

Article

Regularized Solution of Singularly Perturbed Cauchy Problem in the Presence of Rational “Simple” Turning Point in Two-Dimensional Case

Alexander Eliseev ^{†,‡} and Tatjana Ratnikova ^{*,‡}

Research Group on Macro-Structural Modeling of the Russian Economy, National Research University “MPEI”, 111250 Moscow, Russia; predikat@bk.ru

* Correspondence: tatrati@mail.ru; Tel.: +7-916-373-18-03

† Current address: National Research University “MPEI”, Krasnokazarmennaya st., 14, 111250 Moscow, Russia.

‡ These authors contributed equally to this work.

Received: 17 September 2019; Accepted: 30 October 2019; Published: 1 November 2019



Abstract: By Lomov’s S.A. regularization method, we constructed an asymptotic solution of the singularly perturbed Cauchy problem in a two-dimensional case in the case of violation of stability conditions of the limit-operator spectrum. In particular, the problem with a “simple” turning point was considered, i.e., one eigenvalue vanishes for $t = 0$ and has the form $t^{m/n}a(t)$ (limit operator is discretely irreversible). The regularization method allows us to construct an asymptotic solution that is uniform over the entire segment $[0, T]$, and under additional conditions on the parameters of the singularly perturbed problem and its right-hand side, the exact solution.

Keywords: singularly perturbed Cauchy problem; regularized asymptotic solution; rational “simple” turning point

1. Introduction

This work consists of five parts. The first part is an introduction. The second part is nomenclature. The third part presents the formulation of the Cauchy problem in the two-dimensional case if stability conditions for the spectrum of the limit operator are violated (the spectrum-stability condition means that eigenvalues of the operator $A(\tau)$ satisfy conditions $\lambda_1(\tau) \neq \lambda_2(\tau)$, $\tau \in [0, T]$ and $\lambda_i \neq 0$, $i = 1, 2$).

A “simple” pivot point of a limit operator (matrix $A(\tau)$) is understood when one eigenvalue vanishes at one point (i.e., matrix $A(\tau)$ is irreversible at this point). In [1], the case was considered of when one of the eigenvalues that had the form $\tau^n a(\tau)$, $a(\tau) \neq 0$, n was natural; in [2] the features of the solution were identified and described for a rational “simple” turning point in the one-dimensional case (when the eigenvalue had the form $\tau^{m/n}a(\tau)$, $a(\tau) \neq 0$).

In this article, we consider the case with a “simple” turning point when one of the two eigenvalues of the operator vanishes at $\tau = 0$ and has the form $\tau^{m/n}a(\tau)$, $a(\tau) \neq 0$.

The fourth part describes the formalism of the Lomov regularization method [1,3,4] that allows one to construct an asymptotic solution uniform over the entire segment $[0, T]$, under additional conditions on the parameters of a singularly perturbed problem, and its right side is the exact solution. The idea of this paper goes back to [1], in which methods were developed for solving a singularly perturbed Cauchy problem in the case of a “simple” turning point of a limit operator with a natural exponent. A lemma is given on the estimation of basic singular functions, a theorem on the point solvability of iterative problems is proved, and the leading term of the asymptotic behavior of a singularly perturbed Cauchy problem is written out.

In the fifth part of the paper, we prove a theorem on the asymptotic behavior of a regularized series and a theorem on the passage to the limit as a small parameter tends to zero. For a parabolic equation, an example of solving a singularly perturbed Cauchy problem with a fractional turning point $\lambda(\tau) = \tau^{1/2}$ is given.

The sixth part is the conclusion.

2. Problem Formulation

Consider the Cauchy problem:

$$\begin{cases} \bar{\varepsilon} \dot{u}(\tau) = A(\tau)u(\tau) + h(\tau), \\ u(0, \varepsilon) = u^0, \end{cases} \quad (1)$$

where

- (1) τ is a variable, $\tau \in [0, T]$;
- (2) $u(\tau)$ is a function, $u(\tau) \in C^\infty[0, T]$;
- (3) $A(\tau)$ is a matrix of size (2×2) , $A(\tau) \in C^\infty(0, T]$;
- (4) $h(\tau)$ is a function, $h(\tau) \in C^\infty[0, T]$;
- (5) $\lambda_1(\tau), \lambda_2(\tau)$ are eigenvalues of matrix $A(\tau)$; $\lambda_1(\tau) \neq \lambda_2(\tau)$, $\tau \in [0, T]$; $\lambda_2(\tau) = \tau^{m/n}a(\tau)$, where $a(\tau) < 0$, $\tau \in [0, T]$, $a(\tau) \in C^\infty[0, T]$;
- (6) m, n are natural numbers;
- (7) $\operatorname{Re} \lambda_1(\tau) \leq 0$;
- (8) $A(t) \in C^\infty[0, T]$, where $t = \tau^{1/n}$;
- (9) $\bar{\varepsilon}, \varepsilon = \bar{\varepsilon}/n \in \mathbb{R}$ there is a small parameter of the problem.

We make the change of variables in Problem (1): $t = \tau^{1/n}$. Then $\tau^{m/n} = t^m$ and

$$\frac{du}{d\tau} = \frac{du}{dt} \cdot \frac{dt}{d\tau} = \dot{u}(t) \frac{1}{n} \tau^{(1-n)/n} = \dot{u}(t) \frac{1}{n} t^{1-n}.$$

Equation (1) takes the form:

$$\frac{\bar{\varepsilon}}{n} \dot{u}(t) t^{1-n} = A(t^n)u(t) + h(t^n)$$

or

$$\frac{\bar{\varepsilon}}{n} \dot{u}(t) = t^{n-1} A(t^n)u(t) + t^{n-1} h(t^n).$$

Denote $\bar{\varepsilon}/n = \varepsilon$, $t^{n-1} A(t^n) = B(t)$. Task (1) takes the form:

$$\begin{cases} \varepsilon \dot{u}(t) = B(t)u(t) + t^{n-1} h(t^n), \\ u(0, \varepsilon) = u^0. \end{cases} \quad (2)$$

