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Review

Groups, Special Functions and Rigged Hilbert Spaces

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Abstract: We show that Lie groups and their respective algebras, special functions and rigged Hilbert spaces are complementary concepts that coexist together in a common framework and that they are aspects of the same mathematical reality. Special functions serve as bases for infinite dimensional Hilbert spaces supporting linear unitary irreducible representations of a given Lie group. These representations are explicitly given by operators on the Hilbert space $\mathcal{H}$ and the generators of the Lie algebra are represented by unbounded self-adjoint operators. The action of these operators on elements of continuous bases is often considered. These continuous bases do not make sense as vectors in the Hilbert space; instead, they are functionals on the dual space, $\Phi^\times$, of a rigged Hilbert space, $\Phi \subset \mathcal{H} \subset \Phi^\times$. In fact, rigged Hilbert spaces are the structures in which both, discrete orthonormal and continuous bases may coexist. We define the space of test vectors Φ and a topology on it at our convenience, depending on the studied group. The generators of the Lie algebra can often be continuous operators on Φ with its own topology, so that they admit continuous extensions to the dual $\Phi^\times$ and, therefore, act on the elements of the continuous basis. We investigate this formalism for various examples of interest in quantum mechanics. In particular, we consider $SO(2)$ and functions on the unit circle, $SU(2)$ and associated Laguerre functions, Weyl–Heisenberg group and Hermite functions, $SO(3,2)$ and spherical harmonics, $su(1,1)$ and Laguerre functions, $su(2,2)$ and algebraic Jacobi functions and, finally, $su(1,1) \oplus su(1,1)$ and Zernike functions on a circle.

Keywords: rigged Hilbert spaces; discrete and continuous bases; special functions; Lie algebras; representations of Lie groups; harmonic analysis

1. Introduction

Harmonic analysis has undergone strong development since the first work by Fourier [1]. The main idea of the Fourier method is to decompose functions in a superposition of other particular functions, i.e., "special functions". Since the original trigonometric functions used by Fourier many special functions, such as the classical orthogonal polynomials [2], have been used generalizing the original Fourier idea. In many cases, such special functions support representations of groups and in this way group representation theory appears closely linked to harmonic analysis [3]. Another crucial fact is that harmonic analysis is related to linear algebra and functional analysis, in the sense that elements of vector spaces or Hilbert spaces are decomposed in terms of orthogonal bases or operators as linear combinations of their eigenvalues (i.e., applying the spectral theorem, see Theorem 1 in Section 2). On many occasions, continuous bases and discrete bases are involved in the same framework. Hence, the arena where all these objects fit in a precise mathematical way is inside a rigged Hilbert space. Hence, we have a set of mathematical objects: classical orthogonal polynomials, Lie algebras, Fourier analysis, continuous and discrete bases, and rigged Hilbert spaces fully incorporated in a harmonic frame that can bee used in quantum mechanics as well as in signal processing.

In a series of previous articles, we gave some examples showing that Lie groups and algebras, special functions, discrete and continuous bases and rigged Hilbert spaces (RHS) are particular aspects of the same mathematical reality, for which a general theory is needed. As a first step in the construction of this general theory, we want to present a compact review of the results which have thus far been obtained by us and that can be useful in applications where harmonic analysis is involved.

Special functions play often the role of being part of orthonormal bases of Hilbert spaces serving as support of representations of Lie groups of interest in Physics. As is well known, decompositions of vectors of these spaces are given in terms of some sort of continuous basis, which are not normalisable and, hence, outside the Hilbert space. The most popular formulation to allow the coexistence of these continuous bases with the usual discrete bases is the RHS, where the elements of continuous bases are well defined as functionals on a locally convex space densely defined as a subspace of the Hilbert space supporting the representation of the Lie group. This notion of continuous basis has been recently revised and generalised in [4].

Thus, we have the need for a framework that includes Lie algebras, discrete and continuous bases and special functions, as building blocks of these discrete bases. In addition, it would be desirable to have structures in which the generators of the Lie algebras be well defined continuous operators on. The rigged Hilbert space comply with the requirements above mentioned.

All the cases presented here have applications not only in physics but in other sciences. In particular, Hermite functions are related to signal analysis in the real line and also with the fractional Fourier transform [5]. In [6], we introduced a new set of functions in terms of the Hermite functions that give rise bases in $L^2(\mathcal{C})$ and in $l^2(\mathbb{Z})$ where $\mathcal{C}$ is the unit circle. Both bases are related by means of the Fourier transform and the discrete Fourier transform. In [7], we presented a systematic study of these functions as well as the corresponding rigged Hilbert space framework. Recently, spherical harmonics are used in three-dimensional signal processing with applications in geodesy, astronomy, cosmology, graph computation, vision computation, medical images, communications systems, etc. [8–10]. Zernike polynomials are well known for their applications in optics [11–13]. Moreover, all of them can be considered as examples of harmonic analysis where the connection between groups, special functions and RHS fit together perfectly.

The paper is organized as follows. A brief description of RHS and their use in physics and in engineering is given in Section 2. In Section 3, SO(2) is discussed in details, related to the exponential $e^{\mathrm{i}m\phi}$, where the technical aspects are reduced to the minimum. Section 4 considers how Associated Laguerre Functions allow constructing two different RHS's, one related to the integer spin and the other to half-integer spin of SU(2). In Section 5, an analysis is performed of the basic case of the line, where the fundamental ingredient of the RHS are the Fourier Transform, the Hermite Functions and the Weyl–Heisenberg group. Section 6 is devoted to the RHS constructed on Spherical Harmonics in relation with SO(3). In Section 7, Laguerre Functions are used to construct another RHS related to SU(1,1). Jacobi Functions and the 15-dimensional algebra SU(2,2) are the foundation of a larger RHS, which is studied in Section 8. The last example we discuss (Section 9) is the RHS constructed on the Zernike Functions and the algebra $su(1,1) \oplus su(1,1)$ (that should be used also in connection with the Laguerre Functions). Few remarks close the paper in Section 10.

2. Rigged Hilbert Spaces

The less popular among our ingredients is the concept of rigged Hilbert spaces, so that a short section devoted to this concept seems necessary. A rigged Hilbert space, also called Gelfand triplet, is a tern of spaces [14]

$$\Phi \subset \mathcal{H} \subset \Phi^\times\,, \tag{1}$$

where: (i) $\mathcal{H}$ is an infinite dimensional separable Hilbert space; (ii) Φ is a dense subspace of $\mathcal{H}$ endowed with a locally convex topology stronger, i.e., it has more open sets, than the Hilbert space topology that Φ has inherited from $\mathcal{H}$; and (iii) $\Phi^\times$ is the space of all continuous antilinear functionals on Φ. Thus,

$F \in \Phi^\times$ is a mapping $F : \Phi \longmapsto \mathbb{C}$ such that for any pair $\psi, \varphi \in \Phi$ and any pair of complex numbers $\alpha, \beta \in \mathbb{C}$, one has

$$F(\alpha\psi + \beta\varphi) = \alpha^* F(\psi) + \beta^* F(\varphi)\,, \tag{2}$$

where the star denotes complex conjugation. The continuity is given with respect to the locally convex topology on Φ and the usual topology on the complex plane $\mathbb{C}$. Instead, the notation in Equation (2), we henceforth use the Dirac notation, which is quite familiar to physicists:

$$F(\varphi) =: \langle\varphi|F\rangle\,. \tag{3}$$

In general, the topology on Φ is given by a family of seminorms. In the examples we have studied thus far, the topology on Φ is given by a countable set of seminorms, $\{p_n\}_{n\in\mathbb{N}}$ where by countable we mean either finite or denumerable. As the topology on Φ is stronger than the Hilbert space topology, one of these seminorms could be chosen to be the Hilbert space norm.

Seminorms provide a nice criterion to determine whether a linear or antilinear functional over Φ is continuous. The linear or antilinear functional $F : \Phi \longmapsto \mathbb{C}$ is continuous if and only if, there exist a positive number $K > 0$ and finite number of seminorms, $p_1, p_2, \ldots, p_m$, taken from those that define the topology on Φ such that for any $\varphi \in \Phi$, we have [15]

$$|\langle\varphi|F\rangle| \leq K\{p_1(\varphi) + p_2(\varphi) + \cdots + p_m(\varphi)\}\,. \tag{4}$$

A typical example of functional is the following one. Pick an arbitrary $\varphi \in \Phi$ and define F_φ as

$$F_\varphi(\psi) = \langle\psi|F_\varphi\rangle := \langle\psi|\varphi\rangle\,, \tag{5}$$

which is obviously antilinear on Φ. Then, use the Schwarz inequality in $|\langle\psi|F_\varphi\rangle| \leq ||\psi||\,||\varphi||$, take $K = ||\varphi||$, $p_1(\psi) := ||\psi||$, for all $\psi \in \Phi$ and use Equation (4) so as to conclude the continuity of F_φ on Φ. However, not all elements of $\Phi^\times$ lie in this category. A typical counterexample is the Dirac delta.

Analogously, assume that Φ and Ψ are two locally convex spaces with topologies given by the respective families of seminorms $\{p_i\}_{i\in\mathcal{I}}$ and $\{q_j\}_{j\in\mathcal{J}}$. A linear or antilinear mapping $F : \Phi \longmapsto \Psi$ is continuous if and only if for each seminorm q_j on Ψ there exists a positive constant $K > 0$ and a finite number of seminorms, $p_1, p_2, \ldots, p_m$, from those defining the topology on Φ such that

$$q_j(F(\varphi)) \leq K\{p_1(\varphi) + p_2(\varphi) + \cdots + p_m(\varphi)\}\,, \qquad \forall\varphi \in \Phi\,. \tag{6}$$

Both the constant K and the seminorms $p_1, p_2, \ldots, p_m$ depend on q_j, but not on φ. We use these results along the present article.

Less interesting is that the dual space $\Phi^\times$ may be endowed with the weak topology induced by Φ. As is well known, the seminorms for this weak topology are defined as follows: for each $\varphi \in \Phi$, we define the seminorm p_φ as $p_\varphi(F) = |\langle\varphi|F\rangle|$, for all $F \in \Phi^\times$.

Since the topology on Φ is stronger than the Hilbert space topology, the canonical injection $i : \Phi \longmapsto \mathcal{H}$, with $i(\varphi) = \varphi$, for all $\varphi \in \Phi$, is continuous. Furthermore, one may prove that the injection $i : \mathcal{H} \longmapsto \Phi^\times$ given by $i(\varphi) := F_\varphi$ in Equation (5) is one-to-one and continuous with respect to the Hilbert space topology on $\mathcal{H}$ and the weak topology on $\Phi^\times$ [14].

RHS have been introduced in physics with the purpose of giving a rigorous mathematical background to the celebrated Dirac formulation of quantum mechanics, which is widely used by physicists. This mathematical formulation has been the objects of various publications [16–22]. In addition, rigged Hilbert spaces have been used in Physics or mathematics with various purposes that include:

1. A proper mathematical meaning for the Gamow vectors, which are the non-normalizable vectors giving the states of the exponentially decaying part of a quantum scattering resonance [23–26].

2. Using Hardy functions on a half-plane [25–27], we may construct RHS that serve as a framework for an extension of ordinary quantum mechanics that accounts for time asymmetric quantum processes. An example of such processes is the quantum decay [28–33].
3. Providing an appropriate context for the spectral decompositions of Koopman and Frobenius–Perron operators in classical chaotic systems in terms of the so called Pollicot–Ruelle resonances, which are singularities of the power spectrum [34,35].
4. Some situations that arise in quantum statistical mechanics demand the use of generalized states and some singular structures that require the use of rigged Liouville spaces [36–38].
5. A proper definition of some of the structures that appear in the axiomatic theory of quantum fields, such as Wightman functional, Borchers algebra, generalized states, etc., require of structures such as the rigged Fock space [39–41]. Both rigged Liouville and Fock spaces are obvious generalizations of RHS.
6. White noise and other stochastic processes may also be formulated in the context of RHS [42,43] as well as the study of certain solutions of partial differential equations [44].
7. In the last years, the RHS have appeared associated to time-frequency analysis and Gabor analysis that have many applications in physics and engineering related to signal processing [45–52]. In particular, applications in electrical engineering have been introduced in [53–56].
8. Finally, various problems in quantum mechanics require a mathematical description in terms of RHS. Let us quote the following references [57–64], although we do not pretend to be exhaustive.

One of the most interesting properties of RHS is the possibility of extending to the duals certain unbounded operators defined on domains including the space Φ. Let us consider a linear operator $A : \Phi \longmapsto \mathcal{H}$ and let $A^\dagger$ its adjoint, which has the following properties:

1. For any $\varphi \in \Phi$, then, $A^\dagger \varphi \in \Phi$. One says that Φ reduces $A^\dagger$ or, equivalently, that $A^\dagger$ leaves Φ invariant, so that $A^\dagger\Phi \subset \Phi$. Note that we do not assume that $A\Phi \subset \Phi$, in general.
2. The adjoint $A^\dagger$ is continuous on Φ.

Then, the operator A may be extended to a continuous operator on $\Phi^\times$, endowed with the weak topology. For any $F \in \Phi^\times$, one defines AF by means of the following *duality formula*:

$$\langle \varphi | AF \rangle = \langle A^\dagger \varphi | F \rangle \,, \qquad \forall\, \varphi \in \Phi \,, \;\; \forall\, F \in \Phi^\times \,. \tag{7}$$

Moreover, the extension is continuous on $\Phi^\times$ when this space has the weak topology.

In particular, if A is a symmetric operator, Equation (7) read as $\langle \varphi | AF \rangle = \langle A\varphi | F \rangle$. If A were self-adjoint, there is always a subspace Φ with the following properties: (i) Φ dense in $\mathcal{H}$; (ii) Φ is a subspace of the domain of A; and (iii) it is possible to endow Φ with a locally convex topology, finer than the Hilbert space topology, such that A be continuous on Φ. As a consequence, there exists a RHS, $\Phi \subset \mathcal{H} \subset \Phi^\times$ such that the self-adjoint operator A may be extended to the dual $\Phi^\times$ and, henceforth, to a larger space than the original Hilbert space where A is densely defined.

These ideas drive us to the important result known as the Gelfand–Maurin theorem [14,65] that gives a spectral decomposition à la Dirac of a self-adjoint operator with continuous spectrum. We present it here in its simplest form in order not to enter in unnecessary complications and notations.

Theorem 1. *(Gelfand–Maurin) Let A be a self-adjoint operator on a infinite dimensional separable Hilbert space $\mathcal{H}$, with simple absolutely continuous spectrum $\sigma(A) \equiv \mathbb{R}^+ \equiv [0, \infty)$. Then, there exists a rigged Hilbert space $\Phi \subset \mathcal{H} \subset \Phi^\times$, such that:*

1. *$A\Phi \subset \Phi$ and A is continuous on Φ. Therefore, it may be continuously extended to $\Phi^\times$.*
2. *For almost all $\omega \in \mathbb{R}^+$, with respect to the Lebesgue measure, there exists a $|\omega\rangle \in \Phi^\times$ with $A|\omega\rangle = \omega\,|\omega\rangle$.*

3. *(Spectral decomposition) For any pair of vectors $\varphi, \psi \in \Phi$, and any measurable function $f : \mathbb{R}^+ \longmapsto \mathbb{C}$, we have that*
$$\langle \varphi | f(A) \psi \rangle = \int_0^\infty f(\omega)\, \langle \varphi | \omega \rangle \langle \omega | \psi \rangle \, d\omega \,, \tag{8}$$
with $\langle \omega | \psi \rangle = \langle \psi | \omega \rangle^*$.
4. *The above spectral decomposition is implemented by a unitary operator $U : \mathcal{H} \longmapsto L^2(\mathbb{R}^+)$, with $U\psi = \langle \omega | \psi \rangle = \psi(\omega)$ and $[UAU^{-1}]\psi(\omega) = \omega\, \psi(\omega) = \omega \langle \omega | \psi \rangle$ for any $\psi \in \Phi$. This means that UAU^{-1} is the multiplication operator on $U\Phi$.*
5. *For any pre-existent RHS $\Phi \subset \mathcal{H} \subset \Phi^\times$, such that $A\Phi \subset \Phi$ with continuity and A is an operator satisfying our hypothesis, then items 2–4 hold.*

This result has some interest in our forthcoming discussion.

Two rigged Hilbert spaces $\Phi \subset \mathcal{H} \subset \Phi^\times$ and $\Psi \subset \mathcal{G} \subset \Psi^\times$ are unitarily equivalent if there exists a unitary operator $U : \mathcal{H} \longmapsto \mathcal{G}$ such that: (i) U is a *one-to-one* mapping from Φ *onto* Ψ; (ii) $U : \Phi \longmapsto \Psi$ is continuous; and (iii) its inverse $U^{-1} : \Psi \longmapsto \Phi$ is also continuous. Then, by using the duality formula
$$\langle \varphi | F \rangle = \langle U\varphi | UF \rangle \,, \qquad \forall\, \varphi \in \Phi \,,\ \forall\, F \in \Phi^\times \,, \tag{9}$$
we extend U to a one-to-one mapping from $\Phi^\times$ onto $\Psi^\times$, which is continuous with the weak topologies on both duals and which has an inverse with the same properties. Resumming, we have the following diagram
$$\begin{array}{ccccc} \Phi & \subset & \mathcal{H} & \subset & \Phi^\times \\ U\downarrow & & U\downarrow & & U\downarrow \\ \Psi & \subset & \mathcal{G} & \subset & \Psi^\times \end{array} \,.$$

3. SO(2): The Basic Example

To begin with, let us briefly summarize the simplest case that contains some general ingredients to be used in other situations [66].

Consider the unit circle in the plane, defined by $\mathcal{C} := \{(x, y) \in \mathbb{R}^2 \,;\ x^2 + y^2 = 1\}$. As is well known, its group of invariance is $SO(2)$.

The Hilbert space on the unit circle is the space of measurable functions $f(\phi) : \mathcal{C} \longmapsto \mathbb{R}$, which are square integrable. We denote this space as $L^2[0, 2\pi) \equiv L^2(\mathcal{C})$. The set of functions
$$f_m(\phi) := \frac{1}{\sqrt{2\pi}}\, e^{-im\phi} \,, \qquad \forall m \in \mathbb{Z} \,, \tag{10}$$
where $\mathbb{Z}$ is the set of integer numbers, is an orthonormal basis in $L^2[0, 2\pi)$. Then, each $f(\phi) \in L^2(\mathcal{C})$ admits a span of the form,
$$f(\phi) = \sum_{m \in \mathbb{Z}} a_m\, f_m(\phi) \,, \qquad a_m \in \mathbb{C} \tag{11}$$
with
$$\sum_{n \in \mathbb{Z}} |a_m|^2 = ||f(\phi)||^2 \,, \tag{12}$$
where $\mathbb{C}$ is the field of the complex numbers and $||f(\phi)||$ is the norm of the function $f(\phi)$ on $L^2[0, 2\pi)$.

3.1. Rigged Hilbert Spaces Associated to $L^2(\mathcal{C})$

To construct a RHS, let us consider the space Ψ of the functions $f(\phi) \in L^2[0, 2\pi)$ having the property,
$$||f(\phi)||_p^2 := \sum_{m \in \mathbb{Z}} |a_m|^2\, |m + i|^{2p} < \infty \,, \qquad p = 0, 1, 2 \ldots \,. \tag{13}$$

The countably family of norms $||-||_p$ generates a metrizable topology on Ψ. The fact that this family includes $p = 0$ shows that the canonical injection $\Psi \longmapsto L^2(\mathcal{C})$ is continuous. Let $\Psi^\times$ be the dual of Ψ (continuous antilinear functionals on Ψ) with the weak topology induced by the dual pair $\{\Psi, \Psi^\times\}$. Then, $\Psi \subset L^2(\mathcal{C}) \subset \Psi^\times$ is a RHS.

