tau} f(z) = I^1_{\delta,\mu,\lambda,\varsigma,\tau}\left(I^{m-1}_{\delta,\mu,\lambda,\varsigma,\tau} f(z) \right). \tag{3}$$

If f is given by (1), then from (2) and (3), we get (see [5])

$$\begin{aligned} \underset{\delta,\mu,\lambda,\varsigma,\tau}{\overset{m}{I}} f(z) &= z + \sum_{n=2}^{\infty} \left(\frac{\mu+\lambda+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda} \right)^m a_n z^n \\ &+ (-1)^m \sum_{n=2}^{\infty} \left(\frac{\mu+\lambda+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda} \right)^m \overline{b_n z^n}. \end{aligned} \tag{4}$$

The operator $I^m_{\delta,\mu,\lambda,\varsigma,\tau} f(z)$ generalizes the following differential operators:

If $f \in A$, then when we take $\mu = 1,\ \lambda = 0,\ \delta = 0,\ \tau = 1,\ \varsigma = 1$ we obtain $I^m_{0,\tau,\ \delta,\varsigma} f(z)$ was introduced and studied by Ramadan and Darus [6]. By taking different choices of $\mu,\ \lambda,\ \delta,\ \tau$ *and* ς we get $I^m_{1-\lambda,\tau,\ 0,\varsigma} f(z)$ was introduced and studied by Darus and Ibrahim [7], $I^m_{\mu,\ \lambda,0,\ 1,0} f(z)$ was introduced and studied by Swamy [8], $I^m_{1-\lambda,0,\ 1,0} f(z)$ was introduced and studied by Al-Oboudi [9] and $I^m_{0,\ 0,,\ 1,0} f(z)$ was introduced and studied by Salagean [2].

If $f \in H$, then $I^m_{\mu,\ \lambda,0,\ 1,0} f(z)$ becomes the modified Salagean operator introduced by Yasar and Yalçin [10].

A function $f : U \to C$ is subordinate to the function $g : U \to C$ denoted by $f(z) \prec g(z)$, if there exists an analytic function $w : U \to U$ with $w(0) = 0$ such that

$$f(z) = g(w(z)), (z \in U).$$

Moreover, if the function g is univalent in U, then we have (see [11,12]):

$$f(z) \prec g(z) \text{ if and only if } f(0) = g(0),\ f(U) \subset g(U).$$

Denote by $SH^0(\delta,\mu,\lambda,\varsigma,\tau,m,A,B)$ the subclass of SH^0 consisting of functions of the form (1) that satisfy the condition

$$\frac{I^{m+1}_{\delta,\mu,\lambda,\varsigma,\tau} f(z)}{I^m_{\delta,\mu,\lambda,\varsigma,\tau} f(z)} \prec \frac{1+Az}{1+Bz}, -1 \le A < B \le 1 \tag{5}$$

where $I^m_{\delta,\mu,\lambda,\eta,\varsigma,\tau} f(z)$ is defined by (4). For relevant and recent references related to this work, we refer the reader to [13–20].

In this paper we use the same techniques that have been used earlier by Dziok [21] and Dziok et al. [22], to investigate coefficient bound, distortion bounds, and some other properties for the class $SH^0(\delta,\mu,\lambda,\varsigma,\tau,m,A,B)$.

2. Coefficient Bounds

In this section we find the coefficient bound for the class $SH^0(\delta,\mu,\lambda,\varsigma,\tau,m,A,B)$.

Theorem 1. *Let the function* $f(z) = h + \overline{g}$ *be defined by (1). Then* $f \in SH^0(\delta,\mu,\lambda,\varsigma,\tau,m,A,B)$ if

$$\sum_{n=2}^{\infty} (C_n|a_n| + D_n|b_n|) \le B - A \tag{6}$$

where

$$C_n = \left(\frac{\mu + \lambda + (\delta - \varsigma)(\lambda - \tau)(n-1)}{\mu + \lambda} \right)^m \left\{ \frac{(\delta - \varsigma)(\lambda - \tau)(n-1)[B+1] - (\mu + \lambda)(B - A)}{\mu + \lambda} \right\} \tag{7}$$

and

$$D_n = \left(\frac{\mu + \lambda + (\delta - \varsigma)(\lambda - \tau)(n-1)}{\mu + \lambda} \right)^m \left\{ \frac{[A + B(2 + (\delta - \varsigma)(\lambda - \tau)(n-1))](\mu + \lambda)}{\mu + \lambda} \right\}. \tag{8}$$