Operator $B(t)$ has eigenvalues $\bar{\lambda}_1(t) = t^{n-1} \lambda_1(t^n)$, $\bar{\lambda}_2(t) = t^p a(t^n)$, where $p = m + n - 1$, and corresponding vectors $\bar{e}_1(t) = e_1(t^n)$, $\bar{e}_2(t) = e_2(t^n)$, where $e_1(\tau)$, $e_2(\tau)$ are eigenvectors of operator $A(\tau)$, i.e.,

$$\begin{aligned} B(t)\bar{e}_1(t) &= \bar{\lambda}_1(t)\bar{e}_1(t) = t^{n-1} \lambda_1(t^n) e_1(t^n); \\ B(t)\bar{e}_2(t) &= \bar{\lambda}_2(t)\bar{e}_2(t) = t^p a(t^n) e_1(t^n). \end{aligned}$$

Methods for solving the Cauchy problem (2) are described in [1]. Basic singularities (2) have the form:

$$e^{\varphi_i(t)/\varepsilon}, \quad i = 1, 2; \quad \sigma_i(t, \varepsilon) = e^{\varphi_2(t)/\varepsilon} \int_0^t e^{-\varphi_2(s)/\varepsilon} s^i ds, \quad i = \overline{0, p-1}; \quad (3)$$

where $\varphi_1(t) = \int_0^t s^{n-1} \lambda_1(s^n) ds$, $\varphi_2(t) = \int_0^t s^p a(s^n) ds$.

Singularities (3) in the source variables have the form

$$e^{\varphi_1(\tau)/\varepsilon}, e^{\varphi_2(\tau)/\varepsilon}, \sigma_i(\tau, \varepsilon) = e^{\varphi_2(\tau)/\varepsilon} \int_0^\tau e^{-\varphi_2(s)/\varepsilon} s^{(i+1-n)/n} ds, \quad i = \overline{0, p-1};$$

where $\varphi_1(\tau) = \int_0^\tau \lambda_1(s) ds$, $\varphi_2(\tau) = \int_0^\tau a(s) s^{m/n} ds$.

3. Formalism of Regularization Method

Point $\varepsilon = 0$ for Problem (1) is special in the sense that classical existence theorems for the solution of the Cauchy problem do not take place. Therefore, in solving this problem, essentially singular singularities arise. When the stability condition for spectrum $A(t)$ is satisfied, singular singularities are described using exponentials of the form:

$$e^{\varphi_i(t)/\varepsilon}, \quad \varphi_i(t) = \int_0^t \lambda_i(s) ds, \quad i = \overline{1, 2}, \quad \lambda_1(t) \neq \lambda_2(t), \quad \lambda_i(t) \neq 0, \quad t \in [0, T],$$

where $\varphi_i(t)$ is a smooth function (in the general case, complex) of a real variable t . To solve linear homogeneous equations, such singularities have been described by Liouville [5–8].

If stability conditions are violated for at least one point of the spectrum of operator $A(t)$, then besides exponentially essentially singularities in the solution of the inhomogeneous equation, singularities of the following form also appear:

$$\sigma_i = e^{\varphi_1(t)/\varepsilon} \int_0^t e^{-\varphi_1(s)/\varepsilon} s^i ds, \quad i = \overline{0, k-1},$$

(k is the extreme zero of $\lambda_1(t)$), which, for $\varepsilon \rightarrow 0$, has a power character of decreasing under the corresponding restrictions on $\lambda_1(t)$, while it is assumed that the remaining points of the spectrum do not vanish at $t = 0$.

Singularly perturbed problems arise in cases when the domain of definition of the initial operator, depending on ε with $\varepsilon \neq 0$, does not coincide with the domain of definition of the limit operator with $\varepsilon = 0$. When studying problems with a "simple" turning point, additional conditions arise when the domain of values of the original operator does not coincide with the domain of values of the limit operator.

Further, we need estimates of functions describing the basic singularities.

Lemma 1. Let the conditions on the spectrum of operator $A(t)$ 5) ÷ 7) be satisfied. Then, the estimates hold:

(a) if $\forall t \in [0, T] \quad \operatorname{Re} \lambda_i(t) \leq 0, i = \overline{1, 2}$, then

$$\left| e^{\frac{1}{\varepsilon} \int_0^t \bar{\lambda}_i(s) ds} \right| \leq C, \quad |\sigma_k(t, \varepsilon)| \leq C,$$

where C is a constant, $k = \overline{0, p-1}$, $p = m + n - 1$;

(b) if $\operatorname{Re} \lambda_1(t) \leq -\alpha < 0$, $\operatorname{Re} a(t) \leq -\alpha < 0$, then

$$\left| e^{\frac{1}{\varepsilon} \int_0^t \bar{\lambda}_1(s) ds} \right| \leq e^{-\frac{\alpha t^n}{\varepsilon^n}}, \quad \left| e^{\frac{1}{\varepsilon} \int_0^t \bar{\lambda}_2(s) ds} \right| \leq e^{-\frac{\alpha t^{p+1}}{\varepsilon^{(p+1)}}}, \quad |\sigma_k(t, \varepsilon)| \leq C \varepsilon^{\frac{k+1}{p+1}}, \quad k = \overline{0, p-1}, \quad p = m + n - 1.$$

Proof of Lemma 1. (a) In this case, estimates are obvious.