Along this particular and concrete RHS, we consider another one, unitarily equivalent to this and constructed as follows. Let us take an abstract infinite dimensional separable Hilbert space $\mathcal{H}$. We know that there is a unitary mapping $U\mathcal{H} \equiv L^2(\mathcal{C})$, in fact continuous. The sequence of vectors $\{|m\rangle\}_{m\in\mathbb{Z}}$, with $U|m\rangle = f_m(\phi)$, forms a orthonormal basis on $\mathcal{H}$. Then, following the comment at the end of Section 2, we may construct a RHS, $\Phi \subset \mathcal{H} \subset \Phi^\times$ unitarily equivalent to $\Psi \subset L^2(\mathcal{C}) \subset \Psi^\times$, just by defining $\Phi := U^{-1}\Psi$ and extending U^{-1} as a continuous mapping from $\Psi^\times$ onto $\Phi^\times$, using the duality formula in Equation (9). More explicitly,

$$\begin{array}{ccccc} \Psi & \subset & L^2(\mathcal{C}) & \subset & \Psi^\times \\ U^{-1}\downarrow & & U^{-1}\downarrow & & U^{-1}\downarrow \\ \Phi & \subset & \mathcal{H} & \subset & \Phi^\times \end{array} .$$

The mapping U^{-1} also transport topologies, so that if $|f\rangle \in \Phi$ with

$$|f\rangle = \sum_{m\in\mathbb{Z}} a_m\,|m\rangle = \sum_{m=-\infty}^{\infty} a_m\,|m\rangle\,, \tag{14}$$

then the topology on Φ is given by the set of norms $||\,|f\rangle||_p^2 = \sum_{m\in\mathbb{Z}} |a_m|^2\,(m+i)^{2p}$.

One of the most important features of RHS is the possibility of using continuous and discrete bases within the same space. For any $\phi \in [0, 2\pi)$, we define the ket $|\phi\rangle$ as a linear mapping from Φ into $\mathbb{C}$, such that for any $|f\rangle \in \Phi$, with $|f\rangle = \sum_{m\in\mathbb{Z}} a_m\,|m\rangle$, we have

$$\langle f|\phi\rangle := \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} a_m^*\, e^{im\phi} \quad \Longrightarrow \quad \langle m|\phi\rangle = \frac{1}{\sqrt{2\pi}}\, e^{im\phi}\,. \tag{15}$$

Note that $\langle f|\phi\rangle$ is nothing but the evaluation functional $f \longmapsto f(\phi)$, whenever the Fourier series converges pointwise to $f(\phi)$.

To prove that $|\phi\rangle$ is continuous as an antilinear functional on Φ, we use the Cauchy–Schwarz inequality as follows:

$$\begin{aligned} |\langle f|\phi\rangle| &\le \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} |a_m| = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \frac{|a_m|\,|m+i|}{|m+i|} \\ &\le \frac{1}{\sqrt{2\pi}} \sqrt{\sum_{m=-\infty}^{\infty} \frac{1}{|m+i|^2}}\, \sqrt{\sum_{m=-\infty}^{\infty} |a_m|^2\,|m+i|^2} = K\,||\,|f\rangle||_1\,, \end{aligned} \tag{16}$$

where the meaning of the constant K is obvious. Then, continuity follows from Equation (4) and, hence, $|\phi\rangle \in \Phi^\times$. Then, let us write $\langle\phi|f\rangle := \langle f|\phi\rangle^*$. It becomes obvious that $\langle\phi|$ is a continuous *linear* functional on Φ. Note that

$$U|f\rangle = \sum_{m=-\infty}^{\infty} a_m U\,|m\rangle = \sum_{m=-\infty}^{\infty} a_m f_m(\phi) = \sum_{m=-\infty}^{\infty} a_m \frac{1}{\sqrt{2\pi}}\, e^{-im\phi} = \langle\phi|f\rangle = f(\phi)\,. \tag{17}$$

Let us consider two arbitrary vectors $|f\rangle$, $|g\rangle \in \Phi$ and their corresponding images in Ψ by U: $f(\phi) = U\,|f\rangle$, $g(\phi) = U\,|g\rangle$, respectively. Since U is unitary, it preserves scalar products, so that

$$\langle g|f\rangle = \int_0^{2\pi} g^*(\phi)\, f(\phi)\, d\phi = \int_0^{2\pi} \langle g|\phi\rangle\langle\phi|f\rangle\, d\phi\,. \tag{18}$$

Omitting the arbitrary $|f\rangle$, $|g\rangle \in \Phi$, obtains a relation of the type

$$\mathbb{I} = \int_0^{2\pi} |\phi\rangle\langle\phi|\, d\phi\,, \tag{19}$$

which is a resolution of the identity. Then, observe that

$$|f\rangle = \mathbb{I}\,|f\rangle = \int_0^{2\pi} |\phi\rangle\langle\phi|f\rangle\, d\phi = \int_0^{2\pi} f(\phi)\,|\phi\rangle\, d\phi\,. \tag{20}$$

Now, let us compare Equation (14) with Equation (20). While Equation (14) is a span of any vector $|f\rangle \in \Phi$ in terms of a discrete basis, Equation (20) is a span of the same vector in terms of a continuous basis. Both bases belong to the dual space $\Phi^\times$, although the discrete basis is in both Φ and $\Phi^\times$ and the continuous basis only in $\Phi^\times$. The identity $\mathbb{I}$ is obviously the canonical injection from Φ into $\Phi^\times$. It is interesting that it may be inserted in the formal product $\langle\phi|f\rangle$, which is

$$f(\phi) = \langle\phi|f\rangle = \int_0^{2\pi} \langle\phi|\phi'\rangle\langle\phi'|f\rangle\, d\phi'\,, \tag{21}$$

so that

$$\langle\phi|\phi'\rangle = \delta(\phi-\phi')\,. \tag{22}$$

Discrete and continuous bases have clear analogies. Since the basis $\{|m\rangle\}$ is an orthonormal basis in $\mathcal{H}$, it satisfies the following completeness relation:

$$\sum_{m=-\infty}^{\infty} |m\rangle\langle m| = I\,, \tag{23}$$

where I is the identity operator on both $\mathcal{H}$ and Φ, so that it is somehow different to the identity $\mathbb{I}$ in Equation (19). The vectors $|m\rangle$ are in Φ, so that they admit an expansion in terms of the continuous basis as in Equation (20):

$$|m\rangle = \mathbb{I}\,|m\rangle = \int_0^{2\pi} |\phi\rangle\langle\phi|m\rangle\, d\phi = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-im\phi}\,|\phi\rangle\, d\phi\,. \tag{24}$$

We have two identities $\mathbb{I}$ in Equation (19) and I in Equation (23) and both are quite different. First, the definitions of both identities are dissimilar. Furthermore, $\mathbb{I}$ cannot be extended to an identity on $\Phi^\times$, since operations such as $\langle\phi|F\rangle$ for any $F \in \Phi^\times$ cannot be defined in general. As happens with the product of distributions, only some of these brackets are allowed. For example, if $|F\rangle = |\phi'\rangle$, for ϕ' fixed in $[0, 2\pi)$. Then, clearly,

$$\mathbb{I}\,|\phi'\rangle = \int_0^{2\pi} |\phi\rangle\langle\phi|\phi'\rangle\, d\phi = \int_0^{2\pi} |\phi\rangle\,\delta(\phi-\phi')\, d\phi = |\phi'\rangle\,. \tag{25}$$

On the other hand, I in Equation (23) can indeed be extended to the whole $\Phi^\times$. Let us write formally for any $g \in \Phi$ and any $F \in \Phi^\times$,

$$\langle g|F\rangle = \sum_{m=-\infty}^{\infty} \langle g|m\rangle\langle m|F\rangle\,. \tag{26}$$

First, observe that both $\langle g|m\rangle$ and $\langle m|F\rangle$ are well defined. The question is to know whether the sum in the right-hand side of Equation (26) converges. To show that this is indeed the case, we need the following result:

Lemma 1. *For any $F \in \Phi^\times$, there exists a constant $C > 0$ and a natural p, such that $|\langle m|F\rangle| \le C\,|m+i|^p$.*

Proof. It is just a mimic of the proof of Theorem V.14 in [15], page 143. □

After Lemma 1, we may show the absolute convergence of the series in Equation (26). For that, recall that $g = \sum_{m=-\infty}^{\infty} \langle m|g\rangle\,|m\rangle$ and that $g \in \Phi$. Then,

$$\begin{aligned}\sum_{m=-\infty}^{\infty} |\langle g|m\rangle| \cdot |\langle m|F\rangle| &\leq C \sum_{m=-\infty}^{\infty} |\langle g|m\rangle| \cdot |m+i|^p = C \sum_{m=-\infty}^{\infty} \frac{|\langle g|m\rangle|}{|m+i|^p}\, |m+i|^{2p} \\ &\leq C \sqrt{\sum_{m=-\infty}^{\infty} |\langle g|m\rangle|^2 \cdot |m+i|^{4p}} \times \sqrt{\sum_{m=-\infty}^{\infty} \frac{1}{|m+i|^{2p}}} = K\,||g||_{2p}\,,\end{aligned} \tag{27}$$

where $K = C$ times the second square root in Equation (27), which obviously converges. This shows the absolute convergence of Equation (26). In consequence, the formal procedure of inserting the identity in Equation (23) to $\langle g|f\rangle$ as in Equation (26) is rigorously correct. Thus, we see that there exists a substantial difference between the identities in Equations (19) and (23). In addition, Equation (23) gives a span of $|\phi\rangle$ in terms of the discrete basis $\{|m\rangle\}$ as follows:

$$I|\phi\rangle = |\phi\rangle = \sum_{m=-\infty}^{\infty} |m\rangle\langle m|\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{im\phi}\,|m\rangle\,. \tag{28}$$

Compare Equation (28) with the converse relation given by Equation (24). It is easy to prove that the series in the right-hand side of Equation (28) converges in the weak topology on $\Phi^\times$.

3.2. About Representations of SO(2)

We define the regular representation of $SO(2)$, $\mathcal{R}(\theta)$, on $L^2[0,2\pi)$ as

$$[\mathcal{R}(\theta)\,f](\phi) := f(\phi-\theta)\,, \quad \text{mod } 2\pi\,, \qquad \forall f \in L^2[0,2\pi)\,,\ \forall\theta \in [0,2\pi)\,. \tag{29}$$

This induces an equivalent representation, $R(\theta)$, supported on $\mathcal{H}$ by means of the unitary mapping U as

$$R(\theta) := U^{-1}\mathcal{R}(\theta)U\,. \tag{30}$$

These representations preserve the RHS structure due to the following result:

Lemma 2. *For any $\theta \in [0,2\pi)$, $R(\theta)$ is a bicontinuous bijection on Φ.*

Proof. Let $|f\rangle \in \Phi$ with $f = \sum_{m=-\infty}^{\infty} a_m\,|m\rangle$. Then,

$$\begin{aligned}R(\theta) \sum_{m=-\infty}^{\infty} a_m\,|m\rangle &= U^{-1} \sum_{m=-\infty}^{\infty} a_m\,U\,R(\theta)\,U^{-1}\,U\,|m\rangle = U^{-1} \sum_{m=-\infty}^{\infty} a_m\,\mathcal{R}(\theta)\,\frac{1}{\sqrt{2\pi}}\,e^{-im\phi} \\ &= U^{-1} \sum_{m=-\infty}^{\infty} a_m\,\frac{1}{\sqrt{2\pi}}\,e^{-im(\phi-\theta)} = \sum_{m=-\infty}^{\infty} a_m\,e^{im\theta}\,|m\rangle \in \Phi\,.\end{aligned} \tag{31}$$

Hence, $R(\theta)\Phi \subset \Phi$. Since $R^{-1}(\theta) = R(-\theta)$, we have that $R(-\theta)\Phi \subset \Phi$, so that $\Phi \subset R(\theta)\Phi$ and, consequently, $R(\theta)\Phi = \Phi$.

The continuity of $R(\theta)$ on Φ is trivial for any $\theta \in [0,2\pi)$ and, thus, its inverse is also continuous. □

This result has some immediate consequences, such as: (i) $R(\theta)$ can be extended to a continuous bijection on $\Phi^\times$, as a consequence of the duality formula in Equation (9); and (ii) $\mathcal{R}(\theta)$ is a bicontinuous bijection on Ψ and also on $\Psi^\times$. A simple consequence of (i) is the following: since $f(\phi) = \langle\phi|f\rangle$ for all $f(\phi) \in \Psi$, we have that

$$\langle R(\theta)\phi|f\rangle = [R(-\theta)\,f](\phi) = f(\phi+\theta) = \langle\phi+\theta|f\rangle\,, \tag{32}$$

so that for any arbitrarily fixed $\theta \in [0, 2\pi)$,

$$\langle R(\theta)\phi| = \langle\phi|R(\theta) = \langle\phi + \theta| \qquad \Longleftrightarrow \qquad R(\theta)|\phi\rangle = |\theta + \phi\rangle\,, \quad \mathrm{mod}\, 2\pi. \tag{33}$$

In addition to the regular representation, there exists one unitary irreducible representation, UIR in the sequel, on $L^2[0, 2\pi)$ for each value of $m \in \mathbb{Z}$ given by $\mathcal{U}_m(\phi) := e^{-im\phi}$. This induces a UIR on $\mathcal{H}$ given by $U_m(\theta) = U^{-1}\,\mathcal{U}(\phi)\,U = e^{-iJ\phi}$, where J is the self-adjoint generator of all these representations. We know that for all $m \in \mathbb{Z}$, we have that

$$J|m\rangle = m|m\rangle\,. \tag{34}$$

Obviously, J cannot be extended to a bounded operator on $\mathcal{H}$.

Proposition 1. *The self-adjoint operator J is a well defined continuous linear operator on Φ.*

Proof. We define the action of J on any $|f\rangle = \sum_{m=-\infty}^{\infty} a_m\,|m\rangle \in \Phi$ as

$$J|f\rangle := \sum_{m=-\infty}^{\infty} a_m\, m\,|m\rangle\,. \tag{35}$$

Then, for $p = 0, 1, 2, \ldots$, we have that

$$||J|f\rangle||_p^2 = \sum_{m=-\infty}^{\infty} |a_m|^2\, m^2\, |m+i|^{2p} \leq \sum_{m=-\infty}^{\infty} |a_m|^2 \cdot |m+i|^{2p+2} = ||\,|f\rangle||_{p+1}^2\,, \tag{36}$$

which shows that, for any $|f\rangle \in \Phi$, $J|f\rangle$ is a well defined vector on Φ. This also shows the inequality valid for any $p = 0, 1, 2, \ldots$ and all $|f\rangle \in \Phi$,

$$||J|f\rangle||_p \leq ||\,|f\rangle||_{p+1}\,, \tag{37}$$

which proves the continuity of J on Φ, after Equation (6). □

All these properties show that J may be extended to a weakly continuous linear operator on $\Phi^\times$. To determine its action on the functionals $|\phi\rangle$, let us consider the following derivation valid for all $f(\phi) \in \Psi$:

$$i\frac{d}{d\phi}\, f(\phi) = i\frac{d}{d\phi}\sum_{m=-\infty}^{\infty} a_m\, e^{-im\phi} := \sum_{m=-\infty}^{\infty} a_m\, m\, e^{-im\phi}\,. \tag{38}$$

It is a very simple exercise to show that this derivation is a well defined continuous linear operator on Ψ. Then, we define D_ϕ as

$$iD_\phi := U^{-1}\, i\frac{d}{d\phi}\, U\,. \tag{39}$$

The operator iD_ϕ is continuous and linear on Φ. Moreover, it is symmetric on Φ, so that it may be extended to a weakly continuous linear operator on $\Phi^\times$. In addition:

$$J|\phi\rangle = \sum_{m=-\infty}^{\infty} e^{-im\phi}\, J|m\rangle = \sum_{m=-\infty}^{\infty} e^{-im\phi}\, m\,|m\rangle = iD_\phi\,|\phi\rangle\,. \tag{40}$$

This derivation is somehow unnecessary as we know from Equations (35), (38) and (40) that $J = iD_\phi$. Here, we close the discussion on $SO(2)$.

4. $SU(2)$ and Associated Laguerre Functions

In the previous section, we study the relations between the Lie group $SO(2)$ and the special functions $f_m(\phi) = \frac{1}{\sqrt{2\pi}}\, e^{-im\phi}$, for $m \in \mathbb{Z}$. We construct a couple of RHS's, one based on the use of these functions, and the other an abstract RHS unitarily equivalent to the former. In the sequel, we extend a similar formalism using instead the group $SU(2)$ and the associated Laguerre functions [67,68].

The associate Laguerre functions [69–71], $L_n^{(\alpha)}(x)$ are functions depending for $n = 0, 1, 2, \ldots$ on the non-negative real variable $x \in [0, \infty)$ and a fixed complex parameter α, which satisfy the following differential equation:

$$\left[x\,\frac{d^2}{dx^2} + (1+\alpha-x)\,\frac{d}{dx} + n\right] L_n^{(\alpha)}(x) = 0\,, \qquad n = 0,1,2,\ldots\,. \tag{41}$$

Note that, for $\alpha = 0$, we obtain the Laguerre polynomials. In this presentation and for reasons to be clarified below, we are interested in those associated Laguerre functions such that α is an integer number, $\alpha \in \mathbb{Z}$.

4.1. Associated Laguerre Functions

It is also useful to introduce a set of alternative variables, such as $j := n + \alpha/2$ and $m := -\alpha/2$ with $|m| \le j$ and $j - m \in \mathbb{N}$, $\mathbb{N}$ being the set of non-negative integers. Observe that j is either a positive integer or a positive semi-integer, i.e., $n \in \mathbb{N}$, $\alpha \in \mathbb{Z}$ and $\alpha \ge -n$. Then, we define the following sequence of functions:

$$\mathcal{L}_j^m(x) := \sqrt{\frac{(j+m)!}{(j-m)!}}\, x^{-m}\, e^{-x/2}\, L_{j+m}^{(-2m)}(x)\,. \tag{42}$$

These functions are symmetric with respect to the exchange $m \leftrightarrow -m$. In addition, they satisfy the following orthonormality and completeness relations:

$$\int_0^\infty \mathcal{L}_j^m(x)\, \mathcal{L}_{j'}^m(x)\, dx = \delta_{jj'}\,, \qquad \sum_{j=|m|}^{\infty} \mathcal{L}_j^m(x)\, \mathcal{L}_j^m(x') = \delta(x-x')\,. \tag{43}$$

It is also well known that, for a fixed value of m, the functions $\{\mathcal{L}_j^m(x)\}_{j=|m|}^\infty$ form an orthonormal basis for $L^2(\mathbb{R}^+)$, $\mathbb{R}^+ := [0,\infty)$.

We may rewrite the differential Equation (41) in terms of the functions $\mathcal{L}_j^m(x)$ as

$$\left[X\, D_x^2 + D_x - \frac{1}{X}\, M^2 - \frac{X}{4} + J + \frac{1}{2}\right] \mathcal{L}_j^m(x) = 0\,, \tag{44}$$

where

$$X\, \mathcal{L}_j^m(x) := x\, \mathcal{L}_j^m(x)\,, \;\; D_x\, \mathcal{L}_j^m(x) := \frac{d}{dx}\, \mathcal{L}_j^m(x)\,, \;\; J\, \mathcal{L}_j^m(x) := j\, \mathcal{L}_j^m(x)\,, \;\; M\, \mathcal{L}_j^m(x) := m\, \mathcal{L}_j^m(x)\,. \tag{45}$$

The operators in Equation (45) can be extended by linearity and closedness to domains dense in $L^2(\mathbb{R}^+)$.