Proof. Let $a_n \neq 0$ or $b_n \neq 0$ for $n \geq 2$. Since $C_n,\ D_n \geq n(B - A)$ by (6), we obtain

$$\begin{aligned} \left|h'(z)\right| - \left|g'(z)\right| &\geq 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} - \sum_{n=2}^{\infty} n|b_n||z|^{n-1} \\ &\geq 1 - |z| \sum_{n=2}^{\infty} (n|a_n| + n|b_n|) \\ &\geq 1 - \tfrac{|z|}{B-A} \sum_{n=2}^{\infty} (C_n|a_n| + D_n|b_n|) \\ &\geq 1 - |z| > 0. \end{aligned}$$

Therefore, f is univalent in U. To ensure the univalence condition, consider $z_1, z_2 \in U$ so that $z_1 \neq z_2$. Then

$$\left| \frac{z_1^n - z_2^n}{z_1 - z_2} \right| = \left| \sum_{m=1}^{n} z_1^{m-1} - z_2^{n-m} \right| \leq \sum_{m=1}^{n} \left|z_1^{m-1}\right| \left|z_2^{n-m}\right| < n\ ,\ n \geq 2.$$

So, we have

$$\begin{aligned} \left| \tfrac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| &\geq 1 - \left| \tfrac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| = 1 - \left| \tfrac{\sum_{n=2}^{\infty} b_k \left(z_1^n - z_2^n\right)}{z_1 - z_2 + \sum_{n=2}^{\infty} a_n \left(z_1^n - z_2^n\right)} \right| \\ &> 1 - \tfrac{\sum_{n=2}^{\infty} n|b_n|}{1 - \sum_{n=2}^{\infty} n|a_n|} \geq 1 - \tfrac{\sum_{n=2}^{\infty} \frac{D_n}{B-A}|b_n|}{\sum_{n=2}^{\infty} \frac{C_n}{B-A}|a_n|} \geq 0, \end{aligned}$$

which proves univalences.

On the other hand, $f \in SH^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B)$ if and only if there exists a function w; with $w(0) = 0$, and $\left|w(z)\right| < 1 (z \in U)$ such that

$$\frac{I^{m+1}_{\delta,\mu,\lambda,\varsigma,\tau} f(z)}{I^{m}_{\delta,\mu,\lambda,\varsigma,\tau} f(z)} \prec \frac{1 + Az}{1 + Bz}$$

or

$$\frac{I^{m+1}_{\delta,\mu,\lambda,\varsigma,\tau} f(z) - I^{m}_{\delta,\mu,\lambda,\varsigma,\tau} f(z)}{BI^{m+1}_{\delta,\mu,\lambda,\varsigma,\tau} f(z) - AI^{m}_{\delta,\mu,\lambda,\varsigma,\tau} f(z)} < 1, \quad (z \in U). \tag{9}$$

The above inequality (9) holds, since for $|z| = r$ $(0 < r < 1)$ we obtain

$$\left|I^{m+1}_{\delta,\mu,\lambda,\varsigma,\tau}f(z) - I^{m}_{\delta,\mu,\lambda,\varsigma,\tau}f(z)\right| - \left|BI^{m+1}_{\delta,\mu,\lambda,\varsigma,\tau}f(z) - AI^{m}_{\delta,\mu,\lambda,\varsigma,\tau}f(z)\right|$$
$$= \left|\sum_{n=2}^{\infty}\left(\tfrac{\mu+\lambda+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda}\right)^m \tfrac{(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda}a_n z^n\right.$$
$$\left.+(-1)^m\sum_{n=2}^{\infty}\left(\tfrac{\mu+\lambda+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda}\right)^m \tfrac{2(\mu+\lambda)+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda}\overline{b_n z^n}\right|$$
$$-\left|(B-A)z + \sum_{n=2}^{\infty}\left(\tfrac{\mu+\lambda+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda}\right)^m\left(B\tfrac{\mu+\lambda+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda} - A\right)a_n z^n\right.$$
$$\left.-(-1)^m\sum_{n=2}^{\infty}\left(\tfrac{\mu+\lambda+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda}\right)^m\left(B,\tfrac{2(\mu+\lambda)+\delta(-\varsigma)(\lambda-\tau)(1-n)}{\mu+\lambda} + A\right)\overline{b_n z^n}\right|$$
$$\leq \sum_{n=2}^{\infty}\left(\tfrac{\mu+\lambda+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda}\right)^m \tfrac{(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda}|a_n|r^n +$$
$$\sum_{n=2}^{\infty}\left(\tfrac{\mu+\lambda+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda}\right)^m \tfrac{2(\mu+\lambda)+(\delta-\varsigma)(\lambda-\tau)(1-n)}{\mu+\lambda}|b_n|r^n - (B-A)r$$
$$+\sum_{n=2}^{\infty}\left(\tfrac{\mu+\lambda+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda}\right)^m\left(B\tfrac{\mu+\lambda+(\delta-\varsigma)(\lambda-\tau)(n-1)+A}{\mu+\lambda} - A\right)|a_n|r^n$$
$$+\sum_{n=2}^{\infty}\left(\tfrac{\mu+\lambda+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda}\right)^m\left(B\tfrac{2(\mu+\lambda)+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda} + A\right)|b_n|r^n$$
$$\leq r\left\{\sum_{n=2}^{\infty}(C_n|a_n| + D_n|b_n|)r^{n-1} - (B-A)\right\} < 0.$$

