axioms

Article

A Metric for Finite Power Multisets of Positive Real Numbers Based on Minimal Matching

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Abstract: In this article, we show how to define a metric on the finite power multisets of positive real numbers. The metric, based on the minimal matching, consists of two parts: the matched part and the mismatched part. We also give some concrete applications and examples to demonstrate the validity of this metric.

Keywords: metric; minimal matching; positive multisets

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1. Introduction

A multiset, unlike a Cantorian set, is a collection of elements whose instances might be multiple (the number of its instances of an element is named multiplicity). The cardinality of a multiset A is defined by the sum of the multiplicities with respect to their corresponding elements and is denoted by $|A|_m$. For example, the cardinality of multiset $A = \{2, 2, 3, 3, 3, 6, 11\}$ is 7, i.e., $|A|_m = 7$. Though unconventional, the theory of multisets has well been developed (see Reference [1]) and it also has various applications in many situations (see Reference [2]). From practical point of view, multisets are easier to represent or simulate than mathematical objects with multiple instances. In this article, we mainly focus on the finite power multisets of positive real numbers.

Let $\mathbb{R}^+$ denote the set of all positive real numbers. Let $\mathbb{N}_0$ denote the set of natural numbers including 0. Let power multiset $\mathcal{MP}(\mathbb{R}^+)$ denote the set of all the sub-multisets of $\mathbb{R}^+$. Suppose $\mathbb{K} \subseteq \mathcal{MP}(\mathbb{R}^+)$ is an arbitrary set of some sub-multisets in $\mathbb{R}^+$ (each multiset is finite) of $\mathcal{MP}(\mathbb{R}^+)$. We call $\mathbb{K}$ a finite power multiset of positive real numbers. The main result in this article is to define a metric on $\mathbb{K}$ based on the concept of minimal matching. The distance between any two multisets consists of two separated parts: the matched part and mismatched part. Matching has been an important problem and has wide applications in the fields of artificial intelligence, graph theories, and operation research (see References [3–5]). In this article, we come up with a new metric which is based on the concept of minimal matching. This metric is used to measure the distance between any two finite multisets of positive real numbers. Though what we define in this article is a standard metric, the whole setting could also be extended to other generalized metrics, for example, $G-$metric (see Reference [6]).

2. Definitions

In this section, we introduce and present multisets via the forms of functions. The basic concepts could be found in many textbooks or journals (see, e.g., References [7,8]). Let Γ denote the set $\mathbb{R}^+ \to \mathbb{N}_0$, i.e., the set of all the functions from $\mathbb{R}^+$ to $\mathbb{N}_0$. Let D_f denote the domain of a function f. Let set $D^*_f = \{r \in \mathbb{R}^+ : f(r) \neq 0\}$ be the non-zero domain of f.

2.1. Multisets

Let $\Gamma^<$ denote all the finite multi-subsets of $\mathbb{R}^+$, i.e., $\Gamma^< = \{f \in \Gamma : |D_f^*| < \infty\}$. Each element in $\Gamma^<$ is simply named a multiset in this article. If for all $x \in \mathbb{R}^+$, $f(x) \leq g(x)$, we say f is a multi-subset of g, denoted by $f \leq g$. Let $f, g \in \Gamma^<$ be arbitrary.

Definition 1. *(Empty Multiset) We call the zero function in $\Gamma^<$ the empty multiset.*

Definition 2. *(Equality $=$) $f = g$ if and only if $f \leq g$ and $g \leq f$.*

Definition 3. *(Intersection $\wedge$) The intersection of f and g, denoted by the function $f \wedge g : \mathbb{R}^+ \to \mathbb{N}_0$, is defined by $(f \wedge g)(a) := min\{f(a), g(a)\}$ for all $a \in \mathbb{R}^+$.*

Definition 4. *(Union $\vee$) The union of f and g, denoted by the function $f \vee g : \mathbb{R}^+ \to \mathbb{N}_0$, is defined by $(f \vee g)(a) := max\{f(a), g(a)\}$ for all $a \in \mathbb{R}^+$.*

Definition 5. *(Difference $\ominus$) Exclusion of g from f, denoted by the function $f \ominus g : \mathbb{R}^+ \to \mathbb{N}_0$, is defined by $(f \ominus g)(a) := f(a) - (f \wedge g)(a)$.*

Each multiset f in $\Gamma^<$ could be uniquely represented by the following descending form (named a representative descending form):

$$f^- = (a_1^{f(a_1)}, a_2^{f(a_2)}, ..., a_n^{f(a_n)}), \tag{1}$$

or in brief $f^- = a_1^{f(a_1)} a_2^{f(a_2)} ... a_n^{f(a_n)}$; or by the following ascending form (named a representative ascending form):

$$f^+ = (a_n^{f(a_n)} a_{n-1}^{f(a_{n-1})}, ..., a_2^{f(a_2)} a_1^{f(a_1)}), \tag{2}$$

or in brief $f^+ = a_n^{f(a_n)} a_{n-1}^{f(a_{n-1})} ... a_1^{f(a_1)}$, where $a_1 > a_2 > a_3 ... > a_n > 0$ and $a_1, a_2, ..., a_n \in D_f^*$ and $f(a_v) > 0$ for all $1 \leq v \leq n$. Let $|f| = f(a_1) + f(a_2) + ... + f(a_n)$.

Definition 6. *(Descending) Define the $p - th$ element in f by function OD as follows:*

$$OD(p, f) := \begin{cases} a_1 & if\ 1 \leq p \leq f(a_1); \\ a_j & if\ \sum_{l=1}^{j-1} f(a_l) < p \leq \sum_{l=1}^{j} f(a_l)\ and\ |D_f^*| \geq j \geq 2; \\ 0 & otherwise\ . \end{cases}$$

Definition 7. *(Ascending) Define the $p - th$ element in f by function OA as follows:*

$$OA(p, f) := \begin{cases} a_n & if\ 1 \leq p \leq f(a_n); \\ a_{n-j} & if\ \sum_{l=0}^{j-1} f(a_{n-l}) < p \leq \sum_{l=0}^{j} f(a_{n-l})\ and\ |D_f^*| \geq j \geq 1; \\ 0 & otherwise\ . \end{cases}$$

2.2. Background

In this article, we show how to define a metric on $\mathbb{K}$ (see Introduction). For any Cantorian set S, we use $|S|$ to denote the cardinality of S. Let d be an arbitrary metric on $\mathbb{R}^+$ satisfying

$$d(a, b) \leq a + b, \tag{3}$$

$$d(a, b) + a \geq b, \tag{4}$$

$$d(a, b) + b \geq a, \tag{5}$$

for all $a, b, c \in \mathbb{R}^+$. Observe that d is a metric (for our generalization purpose) on $\mathbb{R}^+ \times \mathbb{R}^+$, which lays a foundation for our latter definition of a metric on $\mathbb{K}$. Let $A, B, C \in \mathbb{K}$ be arbitrary. Let $A \to B$ denote the set of all the functions from A to B, in which the repeated elements are deemed distinct.