Next, we formally define the following linear operators:

$$\begin{aligned} K_\pm &:= \mp 2D_x \left(M \pm \tfrac{1}{2}\right) + \tfrac{2}{X}\left(M \pm \tfrac{1}{2}\right) - \left(J + \tfrac{1}{2}\right) \\ K_3 &:= M\,, \end{aligned} \tag{46}$$

which give the following relations:

$$\begin{aligned} K_{\pm}\, \mathcal{L}_j^m(x) &:= \sqrt{(j \mp m)(j \pm m + 1)}\, \mathcal{L}_j^{m\pm 1}(x) \\ K_3\, \mathcal{L}_j^m(x) &:= m\, \mathcal{L}_j^m(x)\,. \end{aligned} \tag{47}$$

On the subspace spanned by linear combinations of the functions $\mathcal{L}_j^m(x)$, we get from Equation (47) the following commutation relations:

$$[K_+, K_-] = 2K_3\,, \qquad [K_3, K_{\pm}] = \pm K_{\pm}\,, \tag{48}$$

which are the commutation relations for the generators of the Lie algebra $su(2)$. For each fixed value of j integer or half-integer and $-j \leq m \leq j$, the space of the linear combinations of the functions $\mathcal{L}_j^m(x)$ support a $2j+1$ dimensional representation of $SU(2)$.

4.2. Associated Laguerre Functions on the Plane

In RHS's, the number of variables is equal to the number of parameters because of the properties of Φ and $\Phi^{\times}$. In Section 3.1, we discuss a RHS based on one parameter m and one continuous variable x. An alternative is to introduce a new continuous variable ϕ and construct a RHS with two parameters j and m and two variables, the old one x and this new one ϕ. This point will be discussed in general in Section 10.

Then, we introduce an angular variable $\phi \in [-\pi, \pi]$ and the new functions:

$$\mathcal{Z}_j^m(r,\phi) := e^{im\phi}\, \mathcal{L}_j^m(r^2)\,. \tag{49}$$

These functions satisfy the property $\mathcal{Z}_j^m(r,\phi+2\pi) = (-1)^{2j}\mathcal{Z}_j^m(r,\phi)$. After Equation (44) and the change of variable $x \to r^2$, we obtain the following differential equation for $\mathcal{Z}_j^m(r,\phi)$:

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4m^2}{r} - r^2 + 4\left(j + \frac{1}{2}\right)\right] \mathcal{Z}_j^m(r,\phi) = 0\,. \tag{50}$$

It is not difficult to obtain the orthonormality and completeness relations for the functions $\mathcal{Z}_j^m(r,\phi)$, which are

$$\begin{aligned} &\frac{1}{\pi}\int_{-\pi}^{\pi} d\phi \int_0^{\infty} r\, dr\, [\mathcal{Z}_j^m(r,\phi)]^*\, \mathcal{Z}_{j'}^{m'}(r,\phi) = \delta_{jj'}\, \delta_{mm'}\,, \\ &\sum_{j,m} [\mathcal{Z}_j^m(r,\phi)]^*\, \mathcal{Z}_j^m(r',\phi') = \frac{\pi}{r}\, \delta(r-r')\, \delta(\phi-\phi')\,. \end{aligned} \tag{51}$$

This shows that the set of functions $\{\mathcal{Z}_j^m(r,\phi)\}$ forms a basis of $L^2(\mathbb{R}^2, d\mu)$ with $d\mu(r,\phi) := r\, dr\, d\phi/\pi$. Observe the similitude with the set of spherical harmonics $\{Y_j^m(\theta,\phi)\}$, which forms a basis of the Hilbert space $L^2(S^2, d\Omega)$.

Let $\mathcal{H}$ be an abstract infinite dimensional separable Hilbert space and U a unitary mapping from $L^2(\mathbb{R}^2)$ onto $\mathcal{H}$, $U : L^2(\mathbb{R}^2) \longmapsto \mathcal{H}$. An orthonormal basis $\{|j,m\rangle\}$ in $\mathcal{H}$ is given by $|j,m\rangle := U\mathcal{Z}_j^m(r,\phi)$, so that $\{|j,m\rangle\}$ satisfy the conditions of orthonormality and completeness:

$$\langle j,m|j',m'\rangle = \delta_{jj'}\, \delta_{mm'}\,, \qquad \sum_{j_{min}}^{\infty} \sum_{m=-j}^{j} |j,m\rangle\langle j,m| = \mathcal{I}\,, \tag{52}$$

where $j_{min} = 0$ for integer spins and $j_{min} = 1/2$ for half-integer spins.

After the two last equations in Equation (45), we define

$$\widetilde{J} := UJU^{-1}\,, \qquad \widetilde{M} = UMU^{-1}\,, \tag{53}$$

so that

$$\widetilde{J}|j,m\rangle = j|j,m\rangle\,, \qquad \widetilde{M}|j,m\rangle = m|j,m\rangle\,, \tag{54}$$

where j is a non-negative integer or half-integer and $m = -j, -j+1, \ldots, j-1, j$.

Let us proceed with the definitions of some new objects. First, the operators $J_\pm$ and J_3 on $L^2(\mathbb{R}^2)$ are

$$J_\pm := e^{\pm i\phi} K_\pm\,, \qquad J_3 := K_3\,. \tag{55}$$

The operators defined in Equation (55) act on the functions $\mathcal{Z}_j^m(r,\phi)$ exactly as $K_\pm$ and K_3 on $\mathcal{L}_j^m(x)$, expressions given in Equation (47). In addition, we define the corresponding operators on $\mathcal{H}$ as

$$\widetilde{J}_\pm := U\,J_\pm\,U^{-1}\,, \qquad \widetilde{J}_3 := U\,J_3\,U^{-1}\,, \tag{56}$$

so that

$$\begin{aligned} &\widetilde{J}_\pm\,|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,|j,m\pm 1\rangle\,, \\ &\widetilde{J}_3\,|j,m\rangle = m\,|j,m\rangle\,. \end{aligned} \tag{57}$$

4.3. Rigged Hilbert Spaces Associated to $L^2(\mathcal{R}^2)$

On $\mathcal{H}$, we have already defined a pair of discrete bases $\{|j,m\rangle\}$, one for integer values of j and the other for half-integer values of j. To define continuous bases, we have to construct a suitable pair of RHS's. Let us consider the space of all $|f\rangle \in \mathcal{H}$,

$$|f\rangle = \sum_{j_{min}}^{\infty} \sum_{m=-j}^{j} a_{j,m}\,|j,m\rangle \qquad \text{with} \quad \sum_{j_{min}}^{\infty} \sum_{m=-j}^{j} |a_{j,m}|^2 < \infty\,, \tag{58}$$

where we have taken one of the choices for j, either integer or half-integer, such that they satisfy the following property:

$$||\,|f\rangle||_p^2 := \sum_{j_{min}}^{\infty} \sum_{m=-j}^{j} \left|a_{j,m}\right|^2 \left(2^{3|m|}(j+|m|+1)!\right)^{2p} < \infty\,, \qquad p = 0,1,2,\ldots\,, \tag{59}$$

where, again, we may use either the basis with j integer or with j half-integer. We call Φ_I and Φ_H the resulting spaces, where the indices I and H mean "integer" and "half-integer", respectively. These spaces are rather small. Nevertheless, they are still dense in $\mathcal{H}$, since they contain the orthonormal basis $\{|j,m\rangle\}$. We need this kind of topology in order to guarantee the continuity of the elements of the continuous basis, as we shall see. Norms $||-||_p$ endow both Φ_I and Φ_H of a structure of metrizable locally convex space and give a pair of unitarily equivalent RHS's

$$\Phi_I \subset \mathcal{H} \subset \Phi_I^\times\,, \qquad \Phi_H \subset \mathcal{H} \subset \Phi_H^\times\,. \tag{60}$$

On these structures, it makes sense the existence of continuous bases, $\{|r,\phi\rangle\}$, as we can show right away. For each pair of values of r and ϕ, we define the following anti-linear mapping $|r,\phi\rangle$ as follows. Let $f(r,\phi) := U^{-1}\,|f\rangle$, $|f\rangle \in \mathcal{H}$ so that

$$f(r,\phi) = \sum_{j_{min}}^{\infty} \sum_{m=-j}^{j} a_{j,m}\,\mathcal{Z}_j^m(r,\phi)\,. \tag{61}$$

Then, define

$$\langle f|r,\phi\rangle := \sum_{j_{min}}^{\infty} \sum_{m=-j}^{j} a_{j,m}^{*}[\mathcal{Z}_j^m(r,\phi)]^* \,. \tag{62}$$

Note that, for $|f\rangle = |j,m\rangle$, we have that

$$\langle j,m|r,\phi\rangle = [\mathcal{Z}_j^m(r,\phi)]^* \quad \text{or} \quad \langle r,\phi|j,m\rangle = \langle j,m|r,\phi\rangle^* = \mathcal{Z}_j^m(r,\phi)\,. \tag{63}$$

As in the previous cases, we may define $\Psi_{I,H} := U^{-1}\Phi_{I,H}$, so as to define two new RHS's, which are unitarily equivalent to Equation (60). These are

$$\Psi_{I,H} \subset L^2(\mathbb{R}^2) \subset \Psi_{I,H}\,. \tag{64}$$

Then, $f(r,\phi)$ as in Equation (61) is in Ψ_I or in Ψ_H, if and only if the coefficients $a_{j,m}$ satisfy the relations in Equation (59). The kets $|r,\phi\rangle$, which are obviously linear on Φ_I and Φ_H, are also continuous under the topologies induced by the norms $||-||_p$. This is a consequence of the next two results.

Lemma 3. *The functions $\mathcal{Z}_j^m(r,\phi)$ have the following upper bound:*

$$\left|\mathcal{Z}_j^m(r,\phi)\right| \leq 2^{3|m|}\,\frac{(j!)^2[(j+|m|)!]^{1/2}}{|m|![(j-|m|)!]^{5/2}}\,. \tag{65}$$

Proof. To begin with, look at Equations (42) and (49). Then, we use the following inequality, which has been given in [72]:

$$|x^k L_n^{(\alpha)}(x)e^{-x/2}| \leq 2^{\min(\alpha,k)}\, 2^k (n+1)^{(k)} \begin{pmatrix} n+\max(\alpha-k,0) \\ n \end{pmatrix}. \tag{66}$$

Here, k and n are natural numbers, $\alpha \geq 0$, $x \geq 0$ and $(n+1)^{(k)} := (n+1)(n+2)\dots(n+k)$ is the Pochhammer symbol.

We have to consider the cases $m < 0$ and $m \geq 0$, as well as the condition $\alpha \geq 0$, which is necessary for the validity of inequality in Equation (4). These two conditions are really only one since $m = -\alpha/2$ and the functions $\mathcal{L}_j^m(x)$ satisfy the following symmetry relation:

$$\mathcal{L}_j^m(x) = (-1)^{2j}\,\mathcal{L}_j^{-m}(x)\,. \tag{67}$$

For this reason, we discuss $m < 0$. Here, we write $-m$ with $m \in \mathbb{N}$ instead. Take Equation (42), where we replace m by $-m$ and use Equation (66). First, we have

$$\left|x^k L_n^{(\alpha)}(x)e^{-x/2}\right| \leq 2^{3m}(j-m+1)^{(m)} \begin{pmatrix} j \\ j-m \end{pmatrix}. \tag{68}$$

Then, complete $\mathcal{L}_j^{-m}(x)$ so as to obtain

$$\left|\mathcal{L}_j^{-m}(x)\right| \leq 2^{3m}\,\frac{(j!)^2[(j+m)!]^{1/2}}{m![(j-m)!]^{5/2}}\,. \tag{69}$$

This result, along (67) and (49) gives (65). □

Theorem 2. *Each of the kets $|r,\phi\rangle$ is a continuous anti-linear functional in both Φ_I and Φ_H.*

Proof. It is a consequence of the previous lemma. From Equations (62) and (65), we have the following inequalities, the first one in the second row being the Cauchy–Schwarz inequality,

$$
\begin{aligned}
|\langle f|r,\phi\rangle| &= \sum_{j_{min}}^{\infty}\sum_{m=-j}^{j}|a_{j,m}|\,|\mathcal{Z}_j^m(r,\phi)| \le \sum_{j_{min}}^{\infty}\sum_{m=-j}^{j}|a_{j,m}|\,2^{3|m|}\,\frac{(j!)^2((j+|m|)!)^{1/2}}{|m|!((j-|m|)!)^{5/2}} \\
&\le \sqrt{\sum_{j_{min}}^{\infty}\sum_{m=-j}^{j}|a_{j,m}|^2\,2^{6|m|}\,(j!)^2\,(j+|m|)!}\;\times\;\sqrt{\sum_{j_{min}}^{\infty}\sum_{m=-j}^{j}\frac{1}{(|m|!)^2\,((j-|m|)!)^5}}\,.
\end{aligned}
\tag{70}
$$

The second row in Equation (70) is the product of two terms. The second one is the root of a convergent series. Let us denote this term by $C > 0$. The expression under the square root in the first factor is bounded by

$$
\sum_{j_{min}}^{\infty}\sum_{m=-j}^{j}|a_{j,m}|^2\,\left(2^{3|m|}(j+|m|+1)!\right)^4 = ||\,|f\rangle||_2^2\,, \tag{71}
$$

so that

$$
|\langle f|r,\phi\rangle| \le C\,||\,|f\rangle||_2\,, \tag{72}
$$

which, along the linearity of $|r,\phi\rangle$ on $\Phi_{I,H}$, proves our assertion. □

Formal relations between discrete $\{|j,m\rangle\}$ and continuous bases $\{|r,\phi\rangle\}$ are easy to find. Let us go back to Equation (62). Due to the unitary relation between $L^2(\mathbb{R}^+)$ and $\mathcal{H}$, we conclude that $a^*_{j,m} = \langle f|j,m\rangle$, so that $\langle f|r,\phi\rangle = \sum_{j=0}^{\infty}\sum_{m=-j}^{j}\langle f|j,m\rangle\,\mathcal{Z}_j^m(r,\phi)$ and, hence, omitting the arbitrary $|f\rangle \in \Phi_{I,H}$, we have that

$$
|r,\phi\rangle = \sum_{j_{min}}^{\infty}\sum_{m=-j}^{j}\mathcal{Z}_j^m(r,\phi)\,|j,m\rangle\,. \tag{73}
$$

The inverse relation may be easily found taking into account the unitary mapping between $L^2(\mathbb{R}^+)$ and $\mathcal{H}$, again. In fact, being given $|f\rangle, |g\rangle \in \Phi_{I,H}$, their scalar product gives:

$$
\langle f|g\rangle = \int_0^{2\pi}d\phi\int_0^{\infty}r\,dr\,\langle f|r,\phi\rangle\langle r,\phi|g\rangle\,. \tag{74}
$$

Then, if we choose $|g\rangle = |j,m\rangle$ and omit the arbitrary $|f\rangle \in \Phi_{I,H}$, we have the converse relation to Equation (73) as

$$
|j,m\rangle = \int_0^{2\pi}d\phi\int_0^{\infty}r\,dr\,|r,\phi\rangle\langle r,\phi|j,m\rangle = \int_0^{2\pi}d\phi\int_0^{\infty}r\,dr\,\mathcal{Z}_j^m(r,\phi)\,|r,\phi\rangle\,. \tag{75}
$$

Although this is implicit in the above expressions, it could be interesting to write the explicit spans of any $|f\rangle \in \Phi_{I,H}$ in terms of the discrete and continuous basis. These are

$$
|f\rangle = \sum_{j_{min}}^{\infty}\sum_{m=-j}^{j}\langle j,m|f\rangle\,|j,m\rangle\,, \tag{76}
$$

and

$$
|f\rangle = \int_0^{2\pi}\int_0^{\infty}r\,dr\,f(r,\phi)\,|r,\phi\rangle\,. \tag{77}
$$

The continuity of the linear operators $\widetilde{J}$, $\widetilde{M}$, $\widetilde{J}_\pm$ and $\widetilde{J}_3$ is rather obvious. For instance, for any $|f\rangle \in \Phi_{I,H}$, we define

$$\widetilde{J}|f\rangle = \sum_{j_{min}}^{\infty} \sum_{m=-j}^{j} a_{j,m}\, j\, |j,m\rangle\,, \tag{78}$$

so that, for any $p = 0, 1, 2, \ldots$

$$\begin{aligned} ||\widetilde{J}\,|f\rangle||_p^2 &:= \sum_{j_{min}}^{\infty} \sum_{m=-j}^{j} |a_{j,m}|^2\, j^2 [2^{3|m|}(j+|m|+1)!]^{2p} \\ &\leq \sum_{j_{min}}^{\infty} \sum_{m=-j}^{j} |a_{j,m}|^2 \left(2^{3|m|}(j+|m|+1)!\right)^{2(p+1)} = ||\,|f\rangle||_{p+1}^2\,. \end{aligned} \tag{79}$$

This relation proves both that $\widetilde{J}\,|f\rangle$ is in either $\Phi_{I,H}$ and the continuity of $\widetilde{J}$ in both spaces. Similar results can be obtained for the other operators: $\widetilde{M}$, $\widetilde{J}_\pm$ and $\widetilde{J}_3$.

In fact, the topology in Equation (59) is too strong, if we only wanted to provide RHS's for which the above operators be continuous. Take for instance the spaces of all $|f\rangle \in \mathcal{H}$ such that

$$q(|f\rangle)^2 = \sum_{j_{min}}^{\infty} \sum_{m=-j}^{j} \left|a_{j,m}\right|^2 (j+|m|+1)^{2p}\,, \qquad p = 0,\, 1,\, 2\, \ldots\,. \tag{80}$$

One of these spaces, Ξ_I, holds for j integer and the other, Ξ_H, holds for j half-integer. The above operators reduce both spaces $\Xi_{I,H}$ and are continuous on them. The proof is essentially identical as in the previous case. Thus, we have two sequences of rigged Hilbert spaces one for j integer, labeled by I, and the other for j half-integer, labeled by H, where all the inclusions are continuous:

$$\Phi_{I,H} \subset \Xi_{I,H} \subset \mathcal{H} \subset \Xi_{I,H}^\times \subset \Phi_{I,H}^\times\,. \tag{81}$$

This type of sequences including several RHS's are nothing else than a particular case of partial inner product spaces, which have been introduced by Antoine and Grossmann [73,74] (cf. [75,76]).

While the operators $\widetilde{J}$, $\widetilde{M}$, $\widetilde{J}_\pm$ and $\widetilde{J}_3$ are continuous on $\Phi_{I,H}$ and $\Xi_{I,H}$, we have introduce the topology in Equation (59) just to make sure of the continuity of the functionals $\{|r,\phi\rangle\}$. All these operators can be continuously extended to the duals. Note that $\widetilde{J}$, $\widetilde{M}$ and $\widetilde{J}_3$ are symmetric, although $\widetilde{J}_\pm$ are formal adjoint of each other.

5. Weyl–Heisenberg Group and Hermite Functions

Possibly, the better studied and the most widely used of the special functions are the Hermite functions. When properly normalized, the Hermite functions form an orthonormal discrete basis for $L^2(\mathbb{R})$ and have the form

$$\psi_n(x) := \frac{e^{-x^2/2}}{\sqrt{2^n n! \sqrt{\pi}}}\, H_n(x)\,, \tag{82}$$

where $H_n(x)$ are the Hermite polynomials [69–71].