Therefore, $f \in SH^0(\delta,\mu,\lambda,\varsigma,\tau,m,A,B)$, and so the proof is completed.

Next we show that the condition (6) is also necessary for the functions $f \in H$ to be in the class $SH^0_T(\delta,\mu,\lambda,\varsigma,\tau,m,A,B) = T^m \cap SH^0(\delta,\mu,\lambda,\varsigma,\tau,m,A,B)$ where T^m is the class of functions $f = h + \overline{g} \in SH^0$ so that

$$f = h + \overline{g} = z - \sum_{n=2}^{\infty} a_n z^n + (-1)^m \sum_{n=2}^{\infty}|b_n|\overline{z^n}(z \in U). \tag{10}$$

□

Theorem 2. *Let $f = h + \overline{g}$ be defined by (10). Then $f \in SH^0_T(\delta,\mu,\lambda,\varsigma,\tau,m,A,B)$ if and only if the condition (6) holds.*

Proof. For this proof, we let the fractions $\frac{(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda} = L$ and $\frac{2(\mu+\lambda)+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda} = K$. The first part "if statement" follows from Theorem 1. Conversely, we suppose that $f \in SH^0_T(\delta,\mu,\lambda,\varsigma,\tau,m,A,B)$, then by (9) we have

$$\left|\frac{\sum_{n=2}^{\infty}\left[(L)^m\frac{(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda}|a_n|z^n + (K)^m\frac{2(\mu+\lambda)+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda}\overline{|b_n|z^n}\right]}{(B-A)z - \sum_{n=2}^{\infty}\left[(L)^m(BL-A)|a_n|z^n + (K)^m(BK+A)|b_n|z^n\right]}\right| < 1.$$

For $|z| = r < 1$, we obtain

$$\frac{\sum\limits_{n=2}^{\infty}\left[(L)^{m}\frac{(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda}|a_n| + (K)^{m}\frac{2(\mu+\lambda)+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda}\overline{|b_n|}\right]r^{n-1}}{(B-A) - \sum\limits_{n=2}^{\infty}\left[(L)^{m}(BL-A)|a_n| + (K)^{m}(BK+A)|b_n|\right]r^{n-1}} < 1.$$

Thus, for C_n and D_n as defined by (7) and (8), we have

$$\sum_{n=2}^{\infty}[C_n|a_n| + D_n|b_n|]r^{n-1} < B - A(0 \le r < 1).(11) \tag{11}$$

Let $\{\rho_n\}$ be the sequence of partial sums of the series

$$\sum_{k=2}^{n}[C_k|a_k| + D_k|b_k|].$$

Then $\{\rho_n\}$ is a non-decreasing sequence and by (11) it is bounded above by $B - A$. Thus, it is convergent and

$$\sum_{n=2}^{\infty}[C_n|a_n| + D_n|b_n|] = \lim_{n\to+\infty}\rho_n \le B - A.$$

This gives us the condition (6). □

3. Compactness and Convex

In this section we obtain the compactness and the convex relation for the class $SH^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B)$.