Example 1. *Suppose* $A = \{2, 2, 3, 5\}$ *and* $B = \{6, 8, 8, 8, 8, 9, 12, 12\}$, *and* $\rho(2) = 9, \rho(2) = 6, \rho(3) = 8$, $\rho(5) = 12$. *Then,* $\rho \in A \to B$. *For clarity, one could simply associate A and B with their ranked multiplicities as follows:* $A = \{(2, 1), (2, 2), (3, 1), (5, 1)\}$ *and* $B = \{(6, 1), (8, 1), (8, 2), (8, 3), (8, 4), (9, 1), (12, 1), (12, 2)\}$ *and* ρ *could also be represented by* $\rho(2, 1) = (9, 1), \rho(2, 2) = (6, 1), \rho(3, 1) = (8, 1), \rho(5, 1) = (12, 1)$. *To save space, we simply use* $\rho(2_1) = 9_1, \rho(2_2) = 6_1, \rho(3_1) = 8_1, \rho(5_1) = 12_1$ *for the representation in this article.*

For any function φ, we use D_φ and R_φ to denote its domain and codomain, respectively. For the previous example, $D_\rho = A$ and $R_\rho = B$. We use $\varphi(S)$ to denote the image $\{\varphi(s) : s \in S\}$, in particular $\varphi(D_\varphi)$ to denote the image of φ and $\varphi^{-1}(S)$ to denote the pre-image of S. If $S \subseteq D_\varphi$, we use $\varphi|S$ to denote φ whose domain is restricted to S. One candidate in mind is $d(a, b) := |a - b|$.

Definition 8. *(Bijective embeddings) Define*

$$BF[A \to B] := \{\varphi \in A \to B : |A|_m = |\varphi(A)|_m\},$$

$$BF[B \to A] := \{\varphi \in B \to A : |B|_m = |\varphi(B)|_m\},$$

$$BF[A, B] := BF[A \to B] \cup BF[B \to A].$$

Example 2. *Suppose* ρ *is defined in Example 1. Since* $|A|_m = 4 = |\rho(A)|_m$, *by the above definition, one has* $\rho \in BF[A \to B]$. *On the other hand, suppose* $\kappa(2_1) = 6_1, \kappa(2_2) = 6_1, \kappa(3_1) = 8, \kappa(5_1) = 12$, *then* $\kappa \notin BF[A \to B]$, *since* $|A|_m = 4 \neq |\kappa(A)|_m = 3$. *Note that* $BF[A, B] = BF[B, A]$. *Moreover, if* $|A|_m > |B|_m$, *then* $BF[A \to B] = \emptyset$; *similarly, if* $|B|_m > |A|_m$, *then* $BF[B \to A] = \emptyset$. *Take A and B in Example 1 for example. One has* $BF[B \to A] = \emptyset$.

Definition 9. *For any function* $\varphi \in BF[A \to B]$, *we call it a a matched function. We call* $(a, \varphi(a))$ *a matched pair. Every remaining element in* $B - \varphi(A)$ *is called a mismatched element.*

On this basis, we could define the distance for the matched elements and the distance for the mismatched elements as follows:

Definition 10. *For any* $\varphi \in BF[A, B]$, *define*

$$||\varphi|| := \sum_{e \in D_\varphi} d(e, \varphi(e)) \text{ and } ||\varphi||^- := \sum_{e \in R_\varphi - \varphi(D_\varphi)} e,$$

where D_φ *and* R_φ *and denote the domain and codomain of* φ, *respectively.* $||\varphi||$ *represents the distance of all the matched elements (or the sum of the distances of all the matched pairs), while* $||\varphi||^-$ *represents the distance of all the mismatched elements in the range.* $||\varphi||^- = 0$ *iff* $|A| = |B|$. *For example, if* $A = \{1, 1, 2, 3, 1\}, B = \{2, 4, 6, 2\}$ *and* $\varphi : A \to B$ *is defined by* $\varphi(1_1) = 2, \varphi(1_2) = 6_1, \varphi(2) = 2_1, \varphi(3) = 2_2$, *then the matched part yields* $||\varphi|| = |1 - 2| + |1 - 6| + |2 - 2| + |3 - 2| = 7$, *where* i_n *denotes the* $n-th$ *repetition of i and the mismatched part* $||\varphi||^- = 1$. *Next, we define the set of all minimal distances consisting of the matched parts and the mismatched parts.*

Definition 11. *(Minimal matched functions) Define*

$$\begin{aligned}&BF_*[A \to B]\\&:= \{\varphi \in BF[A \to B] : ||\varphi|| + ||\varphi||^- \le ||\psi|| + ||\psi||^-, \forall \psi \in BF[A \to B]\},\\&BF_*[B \to A]\\&:= \{\varphi \in BF[B \to A] : ||\varphi|| + ||\varphi||^- \le ||\psi|| + ||\psi||^-, \forall \psi \in BF[B \to A]\},\\&BF_*[A, B]\\&:= \{\varphi \in BF[A, B] : ||\varphi|| + ||\varphi||^- \le ||\psi|| + ||\psi||^-, \forall \psi \in BF[A, B]\}.\end{aligned}$$

Definition 12. *(Distance function) Define* $\delta : \mathbb{K} \times \mathbb{K} \to \mathbb{R}^+$ *by*

$$\delta(A, B) := min\{||\varphi|| + ||\varphi||^- : \varphi \in BF[A, B]\}. \tag{6}$$

By the definition, one has $\delta(A, B) = ||\varphi|| + ||\varphi||^-$ for any $\varphi \in BF_*[A, B]$. In the following, we show that δ is indeed a metric. The reasoning will be proceeded by their relations (i.e., larger, less than and equal to) between cardinalities of A, B, and C, i.e., $|A|_m, |B|_m$, and $|C|_m$. To validate that δ is a metric, we need to consider all the 27 relations between $|A|_m, |B|_m$ and $|C|_m$: for example, $|A|_m > |B|_m > |C|_m, |A|_m = |B|_m < |C|_m, etc.$ In order to facilitate our computing, we encode the 27 relations by the following set

$$\{(n_1, n_2, n_3) : n_1, n_2, n_3 \in \{1, 2, 3\}\},$$

in which each (n_1, n_2, n_3) represents the relation $|A|_m n_1 |B|_m, |B|_m n_2 |C|_m$ and $|A|_m n_3 |C|_m$, respectively, where $1, 2$, and 3 represent the relation $<, =$ and $>$ correspondingly. For example, $(1, 2, 3)$ represents the relation $|A|_m < |B|_m, |B|_m = |C|_m$, and $|A|_m > |C|_m$. By the transitivity of their cardinalities, only 13 of the 27 relations are valid (shown in Lemma 1). Moreover, these 13 relations could be further reduced to 8 relations by the symmetry of δ, i.e.,

$$\delta(A, B) + \delta(B, C) \ge \delta(A, C) \Leftrightarrow \delta(C, B) + \delta(B, A) \ge \delta(C, A), \tag{7}$$

as shown in Corollary 1. If φ is a bijective function, we use φ^{-1} to denote its inverse function. In the following, let $\varphi \in BF_*[A, B], \tilde{\varphi} \in BF_*[B, C]$, and $\tilde{\tilde{\varphi}} \in BF_*[A, C]$ be arbitrary. Before we proceed further, we have the definitions:

1. We use B_A to denote $\varphi(A)$, if $|A|_m \le |B|_m$ and A_B to denote $\varphi(B)$, if $|A|_m > |B|_m$.
2. We use C_B to denote $\tilde{\varphi}(B)$, if $|B|_m \le |C|_m$ and B_C to denote $\tilde{\varphi}(C)$, if $|B|_m > |C|_m$.
3. We use C_A to denote $\tilde{\tilde{\varphi}}(A)$, if $|A|_m \le |C|_m$ and A_C to denote $\tilde{\tilde{\varphi}}(C)$, if $|C|_m < |A|_m$.

Though there are 27 relations between the cardinalities of A, B, and C, only 13 of them are valid as shown in the following lemma.

Lemma 1. *There are only 13 relations which do not violate the transitivity property in terms of their cardinalities:*

$$\begin{aligned}&(1,1,1), (1,2,1), (1,3,1), (1,3,2), (1,3,3), (2,1,1),\\&(2,2,2), (2,3,3), (3,1,1), (3,1,2), (3,1,3), (3,2,3), (3,3,3).\end{aligned}$$

Proof. The result follows immediately from their relations. Take the relation $(1, 1, 1)$ for example. Recall that $(1, 1, 1)$ represents the relation $|A|_m < |B|_m < |C|_m$, in which the property of transitivity $|A|_m < |C|_m$ holds. One could verify that each of the other 12 relations also holds the transitivity property. However, the other 15 relations fail the transitivity property: for example $(1, 1, 3)$ (i.e., $|A|_m < |B|_m, |B|_m < |C|_m, |A|_m > |C|_m$). □

Lemma 2. *(Non-negative, symmetric)*

1. $\delta(A,B) \geq 0$.
2. $\delta(A,B) = 0$ *iff* $A = B$.
3. $\delta(A,B) = \delta(B,A)$.

Proof. The first statement follows immediately from the definition and the third one follows from the fact that $BF[A,B] = BF[B,A]$. Here we show the second one. Suppose $A = B$, then $\delta(A,B) = ||I|| = 0$, where I is the identity function. Suppose $A \neq B$. Then, there are two cases: either $|A|_m \neq |B|_m$ or $|A|_m = |B|_m$. For the former one, one has $\forall \varphi \in BF[A,B](||\varphi||^- > 0)$, i.e., $\delta(A,B) > 0$. For the latter one, one has $I \notin BF[A,B]$ and thus $\forall \varphi \in BF[A,B](||\varphi|| > 0)$, i.e., $\delta(A,B) > 0$. Hence, we have shown $\delta(A,B) = 0$ iff $A = B$. □

In the following, we show the triangle inequality of δ. Let us show the following corollary first.

Corollary 1. *To show δ satisfy the triangle inequality, it suffices to consider the following eight relations:*

$$(2,2,2),(2,3,3),(2,1,1),(3,1,2),$$
$$(1,3,2),(1,1,1),(3,1,1),(1,3,1).$$

Proof. By Equation (7) and Lemma 1, A and C are interchangeable, i.e., the relations

$$(1,1,1),(2,3,3),(1,3,1),(2,1,1),(3,1,1)$$

are equivalent to (respectively)

$$(3,3,3),(1,2,1),(1,3,3),(3,2,3),(3,1,3).$$

□

By this corollary, we only need to consider the triangle inequality of the above-mentioned eight relations.

Lemma 3. (*Relation* $(2,2,2)$)
If $|A|_m = |B|_m = |C|_m$, *then* $\delta(A,B) + \delta(B,C) \geq \delta(A,C)$.

Proof. Since $|A|_m = |B|_m = |C|_m$, it follows

$$\delta(A,B) = ||\varphi|| = \sum_{e \in A} d(e, \varphi(e))$$

and

$$\delta(B,C) = ||\tilde{\varphi}|| = \sum_{h \in A} d(h, \tilde{\varphi}(h)) = \sum_{e \in A} d(\varphi(e), \tilde{\varphi} \circ \varphi(e)).$$

Since $\tilde{\varphi} \circ \varphi \in BF(A,C)$, by the definition of d, it follows

$$\delta(A,B) + \delta(B,C) \geq \sum_{e \in A} d(e, \tilde{\varphi} \circ \varphi(e)) \geq \sum_{e \in A} d(e, \tilde{\tilde{\varphi}}(e)),$$

i.e., $\delta(A,B) + \delta(B,C) \geq \delta(A,C)$. □

Lemma 4. (*Relation* $(2,3,3)$)
If $|A|_m = |B|_m > |C|_m$, *then* $\delta(A,B) + \delta(B,C) \geq \delta(A,C)$.

Proof. By the definition δ, it follows

$$\delta(A,C)=\sum_{a\in A-A_C}a+\sum_{a\in A_C}d(a,\tilde{\tilde{\varphi}}(a))\leq\sum_{a\in A-A_{B_C}}a+\sum_{a\in A_{B_C}}d(a,\tilde{\varphi}\circ\varphi(a)),$$

where A_{B_C} denotes $\varphi^{-1}(B_C)$. Furthermore,

$$\delta(A,B)=\sum_{a\in A-A_{B_C}}d(a,\varphi(a))+\sum_{a\in A_{B_C}}d(a,\varphi(a)),$$

$$\begin{aligned}\delta(B,C)&=\sum_{b\in B-B_C}b+\sum_{b\in B_C}d(b,\tilde{\varphi}(b))\\&=\sum_{a\in A-A_{B_C}}\varphi(a)+\sum_{a\in A_{B_C}}d(\varphi(a),\tilde{\varphi}\circ\varphi(a)).\end{aligned}$$

Henceforth, by the properties of d and the definition of δ

$$\begin{aligned}&\delta(A,B)+\delta(B,C)\\&=\sum_{a\in A-A_{B_C}}[d(a,\varphi(a))+\varphi(a)]+\sum_{a\in A_{B_C}}[d(a,\varphi(a))+d(\varphi(a),\tilde{\varphi}\circ\varphi(a))]\\&\geq\sum_{a\in A-A_{B_C}}a+\sum_{a\in A_{B_C}}d(a,\tilde{\varphi}\circ\varphi(a))\geq\delta(A,C).\end{aligned}$$

□

Lemma 5. (*Relation* $(2,1,1)$)
If $|A|_m=|B|_m<|C|_m$, *then* $\delta(A,B)+\delta(B,C)\geq\delta(A,C)$.