5.1. Continuous and Discrete Bases and RHS

In quantum mechanics for one-dimensional systems [77], one often uses a pair of continuous bases: the continuous bases in the coordinate and momentum representation, denoted as $\{|x\rangle\}$ and $\{|p\rangle\}$ respectively, with $x, p \in \mathbb{R}$. Kets $|x\rangle$ and $|p\rangle$ are eigenkets of the position operator Q and momentum operator P, respectively: $Q|x\rangle = x|x\rangle$ and $P|p\rangle = p|p\rangle$. To define these objects and the continuous bases they produce, we need a RHS [16]. Then, the ingredients in our construction are the following:

- The Schwartz space S of all complex indefinitely differentiable functions of the real variable $x \in \mathbb{R}$, such as they and all their derivatives at all orders go to zero at the infinity faster than the inverse of any polynomial. The Schwartz space S is endowed with a metrizable locally convex topology [15]. It is well known that S is the first element of a RHS $S \subset L^2(\mathbb{R}) \subset S^\times$. Note that the Fourier transform leaves this triplet invariant.
- An abstract infinite dimensional separable Hilbert space $\mathcal{H}$ along a fixed, although arbitrary, unitary operator $U : \mathcal{H} \to L^2(\mathbb{R})$. If $\Phi := U^{-1}S$ and we transport the locally convex topology from S to Φ by U^{-1}, we have a second RHS $\Phi \subset \mathcal{H} \subset \Phi^\times$, unitarily equivalent to $S \subset L^2(\mathbb{R}) \subset S^\times$.
- For any $|f\rangle \in \Phi$ and any $x_0 \in \mathbb{R}$, we define $\langle f|x_0\rangle := f(x_0)$, where $f(x) := U|f\rangle$, so that $f(x) \in S$. Analogously, for any $p_0 \in \mathbb{R}$, we define

$$\langle f|p_0\rangle := \int_{-\infty}^{\infty} e^{-ixp_0} \langle f|x\rangle \, dx = \int_{-\infty}^{\infty} e^{-ixp_0} f(x) \, dx \,. \tag{83}$$

  Vectors $|x\rangle, |p\rangle \in \Phi^\times$ for any $x, p \in \mathbb{R}$ [78].
- Define $\widetilde{Q}f(x) := xf(x)$ and $\widetilde{P}f(x) := -if'(x)$, for all $f(x) \in S$, where the prime means derivative. Let $Q := U^{-1}\widetilde{Q}U$ and $P := U^{-1}\widetilde{P}U$. Then, for given $x_0 \in \mathbb{R}$, $\langle Qf|x_0\rangle = x_0 f(x_0)$, so that $\langle f|Q|x_0\rangle = x_0 f(x_0) = x_0\langle f|x_0\rangle$, which implies that $Q|x_0\rangle = x_0|x_0\rangle$. We use the same notation for Q and its extension to $\Phi^\times$. Analogously, $P|p_0\rangle = p_0|p_0\rangle$, for any $p_0 \in \Phi^\times$.
- Since U is unitary, it preserves scalar products, so that, for arbitrary $|g\rangle, |f\rangle \in \Phi$, we have

$$\langle g|f\rangle = \int_{-\infty}^{\infty} g^*(x) f(x) \, dx = \int_{-\infty}^{\infty} \langle g|x\rangle\langle x|f\rangle \, dx \,, \tag{84}$$

  which defines the following identity:

$$I := \int_{-\infty}^{\infty} |x\rangle\langle x| \, dx \,, \tag{85}$$

  that is the canonical injection $I : \Phi \longmapsto \Phi^\times$ with $I|f\rangle \in \Phi^\times$ for any $|f\rangle \in \Phi$. Another representation of this identity is

$$I = \int_{-\infty}^{\infty} |p\rangle\langle p| \, dp \,. \tag{86}$$

  This means that, for any $|f\rangle \in \Phi$, $I|f\rangle \in \Phi^\times$ can be written as

$$I|f\rangle = \int_{-\infty}^{\infty} |x\rangle\langle x|f\rangle \, dx = \int_{-\infty}^{\infty} f(x) \, |x\rangle \, dx \,, \tag{87}$$

  and

$$I|f\rangle = \int_{-\infty}^{\infty} |p\rangle\langle p|f\rangle \, dp = \int_{-\infty}^{\infty} f(p) \, |p\rangle \, dp \,, \quad \text{with} \quad f(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} f(x) \, dx \,. \tag{88}$$

- The conclusion of the above paragraph is that both sets of vectors $\{|x\rangle\}$ and $\{|p\rangle\}$ form a continuous basis for the vectors in Φ. In addition, we have a discrete basis on $\mathcal{H}$ defined as

$$|n\rangle := U^{-1} \left(\psi_n(x)\right) \,, \qquad n = 0, 1, 2, \ldots, \tag{89}$$

  which has the properties,

$$\mathbb{I} = \sum_{n=0}^{\infty} |n\rangle\langle n| \,, \qquad \langle n|m\rangle = \delta_{n,m} \,, \tag{90}$$

where $\mathbb{I}$ is the identity operator on $\mathcal{H}$. For any $|f\rangle = \sum_{n=0}^{\infty} a_n|n\rangle \in \Phi$, we have that

$$a_n^* = \langle f|n\rangle = \int_{-\infty}^{\infty} \psi_n(x)\, f^*(x)\, dx = \int_{-\infty}^{\infty} \psi_n(x)\, \langle f|x\rangle\, dx\,, \tag{91}$$

so that

$$|n\rangle = \int_{-\infty}^{\infty} \psi_n(x)\, |x\rangle\, dx\,, \qquad n = 0, 1, 2, \ldots, \tag{92}$$

identity that makes sense in $\Phi^\times$. Considering Equations (85) and (91), and that $\langle n|x\rangle = \psi_n(x)$, we may invert Equation (92). Take an arbitrary $|f\rangle \in \Phi$:

$$\begin{aligned} \langle f|x\rangle &= \sum_{n=0}^{\infty} a_n^* \langle n|x\rangle = \sum_{n=0}^{\infty} \left[\int_{-\infty}^{\infty} \psi_n(x')\, \langle f|x'\rangle\, dx'\right] \langle n|x\rangle \\ &= \sum_{n=0}^{\infty} \left[\int_{-\infty}^{\infty} \langle f|x'\rangle\langle x'|n\rangle\, dx'\right] \psi_n(x) = \sum_{n=0}^{\infty} \psi_n(x)\, \langle f|n\rangle\,, \end{aligned} \tag{93}$$

so that, if we omit the arbitrary bra $\langle f|$, we conclude that

$$|x\rangle = \sum_{n=0}^{\infty} \psi_n(x)\, |n\rangle\,, \tag{94}$$

identity that makes sense in $\Phi^\times$. Another property can be easily shown from Equation (69) and $f(x) = \langle x|f\rangle \in \Phi$:

$$f(x') = \langle x'|f\rangle = \int_{-\infty}^{\infty} f(x)\langle x'|x\rangle\, dx \Longleftrightarrow \langle x'|x\rangle = \delta(x - x')\,. \tag{95}$$

- Analogously, in the momentum representation, we have that

$$|n\rangle = (-i)^n \int_{-\infty}^{\infty} \psi_n(p)\, |p\rangle\, dp\,, \tag{96}$$

since the Fourier transform of $\psi_n(x)$ is $(-i)^n \psi_n(p)$

$$|p\rangle = \sum_{n=0}^{\infty} (-i)^n \psi_n(p)\, |n\rangle\,, \tag{97}$$

and $\langle p|p'\rangle = \delta(p - p')$.

5.2. The Weyl–Heisenberg Lie Algebra

Let us consider the following operators, defined by their action on the normalized Hermite functions [79]:

$$\widetilde{Q}\psi_n(x) := x\psi_n(x)\,, \quad \widetilde{P}\psi_n(x) := i\frac{d\psi_n(x)}{dx}\,, \quad \widetilde{N}\psi_n(x) := n\psi_n(x)\,, \quad \mathbb{I}\psi_n(x) := \psi_n(x) \tag{98}$$

for $n = 0, 1, 2, \ldots$. These operators can be uniquely extended to S, and these extensions are essentially self-adjoint and continuous on S with its own topology, so that they are extensible to weakly continuous operators on $S^\times$. The properties of these operators are very well known. Let us name

$$Q := U\,\widetilde{Q}\,U^{-1}\,, \qquad P := U\,\widetilde{P}\,U^{-1}\,, \qquad N := U\,\widetilde{N}\,U^{-1}\,, \qquad \mathbb{I} := U\,\widetilde{\mathbb{I}}\,U^{-1}\,, \tag{99}$$

which have the same properties on Φ. As usual,

$$a := \frac{1}{\sqrt{2}}\,(Q - iP)\,, \qquad a^\dagger := \frac{1}{\sqrt{2}}\,(Q + iP)\,, \tag{100}$$

so that,

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \qquad a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,. \tag{101}$$

Obviously, a and $a^\dagger$ are continuous on Φ and extended with continuity to $\Phi^\times$. The extensions are defined using the duality formula in Equation (7). As a system of generators of the Weyl–Heisenberg Lie algebra, we may use either $\{Q, P, N, \mathbb{I}\}$ or $\{a, a^\dagger, N, \mathbb{I}\}$. Note that

$$N = \frac{1}{2}(\{a, a^\dagger\} - \mathbb{I}) = \frac{1}{2}(Q^2 + P^2 - \mathbb{I})\,, \tag{102}$$

where the brackets mean anti-commutator. On Φ, the Casimir operator vanishes:

$$C := \frac{1}{2}(Q^2 + P^2) - \left(N + \frac{1}{2}\,\mathbb{I}\right) \equiv 0\,. \tag{103}$$

In addition, the universal enveloping algebra of the Weyl–Heisenberg group is irreducible on the RHS $\Phi \subset \mathcal{H} \subset \Phi^\times$.

6. The Group SO(3,2) and the Spherical Harmonics

Let us consider the hollow unit sphere S^2 in $\mathbb{R}^3$. Any point in S^2 is characterized by two angular variables θ and ϕ, with $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$. Let us consider the Hilbert space, $L^2(S^2, d\Omega)$, with $d\Omega := d(\cos\theta)\,d\phi$, of Lebesgue measurable complex functions, $f(\theta,\phi)$, such that

$$\int_0^{2\pi} d\phi \int_0^{\pi} d(\cos\theta)\,|f(\theta,\phi)|^2 < \infty\,. \tag{104}$$

An orthonormal basis for $L^2(S^2, d\Omega)$ is given by $\sqrt{l+1/2}\,Y_l^m(\theta,\phi)$, where $Y_l^m(\theta,\phi)$ are the spherical harmonics [69–71]

$$Y_l^m(\theta,\phi) = \sqrt{\frac{(l-m)!}{2\pi\,(l+m)!}}\,e^{im\phi}\,P_l^m(\cos\theta)\,, \tag{105}$$

where $l \in \mathbb{N}$, the set of natural numbers, $m \in \mathbb{Z}$ the set of integers, with $|m| \le l$ and P_l^m are the associated Legendre functions. This means, in particular, that for any $f(\theta,\phi) \in L^2(S^2, d\Omega)$

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{l,m}\,\sqrt{l+1/2}\,Y_l^m(\theta,\phi)\,, \quad \text{with} \quad \sum_{l=0}^{\infty}\sum_{m=-l}^{l} |f_{l,m}|^2 < \infty\,, \tag{106}$$

and

$$f_{l,m} = \sqrt{l+1/2}\int_0^{2\pi} d\phi \int_0^{\pi} d(\cos\theta)\,Y_l^m(\theta,\phi)^*\,f(\theta,\phi)\,. \tag{107}$$

From the fact that the set of spherical harmonics is an orthonormal basis, we obtain the following relations:

$$\begin{aligned} \int_{S^2} d\Omega\,Y_l^m(\theta,\phi)^*(l+1/2)Y_l^m(\theta',\phi') &= \delta_{l,l'}\,\delta_{m,m'}\,, \\ \sum_{l=|m|}^{\infty}\sum_{m=-\infty}^{\infty} Y_l^m(\theta,\phi)^*(l+1/2)Y_l^m(\theta',\phi') &= \delta(\cos\theta - \cos\theta')\,\delta(\phi-\phi')\,, \end{aligned} \tag{108}$$

with $\delta(\cos\theta - \cos\theta') = \delta(\theta-\theta')/\big|\sin\theta\big|$.

6.1. RHS Associated to the Spherical Harmonics

The Hilbert space $L^2(S^2, d\Omega)$ supports a representation of a UIR of the de-Sitter group $SO(3,2)$ with quadratic Casimir $\mathcal{C}_{so(3,2)} = -5/4$ on the spherical harmonics [80,81]. The action of the generators of the Cartan subalgebra of the Lie algebra $so(3,2)$, L and M, is

$$L\, Y_l^m(\theta,\phi) = l\, Y_l^m(\theta,\phi)\,, \qquad M\, Y_l^m(\theta,\phi) = m\, Y_l^m(\theta,\phi)\,. \tag{109}$$

Once we have established this Hilbert space which supports a representation of the Anti-de-Sitter group, let us consider a unitarily equivalent abstract Hilbert space $\mathcal{H} \equiv U[L^2(S^2, d\Omega)]$, where U is unitary. An orthonormal basis for this space is given by the vectors $\{|l,m\rangle\}$, where for each pair l, m (with $|m| \leq l$), $|l,m\rangle := U\,[\sqrt{1+1/2}\, Y_l^m(\theta,\phi)]$. If we define

$$\widetilde{L} := ULU^{-1}\,, \qquad \widetilde{M} := UMU^{-1}\,, \tag{110}$$

we have

$$\widetilde{L}\,|l,m\rangle = l\,|l,m\rangle\,, \qquad \widetilde{M}\,|l,m\rangle = m\,|l,m\rangle\,. \tag{111}$$

The operators $\widetilde{L}$ and $\widetilde{M}$ on $\mathcal{H}$, as well as L and M on $L^2(S^2, d\Omega)$, are obviously unbounded and self-adjoint on its maximal domain as symmetric generators of a Lie algebra.

Next, we construct a RHS on which they are, in addition, continuous [81]. Let us consider the subspace Φ of all vectors $|f\rangle = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{l,m}\,|l,m\rangle \in \mathcal{H}$, such that

$$\big|\big|\,|f\rangle\big|\big|_p^2 := \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \big|f_{l,m}\big|^2 (l+|m|+1)^{2p}\,, \qquad p = 0,\,1,\,2,\ldots. \tag{112}$$

The objects $\big|\big| - \big|\big|_p$ are indeed norms, which provides Φ of a metrizable locally convex topology. For $p=0$, we have the norm on $\mathcal{H}$, so that the canonical injection $i : \Phi \longmapsto \mathcal{H}$ is continuous. Take the anti-dual space $\Phi^\times$ and endow it with the weak topology compatible with the dual pair $\{\Phi, \Phi^\times\}$. Thus, we have the RHS:

$$\Phi \subset \mathcal{H} \subset \Phi^\times\,. \tag{113}$$

Then, define $\mathcal{D} := U^{-1}\Phi$, and transport the topology from Φ to $\mathcal{D}$. This topology is given by the norms

$$\big|\big|f(\theta,\phi)\big|\big|_p^2 = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \big|f_{l,m}\big|^2 (l+|m|+1)^{2p}\,, \qquad p = 0,\,1,\,2,\ldots. \tag{114}$$

The anti-dual $\mathcal{D}^\times = U^{-1}\Phi^\times$ is defined via the extension of U^{-1} to $\Phi^\times$ via a duality formula of the type in Equation (9). We have the rigged Hilbert space

$$\mathcal{D} \subset L^2(S^2, d\Omega) \subset \mathcal{D}^\times\,, \tag{115}$$

unitarily equivalent to $\Phi \subset \mathcal{H} \subset \Phi^\times$ in Equation (113).

6.2. Continuous Bases Depending on the Angular Variables

Let us begin with $f(\theta,\phi) \in \mathcal{D}$ and $|f\rangle = U[f(\theta,\phi)] \in \Phi$. For fixed angles with values $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$, almost elsewhere, define the following continuous anti-linear functional, $|\theta,\phi\rangle$, on Φ: For arbitrary $|f\rangle \in \Phi$, one defines the mapping $|\theta,\phi\rangle$ as

$$\langle f|\theta,\phi\rangle := \langle\theta,\phi|f\rangle^* := f(\theta,\phi)\,, \tag{116}$$

where the star denotes complex conjugation. The linearity of each $|\theta,\phi\rangle$ on Φ is obvious. To prove the continuity, take

$$\begin{aligned}\langle f|\theta,\phi\rangle = f(\theta,\phi) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{l,m}\,\sqrt{l+1/2}\,Y_l^m(\theta,\phi)\\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{l,m}(l+|m|+1)^p\left(\frac{\sqrt{l+1/2}}{(l+|m|+1)^p}\,Y_l^m(\theta,\phi)\right),\end{aligned}$$

where p is a natural number with $p \geq 3$. Then, take the modulus in Equation (**??**) and use the Schwarz inequality in the right-hand side. We have

$$|\langle f|\theta,\phi\rangle| \leq \sqrt{\sum_{l=0}^{\infty}\sum_{m=-l}^{l} |f_{l,m}|^2\,(l+|m|+1)^{2p}} \times \sqrt{\frac{l+1/2}{(l+|m|+1)^{2p}}\,|Y_l^m(\theta,\phi)|^2}\,. \tag{117}$$

The first factor on the right-hand side of Equation (117) is nothing else than $||\,|f\rangle||_p$, while the second factor converges due to the fact that $|Y_l^m(\theta,\phi)|^2 \leq (2\pi)^{-1}$ for all θ,ϕ [82]. If we call C this second factor, we finally conclude that

$$|\langle f|\theta,\phi\rangle| \leq C\,||\,|f\rangle||_p\,, \tag{118}$$

which, after Equation (4), guarantees the continuity of the functional $|\theta,\phi\rangle$ on Φ, so that $|\theta,\phi\rangle \in \Phi^\times$ for almost all $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$. These functionals have some interesting properties:

- For any $f(\theta,\phi) \in \mathcal{D}$, we can define the operator $\cos\Theta\, f(\theta,\phi) := \cos\theta\, f(\theta,\phi)$. One has that $\cos\theta\, f(\theta,\phi) \in \mathcal{D}$ and $\cos\Theta$ is continuous on $\mathcal{D}$. Therefore, we may define $\widehat{\cos\Theta} := U\,\cos\Theta\,U^{-1}$, which is a symmetric continuous linear operator on Φ and, hence, can be extended into the anti-dual $\Phi^\times$ by the duality formula in Equation (7). For almost all $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$, we can prove that
$$\widehat{\cos\Theta}\,|\theta,\phi\rangle = \cos\theta\,|\theta,\phi\rangle\,. \tag{119}$$
- Analogously, if we define the operator $e^{i\Phi}$ on $f(\theta,\phi) \in \mathcal{D}$ as $e^{i\Phi}\,f(\theta,\phi) := e^{i\phi}\,f(\theta,\phi)$ and $\widehat{e^{i\Phi}} := U\,e^{i\Phi}\,U^{-1}$, we have that
$$\widehat{e^{i\Phi}}\,|\theta,\phi\rangle = e^{i\phi}\,|\theta,\phi\rangle\,. \tag{120}$$
- Let $|g\rangle, |f\rangle \in \Phi$. Their scalar product is
$$\langle g|f\rangle = \langle U^{-1}g|U^{-1}f\rangle = \int_{S^2} d\Omega\, g(\theta,\phi)^*\, f(\theta,\phi) = \int_{S^2} d\Omega\,\langle g|\theta,\phi\rangle\langle\theta,\phi|f\rangle\,. \tag{121}$$

  Then, we may write the following formal identity:
$$\mathbb{I} = \int_{S^2} d\Omega\,|\theta,\phi\rangle\langle\theta,\phi| = \int_0^{2\pi} d\phi \int_0^{\pi} d(\cos\theta)\,|\theta,\phi\rangle\langle\theta,\phi|\,. \tag{122}$$

  We give below the meaning of this $\mathbb{I}$.