Theorem 3. *The class $SH^0_T(\delta, \mu, \lambda, \varsigma, \tau, m, A, B)$ is convex and compact subset of SH.*

Proof. Let $f_t \in SH^0_T(\delta, \mu, \lambda, \varsigma, \tau, m, A, B)$, where

$$f_t(z) = z - \sum_{n=2}^{\infty}|a_{t,n}|z^n + (-1)^m\sum_{n=2}^{\infty}|b_{t,n}|\overline{z^n}(z \in U,\ t \in \mathbb{N}). \tag{12}$$

Then for $0 \le \psi \le 1$, let $f_1, f_2 \in SH^0_T(\delta, \mu, \lambda, \varsigma, \tau, m, A, B)$ be defined by (12). Then

$$\begin{aligned}\xi(z) &= \psi f_1(z) + (1-\psi)f_2(z)\\ &= z - \sum_{n=2}^{\infty}\left(\psi\left|a_{1,n}\right| + (1-\psi)\left|a_{2,n}\right|\right)z^n + (-1)^m\sum_{n=2}^{\infty}\left(\psi\left|b_{1,n}\right| + (1-\psi)\left|b_{2,n}\right|\right)\overline{z^n}\end{aligned}$$

and

$$\begin{aligned}&\sum_{n=2}^{\infty}\left\{C_n\left(\psi\left|a_{1,n}\right| + (1-\psi)\left|a_{2,n}\right|\right) + D_n\left(\psi\left|b_{1,n}\right| + (1-\psi)\left|b_{2,n}\right|\right)\right\}\\ &= \psi\sum_{n=2}^{\infty}\left\{C_n\left|a_{1,n}\right| + D_n\left|b_{1,n}\right|\right\} + (1-\psi)\sum_{n=2}^{\infty}\left\{C_n\left|a_{2,n}\right| + D_n\left|b_{2,n}\right|\right\}\\ &\le \psi(B-A) + (1-\psi)(B-A) = B - A.\end{aligned}$$

Thus, the function $\xi = \psi f_1(z) + (1-\psi)f_2(z)$ is in the class $SH^0_T(\delta, \mu, \lambda, \varsigma, \tau, m, A, B)$. This implies that $SH^0_T(\delta, \mu, \lambda, \varsigma, \tau, m, A, B)$ is convex.

For $f_t \in SH_T^0(\delta,\mu,\lambda,\varsigma,\tau,m,A,B), t \in \mathbb{N}$ and $|z| \le r$ $(0 < r < 1)$, then we have

$$\begin{aligned}\left|f_t(z)\right| &\le r + \sum_{n=2}^{\infty}\left\{\left|a_{t,n}\right| + \left|b_{t,n}\right|\right\}r^n \\ &\le r + \sum_{n=2}^{\infty}\left\{C_n\left|a_{t,n}\right| + D_n\left|b_{t,n}\right|\right\}r^n \\ &\le r + (B-A)r^2.\end{aligned}$$

Therefore, $SH_T^0(\delta,\mu,\lambda,\varsigma,\tau,m,A,B)$ is uniformly bounded. Let

$$f_t(z) = z - \sum_{n=2}^{\infty}\left|a_{t,n}\right|z^n + (-1)^m\sum_{n=2}^{\infty}\left|b_{t,n}\right|\overline{z^n}(z \in U,\ t \in \mathbb{N}).$$

also, let $f = h + \overline{g}$ where h and g are given by (1). Then by Theorem 2 we get

$$\sum_{n=2}^{\infty}\left\{C_n|a_n| + D_n\left|b_{t,n}\right|\right\} \le B - A. \tag{13}$$

If we assume $f_t \to f$, then we get that $\left|a_{t,n}\right| \to |a_n|$ and $\left|b_{t,n}\right| \to |b_n|$ as $n \to +\infty$ $(t \in \mathbb{N})$. Let $\{\rho_n\}$ be the sequence of partial sums of the series $\sum_{n=2}^{\infty}\left\{C_n\left|a_{t,n}\right| + D_n\left|b_{t,n}\right|\right\}$. Then $\{\rho_n\}$ is a non-decreasing sequence and by (13) it is bounded above by $B - A$. Thus, it is convergent and

$$\sum_{n=2}^{\infty}\left\{C_n\left|a_{t,n}\right| + D_n\left|b_{t,n}\right|\right\} = \lim_{n\to\infty}\rho_n \le B - A.$$

Therefore, $f \in SH_T^0(\delta,\mu,\lambda,\varsigma,\tau,m,A,B)$ and therefore the class $SH_T^0(\delta,\mu,\lambda,\varsigma,\tau,m,A,B)$ is closed. As a result, the class is closed, and the class $SH_T^0(\delta,\mu,\lambda,\varsigma,\tau,m,A,B)$ is also compact subset of SH, which completes the proof. □

Lemma 1 [23]. *Let $f = h + \overline{g}$ be so that h and g are given by (1). Furthermore, let*

$$\sum_{n=2}^{\infty}\left\{\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right\} \le 1(z \in U)$$

where $0 \le \alpha < 1$. Then f is harmonic, orientation preserving, univalent in U and f is starlike of order α.