Proof. By the definition δ, it follows

$$\delta(A,C)=\sum_{c\in C-C_A}c+\sum_{a\in A}d(a,\tilde{\tilde{\varphi}}(a))\leq\sum_{c\in C-C_B}c+\sum_{a\in A}d(a,\tilde{\varphi}\circ\varphi(a)),$$

$$\delta(A,B)=\sum_{a\in A}d(a,\varphi(a)),$$

$$\delta(B,C)=\sum_{c\in C-C_B}c+\sum_{b\in B}d(b,\tilde{\varphi}(b))=\sum_{c\in C-C_B}c+\sum_{a\in A}d(\varphi(a),\tilde{\varphi}\circ\varphi(a)).$$

Henceforth, by the triangle inequality of d

$$\begin{aligned}&\delta(A,B)+\delta(B,C)\\&=\sum_{a\in A}d(a,\varphi(a))+\sum_{c\in C-C_B}c+\sum_{b\in B}d(b,\tilde{\varphi}(b))\\&=\sum_{a\in A}d(a,\varphi(a))+\sum_{c\in C-C_B}c+\sum_{a\in A}d(\varphi(a),\tilde{\varphi}\circ\varphi(a))\\&\geq\sum_{c\in C-C_B}c+\sum_{a\in A}d(a,\tilde{\varphi}\circ\varphi(a))\geq\sum_{c\in C-C_A}c+\sum_{a\in A}d(a,\tilde{\tilde{\varphi}}(a))\\&=\delta(A,C).\end{aligned}$$

□

Lemma 6. (*Relation* $(3,1,2)$)
If $|A|_m=|C|_m>|B|_m$, *then* $\delta(A,B)+\delta(B,C)\geq\delta(A,C)$.

Proof. Since

$$\delta(A,C) = \sum_{a\in A} d(a,\tilde{\tilde{\varphi}}(a)),$$

$$\delta(A,B) = \sum_{a\in A-A_B} a + \sum_{a\in A_B} d(a,\varphi(a)),$$

$$\delta(B,C) = \sum_{c\in C-C_B} c + \sum_{b\in B} d(b,\tilde{\varphi}(b)) = \sum_{c\in C-C_B} c + \sum_{a\in A_B} d(\varphi(a),\tilde{\varphi}\circ\varphi(a)).$$

By the triangle inequality of d and the definitions of δ

$$\begin{aligned}
&\delta(A,B)+\delta(B,C)\\
&\geq \sum_{A-A_B} a + \sum_{c\in C-C_B} c + \sum_{a\in A_B} [d(a,\varphi(a)) + d(\varphi(a),\tilde{\varphi}\circ\varphi(a))]\\
&\geq \sum_{a\in A-A_B} [a+\psi(a)] + \sum_{a\in A_B} [d(a,\tilde{\varphi}\circ\varphi(a))] \text{ for some bijective function } \psi\\
&\text{between } A-A_B \text{ and } C-C_B\\
&\geq \sum_{a\in A-A_B} d(a,\psi(a)) + \sum_{a\in A_B} [d(a,\tilde{\varphi}\circ\varphi(a))] \text{ for some bijective function } \psi\\
&\text{between } A-A_B \text{ and } C-C_B\\
&\geq \delta(A,C) (\text{ since the coupling of } \psi|_{A-A_B} \text{ and } \varphi|_{A_B} \text{ lies in } BF(A,C)).
\end{aligned}$$

□

Lemma 7. (*Relation* $(1,3,2)$)
If $|A|_m = |C|_m < |B|_m$, *then* $\delta(A,B)+\delta(B,C) \geq \delta(A,C)$.

Proof. By the definitions,

$$\begin{aligned}
\delta(A,B) &= \sum_{b\in B-B_A} b + \sum_{a\in A} d(a,\varphi(a))\\
&= \sum_{b\in B-B_A} b + \sum_{a\in A_{B_A\cap B_C}} d(a,\varphi(a)) + \sum_{a\in A-A_{B_A\cap B_C}} d(a,\varphi(a))\\
&\geq \sum_{b\in B_C-B_A\cap B_C} b + \sum_{a\in A_{B_A\cap B_C}} d(a,\varphi(a)) + \sum_{a\in A-A_{B_A\cap B_C}} d(a,\varphi(a))\\
&= \sum_{b\in B_C-B_A\cap B_C} b + \sum_{a\in A_{B_A\cap B_C}} d(a,\varphi(a)) + \sum_{a\in A-A_{B_A\cap B_C}} d(a,\varphi(a))
\end{aligned}$$

where $A_{B_A\cap B_C}$ denotes $\varphi^{-1}(B_A\cap B_C)$. Moreover,

$$\begin{aligned}
&\delta(B,C)\\
&= \sum_{b\in B-B_C} b + \sum_{b\in B_C} d(b,\tilde{\varphi}(b))\\
&= \sum_{b\in B-B_C} b + \sum_{b\in B_C-B_A\cap B_C} d(b,\tilde{\varphi}(b)) + \sum_{b\in B_A\cap B_C} d(b,\tilde{\varphi}(b))\\
&\geq \sum_{b\in B_A-B_A\cap B_C} b + \sum_{b\in B_C-B_A\cap B_C} d(b,\tilde{\varphi}(b)) + \sum_{b\in B_A\cap B_C} d(b,\tilde{\varphi}(b))\\
&= \sum_{a\in A-A_{B_A\cap B_C}} \varphi(a) + \sum_{b\in B_C-B_A\cap B_C} d(b,\tilde{\varphi}(b)) + \sum_{a\in A_{B_A\cap B_C}} d(\varphi(a),\tilde{\varphi}\circ\varphi(a)).
\end{aligned}$$

Hence, by the triangle inequality of d and the definitions of δ

$$\begin{aligned}
&\delta(A,B)+\delta(B,C)\\
&\geq \sum_{a\in A-A_{B_A\cap B_C}} [\varphi(a)+d(a,\varphi(a))] + \sum_{b\in B_C-B_A\cap B_C} [b+d(b,\tilde{\varphi}(b))]\\
&\sum_{a\in A_{B_A\cap B_C}} [d(a,\varphi(a))+d(\varphi(a),\tilde{\varphi}\circ\varphi(a))]\\
&\geq \sum_{a\in A-A_{B_A\cap B_C}} a + \sum_{b\in B_C-B_A\cap B_C} \tilde{\varphi}(b) + \sum_{a\in A_{B_A\cap B_C}} d(a,\tilde{\varphi}\circ\varphi(a))\\
&\geq \sum_{a\in A-A_{B_A\cap B_C}} d(a,\psi(a)) + \sum_{a\in A_{B_A\cap B_C}} d(a,\tilde{\varphi}\circ\varphi(a)),\\
&\text{where } \psi\in BF(A-A_{B_A\cap B_C}, B_C-B_A\cap B_C)\\
&\geq \sum_{a\in A} d(a,\tilde{\tilde{\varphi}}(a)) = \delta(A,C).
\end{aligned}$$

□

Lemma 8. (*Relation* (1, 1, 1))
If $|A|_m < |B|_m < |C|_m$, *then* $\delta(A,B)+\delta(B,C)\geq\delta(A,C)$.