Let us take the formal identity $\mathbb{I}$ as in Equation (122) and let us apply it to the arbitrary vector $|f\rangle \in \Phi$. We have

$$|f\rangle = \mathbb{I}\,|f\rangle = \int_{S^2} d\Omega\,|\theta,\phi\rangle\langle\theta,\phi|f\rangle = \int_{S^2} d\Omega\, f(\theta,\phi)\,|\theta,\phi\rangle\,. \tag{123}$$

This gives a span of $|f\rangle$ in terms of the vectors of the form $|\theta,\phi\rangle$. This justifies the name of continuous basis for the set of vectors $\{|\theta,\phi\rangle\}$, $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$. Furthermore, the formal expression in

Equation (123) is indeed a continuous anti-linear functional on Φ. If we apply it to an arbitrary vector $|g\rangle \in \Phi$ and take the modulus, it comes

$$\left|\langle g| \int_{S^2} d\Omega \, |\theta,\phi\rangle\langle\theta,\phi|f\rangle\right| \leq \int_{S^2} d\Omega \, |\langle g|\theta,\phi\rangle| \cdot |\langle\theta,\phi|f\rangle| \leq 4\pi \, C^2 \, ||\,|f\rangle||_p \cdot ||\,|g\rangle||_p = K\,||g\rangle||_p \,, \tag{124}$$

with $K = 4\pi \, C^2 \, ||\,|f\rangle||_p$. Thus, the right-hand side of Equation (123) makes sense as an element of $\Phi^\times$. Consequently, the identity $\mathbb{I}$ represents the canonical identity from Φ into $\Phi^\times$. In particular, Equation (123) gives

$$|l,m\rangle = \int_{S^2} d\Omega \, \sqrt{l+1/2} \, Y_l^m(\theta,\phi) \, |\theta,\phi\rangle \,. \tag{125}$$

Since $\{|l,m\rangle\}$ is a basis for $\mathcal{H}$, the identity $\mathcal{I}$ on $\mathcal{H}$ may be written as

$$\mathcal{I} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} |l,m\rangle\langle l,m| \,. \tag{126}$$

Thus, for each $|f\rangle \in \Phi$, we may write

$$f(\theta,\phi)^* = \langle f|\theta,\phi\rangle = (\langle f|\mathcal{I})|\theta,\phi\rangle = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \langle f|l,m\rangle\langle l,m|\theta,\phi\rangle = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_l^m(\theta,\phi) \, \langle f|l,m\rangle \,, \tag{127}$$

so that, omitting the arbitrary $|f\rangle \in \Phi$, we have that

$$|\theta,\phi\rangle = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_l^m(\theta,\phi) \, |l,m\rangle \,, \tag{128}$$

which may be looked as the inversion formula for Equation (123) . If we multiply Equation (123) to the left by $\langle\theta',\phi'|$, an operation which is legitimate, we immediately realize that

$$\langle\theta',\phi'|\theta,\phi\rangle = \delta(\cos\theta' - \cos\theta') \, \delta(\phi' - \phi) \,, \tag{129}$$

which is a textbook formula.

6.3. Continuity of the Generators of so(3,2)

Along the present section, we use the following definitions for the generators of the $so(3,2)$ Lie algebra, based on the action of these generators on the spherical harmonics [80]:

$$\begin{aligned}
J_\pm \, Y_l^m(\theta,\phi) \quad &:= \quad \sqrt{(l \mp m)(l \pm m + 1)} \, Y_l^{m+1}(\theta,\phi) \,, \\
K_\pm \, Y_l^m(\theta,\phi) \quad &:= \quad \sqrt{\left(l + \frac{1}{2} \pm \frac{1}{2}\right)^2 - m^2} \, Y_{l+1}^m(\theta,\phi) \,, \\
R_\pm \, Y_l^m(\theta,\phi) \quad &:= \quad \sqrt{(l+m+1 \pm 1)(l+m \pm 1)} \, Y_{l+1}^{m+1}(\theta,\phi) \,, \\
S_\pm \, Y_l^m(\theta,\phi) \quad &:= \quad \sqrt{(l-m+1 \pm 1)(l-m \pm 1)} \, Y_{l+1}^{m-1}(\theta,\phi) \,.
\end{aligned} \tag{130}$$

These operators can be extended to closed linear operators on suitable dense domains. In addition, we have the generators of the Cartan subalgebra, since the rank of the $so(3,2)$ Lie algebra is 2 and and its dimension is 10. These generators are precisely the operators L and M defined in Equation (109). Correspondingly, we have analogous operators densely defined on $\mathcal{H}$ as

$$\widehat{J}_\pm = UJ_\pm U^{-1}\,,\;\; \widehat{K}_\pm = UK_\pm U^{-1}\,,\;\; \widehat{R}_\pm = UR_\pm U^{-1}\,,\;\; \widehat{S}_\pm = US_\pm U^{-1}\,. \tag{131}$$

The action of the operators in Equation (131) on the elements of the basis $\{|l,m\rangle\}$ is obvious. The continuity of these operators on Φ and $\Phi^\times$ has been established in [81,82]. For instance, assume that $|f\rangle = \sum_{l=0}^\infty \sum_{m=-l}^l f_{l,m}\,|l,m\rangle \in \Phi$. Then, write

$$\widehat{L}|f\rangle = \sum_{l=0}^\infty \sum_{m=-l}^l l\, f_{l,m}\,|l,m\rangle\,, \tag{132}$$

and

$$||\widehat{L}|f\rangle||_p^2 = \sum_{l=0}^\infty \sum_{m=-l}^l l^2\,|f_{l,m}|^2 (l+|m|+1)^{2p} \le \sum_{l=0}^\infty \sum_{m=-l}^l |f_{l,m}|^2 (l+|m|+1)^{2p+2} = ||\,|f\rangle||_{p+1}^2\,, \tag{133}$$

an expression valid for $p = 0,\,1,\,2,\,\ldots$. This means that $\widehat{L}|f\rangle \in \Phi$ if $|f\rangle \in \Phi$. Due to Equations (6) and (133) and the linearity of $\widehat{L}$, it is continuous. Since $\widehat{L}$ is symmetric, it is extensible to $\Phi^\times$ with continuity under the weak topology.

The proof for the continuity of the operators in Equation (131) on Φ is similar. To extend these operators by continuity to $\Phi^\times$, we have to realize first that all the operators with index $+$ are the formal adjoints of the corresponding operator with index $-$ and vice versa, for instance $\widehat{K}_+$ and $\widehat{K}_-$ are formal adjoint of each other. Therefore, to extend these operators to $\Phi^\times$, we only have to use the duality formula in Equation (7). Needless to say, L, M and the operators in Equation (130) have the same properties on $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$.

7. The $su(1,1)$ Lie Algebra and Laguerre Functions

The associated Laguerre polynomials with index $\alpha \in (-1,\infty)$, $L_n^\alpha(y)$, $n = 0,1,2,\ldots$, are defined on the half-line $\mathbb{R}^+ \equiv [0,\infty)$ [69–71]. An orthonormal basis on the Hilbert space $L^2(\mathbb{R}^+)$ is given by the following functions

$$M_n^\alpha(y) := \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+a+1)}}\, y^{\alpha/2}\, e^{-y/2}\, L_n^\alpha(y)\,, \qquad n = 0,\,1,\,2,\,\ldots\,, \tag{134}$$

and α fixed.

Let us consider the space, $\mathcal{D}_\alpha$, of vectors $f(y) = \sum_{n=0}^\infty a_n\, M_n^\alpha(y) \in L^2(\mathbb{R}^+)$, such that

$$||f(y)||_p^2 := \sum_{n=0}^\infty |a_n|^2 (n+1)^{2p} (n+\alpha+2)^{2p} < \infty, \qquad p = 0,\,1,\,2,\,\ldots, \tag{135}$$

with the topology produced by the norms $||-||_p$. With this topology, the space $\mathcal{D}_\alpha$ is a Fréchet nuclear space and is dense in $L^2(\mathbb{R}^+)$. For $p = 0$, we have the Hilbert space norm, so that the canonical injection $i : \mathcal{D}_\alpha \longmapsto \mathcal{H}$ is continuous. In consequence, for any fixed $\alpha \in (-1,\infty)$,

$$\mathcal{D}_\alpha \subset L^2(\mathbb{R}^+) \subset \mathcal{D}_\alpha^\times\,, \tag{136}$$

is a RHS.

7.1. Symmetries of the Laguerre Functions

The following operators defined on the functions $M_n^\alpha(y)$ as

$$Y\, M_n^\alpha(y) := y\, M_n^\alpha(y)\,, \quad D_y\, M_n^\alpha(y) := \frac{d}{dy}\, M_n^\alpha(y)\,, \quad N\, M_n^\alpha(y) := n\, M_n^\alpha(y)\,, \tag{137}$$

admit closed extensions on $L^2(\mathbb{R}^+)$. In addition, define the following operators [67]:

$$K_\pm := \pm Y\, D_y + N + I + \frac{\alpha - Y}{2}\,, \qquad K_3 := N + \frac{\alpha+1}{2}\, I\,, \tag{138}$$

where I is the identity operator. The action of these operators on the functions of the basis $\{M_n^\alpha(y)\}$ is

$$K_\pm\, M_n^\alpha(y) = \sqrt{(n + \frac{1}{2} \pm \frac{1}{2})(n + \alpha + \frac{1}{2} \pm \frac{1}{2})}\, M_{n+1}^\alpha(y)\,, \qquad K_3\, M_n^\alpha(y) = (n + (\alpha+1)/2)\, M_n^\alpha(y). \tag{139}$$

Note that K_+ and K_- are the formal adjoint of each other (i.e., $(K_\pm)^\dagger = K_\mp$) and

$$Y = -(K_+ + K_-) + 2N + (\alpha+1)I\,. \tag{140}$$

The commutation relations of $K_\pm$ and K_3 are

$$[K_3, K_\pm] = \pm K_\pm\,, \qquad [K_+, K_-] = -2K_3\,, \tag{141}$$

which are the commutation relations for the generators of the $su(1,1)$ Lie algebra [79]. The Casimir is

$$\mathcal{C} = K_3^2 - \frac{1}{2}\{K_+, K_-\} = \frac{\alpha^2 - 1}{4}\, I\,. \tag{142}$$

The next result concerns the continuity of these operators.

Proposition 2. *The operators $K_\pm$, K_3, Y and $Y\, D_y$ are continuous on $\mathcal{D}_\alpha$ for fixed $\alpha \in (-1, \infty)$.*

Proof. Let $f(y) = \sum_{n=0}^\infty a_n\, M_n^\alpha(y) \in \mathcal{D}_\alpha$. Then,

$$K_+\, f(y) = \sum_{n=0}^\infty \sqrt{(n+1)(n+\alpha+1)}\, a_n\, M_n^\alpha(y)\,. \tag{143}$$

We need to show that Equation (143) is well defined on $\mathcal{D}_\alpha$. For all $p = 0, 1, 2, \ldots$, take,

$$\begin{aligned} ||K_+\, f(y)||_p^2 &= \sum_{n=0}^\infty (n+1)(n+\alpha+1)(n+1)^{2p}(n+\alpha+2)^{2p}\, |a_n|^2 \\ &\le \sum_{n=0}^\infty (n+1)^{2p+2}(n+\alpha+1)^{2p+2}\, |a_n|^2 \le ||f(y)||_{p+1}^2\,. \end{aligned} \tag{144}$$

This shows both our claim and the continuity of K_+ on $\mathcal{D}_\alpha$. Proofs for K_- and K_3 are similar. The continuity of Y comes from Equation (140) and the continuity of $Y\, D_y$ from Equation (138). □

7.2. RHS and Continuous Bases

To define the continuous basis, we need an abstract RHS, which is the usual procedure. Let us consider an abstract infinite dimensional separable Hilbert space $\mathcal{H}$ and a unitary operator U from $\mathcal{H}$ to $L^2(\mathbb{R}^+)$. We choose the operator U as that given by the Gelfand–Maurin theorem (Section 2), where the role of A is played by the operator Y defined in Equation (140). This unitary operator U is not necessarily unique, although this is irrelevant, choose any one that makes this job.

Then, define for each $\alpha \in (-1, \infty)$ the space $\Phi_\alpha := U^{-1}\mathcal{D}_\alpha$ and transport the topology from $\mathcal{D}_\alpha$ to Φ_α by U^{-1}. Call $|n, \alpha\rangle := U^{-1}\, M_n^\alpha(y)$. For any $|f\rangle = \sum_{n=0}^\infty a_n\, |n, \alpha\rangle \in \Phi_\alpha$, the norms defining the topology are

$$|| \, |f\rangle ||_p^2 := \sum_{n=0}^\infty |a_n|^2 \, (n+1)^{2p}(n+\alpha+1)^{2p} \,, \qquad p = 0, 1, 2, \ldots . \tag{145}$$

We have the family of RHS's given by $\Phi_\alpha \subset \mathcal{H} \subset \Phi_\alpha^\times$ for each $\alpha \in (-1, \infty)$. Let us define the operator $\widetilde{Y} := U\, Y\, U^{-1}$, which is continuous on each of the Φ_α. After the Gelfand–Maurin theorem, we conclude that there exists a set of functionals $|y\rangle \in \Phi_\alpha^\times$ for $y \in \mathbb{R}^+$, such that $\widetilde{Y}\,|y\rangle = y\,|y\rangle$. In the kets $|y\rangle$, we omit the index α for simplicity. Furthermore, according to Equation (7), for any pair of vectors $|f\rangle, |g\rangle \in \Phi_\alpha$, we have that

$$\langle f|g\rangle = \int_0^\infty \langle f|y\rangle\langle y|g\rangle \, dy \,, \qquad \text{and} \qquad \langle f|n,\alpha\rangle = \int_0^\infty \langle f|y\rangle\langle y|n,\alpha\rangle \, dy = \int_0^\infty f^*(y)\, M_n^\alpha(y)\, dy \,. \tag{146}$$

If we omit the arbitrary bra $\langle f|$ in both identities of Equation (146), we obtain the following information:

1. For each $|g\rangle \in \Phi_\alpha$, we have the decomposition

$$|g\rangle = \int_0^\infty |y\rangle\langle y|g\rangle \, dy = \int_0^\infty g(y)\, |y\rangle \, dy \,, \tag{147}$$

which shows that the functionals $|y\rangle$, for all $y \in \mathbb{R}^+$, form a continuous basis for Φ_α.

2. Vectors in the discrete and continuous basis are related by

$$|n, \alpha\rangle = \int_0^\infty M_n^\alpha(y)\, |y\rangle \, dy \,. \tag{148}$$

3. If, in addition, we omit the arbitrary ket $|g\rangle$ in Equation (146), we obtain the following identity,

$$\mathbb{I} = \int_0^\infty |y\rangle\langle y| \, dy \,, \tag{149}$$

which is the canonical injection $\mathbb{I} : \Phi_\alpha \longmapsto \Phi_\alpha^\times$.

8. The $su(2,2)$ Lie Algebra and Algebraic Jacobi Functions

The Jacobi polynomials of order $n \in \mathbb{N}$, $J_n^{\alpha,\beta}(x)$, are usually defined as

$$J_n^{(\alpha,\beta)}(x) := \sum_{s=0}^n \begin{pmatrix} n+\alpha \\ s \end{pmatrix} \begin{pmatrix} n+\beta \\ n-s \end{pmatrix} \left(\frac{x+1}{2}\right)^s \left(\frac{x-1}{2}\right)^{n-s} \,, \tag{150}$$

with

$$\begin{pmatrix} a \\ s \end{pmatrix} := \frac{(a+1-s)(a+1-s+1)\ldots a}{s!} \,, \tag{151}$$

which are the generalized binomial coefficients, a is an arbitrary number and s a positive integer [69–71]. They verify the following second order differential equation:

$$\left[(1-x^2)\frac{d^2}{dx^2} - ((\alpha+\beta+2)x + (\alpha-\beta))\frac{d}{dx} + n(n+\alpha+\beta+1)\right] J_n^{(\alpha,\beta)}(x) = 0 \,. \tag{152}$$

8.1. Algebraic Jacobi Functions

Jacobi polynomials yield to the main concept of this section, the *algebraic Jacobi functions* [80,83–85], defined as

$$\mathcal{J}_j^{m,q}(x) := \sqrt{\frac{\Gamma(j+m+1)\Gamma(j-m+1)}{\Gamma(j+q+1)\Gamma(j-q+1)}} \left(\frac{1-x}{2}\right)^{\frac{m+q}{2}} \left(\frac{1+x}{2}\right)^{\frac{m-q}{2}} J_{j-m}^{(m+q,m-q)}(x)\,, \tag{153}$$

where

$$j := n + \frac{\alpha+\beta}{2}\,, \qquad m := \frac{\alpha+\beta}{2}\,, \qquad \beta := m-q\,. \tag{154}$$

Considerations derived from the theory of group representations force the following restrictions in the above parameters:

$$j \geq |m|\,, \quad j \geq |q|\,, \quad 2j \in \mathbb{N}\,, \quad j-m \in \mathbb{N}\,, \quad j-q \in \mathbb{N}\,, \tag{155}$$

and the parameters (j, m, q) are all together integers or half-integers. We may rewrite conditions in Equation (155) in terms of the original parameters (n, α, β) as

$$n \in \mathbb{N}\,, \quad \alpha, \beta \in \mathbb{Z}\,, \quad \alpha \geq -n\,, \quad \beta \geq -n\,, \quad \alpha+\beta \geq -n\,. \tag{156}$$

The algebraic Jacobi functions $\mathcal{J}_j^{m,q}(x)$ verify the following differential equation:

$$\left[-(1-x^2)\,\frac{d^2}{dx^2} + 2\,x\frac{d}{dx} + \frac{2\,m\,q\,x + m^2 + q^2}{1-x^2} - j(j+1)\right]\,\mathcal{J}_j^{m,q}(x) = 0\,, \tag{157}$$

where the symmetry under the interchange $m \Leftrightarrow q$ is evident. In addition, for fixed m and q, the algebraic Jacobi functions satisfy the following relations:

$$\begin{aligned} \int_{-1}^{1} \mathcal{J}_j^{m,q}(x)(j+1/2)\,\mathcal{J}_{j'}^{m,q}(x)\,dx &= \delta_{jj'}\,, \\ \sum_{j\geq \sup(|m|,|q|)}^{\infty} \mathcal{J}_j^{m,q}(x)(j+1/2)\,\mathcal{J}_j^{m,q}(y) &= \delta(x-y)\,. \end{aligned} \tag{158}$$

The indices j, m and q are either integer or half-integer. The relations in Equation (158) show that, for fixed m and q, the set of functions given by $\{\sqrt{j+1/2}\,J_j^{m,q}(x)\}_{j\geq\sup(|m|,|q|)}^{\infty}$ form an orthonormal basis of the Hilbert space $L^2[-1,1]$.