Theorem 4. *Let $0 \le \alpha < 1, C_n$ and D_n be defined by (7) and (8). Then*

$$r_\alpha^*\left(SH_T^0(\delta,\mu,\lambda,\varsigma,\tau,n,A,B)\right) = \inf_{n\ge 2}\left[\frac{1-\alpha}{B-A}\min\left\{\frac{C_n}{n+\alpha}, \frac{D_n}{n+\alpha}\right\}\right]^{\frac{1}{n-1}}, \tag{14}$$

where r_α^* is the radius of starlikeness of order α.

Proof. Let $f \in SH_T^0(\delta,\mu,\lambda,\varsigma,\tau,m,A,B)$ be of the form (10). Then, for $|z| = r < 1$, we get

$$\begin{aligned}
&\left|\frac{I_{0,\eta}f(z)-(1+\alpha)f(z)}{I_{0,\eta}f(z)+(1+\alpha)f(z)}\right| \\
&= \left|\frac{-\alpha z-\sum\limits_{n=2}^{\infty}(n-1-\alpha)|a_n|z^n-(-1)^m\sum\limits_{n=2}^{\infty}(n+1+\alpha)|b_n|\overline{z^n}}{(2-\alpha)z-\sum\limits_{n=2}^{\infty}(n-1-\alpha)|a_n|z^n-(-1)^m\sum\limits_{n=2}^{\infty}(n-1+\alpha)|b_n|\overline{z^n}}\right| \\
&\le \frac{\alpha-\sum\limits_{n=2}^{\infty}\left\{(n-1-\alpha)|a_n|-(-1)^m\sum\limits_{n=2}^{\infty}(n+1+\alpha)|b_n|\right\}r^{n-1}}{2-\alpha-\sum\limits_{n=2}^{\infty}\left\{(n-1-\alpha)|a_n|-(-1)^m\sum\limits_{n=2}^{\infty}(n-1+\alpha)|b_n|\right\}}.
\end{aligned}$$

By using Lemma 1, we observe that f is starlike of order α in U_r if and only if

$$\left|\frac{I_{0,\eta}f(z)-(1+\alpha)f(z)}{I_{0,\eta}f(z)+(1+\alpha)f(z)}\right| < 1, z \in U_r$$

or

$$\sum_{n=2}^{\infty}\left\{\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right\}r^{n-1} \le 1. \tag{15}$$

Furthermore, by using Theorem 2, we get

$$\sum_{n=2}^{\infty}\left\{\frac{C_n}{1-\alpha}|a_n| + \frac{D_n}{1-\alpha}|b_n|\right\}r^{n-1} \le 1.$$

Condition (15) is true if

$$\frac{n-\alpha}{1-\alpha}r^{n-1} \le \frac{C_n}{B-A}r^{n-1}.$$

This proves

$$\frac{n+\alpha}{1-\alpha}r^{n-1} \le \frac{D_n}{B-A}r^{n-1} (n = 2, 3 \ldots).$$

So, the function f is starlike of order α in the disk $U_{r_\alpha}^*$ where

$$r_\alpha^* = \inf_{n\ge 2}\left[\frac{1-\alpha}{B-A}\min\left\{\frac{C_n}{n+\alpha}, \frac{D_n}{n+\alpha}\right\}\right]^{\frac{1}{n-1}},$$

and the function

$$f_n(z) = h_n(z) + \overline{g_n(z)} = z - \frac{B-A}{C_n}z^n + (-1)^m\frac{B-A}{D_n}\overline{z^n}.$$

So, the radius r_α^* cannot be larger. Then we get (14). □

4. Extreme Points

In this section we find the extreme points for the class $SH^0(\delta,\mu,\lambda,\varsigma,\tau,m,A,B)$.