$$(b) \left| e^{\frac{1}{\varepsilon} \int_0^t \bar{\lambda}_1(s) ds} \right| \leq e^{-\frac{\alpha}{\varepsilon} \int_0^t s^{n-1} ds} = e^{-\frac{\alpha t^n}{\varepsilon^n}}, \quad \left| e^{\frac{1}{\varepsilon} \int_0^t \bar{\lambda}_2(s) ds} \right| \leq e^{-\frac{\alpha}{\varepsilon} \int_0^t s^p ds} = e^{-\frac{\alpha t^{p+1}}{\varepsilon(p+1)}},$$

$$\begin{aligned} |\sigma_k(t, s)| &= \left| \int_0^t e^{\frac{1}{\varepsilon} \int_s^t \bar{\lambda}_2(s) ds} s^k ds \right| \leq \int_0^t e^{-\frac{\alpha}{\varepsilon} \int_s^t s_1^p ds_1} s^k ds = \int_0^t e^{-\frac{\alpha(t^{p+1}-s^{p+1})}{\varepsilon(p+1)}} s^k ds = \frac{\int_0^t e^{\frac{\alpha s^{p+1}}{\varepsilon(p+1)}} s^k ds}{e^{\frac{\alpha t^{p+1}}{\varepsilon(p+1)}}} = \\ &= \frac{\frac{k+1}{\varepsilon^{p+1}} \int_0^t e^{\frac{\alpha s^{p+1}}{\varepsilon(p+1)}} s^k ds}{e^{\frac{\alpha t^{p+1}}{\varepsilon(p+1)}}}. \end{aligned}$$

Denote $\tau = \frac{t}{\varepsilon^{1/(p+1)}}$. Consider a fraction when $\tau \rightarrow \infty$; then, we have

$$\frac{\int_0^\tau e^{\frac{\alpha \xi^{p+1}}{\varepsilon^{p+1}}} \xi^k d\xi}{e^{\frac{\alpha \tau^{p+1}}{\varepsilon^{p+1}}}} \sim \frac{\tau^k}{\alpha \tau^p} \xrightarrow{\tau \rightarrow \infty} 0, \text{ as } k < p.$$

Consequently, $\sigma_k(t, \varepsilon) = \underline{O}(\varepsilon^{\frac{k+1}{p+1}})$. $\square$

Remark 1. Estimates in the source variables have the form:

$$\left| e^{\frac{1}{\varepsilon} \int_0^t \lambda_1(s) ds} \right| \leq e^{-\frac{\alpha t}{\varepsilon}}, \quad \left| e^{\frac{1}{\varepsilon} \int_0^t \lambda_2(s) ds} \right| \leq e^{-\frac{\alpha t^{m/n+1}}{\varepsilon(m/n+1)}}, \quad \sigma_k(t, \varepsilon) = \underline{O}(\varepsilon^{\frac{k+1}{m/n+1}}).$$

According to the regularization method, we seek a solution of Problem (2) in the form

$$u(t, \varepsilon) = x(t, \varepsilon)e^{\varphi_1(t)/\varepsilon} + y(t, \varepsilon)e^{\varphi_2(t)/\varepsilon} + \sum_{i=0}^{p-1} z^i(t, \varepsilon)\sigma_i(t, \varepsilon) + W(t, \varepsilon), \quad (4)$$

where $x(t, \varepsilon)$, $y(t, \varepsilon)$, $W(t, \varepsilon)$, $z^i(t, \varepsilon)$, $i = \overline{0, p-1}$ are smooth with respect to t functions that depend on power on ε . Substituting Problem (4) into Problem (2), we get system

$$\begin{cases} (B(t) - \bar{\lambda}_1(t))x(t, \varepsilon) = \varepsilon \dot{x}(t, \varepsilon), \\ (B(t) - \bar{\lambda}_2(t))y(t, \varepsilon) = \varepsilon \dot{y}(t, \varepsilon), \\ (B(t) - \bar{\lambda}_2(t))z^i(t, \varepsilon) = \varepsilon \dot{z}^i(t, \varepsilon), \quad i = \overline{0, p-1}, \\ B(t)W(t, \varepsilon) = \varepsilon \dot{W}(t, \varepsilon) - t^{n-1}h(t^n) + \sum_{i=0}^{p-1} t^i z^i(t, \varepsilon), \\ x(0, \varepsilon) + y(0, \varepsilon) + W(0, \varepsilon) = u^0. \end{cases} \quad (5)$$

Decomposing the unknown vector functions in a series in powers of ε , we obtain a series of iterative problems:

$$\begin{cases} (B(t) - \bar{\lambda}_1(t))x_k(t) = \dot{x}_{k-1}(t), \\ (B(t) - \bar{\lambda}_2(t))y_k(t) = \dot{y}_{k-1}(t), \\ (B(t) - \bar{\lambda}_2(t))z_k^i(t) = \dot{z}_{k-1}^i(t), \quad i = \overline{0, p-1}, \\ B(t)W_k(t) = \dot{W}_{k-1}(t) - \delta_0^k t^{n-1}h(t^n) + \sum_{i=0}^{p-1} t^i \dot{z}_{k-1}^i(t), \\ x_k(0) + y_k(0) + W_k(0) = \delta_k^0 u^0. \end{cases} \quad (6)$$

To solve iterative Problems (6), we formulate a point-solvability theorem.

Theorem 1. Let the following equation be given:

$$B(t)u(t) \equiv t^{n-1}A(t^n) = t^{n-s}h(t^n), \quad 0 \leq s \leq n-1 \quad (7)$$

and let the following conditions are met:

- (1) $B(t)$ has eigenvalues $\bar{\lambda}_1(t) = t^{n-1}\lambda_1(t^n)$, $\bar{\lambda}_2(t) = t^p a(t^n)$ and eigenvectors $\bar{e}_1(t)$, $\bar{e}_2(t)$;
- (2) $h(t^n) \in C^\infty[0, T]$.

Then, Problem (7) is solvable if and only if

- (a) $h_1(0) = 0$, $s = \overline{2, n-1}$;
- (b) $h_2^{(k)} = 0$, $k = 0, \left[\frac{m+s-1}{n} \right]$, $s = \overline{0, n-1}$,

where $h_1(t^n)$, $h_2(t^n)$ are the components of decomposition $h(t)$ on the basis of eigenvectors of operator $B(t)$; $u_1(t^n)$, $u_2(t^n)$ are the components of the expansion of $u(t)$ on the basis of eigenvectors of operator $B(t)$.