Proof. By the definitions of δ

$$\delta(A,C) = \sum_{c\in C-C_A} c + \sum_{a\in A} d(a,\tilde{\tilde{\varphi}}(a)) \leq \sum_{c\in C-C_{B_A}} c + \sum_{a\in A} d(a,\tilde{\varphi}\circ\varphi(a)),$$

where C_{B_A} denotes $\tilde{\varphi}(B_A)$;

$$\delta(A,B) = \sum_{b\in B-B_A} b + \sum_{a\in A} d(a,\varphi(a)).$$

Furthermore,

$$\begin{aligned}
\delta(B,C) &= \sum_{c\in C-C_B} c + \sum_{b\in B} d(b,\tilde{\varphi}(b))\\
&= \sum_{c\in C-C_B} c + \sum_{b\in B_A} d(b,\tilde{\varphi}(b)) + \sum_{b\in B-B_A} d(b,\tilde{\varphi}(b))\\
&= \sum_{c\in C-C_B} c + \sum_{a\in A} d(\varphi(a),\tilde{\varphi}\circ\varphi(a)) + \sum_{b\in B-B_A} d(b,\tilde{\varphi}(b)).
\end{aligned}$$

Then, by the triangle inequality of d and the definitions of δ

$$\begin{aligned}
&\delta(A,B)+\delta(B,C)\\
&\geq \sum_{a\in A} d(a,\tilde{\varphi}\circ\varphi(a)) + \sum_{b\in B-B_A} [b+d(b,\tilde{\varphi}(b))] + \sum_{c\in C-C_B} c\\
&\geq \sum_{a\in A} d(a,\tilde{\varphi}\circ\varphi(a)) + \sum_{b\in B-B_A} \tilde{\varphi}(b) + \sum_{c\in C-C_B} c(\text{ by Equation (4)})\\
&= \sum_{a\in A} d(a,\tilde{\varphi}\circ\varphi(a)) + \sum_{c\in C_B-C_{B_A}} c + \sum_{c\in C-C_B} c\\
&\geq \sum_{a\in A} d(a,\tilde{\varphi}\circ\varphi(a)) + \sum_{c\in C-C_{B_A}} c \geq \delta(A,C).
\end{aligned}$$

□

Lemma 9. (*Relation* $(3,1,1)$)
If $|C|_m > |A|_m > |B|_m$*, then* $\delta(A,B)+\delta(B,C) \geq \delta(A,C)$.

Proof. We derive the three components one by one. Firstly, suppose $\bar{\bar{\varphi}} \in BM(A \to B)$ is a function satisfying $\bar{\bar{\varphi}}|(A_B) = C_B$ (as shown in Figure 1), i.e., $\bar{\bar{\varphi}}(a) = \tilde{\varphi} \circ \varphi^{-1}(a)$ for all $a \in A_B$. By the definitions of δ

$$\begin{aligned}\delta(A,C) &= \sum_{c \in C - C_A} c + \sum_{a \in A} d(a, \tilde{\tilde{\varphi}}(a)) \\ &\leq \sum_{c \in C - \bar{\bar{\varphi}}(A)} c + \sum_{a \in A} d(a, \bar{\bar{\varphi}}(a)) \\ &= \sum_{c \in C - \bar{\bar{\varphi}}(A)} c + \sum_{a \in A_B} d(a, \bar{\bar{\varphi}}(a)) + \sum_{a \in A - A_B} d(a, \bar{\bar{\varphi}}(a))\end{aligned}$$

Secondly,

$$\delta(A,B) = \sum_{a \in A - A_B} a + \sum_{a \in A_B} d(a, \varphi^{-1}(a)).$$

Thirdly,

$$\begin{aligned}\delta(B,C) &= \sum_{c \in C - C_B} c + \sum_{b \in B} d(b, \tilde{\varphi}(b)) \\ &= \sum_{c \in C - \bar{\bar{\varphi}}(A)} c + \sum_{c \in \bar{\bar{\varphi}}(A) - \bar{\bar{\varphi}}(A_B)} c + \sum_{b \in B} d(b, \tilde{\varphi}(b)) \\ &= \sum_{c \in C - \bar{\bar{\varphi}}(A)} c + \sum_{c \in \bar{\bar{\varphi}}(A) - \bar{\bar{\varphi}}(A_B)} c + \sum_{a \in A_B} d(\varphi^{-1}(a), \tilde{\varphi} \circ \varphi^{-1}(a)) \\ &= \sum_{c \in C - \bar{\bar{\varphi}}(A)} c + \sum_{c \in \bar{\bar{\varphi}}(A) - \bar{\bar{\varphi}}(A_B)} c + \sum_{a \in A_B} d(\varphi^{-1}(a), \bar{\bar{\varphi}}(a)) \\ &= \sum_{c \in C - \bar{\bar{\varphi}}(A)} c + \sum_{a \in A - A_B} \bar{\bar{\varphi}}(a) + \sum_{a \in A_B} d(\varphi^{-1}(a), \bar{\bar{\varphi}}(a))\end{aligned}$$

Hence,

$$\begin{aligned}&\delta(A,B) + \delta(B,C) \geq \sum_{c \in C - \bar{\bar{\varphi}}(A)} c + \sum_{a \in A_B} d(a, \bar{\bar{\varphi}}(a)) \\ &+ \sum_{a \in A - A_B} [a + \bar{\bar{\varphi}}(a)] \\ &\geq \sum_{c \in C - \bar{\bar{\varphi}}(A)} c + \sum_{a \in A_B} d(a, \bar{\bar{\varphi}}(a)) + \sum_{a \in A - A_B} d(a, \bar{\bar{\varphi}}(a)) (\text{Equation}(3)) \\ &\geq \delta(A,C).\end{aligned}$$

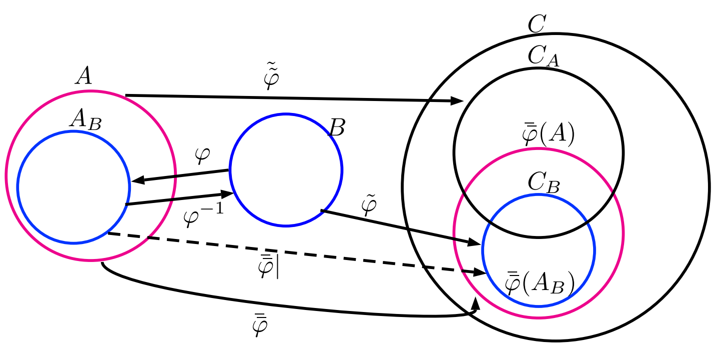

Figure 1. Triangle Inequality for $(3,1,1)$ case.