We may comment in passing the existence of a relation between the Legendre functions and some of the algebraic Jacobi functions, which is

$$P_l^m(x) = (-1)^m \sqrt{\frac{(l+m)!}{(l-m)!}}\,\mathcal{J}_l^{m,0}(x)\,. \tag{159}$$

8.2. Symmetries of the Algebraic Jacobi Functions

In addition, the ladder operators, $A_\pm, B_\pm, C_\pm, D_\pm, E_\pm, F_\pm$ that appear in the theory of algebraic Jacobi functions are generators of the $su(2,2)$ Lie algebra [83,84]. Their action on the algebraic Jacobi functions is given by

$$\begin{aligned}
A_\pm\, \mathcal{J}_j^{m,q}(x) &= \sqrt{(j\mp m)(j\pm m+1)}\, \mathcal{J}_j^{m\pm1,q}(x)\,,\\
B_\pm\, \mathcal{J}_j^{m,q}(x) &= \sqrt{(j\mp q)(j\pm q+1)}\, \mathcal{J}_j^{m,q\pm1}(x)\,,\\
C_\pm\, \mathcal{J}_j^{m,q}(x) &= \sqrt{(j+m+\tfrac12\pm\tfrac12)(j+q+\tfrac12\pm\tfrac12)}\; \mathcal{J}_{j\pm1/2}^{m\pm1/2,\,q\pm1/2}(x),\\
D_\pm\, \mathcal{J}_j^{m,q}(x) &= \sqrt{(j+m+\tfrac12\pm\tfrac12)(j-q+\tfrac12\pm\tfrac12)}\; \mathcal{J}_{j\pm1/2}^{m\pm1/2,\,q\mp1/2}(x)\\
E_\pm\, \mathcal{J}_j^{m,q}(x) &= \sqrt{(j-m+\tfrac12\pm\tfrac12)\,(j+q+\tfrac12\pm\tfrac12)}\; \mathcal{J}_{j\pm1/2}^{m\mp1/2,\,q\pm1/2}(x),\\
F_\pm\, \mathcal{J}_j^{m,q}(x) &= \sqrt{(j-m+\tfrac12\pm\tfrac12)\,(j-q+\tfrac12\pm\tfrac12)}\; \mathcal{J}_{j\pm1/2}^{m\mp1/2,q\mp1/2}(x)\,.
\end{aligned} \tag{160}$$

The generators of the Cartan subalgebra, J, M and Q, act on the algebraic Jacobi functions as follows:

$$J\, \mathcal{J}_j^{m,q}(x) = j\, \mathcal{J}_j^{m,q}(x)\,, \quad M\, \mathcal{J}_j^{m,q}(x) = m\, \mathcal{J}_j^{m,q}(x)\,, \quad Q\, \mathcal{J}_j^{m,q}(x) = q\, \mathcal{J}_j^{m,q}(x)\,. \tag{161}$$

All these operators can be extended to unbounded closed operators on $L^2[-1,1]$. In the case of J, M and Q, they admit self-adjoint extensions. Operators denoted with the same capital letter and different sign are formal adjoint (conjugate Hermitian) of each other (i.e., $(X_\pm)^\dagger = X_\mp$). On functions $\mathcal{J}_j^{m,q}(x)$ with $j \geq |m| > |q|$, one may define the following pair of mutually Hermitian formal adjoint operators:

$$K_\pm := F_\mp C_\pm \frac{1}{\sqrt{(J+1/2\pm1/2)^2 - Q^2}}\,, \tag{162}$$

so that

$$K_\pm\; \mathcal{J}_j^{m,q}(x) = \sqrt{(j+1/2\pm1/2)^2 - m^2}\, \mathcal{J}_{j\pm1}^{m,q}(x)\,. \tag{163}$$

These operators $K_\pm$ along $K_3 := J + 1/2$ close a $su(1,1)$ Lie algebra, since

$$[K_+, K_-] = -2K_3, \qquad [K_3, K_\pm] = \pm K_\pm\,, \tag{164}$$

and the set of functions $\{\mathcal{J}_j^{m,q}(x)\}_{j\geq|m|>|q|}^{m,q\,\text{fixed}}$ with $|m| > |q|$ is a basis of the space supporting a UIR of the group $SU(1,1)$ with Casimir $\mathcal{C} = m^2 - 1/4$.

8.3. Algebraic Jacobi Functions on the Hypersphere $\mathcal{S}^3$

In the preceding analysis, we deal with situations in which the number of discrete and continuous variables is the same. This idea revealed to be of importance in the analysis of the spaces which make continuous the above operators, if we are really interested in a description encompassing the maximal generality. To this end, we define the following functions:

$$\mathcal{N}_j^{m,q}(x,\phi,\chi) := \sqrt{j+1/2}\, \mathcal{J}_j^{m,q}(x)\, e^{im\phi}\, e^{iq\chi}\,, \tag{165}$$

where ϕ and χ are two angular variables, $\phi \in [0,2\pi)$ and $\chi \in [0,\pi]$ (x could be considered as $x = \cos\theta$ with $\theta \in [0,\pi]$ and in this case the Jacobi functions will live in the hypersphere $\mathcal{S}^3$). Thus, the $\mathcal{N}$-functions defined in Equation (165) depend on the variables, x, ϕ, χ, and the discrete parameters

j, m, q. The properties of the Jacobi functions yield to the following orthogonality relations valid for *either* j integer *or* j half-integer, with $m, q = -j, -j+1, \ldots, j-1, j$ and $m', q' = -j', -j'+1, \ldots, j'-1, j'$:

$$\frac{1}{2\pi^2} \int_0^{2\pi} d\phi \int_0^{\pi} d\chi \int_{-1}^{1} dx\, \mathcal{N}_j^{m,q}(x,\phi,\chi)\, \mathcal{N}_{j'}^{m',q'\,*}(x,\phi,\chi) = \delta_{jj'}\, \delta_{mm'}\, \delta_{qq'}\,. \tag{166}$$

These functions satisfy a completeness relation of the type:

$$\sum_{j_{min}}^{\infty} \sum_{m,q=-j}^{j} |\mathcal{N}_j^{m,q}(x,\phi,\chi)|^2 = I\,, \tag{167}$$

where $j_{min} = 0$ if j is integer and $j_{min} = 1/2$ if j is half-integer and I is an identity. Note that we have two different situations, one when j is an integer and the other when j is a half-integer. In both cases, either $e^{im\phi}$ or $e^{iq\chi}$ span respective vector spaces of dimension $2j+1$. These spaces, being isomorphic to $\mathbb{C}^{2j+1}$, may be identified with it. Then, for either j integer or half-integer, the set of functions $\mathcal{N}_j^{m,q}(x,\phi,\chi)$ with $m, q = -j, -j+1, \ldots, j-1, j$ is the basis for the following Hilbert spaces:

$$\mathcal{H}_I := \bigoplus_{j=0}^{\infty} L^2[-1,1] \otimes \mathbb{C}^{2j+1} \otimes \mathbb{C}^{2j+1}\,, \qquad \mathcal{H}_H := \bigoplus_{j=1/2}^{\infty} L^2[-1,1] \otimes \mathbb{C}^{2j+1} \otimes \mathbb{C}^{2j+1}\,, \tag{168}$$

respectively. The subindices I and H stand for integer and half-integer, respectively. Then, let us take $f_I(x,\phi,\chi) \in \mathcal{H}_I$ and $f_H(x,\phi,\chi) \in \mathcal{H}_H$, so that

$$f_I(x,\phi,\chi) = \sum_{j=0}^{\infty} \sum_{m,q=-j}^{j} a_{j,m,q}\, \mathcal{N}_j^{m,q}(x,\phi,\chi)\,, \quad f_H(x,\phi,\chi) = \sum_{j=1/2}^{\infty} \sum_{m,q=-j}^{j} b_{j,m,q}\, \mathcal{N}_j^{m,q}(x,\phi,\chi)\,. \tag{169}$$

8.4. RHS Associated to the Algebraic Jacobi Functions

Next, we define two new rigged Hilbert spaces. The spaces of test functions Φ_I and Φ_H are the functions in $\mathcal{H}_I$ and $\mathcal{H}_H$, respectively, such that

$$\left[p_{r,s}^I(f_I)\right]^2 := \sum_{j=0}^{\infty} \sum_{m,q=-j}^{j} |a_{j,m,q}|^2\, (j+|m|+1)^{2r}\, (j+|q|+1)^{2s} < \infty\,, \tag{170}$$

and

$$\left[p_{r,s}^H(f_H)\right]^2 := \sum_{j=0}^{\infty} \sum_{m,q=-j}^{j} |b_{j,m,q}|^2\, (j+|m|+1)^{2r}\, (j+|q|+1)^{2s} < \infty\,, \tag{171}$$

respectively, with $r, s = 0, 1, 2, \ldots$. Observe that Equations (170) and (171) define norms on $\mathcal{H}_I$ and $\mathcal{H}_H$, respectively, and they generate respective topologies on Φ_I and Φ_H. For $r = s = 0$, we recover the Hilbert space topology, which shows that the canonical injections $\Phi_{I,H} \longmapsto \mathcal{H}_{I,H}$ are continuous, so that

$$\Phi_I \subset \mathcal{H}_I \subset \Phi_I^\times\,, \qquad \text{and} \qquad \Phi_H \subset \mathcal{H}_H \subset \Phi_H^\times\,, \tag{172}$$

are rigged Hilbert spaces.

Analogously, we define the spaces Ξ_I and Ξ_H as the spaces of functions in $\mathcal{H}_I$ and $\mathcal{H}_H$ verifying the following relations:

$$t_{r,s}^I(f_I) := \sum_{j=0}^{\infty} \sum_{m,q=-j}^{j} |a_{j,m,q}|\, (j+|m|+1)^{r}\, (j+|q|+1)^{s} < \infty\,, \tag{173}$$

and

$$t^H_{r,s}(f_H) := \sum_{j=1/2}^{\infty} \sum_{m,q=-j}^{j} |b_{j,m,q}|\,(j+|m|+1)^r\,(j+|q|+1)^s < \infty\,, \tag{174}$$

respectively, with $r,s = 0,1,2,\ldots$. These are also norms that endow respective topologies on Ξ_I and Ξ_H. Since

$$p^{I,H}_{r,s}(f_{I,H}) \le t^{I,H}_{r,s}(f_{I,H})\,, \qquad r,s = 0,1,2,\ldots\,, \tag{175}$$

we conclude that $\Xi_{I,H} \subset \Phi_{I,H}$ and that the canonical injections $\Xi_{I,H} \longmapsto \Phi_{I,H}$ are continuous. Thus, we have two new RHS's, and, in addition, we have the following subordinate relations with continuity

$$\Xi_{I,H} \subset \Phi_{I,H} \subset \mathcal{H}_{I,H} \subset \Phi^\times_{I,H} \subset \Xi^\times_{I,H}\,, \tag{176}$$

where, in each sequence in Equation (176), we should keep either the subindex I or H.

8.5. Continuity of the $su(2,2)$ Operators

The operators J, M and Q, defined above in this section, admit obvious extensions to respective dense subspaces of $\mathcal{H}_{I,H}$. For instance,

$$(Jf_I)(x,\phi,\chi) = \sum_{j=0}^{\infty} \sum_{m,q=-j}^{j} j\, a_{j,m,q}\, \mathcal{N}_j^{m,q}(x,\phi,\chi)\,. \tag{177}$$

Thus,

$$\begin{aligned} \left[p^I_{r,s}(Jf_I)\right]^2 &= \sum_{j=0}^{\infty} \sum_{m,q=-j}^{j} |a_{j,m,q}|^2\, j^2\,(j+|m|+1)^{2r}\,(j+|q|+1)^{2s} \\ &\le \sum_{j=0}^{\infty} \sum_{m,q=-j}^{j} |a_{j,m,q}|^2\,(j+|m|+1)^{2(r+1)}\,(j+|q|+1)^{2s} = [p^I_{r+1,s}(f_I)]^2\,, \end{aligned} \tag{178}$$

for $r,s = 0,1,2,\ldots$, which proves that $J\Phi_I \subset \Phi_I$ with continuity. Analogously,

$$\begin{aligned} t^I_{r,s}(Jf_I) &= \sum_{j=0}^{\infty} \sum_{m,q=-j}^{j} |a_{j,m,q}|\, j\,(j+|m|+1)^{r}\,(j+|q|+1)^{s} \\ &\le \sum_{j=0}^{\infty} \sum_{m,q=-j}^{j} |a_{j,m,q}|\,(j+|m|+1)^{r+1}\,(j+|q|+1)^{s} = t^I_{r+1,s}(f_I)\,, \end{aligned} \tag{179}$$

for $r,s = 0,\,1,\,2,\,\ldots$, which proves that $J\,\Xi_I \subset \Xi_I$ with continuity. It is the same for J on Φ_H and Ξ_H and for M and Q in these four spaces. Since these operators are symmetric and self-adjoint on a proper domain, they may be extended by continuity to the duals. A similar proof is also valid to show the continuity of the ladder operators $A_\pm$ and $B_\pm$, defined in Equation (160) and $K_\pm$ in Equation (162) on all the spaces $\Xi_{I,H}$ and $\Phi_{I,H}$ and therefore their extensions by continuity to the duals.

However, the ladder operators $C_\pm, D_\pm, E_\pm, F_\pm$ have a different nature, as they transform algebraic Jacobi functions of integer indices into the same type of functions with half-integer indices and vice versa. Under the assumption that $C_-\,\mathcal{N}_j^{0,0}(x,\phi,\chi) = 0$ and the same for D_-, E_- and F_-, we can easily prove that all these operators are continuous from Φ_I into Φ_H and vice versa and the same from Ξ_I into Ξ_H and vice versa. As they are the formal adjoint of each other, we conclude that they can also be continuously extended as analogous relations between the duals.

8.6. Discrete and Continuous Basis

In the sequel, we omit the subindices I and H for simplicity. All results are valid for both cases. As done in all precedent examples, let us consider an abstract infinite dimensional separable Hilbert space $\mathcal{G}$ and a unitary mapping $U : \mathcal{G} \longmapsto \mathcal{H}$. In fact, there are two of each: $U_{I,H} : \mathcal{G}_{I,H} \longmapsto \mathcal{H}_{I,H}$, although we omit the subindices, as mentioned. Take $\Theta := U^{-1}\Xi$ and $\Psi := U^{-1}\Phi$, and endow Θ and Φ with the topologies transported by U^{-1} from Ξ and Φ, respectively. Then, we have two new RHS's, $\Theta \subset \mathcal{G} \subset \Theta^\times$ and $\Psi \subset \mathcal{G} \subset \Psi^\times$. We focus our attention on the former.

For any $|f\rangle \in \Theta$, we define the action of the ket $|x, m, q\rangle$, $x \in [-1, 1]$, m and q being fixed, as

$$\langle f|x, m, q\rangle := \sum_{j_{min}}^{\infty} a_{j,m,q}\, \mathcal{N}_j^{m,q}(x, 0, 0) = \sum_{j=0}^{\infty} a_{j,m,q}\, \sqrt{j + 1/2}\, \mathcal{J}_j^{m,q}(x)\,. \tag{180}$$

This definition shows that $|x, m, q\rangle$ is an antilinear mapping on Θ, which is also continuous since,

$$\begin{aligned} |\langle f|x, m, q\rangle| &\leq \sum_j |a_{j,m,q}|\,(j + |m| + 1)^2(j + |q| + 1) \\ &\leq \sum_j \sum_{m,q=-j}^{j} |a_{j,m,q}|\,(j + |m| + 1)^2(j + |q| + 1) = t_{2,1}(|f\rangle)\,, \end{aligned} \tag{181}$$

with

$$U\,|f\rangle = f(x, \phi, \chi) = \sum_{j_{min}}^{\infty} \sum_{m,q=-j}^{j} |a_{j,m,q}|\, \mathcal{N}_j^{m,q}(x, \phi, \chi)\,. \tag{182}$$

Next, let us define the kets $|j, m, q\rangle$ for any j and any $m, q = -j, -j + 1, \ldots, j - 1, j$ as

$$|j, m, q\rangle := U^{-1}\, \mathcal{N}_j^{m,q}(x, \phi, \chi)\,, \tag{183}$$

so that Equation (180) gives

$$\langle j, m', q'|x, m, q\rangle = \sqrt{j + 1/2}\, \mathcal{J}_j^{m',q'}(x)\, \delta_{mm'}\, \delta_{qq'} = \langle x, m, q|j, m', q'\rangle\,, \tag{184}$$

since Equation (184) is real. Observe that there exists the following formal relation between $|x, m, q\rangle$ and $|j, m, q\rangle$:

$$|x, m, q\rangle = \sum_{j_{min}}^{\infty} |j, m, q\rangle\, \sqrt{j + 1/2}\, \mathcal{J}_j^{m,q}(x)\,. \tag{185}$$

This is easily justified by multiplying Equation (185) by $\langle j, m', q'|$:

$$\begin{aligned} \langle j', m', q'|x, m, q\rangle &= \sum_{j_{min}}^{\infty} \langle j', m', q'|j, m, q\rangle \sqrt{j + 1/2}\, \mathcal{J}_j^{m,q}(x) \\ &= \sum_{j_{min}}^{\infty} \delta_{jj'}\, \sqrt{j + 1/2}\, \mathcal{J}_j^{m,q}(x)\, \delta_{mm'}\, \delta_{qq'} = \sqrt{j + 1/2}\, \mathcal{J}_j^{m,q}(x)\, \delta_{mm'}\, \delta_{qq'}\,, \end{aligned} \tag{186}$$

which coincides with Equation (184). There are some other formal relations that can be easily obtained. Proofs are published elsewhere [85]; they are simple notwithstanding. First, we have

$$\langle x', m', q'|x, m, q\rangle = \sum_j \mathcal{N}_j^{m',q}(x, \phi, \chi)\, \mathcal{N}_j^{m,q}(x, \phi, \chi)\, \delta_{mm'}\, \delta_{qq'} = \delta(x - x')\, \delta_{mm'}\, \delta_{qq'}\,. \tag{187}$$

For any $|f\rangle \in \Theta$, we have the following relation:

$$\langle j', m', q'|f\rangle = \sum_{m,q=-\infty}^{\infty} \int_{-1}^{1} \langle j', m', q'|x, m, q\rangle\, f^{m,q}(x)\, dx\,, \tag{188}$$

where if $|f\rangle = \sum_{j=0}^{\infty} \sum_{m,q=-j}^{j} a_{j,m,q} \sqrt{j+1/2}\, \mathcal{J}_j^{m,q}(x)$, we have that

$$f^{m,q}(x) = \sum_{j_{min}}^{\infty} a_{j,m,q} \sqrt{j+1/2}\, \mathcal{J}_j^{m,q}(x)\,, \tag{189}$$

so that

$$|f\rangle = \sum_{m,q=-\infty}^{\infty} \int_{-1}^{1} |x,m,q\rangle\, f^{m,q}(x)\, dx\,, \tag{190}$$

which shows that any $|f\rangle \in \Theta$ may be written formally in terms of the elements of the set of functionals $\{|x,m,q\rangle\}$, which acquires the category of continuous basis due to this fact. Here, $x \in [-1,1]$, m,q being the set either of the integers or the half-integers, either positive or negative.

For $|j,m,q\rangle$, the functions $f^{m,q}(x)$ are equal to $\sqrt{j+1/2}\, \mathcal{J}_j^{m,q}(x)$ that, after Equation (190), gives

$$|j,m,q\rangle = \sum_{m,q=-\infty}^{\infty} \int_{-1}^{1} |x,m,q\rangle\, \sqrt{j+1/2}\, \mathcal{J}_j^{m,q}(x)\, dx\,, \tag{191}$$

which gives the inversion formula for Equation (185). We have completed the relation between discrete and continuous basis. Moreover, note that

$$f^{m,q}(x) = \langle x,m,q|f\rangle\,, \tag{192}$$

and

$$\sum_{m,q=-j}^{j} \int_{-1}^{1} dx\, |x,m,q\rangle\langle x,m,q| = \mathcal{I}\,, \tag{193}$$

where $\mathcal{I} : \Theta \longmapsto \Theta^{\times}$ is the canonical injection relating this dual pair. We close here the discussion on Jacobi algebraic functions.

9. su(1,1)⊕su(1,1), Zernike Functions and RHS

The so-called Zernike polynomials were introduced by Zernike in 1934 in connection with some applications in the analysis of optical images [11]. These Zernike polynomials $R_n^m(r)$, also called Zernike radial polynomials [86], as usually one takes $0 \le r \le 1$ in applications, are the solutions of the differential equation,

$$\left[(1-r^2)\,\frac{d^2}{dr^2} - \left(3r - \frac{1}{r}\right)\frac{d}{dr} + n(n+2) - \frac{m^2}{r^2}\right] R_n^m(r) = 0\,, \tag{194}$$

verifying

$$R_n^m(1) = 1\,, \qquad R_n^m(r) = R_n^{-m}(r)\,. \tag{195}$$

Explicitly,

$$R_n^m(r) = \sum_{k=0}^{\frac{n-m}{2}} (-1)^k \begin{pmatrix} n-k \\ k \end{pmatrix} \begin{pmatrix} n-2k \\ \frac{n-m}{2}-k \end{pmatrix} r^{n-2k}\,. \tag{196}$$

For each value of m, Zernike polynomials show orthogonality properties:

$$\int_0^1 R_n^m(r)\, R_{n'}^m(r)\, r\, dr = \frac{\delta_{nn'}}{2(n+1)}\,, \tag{197}$$

as well as a completeness relation such as

$$\sum_{\substack{n=|m| \\ n\equiv m\,(\mathrm{mod}\,2)}}^{\infty} R_n^m(r)\, R_n^m(r')\,(n+1) = \frac{\delta(r-r')}{2r}\,. \tag{198}$$

They are also related to the Jacobi polynomials according to the following formula:

$$R_n^m(r) = (-1)^{(n-m)/2}\, r^m\, J_n^{(m,0)}(1-2r^2)\,. \tag{199}$$

Along Zernike polynomials, there exist the Zernike functions $\mathcal{Z}_n^m(r,\phi)$, which are defined on the closed unit circle

$$\mathcal{D} \equiv \{(r,\phi)\,,\ \ 0 \le r \le 1\,,\ \ \phi \in [0,2\pi)\ \}\,, \tag{200}$$

as follows:

$$\mathcal{Z}_n^m(r,\phi) := R_n^m(r)\, e^{im\phi}\,, \qquad n \in \mathbb{N}\,,\ m \in \mathbb{Z}\,, \tag{201}$$

with the conditions $|m| \le n$ and $\frac{n-|m|}{2} \in \mathbb{N}$.