Proof of Theorem 1. Let us prove the need. Let system

$$\begin{cases} t^{n-1}\lambda_1(t^n)u_1(t) = t^{n-s}h_1(t^n), \\ t^p a(t^n)u_2(t) = t^{n-s}h_2(t^n) \end{cases} \quad (8)$$

have a solution. Then,

- (1) the first equation of System (8) is solvable:
 - (a) if $s = 0, 1$, then $u_1(t) = t^{1-s} \frac{h_1(t^n)}{\lambda_1(t^n)}$,
 - (b) if $s = \overline{2, n-1}$, then $h_1(0) = 0$ and $u_1(t) = t^{n+1-s} \frac{\bar{h}_1(t^n)}{\lambda_1(t^n)}$, where $h_1(t^n) = t^n \bar{h}_1(t^n)$;
- (2) the second equation of System (8) is solvable if $(k+1)n-s \leq p < (k+2)n-s$, which is equivalent to $h_2^{(k)}(0) = 0$, $k = 0, \left[\frac{m+s-1}{n} \right]$ and $u_2(t) = t^{n-j} \frac{\bar{h}_2(t^n)}{a(t^n)}$, $0 \leq j \leq n-1$.

Sufficiency is obvious. $\square$

Consider Problem (6) as $k = -1$:

$$\begin{cases} (B(t) - \bar{\lambda}_1(t))x_{-1}(t) = 0, \\ (B(t) - \bar{\lambda}_2(t))y_{-1}(t) = 0, \\ (B(t) - \bar{\lambda}_2(t))z_{-1}^i(t) = 0, \quad i = \overline{0, p-1}, \\ B(t)W_{-1}(t) = 0, \\ x_{-1}(0) + y_{-1}(0) + W_{-1}(0) = 0. \end{cases} \quad (9)$$

Solution (9) has the form

$$x_{-1}(t) = \alpha_{-1}^1(t)\bar{e}_1(t), \quad y_{-1}(t) = \beta_{-1}^2(t)\bar{e}_2(t), \quad z_{-1}^i(t) = \gamma_{-1}^{i,2}(t)\bar{e}_2(t), \\ W_{-1}(t) \equiv 0, \quad \alpha_{-1}^1(0) = 0, \quad \beta_{-1}^2(0) = 0.$$

Functions $x_{-1}(t)$, $y_{-1}(t)$, $z_{-1}^i(t)$ are determined at the next iteration step $k = 0$ from the solvability conditions:

$$\begin{cases} (B(t) - \bar{\lambda}_1(t))x_0(t) = \dot{x}_{-1}(t), \\ (B(t) - \bar{\lambda}_2(t))y_0(t) = \dot{y}_{-1}(t), \\ (B(t) - \bar{\lambda}_2(t))z_0^i(t) = \dot{z}_{-1}^i(t), \quad i = \overline{0, p-1}, \\ B(t)W_0(t) = -t^n h(t^n) + \sum_{i=0}^{p-1} t^i z_{-1}^i(t), \\ x_0(0) + y_0(0) + W_0(0) = u^0. \end{cases} \quad (10)$$

Let be

$$\dot{e}_i(t) = \dot{e}_i(t^n) = nt^{n-1}C_i^1(t^n)\bar{e}_1(t) + nt^{n-1}C_i^2(t^n)\bar{e}_2(t), \quad i = 1, 2.$$

Denote by $\bar{C}_i^j(t) = nt^{n-1}C_i^j(t^n)$, $i, j = 1, 2$. Then

$$\dot{e}_i(t) = \sum_{j=1}^2 \bar{C}_i^j(t)\bar{e}_j(t), \quad i = 1, 2.$$

System (10) takes the form:

$$\begin{cases} (B(t) - \bar{\lambda}_1(t))x_0(t) = (\dot{\alpha}_{-1}^1(t) + \bar{C}_1^1(t)\alpha_{-1}^1(t))\bar{e}_1(t) + \alpha_{-1}^1(t)\bar{C}_1^2(t)\bar{e}_2(t), \\ (B(t) - \bar{\lambda}_2(t))y_0(t) = (\dot{\beta}_{-1}^2(t) + \bar{C}_2^2(t)\beta_{-1}^2(t))\bar{e}_2(t) + \beta_{-1}^2(t)\bar{C}_2^1(t)\bar{e}_1(t), \\ (B(t) - \bar{\lambda}_2(t))z_0^i(t) = (\dot{\gamma}_{-1}^{i,2}(t) + \bar{C}_2^2(t)\gamma_{-1}^{i,2}(t))\bar{e}_2(t) + \gamma_{-1}^{i,2}(t)\bar{C}_2^1(t)\bar{e}_1(t), \quad i = \overline{0, p-1}, \\ B(t)W_0(t) = -t^n h(t^n) + \sum_{i=0}^{p-1} t^i z_{-1}^i(t), \\ x_0(0) + y_0(0) + W_0(0) = u^0. \end{cases} \quad (11)$$

The conditions for the solvability of System (11) and the initial conditions at the $k = -1$ step imply that $\alpha_{-1}^1(t) \equiv 0$, $\beta_{-1}^2(t) \equiv 0$. To determine $z_{-1}^i(t)$, we wrote by coordinate the equation for the $W_0(t)$ of System (11):

$$\begin{cases} \bar{\lambda}_1(t)W_0^1(t) \equiv t^{n-1}\lambda_1(t^n)W_0^1(t) = -t^{n-1}h_1(t^n), \\ \bar{\lambda}_2(t)W_0^2(t) \equiv t^p a(t^n)W_0^2(t) = -t^{n-1}h_2(t^n) + \sum_{i=0}^{p-1} t^i \gamma_{-1}^{i,2}(t). \end{cases} \quad (12)$$

Then, $W_0^1(t) = -\frac{h_1(t^n)}{\lambda_1(t^n)}$. On the basis of the point-solvability theorem, we obtained:

$$\gamma_{-1}^{n-1,2}(0) = h_2(0), \quad \gamma_{-1}^{2n-1,2}(0) = \dot{h}_2(0), \quad \dots, \quad \gamma_{-1}^{(k+1)n-1,2}(0) = \frac{h_2^{(k)}(0)}{k!} e^{-\int_0^t \bar{C}_2^2(s)ds},$$

where $k = [m/n]$ is the integer part, so when order $\text{ord}(t^i)$ is equal to order $\text{ord}(t^{(j+1)n-1})$ in the expansion of $t^{n-1}h(t^n)$ in Taylor–Maclaurin series, other $\gamma_{-1}^{i,2}(0) = 0$. Thus, the solution is determined at step $k = -1$:

$$u_{-1}(t, \varepsilon) = \sum_{i=0}^{[m/n]} \gamma_{-1}^{(i+1)n-1,2}(t)\bar{e}_2(t)\sigma_{(i+1)n-1}(t, \varepsilon), \quad (13)$$

where $\gamma_{-1}^{(i+1)n-1,2}(t) = \frac{h_2^{(i)}(0)}{i!} e^{-\int_0^t \bar{C}_2^2(s)ds}$.