□

Lemma 10. (*Relation* $(1,3,1)$)

If $|B|_m > |C|_m > |A|_m$, *then* $\delta(A,B) + \delta(B,C) \geq \delta(A,C)$.

Proof. Suppose $A_2 = \varphi^{-1}(B_A \cap B_C) \equiv A_{B_A \cap B_C}$. Suppose $A_1 = A - A_2$. Suppose $B_{A_1} = \varphi(A_1) = B_A - B_A \cap B_C, B_{A_2} = \varphi(A_2) = B_A \cap B_C$. Choose a function $\psi \in BF(A, B_C)$, in which $\forall a \in A_2[\psi(a) = \varphi(a)]$, i.e., $\psi|A_2 = \varphi|A_2$. Suppose $\bar{B} = B_{A_2} \cup \psi(A_1) \equiv B_{A_2} \cup B^*_{A_1}$, where $B^*_{A_1} \equiv \psi(A_1)$. Let ρ denote the composition $\tilde{\varphi}^{-1}|B \circ \psi$ (or simply $\tilde{\varphi}^{-1} \circ \psi$). Then, $\rho \in BF[A, C]$ and

$$\rho(A) = \tilde{\varphi}^{-1} \circ \psi(A) = \tilde{\varphi}^{-1} \circ \psi(A_1) \cup \tilde{\varphi}^{-1} \circ \psi(A_2) = \tilde{\varphi}^{-1} \circ \psi(A_1) \cup \tilde{\varphi}^{-1}(B_{A_2}),$$

as shown in Figure 2. Furthermore, by the definition of δ,

$$\begin{aligned}\delta(A,C) &= \sum_{c \in C - C_A} c + \sum_{c \in C_A} d(a, \tilde{\tilde{\varphi}}(a)) \leq \sum_{c \in C - \rho(A)} c + \sum_{a \in A} d(a, \rho(a)) \\ &= \sum_{c \in C - \rho(A)} c + \sum_{a \in A_1} d(a, \rho(a)) + \sum_{a \in A_2} d(a, \rho(a));\end{aligned}$$

$$\begin{aligned}\delta(A,B) &= \sum_{b \in B - B_A} b + \sum_{a \in A} d(a, \varphi(a)) \\ &= \sum_{b \in B - B_A} b + \sum_{a \in A_1} d(a, \varphi(a)) + \sum_{a \in A_2} d(a, \varphi(a)) \\ &= \sum_{b \in B - B_A \cup B_C} b + \sum_{b \in B_C - B_{A_1} \cup B_{A_2}} b + \sum_{b \in B_{A_1}} b \\ &+ \sum_{a \in A_1} d(a, \varphi(a)) + \sum_{a \in A_2} d(a, \varphi(a)) \\ &\geq \sum_{b \in B_C - \bar{B}} b + \sum_{b \in B^*_{A_1}} b + \sum_{a \in A_1} d(a, \varphi(a)) + \sum_{a \in A_2} d(a, \varphi(a))\end{aligned}$$

$$\begin{aligned}\delta(B,C) &= \sum_{b \in B - B_C} b + \sum_{b \in B_C} d(b, \tilde{\varphi}^{-1}(b)) \\ &= \sum_{b \in B - B_C} b + \sum_{b \in B_{A_2}} d(b, \tilde{\varphi}^{-1}(b)) + \sum_{b \in B^*_{A_1}} d(b, \tilde{\varphi}^{-1}(b)) + \sum_{b \in B_C - \bar{B}} d(b, \tilde{\varphi}^{-1}(b)) \\ &\geq \sum_{a \in A_1} \varphi(a) + \sum_{b \in B_{A_2}} d(b, \tilde{\varphi}^{-1}(b)) + \sum_{b \in B^*_{A_1}} d(b, \tilde{\varphi}^{-1}(b)) + \sum_{b \in B_C - \bar{B}} d(b, \tilde{\varphi}^{-1}(b)) \\ &= \sum_{a \in A_1} \varphi(a) + \sum_{a \in A_2} d(\varphi(a), \rho(a)) + \sum_{b \in B^*_{A_1}} d(b, \tilde{\varphi}^{-1}(b)) + \sum_{b \in B_C - \bar{B}} d(b, \tilde{\varphi}^{-1}(b)).\end{aligned}$$

Henceforth, by Equations (3)–(5), it follows

$$\begin{aligned}\delta(A,B) + \delta(B,C) &\geq \sum_{b \in B_C - \bar{B}} [b + d(b, \tilde{\varphi}^{-1}(b))] \\ &+ \sum_{b \in B^*_{A_1}} [b + d(b, \tilde{\varphi}^{-1}(b))] + \sum_{a \in A_1} [d(a, \varphi(a)) + \varphi(a)] \\ &+ \sum_{a \in A_2} [d(a, \varphi(a)) + d(\varphi(a), \rho(a))] \\ &\geq \sum_{b \in B_C - \bar{B}} \tilde{\varphi}^{-1}(b) + \sum_{b \in B^*_{A_1}} \tilde{\varphi}^{-1}(b) + \sum_{a \in A_1} a + \sum_{a \in A_2} d(a, \rho(a))\end{aligned}$$

$$
\begin{aligned}
&= \sum_{b \in B_C - \bar{B}} \tilde{\varphi}^{-1}(b) + \sum_{a \in A_1} [(\rho(a) + a] + \sum_{a \in A_2} d(a, \rho(a)) \\
&\geq \sum_{b \in B_C - \bar{B}} \tilde{\varphi}^{-1}(b) + \sum_{a \in A_1} d(a, \rho(a)) + \sum_{a \in A_2} d(a, \rho(a)) \\
&= \sum_{c \in C - \rho(A)} c + \sum_{a \in A_1} d(a, \rho(a)) + \sum_{a \in A_2} d(a, \rho(a)) \\
&\geq \delta(A, C).
\end{aligned}
$$