Theorem 5. *The extreme points of $SH_T^0(\delta,\mu,\lambda,\varsigma,\tau,m,A,B)$ are the functions f of the form (1) where $h = h_k$ and $g = g_k$ are of the form*

$$\begin{aligned}
h_1(z) &= z, \\
h_n(z) &= z - \tfrac{B-A}{C_n}z^n, \\
g_n(z) &= (-1)^m\tfrac{B-A}{D_n}\overline{z^n}, \ (z \in U, \ n \ge 2).
\end{aligned} \tag{16}$$

Proof. Suppose that $g_n = \psi f_1 + (1-\psi) f_2$ where $0 < \psi < 1$ and $f_1, f_2 \in SH_T^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B)$ are written in the form

$$f_t(z) = z - \sum_{n=2}^{\infty} \left| a_{t,n} \right| z^n + (-1)^m \sum_{n=2}^{\infty} \left| b_{t,n} \right| \overline{z^n} (z \in U,\ t \in \{1,2\}).$$

Then, by (16), we get

$$\left| b_{1,n} \right| = \left| b_{2,n} \right| = \frac{B-A}{D_n},$$

and $a_{1,t} = a_{2,t} = 0$ for $t \in \{2,3\ldots\}$ and $b_{1,t} = b_{2,t} = 0$ for $t \in \{2,3\ldots\} \setminus \{n\}$. It follows that $g_n(z) = f_1(z) = f_2(z)$ and g_n are in the class of extreme points of the class $SH_T^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B)$. We also can ensure that the functions $h_n(z)$ are the extreme points of the class $SH_T^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B)$. Now, assume that a function f of the form (1) is in the class of the extreme points of the class $SH_T^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B)$ and f is not of the form (16). Then there exists $k \in \{2,3\ldots\}$ such that

$$0 < |a_k| < \frac{B-A}{\left(\frac{\mu+\lambda+(\delta-\varsigma)(\lambda-\tau)(k-1)}{\mu+\lambda}\right)^m \left\{\frac{(\delta-\varsigma)(\lambda-\tau)[(k-1)(B+1)]+(\mu+\lambda)(B-A)}{\mu+\lambda}\right\}}$$

or

$$0 < |b_k| < \frac{B-A}{\left(\frac{\mu+\lambda-(\delta-\varsigma)(\lambda-\tau)(n+1)}{\mu+\lambda}\right)^m \left\{\frac{[A+B(2+(\delta-\varsigma)(\lambda-\tau)(n-1))](\mu+\lambda)}{\mu+\lambda}\right\}}.$$

If

$$0 < |a_k| < \frac{B-A}{\left(\frac{\mu+\lambda+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda}\right)^m \left\{\frac{(\delta-\varsigma)(\lambda-\tau)(n-1)[B+1]-(\mu+\lambda)(B-A)}{\mu+\lambda}\right\}}$$

then putting

$$\psi = \frac{|a_k|\left[\left(\frac{\mu+\lambda+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda}\right)^m \left\{\frac{(\delta-\varsigma)(\lambda-\tau)(n-1)[B+1]-(\mu+\lambda)(B-A)}{\mu+\lambda}\right\}\right]}{B-A}$$

and

$$\chi = \frac{f-\psi h_k}{1-\psi},$$

we have $0 < \psi < 1$, $h_k \neq \chi$. Therefore, f is not in the class of the extreme points of the class $SH_T^0(\delta, \mu, \lambda, \eta, \varsigma, \tau, m, A, B)$. Similarly, if

$$0 < |b_k| < \frac{B-A}{\left(\frac{\mu+\lambda+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda}\right)^m \left\{\frac{[A+B(2+(\delta-\varsigma)(\lambda-\tau)(n-1))](\mu+\lambda)}{\mu+\lambda}\right\}}$$

then putting

$$\psi = \frac{|b_k|\left(\frac{\mu+\lambda+(\delta-\varsigma)(\lambda-\tau)(n-1)}{\mu+\lambda}\right)^m \left\{\frac{[A+B(2+(\delta-\varsigma)(\lambda-\tau)(n-1))](\mu+\lambda)}{\mu+\lambda}\right\}}{B-A}$$

and

$$\chi = \frac{f-\psi g_k}{1-\psi},$$

we have $0 < \psi < 1$, $g_k \neq \chi$. It follows that f is not in the family of extreme points of the class $SH_T^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B)$ and so the proof is completed. □

Author Contributions: Conceptualization, A.T.Y. and Z.S.; methodology, A.T.Y.; software, A.T.Y.; validation, A.T.Y. and Z.S.; formal analysis, A.T.Y.; investigation, A.T.Y.; resources, A.T.Y.; data curation, A.T.Y.; writing—original draft preparation, A.T.Y.; writing—review and editing, Z.S.; visualization, Z.S.; supervision, Z.S.; project administration, Z.S.; funding acquisition, Z.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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