The solution at zero step $k = 0$ is written in the form

$$\begin{cases} x_0(t) = \alpha_0^1(t)\bar{e}_1(t), \quad y_0(t) = \beta_0^2(t)\bar{e}_2(t), \\ z_0^i(t) = \begin{cases} \gamma_0^{i,2}(t)\bar{e}_2(t), & \text{ord}(t^i) \neq \text{ord}(t^{(j+1)n-1}), \\ \gamma_0^{i,2}(t)\bar{e}_2(t) - \gamma_{-1}^{i,2}(t)\frac{\bar{C}_2^1(t)}{\bar{\lambda}_1(t) - \bar{\lambda}_2(t)}\bar{e}_1(t), & \text{ord}(t^i) = \text{ord}(t^{(j+1)n-1}), \end{cases} \\ i = \overline{0, p-1}, \quad j = \overline{0, [m/n]}, \\ W_0(t) = -\frac{h_1(t^n)}{\lambda_1(t^n)}\bar{e}_1(t) + t^{n-s}H_0(t^n)\bar{e}_2(t), \end{cases} \quad (14)$$

where

(a) $s = \left\{ \frac{m}{n} \right\} n$ is the remainder of dividing m by n ;

$$(b) \quad t^{n-s} H_0(t^n) \equiv \frac{-t^{n-1} h(t^n) + \sum_{i=0}^{[m/n]} t^{(i+1)n-1} \gamma_{-1}^{(i+1)n-1,2}(t)}{t^p a(t^n)}.$$

Arbitrary functions $\alpha_0^1(t)$, $\beta_0^2(t)$, $\gamma_0^{i,2}(t)$ are determined from the conditions for the solvability of the system at step $k = 1$:

$$\left\{ \begin{array}{l} (B(t) - \bar{\lambda}_1(t))x_1(t) = (\dot{\alpha}_0^1(t) + \bar{C}_1^1(t)\alpha_0^1(t))\bar{e}_1(t) + \alpha_0^1(t)\bar{C}_1^2(t)\bar{e}_2(t), \\ (B(t) - \bar{\lambda}_2(t))y_1(t) = (\dot{\beta}_0^2(t) + \bar{C}_2^2(t)\beta_0^2(t))\bar{e}_2(t) + \beta_0^2(t)\bar{C}_2^1(t)\bar{e}_1(t), \\ (B(t) - \bar{\lambda}_1(t))z_1^i(t) = (\dot{\gamma}_0^{i,2}(t) + \bar{C}_2^2(t)\gamma_0^{i,2}(t))\bar{e}_2(t) + \gamma_0^{i,2}(t)\bar{C}_2^1(t)\bar{e}_1(t), \\ \quad \text{ord}(t^i) \neq \text{ord}(t^{(j+1)n-1}), \\ (B(t) - \bar{\lambda}_2(t))z_1^i(t) = \left(\dot{\gamma}_0^{i,2}(t) + \bar{C}_2^2(t)\gamma_0^{i,2}(t) - \gamma_{-1}^{i,2}(t) \frac{\bar{C}_2^1(t)\bar{C}_1^2(t)}{\bar{\lambda}_1(t) - \bar{\lambda}_2(t)} \right) \bar{e}_2(t) + \\ \quad + \left(\dot{\gamma}_0^{i,2}(t)\bar{C}_2^1(t) - \left(\gamma_{-1}^{i,2}(t) \frac{\bar{C}_2^1(t)}{\bar{\lambda}_1(t) - \bar{\lambda}_2(t)} \right) - \gamma_{-1}^{i,2}(t) \frac{\bar{C}_2^1(t)\bar{C}_1^1(t)}{\bar{\lambda}_1(t) - \bar{\lambda}_2(t)} \right) \bar{e}_1(t), \\ \quad \text{ord}(t^i) = \text{ord}(t^{(j+1)n-1}), \quad i = 0, p-1, \quad j = 0, [m/n], \\ B(t)W_1(t) = \dot{W}_0(t) + \sum_{i=0}^{p-1} t^i z_0^i(t), \\ x_1(0) + y_1(0) + W_1(0) = 0. \end{array} \right. \quad (15)$$

The solvability theorem of System (15) gives

$$\alpha_0^1(t) = \left(u_1^0 + \frac{h_1(0)}{\lambda_1(0)} \right) e^{-\int_0^t \bar{C}_1^1(s) ds}, \quad \beta_0^1(t) \equiv u_2^0 e^{-\int_0^t \bar{C}_2^2(s) ds}.$$

Consider the equation for $W_1(t)$. Given the expression for $\bar{C}_i^j(t) = nt^{n-1}C_i^j(t^n)$, this equation can be written as follows:

$$B(t)W_1(t) = t^{n-1}(\dot{W}_0)_1(t)\bar{e}_1(t) + t^{n-1-s}(\dot{W}_0)_2(t)\bar{e}_2(t) + \sum_{i=0}^{p-1} t^i z_0^i(t). \quad (16)$$

Consider Equation (16) component-wise:

$$\left\{ \begin{array}{l} \bar{\lambda}_1(t)W_1^1(t) = t^{n-1}(\dot{W}_0)_1(t) - \sum_{i=0}^{[m/n]} \gamma_{-1}^{(i+1)n-1,2}(t) \frac{\bar{C}_2^1(t)t^{(i+1)n-1}}{\bar{\lambda}_1(t) - \bar{\lambda}_2(t)}, \\ \bar{\lambda}_2(t)W_1^2(t) = t^{n-1-s}(\dot{W}_0)_2(t) + \sum_{i=0}^{p-1} t^i \gamma_0^{i,2}(t). \end{array} \right. \quad (17)$$