9.1. W-Zernike Functions

From Equation (201), we define the W-Zernike functions, $W_{u,v}(r,\phi)$, using the following procedure [87]. First, introduce the parameters u and v, defined as

$$u := \frac{n+m}{2}\,, \qquad v := \frac{n-m}{2}\,, \tag{202}$$

which are positive integers and independent of each other, $u,v = 0,\,1,\,2,\,\ldots$. With this notation,

$$R_n^m(r) \equiv R_{u+v}^{|u-v|}(r)\,. \tag{203}$$

The W-Zernike functions, $W_{u,v}(r,\phi)$, are functions on the closed unit circle $\mathcal{D}$, verifying the relation

$$W_{u,v}(r,\phi) = \sqrt{\frac{u+v+1}{\pi}}\ \mathcal{Z}_{u+v}^{u-v}(r,\phi) = \sqrt{\frac{u+v+1}{\pi}}\ R_{u+v}^{|u-v|}(r)\, e^{i(u-v)\phi}\,. \tag{204}$$

In addition, the W-Zernike functions have some interesting properties:

- They are square integrable on $\mathcal{D}$, so that they belong to the Hilbert space $L^2(\mathcal{D}, r dr d\phi) \equiv L^2(\mathcal{D})$.
- They fulfill some symmetry relations such as

  $$W_{v,u}(r,\phi) = W_{u,v}(r,\phi)^* = W_{u,v}(r,-\phi)\,, \tag{205}$$

  where the star denotes complex conjugation.
- They are orthonormal on $L^2(\mathcal{D})$:

  $$\langle W_{u',v'}, W_{u,v}\rangle = \int_0^{2\pi} d\phi \int_0^1 dr\, r\, W_{u',v'}(r,\phi)^*\, W_{u,v}(r,\phi) = \delta_{uu'}\,\delta_{vv'}\,, \tag{206}$$

  where $\langle\cdot,\cdot\rangle$ denotes scalar product on $L^2(\mathcal{D})$.
- A completeness relation holds:

  $$\sum_{u,v=0}^{\infty} W_{u,v}(r,\phi)\, W_{u,v}^*(r',\phi') = \frac{1}{2r}\,\delta(r-r')\,\delta(\phi-\phi')\,. \tag{207}$$

- The fact that Zernike polynomials are bounded, $|R_n^m(r)| \leq 1$, on the interval $0 \leq r \leq 1$, implies an interesting upper bound for the W-Zernike functions:

$$\left|W_{u,v}(r,\phi)\right| \leq \sqrt{\frac{u+v+1}{\pi}}\,, \qquad \forall\,(r,\phi) \in \mathcal{D}\,. \tag{208}$$

9.2. Rigged Hilbert Spaces and W-Zernike Functions

The set of W-Zernike functions forms an orthonormal basis for $L^2(\mathcal{D})$ so that, for any square integrable function $f(r,\phi) \in L^2(\mathcal{D})$, we have that

$$f(r,\phi) = \sum_{u,v=0}^{\infty} f_{u,v}\, W_{u,v}(r,\phi)\,, \tag{209}$$

with

$$f_{u,v} = \int_0^{2\pi} d\phi \int_0^1 dr\, r\, W^*_{u,v}(r,\phi)\, f(r,\phi)\,. \tag{210}$$

Let us define two different spaces, which are the spaces of test functions for respective RHS. The first one is

$$\Psi_1 := \left\{ f(r,\phi) \in L^2(\mathcal{D}) \;\middle|\; \sum_{u,v=0}^{\infty} |f_{u,v}|^2\, (u+v+1)^{2p} < \infty\,,\;\; p = 0, 1, 2, \ldots \right\}. \tag{211}$$

The space Ψ_1 is endowed with the Frèchet topology given by the following family of norms

$$||f(r,\phi)||_p^2 := \sum_{u,v=0}^{\infty} |f_{u,v}|^2\, (u+v+1)^{2p} < \infty\,, \quad p = 0, 1, 2, \ldots. \tag{212}$$

The second space of test functions is defined by the following condition:

$$\Psi_2 := \left\{ f(r,\phi) \in L^2(\mathcal{D}) \;\middle|\; \sum_{u,v=0}^{\infty} |f_{u,v}|\, (u+v+1)^{q} < \infty\,,\;\; q = 0, 1, 2, \ldots \right\}. \tag{213}$$

Its topology is given by the following sequence of norms:

$$||f(r,\phi)||_{1,q} := \sum_{u,v=0}^{\infty} |f_{u,v}|\, (u+v+1)^{q}\,, \quad q = 0, 1, 2, \ldots. \tag{214}$$

Let us consider a sequence of complex numbers $\{a_n\}$ such that the series $\sum_{n=0}^{\infty} |a_n| < \infty$. Clearly,

$$\sqrt{\sum_{n=0}^{\infty} |a_n|^2} \leq \sum_{n=0}^{\infty} |a_n|\,, \tag{215}$$

which shows that

$$\big|\big|f(r,\phi)\big|\big|_p = \sqrt{\sum_{u,v=0}^{\infty} |f_{u,v}|^2\, (u+v+1)^{2p}} \leq \sum_{u,v=0}^{\infty} \big|f_{u,v}\big|\, (u+v+1)^{p} = \big|\big|f(r,\phi)\big|\big|_{1,p}\,, \tag{216}$$

for $p = 0, 1, 2, \ldots$. This shows that $\Psi_2 \subset \Psi_1$ and that the canonical injection $i : \Psi_2 \longmapsto \Psi_1$ is continuous. This gives a couple of RHS's where injections in all inclusions are continuous:

$$\Psi_2 \subset \Psi_1 \subset L^2(\mathcal{D}) \subset \Psi_1^\times \subset \Psi_2^\times\,. \tag{217}$$

An important property for the span of the functions $f(r,\phi) \in \Psi_2$ in terms of the W-Zernike functions is given by the following result:

Theorem 3. *For any $f(r,\phi) \in \Psi_2$, the series*

$$f(r,\phi) = \sum_{u,v}^{\infty} f_{u,v}\, W_{u,v}(r,\phi)\,, \tag{218}$$

converges absolutely and uniformly and hence pointwise.

Proof. The proof is based on the bound in Equation (208) valid for the W-Zernike functions. Thus, using Equation (208) and considering Equation (213), we have that

$$\sum_{u,v=0}^{\infty} |f_{u,v}|\, |W_{u,v}(r,\phi)| \leq \sum_{u,v=0}^{\infty} |f_{u,v}| \sqrt{\frac{u+v+1}{\pi}} \leq \frac{1}{\sqrt{\pi}} \sum_{u,v=0}^{\infty} |f_{u,v}|\, (u+v+1) < \infty\,. \tag{219}$$

Then, the Weiersstrass M-Theorem guarantees the absolute and uniform convergence of the series. □

9.3. Continuity of Relevant Operators Acting on the W-Zernike Functions

In the discussion on the continuous basis below, we see the relevance of the following operator on $L^2(\mathcal{D})$:

$$P\, f(r,\phi) = r\, e^{i\phi}\, f(r,\phi)\,. \tag{220}$$

In [87], we prove that

$$P\, W_{u,v}(r,\phi) = \alpha_u^v\, W_{u+1,v}(r,\phi) + \beta_u^v\, W_{u,v-1}(r,\phi)\,, \tag{221}$$

with

$$\alpha_u^v = \frac{u+1}{\sqrt{(u+v+1)(u+v+2)}}\,, \qquad \beta_u^v = \frac{v}{\sqrt{(u+v)(u+v+1)}}\,. \tag{222}$$

Note that $0 \leq \alpha_u^v, \beta_u^v \leq 1$ and $f_{-1,0} = 0$.

We want to show that $P\Psi_2 \subset \Psi_2$ with continuity. Let us take $f(r,\phi) = \sum_{u,v=0}^{\infty} f_{u,v}\, W_{u,v}(r,\phi) \in \Psi_2$, so that

$$\begin{aligned} \left|\left| P \textstyle\sum_{u,v=0}^{\infty} f_{u,v}\, W_{u,v}(r,\phi) \right|\right|_{1,r} &= \sum_{u,v=0}^{\infty} \left| \alpha_{u-1}^v\, f_{u-1,v} + \beta_u^{v+1}\, f_{u,v+1} \right| (u+v+1)^r \\ &\leq \sum_{u,v=0}^{\infty} |f_{u-1,v}|(u+v+1)^r + \sum_{u,v=0}^{\infty} |f_{u,v+1}|(u+v+1)^r\,. \end{aligned} \tag{223}$$

Since $f_{-1,0} = 0$, the first term of the second row in Equation (205) gives

$$\begin{aligned} \sum_{u,v=0}^{\infty} |f_{u-1,v}|(u+v+1)^r &= \sum_{u,v=0}^{\infty} |f_{u,v}|(u+v+2)^r \leq 2^r \sum_{u,v=0}^{\infty} |f_{u-1,v}|(u+v+1)^r \\ &= 2^r \left|\left| \sum_{u,v=0}^{\infty} f_{u,v}\, W_{u,v}(r,\phi) \right|\right|_{1,r}\,. \end{aligned} \tag{224}$$

The second term in the same row gives,

$$\begin{aligned} \sum_{u,v=0}^{\infty} |f_{u,v+1}|(u+v+1)^r &\leq \sum_{u,v=0}^{\infty} |f_{u,v}|(u+v)^r \leq \sum_{u,v=0}^{\infty} |f_{u,v}|(u+v+1)^r \\ &= \left|\left| \sum_{u,v=0}^{\infty} f_{u,v}\, W_{u,v}(r,\phi) \right|\right|_{1,r}\,. \end{aligned} \tag{225}$$

Equations (224) and (225) together show that

$$\left|\left|P\sum_{u,v=0}^{\infty} f_{u,v}\,W_{u,v}(r,\phi)\right|\right|_{1,r} \leq (2^r+1)\left|\left|\sum_{u,v=0}^{\infty} f_{u,v}\,W_{u,v}(r,\phi)\right|\right|_{1,r}, \tag{226}$$

which shows our claim.

Other important operators are the generators of the Lie algebra $su(1,1)\oplus su(1,1)$, U, V, $A_\pm$, $B_\pm$, $A_3 = U+1/2$ and $B_3 = V+1/2$. Their commutation relations are the following:

$$\begin{aligned} &[U,A_\pm] = \pm A_\pm\,, \quad [V,B_\pm] = \pm B_\pm\,, \quad [A_+,A_-] = -2A_3\,, \\ &[A_3,A_\pm] = \pm A_\pm\,, \quad [B_+,B_-] = -2B_3\,, \quad [B_3,B_\pm] = \pm B_\pm\,. \end{aligned} \tag{227}$$

All the A operators commute with all the B operators. The Casimirs are

$$C_A = A_3^2 - \frac{1}{2}\{A_+,A_-\}\,, \qquad C_B = B_3^2 - \frac{1}{2}\{B_+,B_-\}\,, \tag{228}$$

with $\{X,Y\} = XY+YX$.

On the W-Zernike functions, all these operators act as follows [87]:

$$\begin{array}{rclrcl} U\,W_{u,v}(r,\phi) &=& u\,W_{u,v}(r,\phi)\,, & V\,W_{u,v}(r,\phi) &=& v\,W_{u,v}(r,\phi)\,, \\ A_+\,W_{u,v}(r,\phi) &=& (u+1)\,W_{u+1,v}(r,\phi)\,, & A_-\,W_{u,v}(r,\phi) &=& u\,W_{u-1,v}(r,\phi)\,, \\ B_+\,W_{u,v}(r,\phi) &=& (v+1)\,W_{u,v+1}(r,\phi)\,, & B_-\,W_{u,v}(r,\phi) &=& v\,W_{u,v-1}(r,\phi)\,. \end{array} \tag{229}$$

All these operators are densely defined and unbounded on $L^2(\mathcal{D})$. Furthermore,

Proposition 3. *The operators U, V, $A_\pm$ and $B_\pm$ are continuous on Ψ_2. In addition, A_+ and A_- are formal adjoint of each other and same for B_+ and B_- and U and V are essentially self-adjoint on Ψ_2.*

Proof. That A_+ and A_- and also B_+ and B_- are formal adjoint of each other is obvious from Equation (212). The proof of the continuity on Ψ_2 of all these operators is the same. Take for instance A_+. The formal action of A_+ on $f(r,\phi)\in\Psi_2$ is given by

$$A_+\,f(r,\phi) = \sum_{u,v=0}^{\infty} f_{u,v}\,(u+1)\,W_{u+1,v}(r,\phi)\,. \tag{230}$$

Then,

$$\begin{aligned} ||A_+\,f(r,\phi)||_{1,r} &= \sum_{u=1,v=0}^{\infty} |f_{u,v}|\,(u+1)(u+v+1)^r \\ &\leq \sum_{u,v=0}^{\infty} |f_{u,v}|(u+v+1)^{r+1} = ||f(r,\phi)||_{1,r+1}\,, \end{aligned} \tag{231}$$

which proves that $A_+\Psi_2 \subset \Psi_2$ with continuity. It is the same for all other operators. Finally, U and V are obviously symmetric on Ψ_2 and the ranges of $U\pm iI$ and $V\pm iI$ on Ψ_2 are Ψ_2 itself, so that U and V are essentially self-adjoint with domain Ψ_2. □

9.4. Continuous Bases and RHS

Let $\mathcal{H}$ be an arbitrary infinite dimensional separable Hilbert space and U a unitary operator $U:\mathcal{H}\longmapsto L^2(\mathcal{D})$. As in previous cases, we define $\Phi_i := U^{-1}\Psi_i$, $i=1,2$ (217), and transport the topologies on Ψ_i to Φ_i by U^{-1}. We have a couple of RHS's in correspondence. Thus, we have the following diagram

$$\begin{array}{ccccccccc} \Psi_2 & \subset & \Psi_1 & \subset & L^2(\mathcal{D}) & \subset & \Psi_1^\times & \subset & \Psi_2^\times \\ U^{-1}\downarrow & & U^{-1}\downarrow & & U^{-1}\downarrow & & U^{-1}\downarrow & & U^{-1}\downarrow \\ \Phi_2 & \subset & \Phi_1 & \subset & \mathcal{H} & \subset & \Phi_1^\times & \subset & \Phi_2^\times \end{array} .$$

Nevertheless, our rigged Hilbert space of reference here is: $\Phi_2 \subset \mathcal{H} \subset \Phi_2^\times$. Take any vector $U^{-1}f(r,\phi) = |f\rangle \in \Phi_2$, and for (almost with respect to the Lebesgue measure) each $0 \le r \le 1$ and $0 \le \phi \le 2\pi$ define the mapping $|r,\phi\rangle$ by $\langle f|r,\phi\rangle := f^*(r,\phi) = \langle r,\phi|f\rangle^*$. Clearly, $|r,\phi\rangle$ is linear for each r and ϕ. In addition, this is continuous so that $|r,\phi\rangle \in \Phi_2^\times$. To prove the continuity, note that U^{-1} transports the given topology from Ψ_2 to Φ_2. Let $|u,v\rangle := U^{-1}\, W_{u,v}(r,\phi)$ for each $u,v = 0, 1, 2, \dots$. Then, if $|f\rangle = \sum_{u,v=0}^\infty f_{u,v}\,|u,v\rangle$, we have that the norms $||\,|f\rangle||_{1,q}$ defining the topology on Ψ_2 are identical to Equation (214). Thus, considering Equation (208), we have

$$\begin{aligned} |\langle f|r,\phi\rangle| &= |\langle r,\phi|f\rangle^*| = |f(r,\phi)| \le \sum_{u,v=0}^\infty |f_{u,v}|\,|W_{u,v}(r,\phi)| \le \sum_{u,v=0}^\infty |f_{u,v}| \sqrt{\frac{u+v+1}{\pi}} \\ &\le \frac{1}{\sqrt{\pi}} \sum_{u,v=0}^\infty |f_{u,v}|\,(u+v+1) = \frac{1}{\sqrt{\pi}}\,||\,|f\rangle||_{1,1}\,. \end{aligned} \tag{232}$$

The scalar product of two vectors $|f\rangle\,, |g\rangle \in \Phi_2$ is given by

$$\langle f|g\rangle = \int_0^{2\pi} d\phi \int_0^1 dr\, r\, f^*(r,\phi)\, g(r,\phi) = \int_0^{2\pi} d\phi \int_0^1 dr\, r\, \langle f|r,\phi\rangle\langle r,\phi|g\rangle\,, \tag{233}$$

so that we have the identity

$$\mathcal{I} := \int_0^{2\pi} d\phi \int_0^1 dr\, r\, |r,\phi\rangle\langle r,\phi|\,, \tag{234}$$

which should be interpreted as the canonical injection $\mathcal{I} : \Phi_2 \longmapsto \Phi_2^\times$. In particular, if we apply Equation (234) to $|u,v\rangle$, we have that

$$|u,v\rangle = \int_0^{2\pi} d\phi \int_0^1 dr\, r\, |r,\phi\rangle\langle r,\phi|u,v\rangle = \int_0^{2\pi} d\phi \int_0^1 dr\, r\, |r,\phi\rangle\, W_{u,v}(r,\phi)\,, \tag{235}$$

which may be looked as a relation between the discrete basis $\{|u,v\rangle\}$ in $\mathcal{H}$ and the continuous basis $\{|r,\phi\rangle\}$. Note that, according to our definition, $\langle r,\phi|u,v\rangle = W_{u,v}(r,\phi)$. If we multiply Equation (235) to the left by $\langle r,\phi|$, we have:

$$\langle r',\phi'|u,v\rangle = W_{u,v}(r',\phi') = \int_0^{2\pi} d\phi \int_0^1 dr\, r\, \langle r',\phi'|r,\phi\rangle\, W_{u,v}(r,\phi)\,, \tag{236}$$

so that

$$\langle r',\phi'|r,\phi\rangle = \frac{1}{r}\,\delta(r-r')\,\delta(\phi-\phi')\,. \tag{237}$$

The relation in Equation (237) suggests an inversion formula for Equation (235). As $\{|u,v\rangle\}$ is an orthonormal basis for $\mathcal{H}$, we may write the identity on $\mathcal{H}$ as

$$\mathbb{I} = \sum_{u,v=0}^\infty |u,v\rangle\langle u,v|\,. \tag{238}$$

As $|u,v\rangle \in \Phi_2$, we may write

$$|r,\phi\rangle = \sum_{u,v=0}^\infty |u,v\rangle\langle u,v|r,\phi\rangle = \sum_{u,v=0}^\infty |u,v\rangle\, W^*_{u,v}(r,\phi)\,. \tag{239}$$

This inversion formula is totally consistent, as one may check by formal multiplication to the left by $\langle r', \phi'|$ and the comparison of the given result with Equation (237) on one side and Equation (207) on the other. In conclusion, each $|f\rangle \in \Phi_2$ admits two different expansions in terms of the discrete basis $\{|u,v\rangle\}$ and the continuous basis $\{|r,\phi\rangle\}$. They are, respectively,

$$|f\rangle = \sum_{u,v=0}^{\infty} |u,v\rangle\langle u,v|f\rangle = \sum_{u,v=0}^{\infty} |u,v\rangle\, f_{u,v}\,, \tag{240}$$

and

$$|f\rangle = \int_0^{2\pi} d\phi \int_0^1 dr\, r\, |r,\phi\rangle\langle r,\phi|f\rangle = \int_0^{2\pi} d\phi \int_0^1 dr\, r\, |r,\phi\rangle\, f(r,\phi)\,. \tag{241}$$

As a final remark, all operators in Equation (227) have their counterparts as operators on $\mathcal{H}$ with exactly the same properties. In particular, they are continuous on Φ_2.