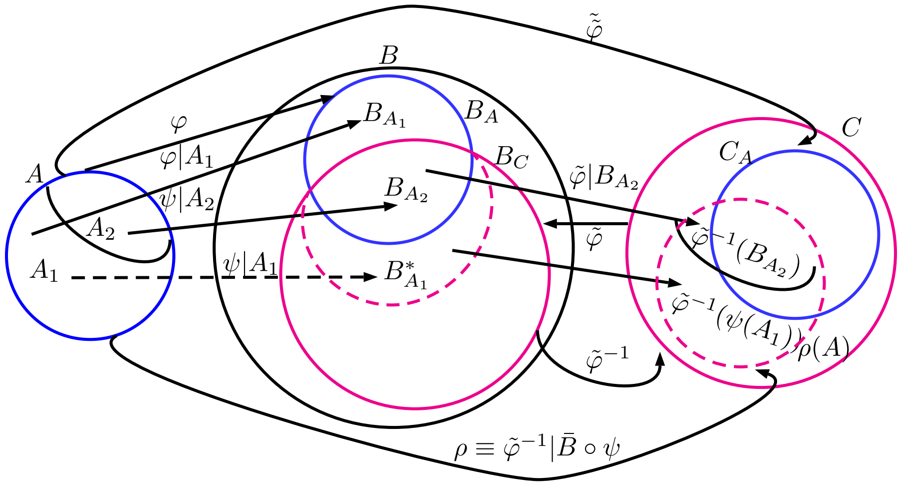

Figure 2. Triangle Inequality for $(1,3,1)$ case.

□

Theorem 1. *$(\mathbb{K}, \delta)$ is a metric space.*

Proof. By Lemmas 2–10 and Corollary 1, the result follows immediately. □

3. Applications and Computations

In this section, we give a group of numerical data and demonstrate how to compute their distances (or adjacency matrix) via the metric δ. In order to facilitate our computing, we show the following lemmas first. Let $a_1, a_2, b_1, b_2 \in \mathbb{R}$ be arbitrary.

3.1. Lemmas

Lemma 11. *If $a_1 \leq a_2$ and $b_1 \leq b_2$, then $|a_1 - b_1| + |a_2 - b_2| \leq |a_1 - b_2| + |a_2 - b_1|$.*

Proof. Suppose $a_2 = a_1 + \lambda_a$, suppose $b_2 = b_1 + \lambda_b$, where $\lambda_a, \lambda_b \geq 0$. Let $k = a_1 - b_1$. Then,

$$
\begin{aligned}
&|a_1 - b_2| + |a_2 - b_1| - |a_1 - b_1| - |a_2 - b_2| \\
&= |a_1 - b_1 - \lambda_b| + |a_1 - b_1 + \lambda_a| - |a_1 - b_1| - |a_1 - b_1 + \lambda_a - \lambda_b| \\
&= |k - \lambda_b| + |k + \lambda_a| - |k| - |k + \lambda_a - \lambda_b|.
\end{aligned}
$$

Furthermore, we consider the following cases:

1. $k = 0$: Then,

$$
\begin{aligned}
&|k - \lambda_b| + |k + \lambda_a| - |k| - |k + \lambda_a - \lambda_b| \\
&= \lambda_b + \lambda_a - |\lambda_a - \lambda_b| \geq 0;
\end{aligned}
$$

2. $k > 0$: Then,

$$|k-\lambda_b| + |k+\lambda_a| - |k| - |k+\lambda_a-\lambda_b|$$
$$= |k-\lambda_b| + k + \lambda_a - k - |k+\lambda_a-\lambda_b| \geq 0;$$

3. $k < 0$: Then,

$$|k-\lambda_b| + |k+\lambda_a| - |k| - |k+\lambda_a-\lambda_b|$$
$$= -k + \lambda_b + |k+\lambda_a| + k - |k+\lambda_a-\lambda_b| \geq 0.$$

Hence, we have shown $|a_1 - b_2| + |a_2 - b_1| - |a_1 - b_1| - |a_2 - b_2| \geq 0$.
□

Lemma 12. *Let $\rho \in BF_*(A,B)$ be arbitrary. Let $e_1, e_2 \in D_\rho$ be arbitrary such that $e_1 \leq e_2$. Then, $\exists \eta \in BF_*[A,B]$ such that $\eta(e) = \rho(e)$ for all $e \in D_\rho - \{e_1, e_2\}$ and $\eta(e_1) \leq \eta(e_2)$.*

Proof. If $\rho(e_1) \leq \rho(e_2)$, then one simply chooses η to be ρ. If $\rho(e_1) > \rho(e_2)$, then one could choose $\eta(e) = \rho(e)$ for all $e \in D_\rho - \{e_1, e_2\}$ and $\eta(e_1) := \rho(e_2)$ and $\eta(e_2) := \rho(e_1)$. Then, one has $\eta(e_1) < \eta(e_2)$. By Lemma 11, one has

$$|e_1 - \rho(e_2)| + |e_2 - \rho(e_1)| \leq |e_1 - \rho(e_1)| + |e_2 - \rho(e_2)|,$$

which together with $\rho \in BF_*[A,B]$ yields

$$|e_1 - \rho(e_2)| + |e_2 - \rho(e_1)| = |e_1 - \rho(e_1)| + |e_2 - \rho(e_2)|,$$

i.e.,

$$|e_1 - \eta(e_1)| + |e_2 - \eta(e_2)| = |e_1 - \rho(e_1)| + |e_2 - \rho(e_2)|,$$

i.e., $||\eta|| + ||\eta||^- = ||\rho|| + ||\rho||^-$, i.e., $\eta \in BF_*[A,B]$. □

Corollary 2. *If $\rho \in BF_*[A,B]$ and $D_\rho = \{e_1, e_2, ..., e_n\}$ with $e_1 \leq e_2 \leq ... \leq e_n$, then $\eta \in BF_*[A,B]$, where $D_\eta = D_\rho$ and $\eta(e_1) = min[\rho(D_\rho)], \eta(e_2) = min[\rho(D_\rho) - \{\eta(e_1)\}], ..., \eta(k+1) = min[\rho(D_\rho) - \{\eta(e_1), \eta(e_2), ..., \eta(e_k)\}]$ for all $k \leq n-1$.*

Proof. By applying Lemma 12 repeatedly, the result follows immediately. □

This corollary directly facilitates our computation in the next section. In addition, one could also simplify and redefine the metric δ by the result of this corollary.

3.2. Computation

In the following, we demonstrate the computation of our metric δ via a group of simulated data. Suppose

$$\mathbb{K} = \{A_1, A_2, A_3, A_4, A_5, A_6\} \subseteq \mathcal{MP}(\mathbb{R}^+)$$

is defined as follows:

- $A_1 = \{91.67, 2, 39.53, 98.34, 8.78\}$;
- $A_2 = \{1.99, 62, 7, 9.52, 9, 8.11\}$;
- $A_3 = \{2.1, 6.22, 27.1, 9.67, 9.19, 81.29, 5.55, 12.41, 1.67, 11.08, 51.15, 0.33\}$;
- $A_4 = \{22.21, 61.26, 71.12, 29.61, 29.19, 29.29, 35.3, 40\}$;
- $A_5 = \{17.19, 2, 70.56, 9.52, 9.45, 18.16, 40\}$;
- $A_6 = \{1.26, 0.19, 2, 4.70, 8.56, 9.09\}$.