Solution of the first equation of System (17) is written as follows:

$$W_1^1(t) = \frac{(\dot{W}_0)_1(t)}{\lambda_1(t^n)} - \sum_{i=0}^{[m/n]} \gamma_{-1}^{(i+1)n-1,2}(t) \frac{nC_2^1(t^n)t^{in}}{\lambda_1(t^n)(\lambda_1(t^n) - t^m a(t^n))}.$$

For the solvability of the second equation of System (17), it is necessary and sufficient that

$$\gamma_0^{(i+1)n-1-s,2}(0) = -\frac{(\dot{W}_0)_2^{(i)}(0)}{i!}, \quad i = 0, \left[\frac{m+s}{n} \right],$$

$$\text{here } \left[\frac{m+s}{n} \right] = \left[\left[\frac{m}{n} \right] + \frac{2s}{n} \right] = \left[\frac{m}{n} \right] + \left[\frac{2s}{n} \right].$$

The other $\gamma_0^{j,2}(0) = 0, j \neq (i+1)n-1-s, j = \overline{0, p-1}$. Defining $\gamma_0^{i,2}(0)$, we can write the expression for $z_0^i(t)$:

(a) if $j = (i+1)n-1-s, i = 0, \left[\frac{m}{n}\right] + \left[\frac{2s}{n}\right]$, then

$$\gamma_0^{j,2}(t) = -\frac{(\dot{W}_0)_2^{(i)}(0)}{i!} e^{-\int_0^t \bar{C}_2^2(s) ds}, \quad z_0^j(t) = \gamma_0^{j,2}(t) \bar{C}_2(t);$$

(b) if $j \neq (i+1)n-1-s, j = (i+1)n-1$, then

$$\gamma_0^{j,2}(t) = e^{-\int_0^t \bar{C}_2^2(s) ds} \int_0^t e^{\int_0^s \bar{C}_2^2(z) ds} \gamma_{-1}^{j,2}(s) \frac{\bar{C}_2^1(s) \bar{C}_1^2(s)}{\bar{\lambda}_1(s) - \bar{\lambda}_2(s)} ds,$$

$$z_0^j(t) = \gamma_0^{j,2}(t) \bar{e}_2(t) - \gamma_{-1}^{j,2}(t) \frac{\bar{C}_2^1(t)}{\bar{\lambda}_1(t) - \bar{\lambda}_2(t)} \bar{e}_1(t);$$

(c) if $j \neq (i+1)n-1-s, j \neq (i+1)n-1$, then

$$\gamma_0^{j,2}(t) \equiv 0, \quad z_0^{j,2}(t) \equiv 0.$$

The solution of the second equation of System (17) is written as follows:

$$W_1^2(t) = t^{n(1-\{\frac{2s}{n}\})} H_1(t^n),$$

$$\text{where } H_1(t^n) = \frac{t^{n-s-1} (\dot{W}_0)_2 - \sum_{i=0}^{p-1} t^i \gamma_0^{i,2}(t)}{t^p a(t^n)}.$$

Thus, the solution is determined at the zero iterative step:

$$\begin{aligned} u_0(t, \varepsilon) = & \alpha_0^1(t) \bar{e}_1(t) e^{-\varphi_1(t)/\varepsilon} + \beta_0^2(t) \bar{e}_2(t) e^{\varphi_2(t)/\varepsilon} + \sum_{i=0}^{\left[\frac{m}{n}\right] + \left[\frac{2s}{n}\right]} z_0^{(i+1)n-1-s}(t) \sigma_{(i+1)n-1-s}(t, \varepsilon) \\ & + \sum_{i=0}^{\left[\frac{m}{n}\right]} z_0^{(i+1)n-1}(t) \sigma_{(i+1)n-1}(t, \varepsilon) - \frac{h_1(t^n)}{\lambda_1(t^n)} \bar{e}_1(t) + t^{n-s} H_0(t^n) \bar{e}_2(t). \end{aligned}$$

Similarly, according to this scheme, the solutions of subsequent iteration problems are determined. Thus, we can get an expression for any member of a regularized series.

We write the main term of the asymptotics of Problem (2):

$$u_{\text{main}} = \frac{1}{\varepsilon} u_{-1}(t, \varepsilon) + u_0(t, \varepsilon).$$

4. Limit-Transition Theorem

To prove the asymptoticity of a regularized series, we prove a theorem on estimating the remainder term for $\varepsilon \rightarrow 0$.

Let be $u(t, \varepsilon) = \sum_{k=-1}^n \varepsilon^k u_k(t, \varepsilon) + \varepsilon^{n+1} R_n(t, \varepsilon)$, where

$$u_k(t, \varepsilon) = x_k(t) e^{\varphi_1(t)/\varepsilon} + y_k(t) e^{\varphi_2(t)/\varepsilon} + \sum_{i=0}^{p-1} z_k^i(t) \sigma_i(t, \varepsilon) + W_k(t). \quad (18)$$

Substituting Problem (18) into Problem (1), we obtain the Cauchy problem for the remainder $R_n(t, \varepsilon)$:

$$\begin{cases} \varepsilon \dot{R}_n(t, \varepsilon) = B(t)R_n(t, \varepsilon) + H(t, \varepsilon), \\ R(0, \varepsilon) = 0, \end{cases} \quad (19)$$

where

$$H(t, \varepsilon) = - \left(\dot{x}_n(t)e^{\varphi_1(t)/\varepsilon} + \dot{y}_n(t)e^{\varphi_2(t)/\varepsilon} + \sum_{i=0}^{p-1} \dot{z}_n(t)\sigma_i(t, \varepsilon) + \left(\dot{W}_n(t) + \sum_{i=0}^{p-1} t^i \dot{z}_n^i(t) \right) \right),$$

in this case, it is assumed that $H(t, \varepsilon)$ satisfies the conditions of the solvability theorem.