10. Concluding Remarks

Specific RHS's are constructed starting from well defined special functions and a particular UIR of a Lie group, which is the symmetry group of the corresponding special functions. The Lie generators of these groups are continuous operators with the topologies carried by the RHS's.

It is a general property that in a RHS the variables and the parameters are one-to-one related. This implies that, starting from special functions with n_p parameters and n_v continuous variables, it is possible to construct different RHS's. Indeed, when $n_p = n_v$, we can construct not only a RHS involving all parameters and variables but also RHS's involving subsets of equal number of parameters and variables, saving the role of spectators for the remaining ones. If $n_p > n_v$, the possible RHS's are limited to n_v and the exceeding parameters remain spectators (as happens with j in Section 4 and α in Section 7) but it is impossible to construct a RHS based on the $\Gamma(z)$ functions where we do not have parameters at all. An alternative is shown by the Spherical Harmonics where a new variable ϕ is added to the Associated Legendre polynomials, by the extension of Jacobi polynomials to the Jacobi functions defined on the hypersphere $\mathcal{S}^3$ in Section 8.3 and by the generalization of Zernike polynomials defined on the interval $[0,1]$ to Zernike functions defined on the unit circle in Section 9.

Special functions are transition matrices between discrete and continuous bases (for instance, generalization of the exponential, i.e., $e^{\mathbf{i}m\phi} = \langle m|\phi\rangle$, in Section 3 and spherical harmonics, given by $Y_l^m(\theta,\phi) = \langle l,m|\theta,\phi\rangle$, in Section 6).

The UIR of the corresponding Lie group defines the basis vectors of the discrete basis in the space Φ, while the regular representation of the Lie group defines the basis vectors of the continuous basis in Φ^x of the RHS $\Phi \subset \mathcal{H} \subset \Phi^\times$.

Special functions determine a basis in the related space of square integrable functions. As they define a basis also of a unitary irreducible representation of the group, all other bases of the space are simply obtained applying on them an arbitrary element of the group.

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References

1. Fourier, J.B.J. *Théorie Analytique de la Chaleur*; F. Didot: Paris, France, 1822.
2. Folland, G.B. *Fourier Analysis and Its Applications*; Wadsworth Inc.: Belmont, CA, USA, 1992.
3. Folland, G.B. *A Course in Abstract Harmonic Analysis*; CRC Press Inc.: Boca Raton, FL, USA, 1995.
4. Trapani, C.; Triolo, S.; Tschinke, F. Distribution frames and bases. *J. Fourier Anal. Appl.* **2019**, *25*, 2109–2140. [CrossRef]
5. Ozaktas, H.M.; Zalevsky, Z.; Alper Kutay, M. *The Fractional Fourier Transform*; Wiley: Chichester, UK, 2001.

6. Celeghini, E.; Gadella, M.; del Olmo, M.A. Hermite Functions, Lie Groups and Fourier Analysis. *Entropy* **2018**, *20*, 816. [CrossRef]
7. Celeghini, E.; Gadella, M.; del Olmo, M.A. Hermite Functions and Fourier Series. **2019**, in preparation.
8. Kennedy, R.A.; Sadeghi, P. *Hilbert Space Methods in Signal Processing*; Cambridge University Press: Cambridge, UK, 2013.
9. Ramamoorthi, R.; Hanrahan, P. An efficient representation for irradiance environment maps. In Proceedings of the 28th Annual Conference on Computer Graphics and Interactive Techniques (SIGGRAPH'01), Los Angeles, CA, USA, 12–17 August 2001; ACM: New York, NY, USA 2001; pp. 117–128.
10. Mahajan, D.; Ramamoorthi, R.; Curless, B. A theory of frequency domain invariants: spherical harmonic identities for BRDF/lighting transfer and image consistency. *IEEE Trans. Pattern Anal. Mach. Intell.* **2008**, *30*, 197–213. [CrossRef] [PubMed]
11. Zernike, F. Inflection theory of the cutting method and its improved form, the phase contrast method. *Physica* **1934**, *1*, 689–704. [CrossRef]
12. Mahajan, V.N.; Aftab, M. Systematic comparison of the use of annular and Zernike circle polynomials for annular wavefronts, case where the circle coefficients. *Appl. Opt.* **2010**, *49*, 6489–6501. [CrossRef] [PubMed]
13. Lakshminarayanana, V.; Fleck, A. Zernike polynomials: A guide. *J. Mod. Opt.* **2011**, *58*, 545–561. [CrossRef]
14. Gelfand, I.M.; Vilenkin, N.Y. *Generalized Functions: Applications to the Harmonic Analysis*; Academic Press: New York, NY, USA, 1964.
15. Reed, M.; Simon, B. *Functional Analysis*; Academic Press: New York, NY, USA, 1972.
16. Bohm, A. *The Rigged Hilbert Space and Quantum Mechanics*; Lecture Notes in Physics 78; Springer: Berlin, Germany, 1978.
17. Roberts, J.E. Rigged Hilbert spaces in quantum mechanics. *Commun. Math. Phys.* **1966**, *2*, 98–119. [CrossRef]
18. Antoine, J.P. Dirac formalism and symmetry problems in quantum mechanics. I. General Dirac formalism. *J. Math. Phys.* **1969**, *10*, 53–69. [CrossRef]
19. Melsheimer, O. Rigged Hilbert space formalism as an extended mathematical formalism for quantum systems. I. General theory. *J. Math. Phys.* **1973**, *15*, 902–916. [CrossRef]
20. Gadella, M.; Gómez, F. A unified mathematical formalism for the Dirac formulation of quantum mechanics. *Found. Phys.* **2002**, *32*, 815–869. [CrossRef]
21. Gadella, M.; Gómez, F. On the mathematical basis of the Dirac formulation of Quantum Mechanics. *Int. J. Theor. Phys.* **2003**, *42*, 2225–2254. [CrossRef]
22. Gadella, M.; Gómez-Cubillo, F. Eigenfunction Expansions and Transformation Theory. *Acta Appl. Math.* **2010**, *109*, 721–742. [CrossRef]
23. Bohm, A. Decaying states in the rigged Hilbert space formulation of quantum mechanics. *J. Math. Phys.* **1979**, *21*, 2813–2823.
24. Bohm, A. Resonance poles and Gamow vectors in the rigged Hilbert space formulation of quantum mechanics. *J. Math. Phys.* **1980**, *22*, 2813–2823. [CrossRef]
25. Bohm, A.; Gadella, M. *Dirac Kets, Gamow Vectors and Gelfand Triplets*; Lecture Notes in Physics 348; Springer: Berlin, Germany, 1989.
26. Civitarese, O.; Gadella, M. Physical and Mathematical Aspects of Gamow States. *Phys. Rep.* **2004**, *396*, 41–113. [CrossRef]
27. Gadella, M. A rigged Hilbert space of Hardy class functions: Applications to resonances. *J. Math. Phys.* **1983**, *24*, 1462–1469. [CrossRef]
28. Bohm, A. Time-asymmetric quantum physics. *Phys. Rev. A* **1999**, *50*, 861–876. [CrossRef]
29. Bohm, A.; Harshman, N.L.; Kaldass, H.; Wickramasekara, S. Time asymmetric quantum theory and the ambiguity of the Z-boson mass and width. *Eur. Phys. J. C* **2000**, *18*, 333–342. [CrossRef]
30. Bohm, A.; Loewe, M.; Van de Ven, B. Time asymmetric quantum theory—I. Modifying an axiom of quantum physics. *Fort. Phys.* **2003**, *51*, 551–568. [CrossRef]
31. Bohm, A.; Kaldass, H.; Wickramasekara, S. Time asymmetric quantum theory—II. Relativistic resonances from S-matrix poles. *Fort. Phys.* **2003**, *51*, 569–603. [CrossRef]
32. Bohm, A.; Kaldass, H.; Wickramasekara, S. Time asymmetric quantum theory—III. Decaying states and the causal Poincaré semigroup. *Fort. Phys.* **2003**, *51*, 604–634. [CrossRef]
33. Bohm, A.; Gadella, M.; Kielanowski, P. Time asymmetric quantum mechanics. *SIGMA* **2011**, *8*, 086. [CrossRef]

34. Antoniou, I.E.; Tasaki, S. Generalized spectral decompositions of mixing dynamical systems. *Int. J. Quantum Chem.* **1993**, *46*, 425–474. [CrossRef]
35. Antoniou, I.E.; Tasaki, S. Generalized spectral decomposition of the β-adic baker's transformation and intrinsic irreversibility. *Phys. A Stat. Mech. Appl.* **1992**, *190*, 303–329. [CrossRef]
36. Antoniou, I.E.; Gadella, M.; Suchanecki, Z. General properties of the Liouville operator. *Int. J. Theor. Phys.* **1998**, *37*, 1641–1654. [CrossRef]
37. Antoniou, I.E.; Gadella, M.; Suchanecki, Z. Some general properties of Liouville Spaces. In *Irreversibility and Causality*; Bohm, A., Doebner, H.D., Kielanowski, P., Eds.; Lecture Notes in Physics 504; Springer: Berlin, Germany, 1998; pp. 38–56.
38. Antoniou, I.E.; Gadella, M. Irreversibility, resonances and rigged Hilbert spaces. In *Irreversible Quantum Dynamics*; Benatti, F., Floreanini, R., Eds.; Lecture Notes in Physics 622; Springer: Berlin, Germany, 2003; pp. 245–302.
39. Bogolubov, N.N.; Logunov, A.A.; Todorov, I.T. *Introduction to Axiomatic Quantum Field Theory*; Benjamin: Reading, MA, USA, 1975.
40. Antoniou, I.E.; Gadella, M.; Prigogine, I.; Pronko, G.P. Relativistic Gamow vectors. *J. Math. Phys.* **1998**, *39*, 2995–3018. [CrossRef]
41. Gadella, M. A RHS for the free radiation field. *J. Math. Phys.* **1985**, *26*, 725–727. [CrossRef]
42. Hida, T. *Stationary Stochastic Processes*; Princeton University Press: Princeton, NJ, USA, 1970.
43. Hida, T. *Brownian Motion*; Springer: Berlin, Germany, 1980.
44. Hormander, L. *The Analysis of Partial Differential Equations I: Distribution Theory and Fourier Analysis*, 2nd ed.; Springer: Berlin, Germany, 1990.
45. Feichtinger, H.G.; Zimmermann, G. A Banach space of test functions for Gabor analysis. In *Gabor Analysis and Algorithms: Theory and Applications, Applied and Numerical Harmonic Analysis*; Feichtinger, H.G., Strohmer, T., Eds.; Birkhäuser: Boston, MA, USA, 1998; pp. 123–170.
46. Cordero, E.; Feichtinger, H.G.; Luef, F. Banach Gelfand triples for Gabor analysis. In *Pseudo-Differential Operators*; Lecture Notes in Mathematics 949; Springer: Berlin, Germany, 2008; pp. 1–33.
47. Feichtinger, H.G. Banach Gelfand triples for applications in physics and engineering. In Proceedings of the AIP Conference Proceedings, Agra, India, 14–16 January 2009; Volume 1146, pp. 189–228.
48. Bannert, S. Banach-Gelfand Triples and Applications in Time-Frequency Analysis. Master's Thesis, University of Vienna, Vienna, Austria, 2010.
49. Feichtinger, H.G.; Jakobsen, M.S. The inner kernel theorem for a certain Segal algebra. *arXiv* **2018**, arXiv:1806.06307.
50. Feichtinger, H.G. Banach Gelfand Triples and some Applications in Harmonic Analysis. In Proceedings of the Conference Harmonic Analysis, Abidjan, Côte d'Ivoire, 23–26 May 2018.
51. Feichtinger, H.G.; Jakobsen, M.S. Distribution theory by Riemann integrals. *arXiv* **2019**, arXiv:1810.04420.
52. Feichtinger, H.G.; Franz Luef, L.; Jakobsen. S.M. Banach Gelfand Triples for analysis. *Notices Am. Math. Soc.***2019**, in press.
53. Heredia-Juesas, J.; Gago-Ribas, E. A new view of spectral analysis of linear systems. In *Progress in Electromagnetics Research Symposium, Proceedings of the PIERS 2012, Kuala Lumpur, Malaysia, 27–30 March 2012*; Electromagnetics Acad: Cambridge, MA, USA, 2012.
54. Heredia-Juesas, J.; Gago-Ribas, E.; Ganoza-Quintana, J.L. A new version of a generalized signals & systems scheme to parameterize and analyze physical problems. In Proceedings of the 2014 International Conference on Electromagnetics in Advanced Applications (ICEAA), Palm Beach, The Netherlands, 3–8 August 2014; IEEE: New York, NY, USA, 2014.
55. Heredia-Juesas, J.; Gago-Ribas, E.; Vidal-García, P. Application of the rigged Hilbert spaces into the generalized signals & systems theory. In Proceedings of the 2015 International Conference on Electromagnetics in Advanced Applications (ICEAA), Turin, Italy, 7–11 September 2015; IEEE: New York, NY, USA, 2015.
56. Heredia-Juesas, J.; Gago-Ribas, E.; Vidal-García, P. Application of the Rigged Hilbert Spaces into the Generalized Signals and Systems Theory: Practical Example. In Proceedings of the 2016 Progress in Electromagnetic Research Symposium (PIERS), Shanghai, China, 8–11 August 2016; IEEE: New York, NY, USA, 2016.

57. Parravicini, G.; Gorini, V.; Sudarshan, E.C.G. Resonances, scattering theory, and rigged Hilbert spaces. *J. Math. Phys.* **1980**, *21*, 2208–2226. [CrossRef]
58. Costin, O.; Soffer, A. Resonance theorey for Schrödinger operators. *Commun. Math. Phys.* **2001**, *224*, 133–152. [CrossRef]
59. De la Madrid, R. Rigged Hilbert space approach to the Schrödinger equation. *J. Phys. A Math. Gen.* **2002**, *35*, 319–342. [CrossRef]
60. Baumgártel, H. Generalized Eigenvectors for Resonances in the Friedrichs Model and Their Associated Gamov Vectors. *Rev. Math. Phys.* **2006**, *18*, 61–78. [CrossRef]
61. Bellomonte, G.; Trapani, C. Rigged Hilbert spaces and contractive families of Hilbert spaces. *Monatshefte Math.* **2011**, *164*, 271–285. [CrossRef]
62. Bellomonte, G.; di Bella, S.; Trapani, C. Operators in rigged Hilbert spaces: some spectral properties. *J. Math. Anal. Appl.* **2014**, *411*, 931–946. [CrossRef]
63. Chiba, H. A spectral theory of linear operators on rigged Hilbert spaces under analyticity conditions. *Adv. Math.* **2015**, *273*, 324–379. [CrossRef]
64. Chiba, H. A spectral theory of linear operators on rigged Hilbert spaces under analyticity conditions II: Applications to Schrödinger operators. *Kyushu J. Math.* **2018**, *72*, 375–405. [CrossRef]
65. Maurin, K. *General Eiegenfunction Expansions and Unitary Representations of Topological Groups*; Polish Scientific Publisheres: Warsaw, Poland, 1968.
66. Celeghini, E.; Gadella, M.; del Olmo, M.A. Lie algebra representations and rigged Hilbert spaces: The $SO(2)$ case. *Acta Polytech.* **2017**, *56*, 379–384. [CrossRef]
67. Celeghini, E.; del Olmo, M.A. Group theoretical aspects of $L^2(\mathbb{R}^+)$ and the associated Laguerre polynomials. In *Physical and Mathematical Aspects of Symmetries*; Duarte, S., Gazeau, J.P., Faci, S., Micklitz, T., Scherer, R., Toppan, F., Eds.; Springer: New York, NY, USA, 2017; pp. 133–138.
68. Celeghini, E.; Gadella, M.; del Olmo, M.A. SU(2), Associated Laguerre Polynomials and Rigged Hilbert Spaces. In *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics*; Springer Proceedings in Mathematics & Statistics; Dobrev, V., Ed.; Springer: Berlin, Germany, 2018; Volume 2, pp. 373–383.
69. Szegö, G. *Orthogonal Polynomials*; American Mathematical Society: Providence, RI, USA, 2003.
70. Abramovich, M.; Stegun, I.A. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*; Dover: New York, NY, USA, 1972.
71. Olver, F.W.J.; Lozier, D.W.; Boisiert, R.F.; Clark, C.W. (Eds.) *NIST Handbook of Mathematical Functions*; Cambridge University Press: Cambridge, MA, USA, 2010.
72. Durán, J.A. A bound on the Laguerre polynomials. *Studia Math.* **1991**, *100*, 169–181. [CrossRef]
73. Antoine, J.P.; Grossmann, A. Partial inner product spaces 1. General properties. *J. Func. Anal.* **1976**, *23*, 369–378. [CrossRef]
74. Antoine, J.P.; Grossmann, A. Partial inner product spaces 2. Operators. *J. Func. Anal.* **1976**, *23*, 379–391. [CrossRef]
75. Antoine, J.P.; Trapani, C. *Partial Inner Product Spaces: Theory and Applications*; Lecture Notes in Mathematics; Springer: Berlin, Germany, 2009; Volume 1986.
76. Antoine, J.P.; Trapani, C. PIP-Space Valued Reproducing Pairs of Measurable Functions. *Axioms* **2019**, *8*, 52. [CrossRef]
77. Cohen-Tanudji, C.; Diu, B.; Laloe, F. *Quantum Mechanics*; Wiley: NewYork, NY, USA; Hermann: Paris, France, 1991.
78. Celeghini, E.; Gadella, M.; del Olmo, M.A. Applications of Rigged Hilbert Spaces in Quantum Mechanics and Signal Procesing. *J. Math. Phys.* **2016**, *57*, 072105. [CrossRef]
79. Celeghini, E.; del Olmo, M.A. Coherent orthogonal polynomials. *Ann. Phys.* **2013**, *335*, 78–85. [CrossRef]
80. Celeghini, E.; del Olmo, M.A. Algebraic special functions and $SO(3,2)$. *Ann. Phys.* **2013**, *333*, 90–103. [CrossRef]
81. Celeghini, E.; Gadella, M.; del Olmo, M.A. Spherical Harmonics and Rigged Hilbert Spaces. *J. Math. Phys.* **2015**, *59*, 053502. [CrossRef]
82. Atkinson, K.; Hang, W. *Spherical Harmonics Approximations on the Unit Sphere*; Springer: Berlin, Germany, 2012.
83. Celeghini, E.; del Olmo, M.A.; Velasco, M.A. Lie groups, algebraic special functions and Jacobi polynomials. *J. Phys. Conf. Ser.* **2015**, *597*, 012023. [CrossRef]

84. Celeghini, E.; del Olmo, M.A.; Velasco, M.A. Jacobi polynomials as infinite-dimensional irreducible representation of su(2; 2). In *Integrability, Supersymmetry and Coherent States*; Kuru, S., Ed.; Springer: Berlin, Germany, 2019.
85. Celeghini, E.; Gadella, M.; del Olmo, M.A. Groups, Jacobi Functions and Rigged Hilbert Spaces. *arXiv* **2019**, arXiv:1907.01281.
86. Born, M.; Wolf, E. *Principles of Optics*; Cambridge University Press: Cambridge, UK, 1999.
87. Celeghini, E.; Gadella, M.; del Olmo, M.A. Zernike functions, rigged Hilbert spaces and potential applications. *arXiv* **2019**, arXiv:1902.08017.