Suppose the distance function over $\mathbb{R}^+$ is defined by

$$d(e, \varphi(e)) := |e - \varphi(e)|.$$

Then, our metric (defined in Equation (6)) derived from this d could be applied. We could then obtain the adjacency matrix of $\mathbb{K}$ via the following two methods:

1. (Method One) List all the permutations and find the optimal permutation and its associated distance, which is the summation of the matched and mismatched parts.
2. (Method Two) List all the combinations and find the optimal combination and its associated distance, which is the summation of the matched and mismatched parts.

Method One comes directly from the definition. By Corollary 2, Method Two is also justified. To demonstrate this, let us first compute $\delta(A_2, A_3)$. If Method One is applied, then one has to compute all the $P(12, 6) = 665{,}280$ permutations, and measure the matched distances between these permutations and A_2 and the mismatched distances between these permutations and A_3. If Method Two is applied, then one sorts the set A_2 first, and then sorts each of the $C(12, 6) = 924$ combinations to measure the matched part between each sorted combination and sorted A_2, and the mismatched part between the sorted combination and A_3. Both methods agree as follows:

$$\delta(A_2, A_3) = min\{||\varphi|| + ||\varphi||^- : \varphi \in BF(A_2, A_3)\} = 120.14,$$

in which the matched distance is $||\varphi|| = 69.6$ and the mismatched distance $||\varphi||^- = 50.54$, where the optimal φ is defined as follows: $\varphi(1.99) = 2.1, \varphi(62) = 81.29, \varphi(7) = 9.19, \varphi(9.52) = 51.15,$ $\varphi(9) = 12.41, \varphi(8.11) = 11.08$. Proceed similarly, the distances for other pairs (A_i, A_j) could also be obtained and the resulting adjacency matrix is demonstrated in Figure 3.

One could verify that this adjacency matrix satisfies all the metric axioms, in particular, $\delta(A_i, A_j) + \delta(A_J, A_k) \geq \delta(A_i, A_k)$ for all $i, j, k \in \{1, 2, 3, 4, 5, 6\}$.

0	156.6800	117.4400	192.9200	128.2000	214.9000
156.6800	0	120.1400	220.3600	69.2600	71.8200
117.4400	120.1400	0	139.8600	60.4400	191.9600
192.9200	220.3600	139.8600	0	151.1000	292.1800
128.2000	69.2600	60.4400	151.1000	0	141.0800
214.9000	71.8200	191.9600	292.1800	141.0800	0

Figure 3. Adjacency Matrix $[\delta(A_i, A_j)]_{i,j=1}^6$.

4. Real World Applications

In addition to some trivial applications, one could consider other handy applications, for example, by replacing the usual Euclidean metric with our metric in the following fields: $k-$means, clustering analysis, graph comparisons, etc. (see Reference [2]). These are frequently-used techniques in analyzing data or theoretical computations. The author has also succeeded in defining a novel metric for graphs based on the metric defined in this article. This enables one to measure the distances between any two graphical structures or networks. The idea for the derived metric is to measure the differences between any two graphs by induction on vertexes. Suppose there are two graphs G_1 and G_2 with the same set of vertices V. For each vertex $v \in V$, one could then generate two multisets whose elements are the lengths between v and its respective set of endpoints in G_1 and G_2. Then, he could compute the distances via the minimal sum of matched elements and mismatched elements as defined in this article. This approach yields a new metric for graphs.

4.1. Example

Let us consider a concrete example. Suppose the government in a country is trying to associate a village (among three candidate villages: VL1,VL2,VL3) which produces maize with the wholesalers which sell maize. In VL1, there are five farmers; in VL2, there are six farmers; in VL3, there are 10 farmers. The expected annual yields of maize for each farmer in VL1 are $3.2, 5.1, 7.6, 3.2, 8.8$ tons; the ones in VL2 are $1.2, 2.1, 3.6, 7.9, 12.1, 6.4$ tons; and the ones in VL3 are $2.6, 4.6, 8.1, 5.1, 2.2, 5, 7.9, 11.1, 12, 4.5$ tons. On the other hand, suppose there are four wholesalers whose annual demands are $7.9, 9.2, 11.6, 8.3$ tons, respectively. The government policy is to associate a village with the wholesalers based on the criterion that the total discrepancy between the village and the wholesalers must be minimal and the condition that each farmer could only exclusively sign the contract with exactly one wholesaler. Assume the government adopts the metric defined in this article. The results could be computed as follows Table 1.

Table 1. Analysis of Optimal Matchings

Villages	Expected Annual Yield (tons)	Matched	Mismatched	Total Discrepancy
VL1	$\{3.2, 5.1, 7.6, 3.2, 8.8\}$	12.3	3.2	15.5
VL2	$\{1.2, 2.1, 3.6, 7.9, 12.1, 6.4\}$	8.0	3.3	11.3
VL3	$\{2.6, 4.6, 8.1, 5.1, 2.2, 5, 7.9, 11.1, 12, 4.5\}$	2.5	24	26.5

Since the total discrepancy (i.e., matched part plus mismatched part) between VL2 and the wholesalers is minimal (or 11.3), the government should associate VL2 with the wholesalers. Henceforth, the government should pick VL2 to sign the contract with the four wholesalers exclusively. In doing so, the total dissatisfaction (or discrepancy) from both the farmer and the wholesalers would be minimal.

4.2. Characteristic and Analysis

The main characteristic of our metric is that it takes the minimal discrepancy into consideration. For the usual metrics, one hardly associates a metric with the minimal matching via combinations or permutations of all sorts of choices. Our method successfully combines the usual definition of a metric with the concept of an optimal choice. With these two concepts combined, one could pick up an optimal decision purely based on the metric defined in this article. This approach gives one a much more direct decision-making process. In addition, since this metric consists of two parts: the matched and mismatched parts, it would provide one with much more insightful knowledge of the discrepancy between mathematical objects.

5. Conclusions

We have defined a metric on a finite power multiset of positive real numbers via the concept of minimal matchings, in which the distances of any two multisets consist of two parts: the distance of the matched part and the distance of the mismatched part. We also implement this metric by an adjacency matrix. A concrete example is also included in this article. In addition to the adjacency matrix, we show another definitional computation to facilitate our computing of the metric. The metric defined in this article could be further applied in some real problems regarding artificial intelligence, clustering, or some other theoretical mathematical research.

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