Theorem 2. Let Cauchy Problem (1) be given and Conditions 1–9 be satisfied. Then, the estimate is correct

$$\left\| u(t, \varepsilon) - \sum_{k=-1}^n \varepsilon^k u_k(t, \varepsilon) \right\| \leq C \varepsilon^{n+1},$$

where $C > 0$ in the norm $\mathbb{C}[0, T]$ for any $(t, \varepsilon) \in [0, T] \times (0, \varepsilon_0]$, $\|x(t)\|_{\mathbb{C}[0, T]} = \max_{t \in [0, T]} |x(t)|$.

Proof of Theorem 2. Solution (19) is written as follows:

$$R_n(t, \varepsilon) = \frac{1}{\varepsilon} \int_0^t U_\varepsilon(t, s) H(s, \varepsilon) ds, \quad (20)$$

where $U_\varepsilon(t, s)$ is resolving operator (fundamental solution system) satisfying system

$$\begin{cases} \varepsilon \dot{U}_\varepsilon(t, s) = B(t)U_\varepsilon(t, s), \\ U_\varepsilon(t, s)|_{s=t} = I. \end{cases} \quad (21)$$

Let $S(t)$ be a matrix of eigenvectors $\bar{e}_1(t)$, $\bar{e}_2(t)$ of operator $B(t)$. Then, System (21) is equivalent to system

$$\begin{cases} \varepsilon \dot{V}_\varepsilon(t, s) = \Lambda(t)V_\varepsilon(t, s) - \varepsilon S^{-1}(t)\dot{S}(t)V_\varepsilon(t, \varepsilon), \\ V_\varepsilon(t, s)|_{s=t} = S^{-1}(0), \end{cases} \quad (22)$$

here, $\Lambda(t) = \begin{pmatrix} \bar{\lambda}_1(t) & 0 \\ 0 & \bar{\lambda}_2(t) \end{pmatrix}$, $V_\varepsilon(t, s) = S^{-1}(t)U_\varepsilon(t, s)$. We reduce System (22) to an integral equation

$$V_\varepsilon(t, s) = e^{\frac{1}{\varepsilon} \int_s^t \Lambda(s_1) ds_1} S^{-1}(0) - \int_s^t e^{\frac{1}{\varepsilon} \int_{s_1}^t \Lambda(s_2) ds_2} S^{-1}(s_1) \dot{S}(s_1) V_\varepsilon(s_1, \varepsilon) ds_1. \quad (23)$$

Let us estimate Equation (23) at the norm $\mathbb{C}[0, T]$. Using the conditions on the spectrum of operator $B(t)$, we obtain

$$\|V_\varepsilon(t, s)\| \leq C_1 \|S^{-1}(0)\| + C_2 \int_s^t \|V_\varepsilon(s_1, s)\| ds_1.$$

Using the Bellman–Gronuola inequality, we obtain $\|U_\varepsilon(t, s)\| \leq C$ on $[0, T]$.

To estimate the remaining term, it is important to take into account that operator $B(t)$ is invertible on vector functions that satisfy the conditions of the solvability theorem. Then, integrating over parts of Solution (20), we obtain chain of equalities

$$\begin{aligned} R_n(t, \varepsilon) &= \frac{1}{\varepsilon} \int_0^t U_\varepsilon(t, s) H(s, \varepsilon) ds = \frac{1}{\varepsilon} \int_0^t U_\varepsilon(t, s) B(s) B^{-1}(s) H(s, \varepsilon) ds = \\ &= -U_\varepsilon(t, s) B^{-1}(s) H(s, \varepsilon) \Big|_0^t + \int_0^t U_\varepsilon(t, s) \frac{d}{ds} B^{-1}(s) H(s, \varepsilon) ds = \\ &= -B^{-1}(t) H(t, \varepsilon) + U_\varepsilon(t, s) B^{-1}(s) H(s, \varepsilon) \Big|_{s=0} + \int_0^t U_\varepsilon(t, s) \frac{d}{ds} B^{-1}(s) H(s, \varepsilon) ds. \end{aligned}$$

Since, by virtue of Conditions 1–9, $H(t, \varepsilon)$ admits estimate $\|H(t, \varepsilon)\| \leq C_1$ in norm $\mathbb{C}[0, T]$, then remainder $R_n(t, \varepsilon)$ satisfies estimate

$$\|R_n(t, \varepsilon)\| \leq C_2 \quad \forall (t, \varepsilon) \in [0, T] \times (0, \varepsilon_0].$$

Therefore, the asymptoticity of series $\sum_{k=-1}^{\infty} \varepsilon^k u_k(t, \varepsilon)$ is proved. $\square$

Theorem 3 (The limit theorem). Let Cauchy Problem (1) be given and have satisfied the conditions:

- (1) Conditions 1–9;
- (2) $h_2^{(i)}(0) = 0, i = \overline{0, [m/n]}$, where $h_2(t)$ is the second coordinate in the expansion of $h(t) = h_1(t)e_1(t) + h_2(t)e_2(t)$ in eigenvectors of the original matrix.

Then,

- (1) for any $\delta > 0, t \in [\delta, T], \operatorname{Re} \lambda_i(t) \leq -\alpha < 0$

$$\lim_{\varepsilon \rightarrow 0} u(t, \varepsilon) = -A^{-1}(t)h(t);$$

- (2) if $\operatorname{Re} \lambda_i(t) = 0$, then

$$u(t, \varepsilon) \xrightarrow[\varepsilon \rightarrow 0]{\text{weak}} -A^{-1}(t)h(t) \text{ in a weak sense.}$$

Proof of Theorem 3. (1) Conditions $h_2^{(i)}(0) = 0, i = \overline{0, [m/n]}$ cause $u_{-1}(t, \varepsilon) = 0$. Then,

$$u_{\text{main}}(t) = u_0(t, \varepsilon).$$

By virtue of the singularity estimates described in the lemma, it follows that for any $\delta > 0, t \in [\delta, T]$

$$\lim_{\varepsilon \rightarrow 0} u_0(t, \varepsilon) = -B^{-1}(t)t^{n-1}h(t^n),$$

equivalent in source variables $\lim_{\varepsilon \rightarrow 0} u_0(t, \varepsilon) = -A^{-1}(t)h(t)$.

- (2) If $\operatorname{Re} \lambda_i(t) \equiv 0, i = 1, 2$, then singularities are rapidly oscillating exponents as $\varepsilon \rightarrow 0$. From here, according to Lebesgue's lemma, for any $\varphi(t) \in \mathbb{C}(0, T)$

$$\int_0^T (u_0(t, \varepsilon) + A^{-1}(t)h(t))\varphi(s)dt \xrightarrow[\varepsilon \rightarrow 0]{} 0.$$

$\square$

Example 1. Consider the Cauchy problem for a parabolic equation

$$\begin{cases} \varepsilon \frac{\partial u}{\partial t} - \varepsilon^2 \frac{\partial^2 u}{\partial x^2} = -\sqrt{t}u + h(x, t), \\ u(x, 0) = \varphi(x), \quad -\infty < x < \infty, \end{cases}$$

where $\varphi(x), h(x, t) \in C_0^\infty(-\infty, \infty)$ are smooth functions with compact support.

Using the technique of the regularization method outlined above, we obtain the principal term of the asymptotics of the solution:

$$\begin{aligned} u(x, t) = & \frac{1}{\varepsilon} h(x, 0) e^{-\frac{2}{3\varepsilon} t^{3/2}} \int_0^t e^{\frac{2s^{3/2}}{3\varepsilon}} ds + \varphi(x) e^{-\frac{2t^{3/2}}{3\varepsilon}} - \frac{h(x, 0)}{2} e^{-\frac{2t^{3/2}}{3\varepsilon}} \int_0^t e^{\frac{2s^{3/2}}{3\varepsilon}} \frac{ds}{\sqrt{s}} + \\ & + t h''(x, 0) e^{-\frac{2t^{3/2}}{3\varepsilon}} \int_0^t e^{\frac{2s^{3/2}}{3\varepsilon}} ds + \frac{h(x, t) - h(x, 0)}{\sqrt{t}}. \end{aligned}$$

5. Conclusions

In this paper, the regularization method was developed into the class of singularly perturbed Cauchy problems in the case of a simple rational turning point for the limit operator (for $\varepsilon = 0$). The main singularities of the solution are highlighted:

$$e^{\varphi_1(t)/\varepsilon}, e^{\varphi_2(t)/\varepsilon}, \sigma_i(t, \varepsilon) = e^{\varphi_2(t)/\varepsilon} \int_0^t e^{-\varphi_2(s)/\varepsilon} s^{(i+1-n)/n} ds, \quad i = \overline{0, p-1},$$

which allowed us to present the solution in the form:

$$u(t, \varepsilon) = x(t, \varepsilon) e^{\varphi_1(t)/\varepsilon} + y(t, \varepsilon) e^{\varphi_2(t)/\varepsilon} + \sum_{i=0}^{p-1} z^i(t, \varepsilon) \sigma_i(t, \varepsilon) + W(t, \varepsilon),$$

where $x(t, \varepsilon), y(t, \varepsilon), W(t, \varepsilon), z^i(t, \varepsilon), i = \overline{0, p-1}$ are t smooth functions that depend on power ε .

Estimates of the main singularities for $\varepsilon \rightarrow 0$ were given, and theorems on the solvability of iterative problems were proved. A theorem on the asymptotic convergence of the solution of the problem was proved, and conditions on the right-hand side of $h(t)$ were described, under which the passage to the limit theorem is valid. An example of solving the Cauchy problem for a parabolic equation with a fractional turning point $\lambda(\tau) = \tau^{1/2}$ was given.

Author Contributions: Conceptualization and writing—review and editing by A.E. and T.R.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

Nomenclature

$u(t), x(t), y(t), z^i(t), i = \overline{0, m+n-1}, w(t), h(t)$	a vector of a function of a real variable
$A(t), B(t)$	matrices of order 2×2
λ_1, λ_2	eigenvalues of matrix A
$\bar{\lambda}_1, \bar{\lambda}_2$	eigenvalues of matrix B
e_1, e_2	eigenvectors of matrix A
$\bar{e}_1, \bar{e}_2$	eigenvectors of matrix B
ε	a small task parameter
$S(t)$	a matrix of eigenvectors $\bar{e}_1, \bar{e}_2$
$\Lambda(t)$	a matrix of eigenvalues of matrix B
R_n	the remainder term of the asymptotic series

References

1. Eliseev, A.G.; Lomov, S.A. The theory of singular perturbations in the case of spectral singularities of the limit operator. *Sb. Math.* **1986**, *131*, 544–557. [\[CrossRef\]](#)
2. Eliseev, A.G.; Ratnikova, T.A. A singularly perturbed Cauchy problem in the presence of a rational "simple" turning point for the limit operator. *Differ. Equ. Control Process.* **2019**, *3*, 63–73.
3. Lomov, S.A. *Introduction to the General Theory of Singular Perturbations*; Nauka: Moscow, Russia, 1981.
4. Safonov, V.F.; Bobodzhanov, A.A. Course of higher mathematics. In *Singularly Perturbed Equations and the Regularization Method: Textbook*; Izdatelstvo MPEI: Moscow, Russia, 2012.
5. Liouville, J. Second memoire sur le development des fonctions en series dont divers termes sont assujettis, a une meme equation. *J. Math. Pure Appl.* **1837**, *2*, 16–35.
6. Lomov, S.A.; Eliseev, A.G. Asymptotic integration of singularly perturbed problems. *Russ. Math. Surv.* **1988**, *43*, 1–63. [\[CrossRef\]](#)
7. Eliseev, A.G. Singular perturbation theory for systems of differential equations in the case of multiple spectrum of the limit operator. III. *Math. USSR-Izv.* **1985**, *25*, 475–500. [\[CrossRef\]](#)
8. Tursunov, D.A.; Kozhbekov, K.G. Asymptotics of the solution of singularly perturbed differential equations with a fractional turning point. *News Irkutsk State Univ.* **2017**, *21*, 108–121.



© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).