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Fundamental Results for Pseudo-Differential Operators of Type $1, 1$

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Abstract: This paper develops some deeper consequences of an extended definition, proposed previously by the author, of pseudo-differential operators that are of type $1, 1$ in Hörmander's sense. Thus, it contributes to the long-standing problem of creating a systematic theory of such operators. It is shown that type $1, 1$-operators are defined and continuous on the full space of temperate distributions, if they fulfil Hörmander's twisted diagonal condition, or more generally if they belong to the self-adjoint subclass; and that they are always defined on the temperate smooth functions. As a main tool the paradifferential decomposition is derived for type $1, 1$-operators, and to confirm a natural hypothesis the symmetric term is shown to cause the domain restrictions; whereas the other terms are shown to define nice type $1, 1$-operators fulfilling the twisted diagonal condition. The decomposition is analysed in the type $1, 1$-context by combining the Spectral Support Rule and the factorisation inequality, which gives pointwise estimates of pseudo-differential operators in terms of maximal functions.

Keywords: pseudo-differential operator of type $1, 1$; twisted diagonal condition; paradifferential decomposition; Spectral Support Rule; factorisation inequality

MSC: 35S05; 47G30

1. Introduction

Pseudo-differential operators $a(x, D)$ of type $1, 1$ have long been known to have peculiar properties, almost since their invention by Hörmander [1]. This is due to initial investigations in 1972 in the thesis of Ching [2] and in lecture notes of Stein (made available in [3] (Chapter VII§1.3)); and again in 1978 by Parenti and Rodino [4].

The understanding of their unusual theory, and of the applications of these linear operators to non-linear problems in partial differential equations, grew crucially in the 1980's through works of Meyer [5,6], Bony [7], Bourdaud [8–11], Hörmander [12,13]. *cf.* also the expositions of Hörmander [14] (Chapter 9) and Taylor [15].

However, the first formal definition of general type $1, 1$-operators was put forward in 2008 by the author [16]. It would not be unjust to view this as an axiomatization of the type $1, 1$-theory, for whereas the previous contributions did not attempt to crystallise what a type $1, 1$-operator *is* or how it can be *characterised* in general, the definition from [16] has been a fruitful framework for raising questions and seeking answers about type $1, 1$-operators.

Indeed, being based on an operator theoretical approach, mimicking unbounded operators in Hilbert space, the definition gave from the outset a rigorous discussion of, e.g., unclosability, pseudo-locality, non-preservation of wavefront sets and the Spectral Support Rule [16]. This was followed up with a systematic L_p-theory of type $1, 1$-operators in [17], where a main theorem relied on a symbol analysis proved in full detail in the present paper.

Meanwhile, Métivier also treated type 1, 1-operators in 2008 in Chapter 4 of [18], but took recourse to the space dependent extensions of Stein [3] (Chapter VII§1.3). Type 1, 1-operators have also been investigated, or played a role, in works of, e.g., Torres [19], Marschall [20], Grafakos and Torres [21], Taylor [22], Hérau [23], Lannes [24], Johnsen [25], Hounie and dos Santos Kapp [26]; and for bilinear operators in Bernicot and Torres [27]. Implicitly type 1, 1-operators also enter many works treating partial differential equations with Bony's paradifferential calculus; but this would lead too far to recall here.

The present paper goes into a deeper, systematic study of type 1, 1-operators on $\mathcal{S}'(\mathbb{R}^n)$ and its subspaces. Indeed, the definition in [16] is shown here to give operators always defined on the maximal smooth subspace $C^\infty \cap \mathcal{S}'$, generalising results of Bourdaud [11] and David and Journé [28]—and shown to be defined on the entire $\mathcal{S}'$ if they belong to the self-adjoint subclass, by an extension of Hörmander's analysis of this class [12,13]. Moreover, the pointwise estimates in [29] are applied to the paradifferential decompositions, which are analysed in the type 1, 1-context here. The decomposition gives 3 other type 1, 1-operators, of which the so-called symmetric term is responsible for the possible domain restrictions, which occur when its infinite series diverges.

Altogether this should bring the theory of type 1, 1-operators to a rather more mature level.

1.1. Background

Recall that the symbol $a(x,\eta)$ of a type 1, 1-operator of order $d \in \mathbb{R}$ fulfils

$$|D_\eta^\alpha D_x^\beta a(x,\eta)| \le C_{\alpha,\beta}(1+|\eta|)^{d-|\alpha|+|\beta|} \quad \text{for} \quad x,\eta \in \mathbb{R}^n. \tag{1}$$

Classical pseudo-differential operators are, e.g., partial differential operators $\sum_{|\alpha|\le d} a_\alpha(x) D^\beta$, having such symbols simply with $d-|\alpha|$ as exponents. The presence of $|\beta|$ allows for a higher growth with respect to η, which has attracted attention for a number of reasons.

The operator corresponding to (1) is for Schwartz functions $u(x)$, *i.e.*, $u \in \mathcal{S}(\mathbb{R}^n)$,

$$a(x,D)u = (2\pi)^{-n} \int e^{\mathrm{i}\, x\cdot\eta} a(x,\eta)\hat{u}(\eta)\, d\eta. \tag{2}$$

But for $u \in \mathcal{S}' \setminus \mathcal{S}$ it requires another definition to settle whether u belongs to the domain of $a(x,D)$ or not. This is indeed a main subject of the present paper, which exploits the general definition of $a(x,D)$ presented in [16]; it is recalled in (10) below.

The non-triviality of the above task was discovered already by Ching [2], who showed unboundedness on L_2 for certain $a_\theta(x,D)$ with $d=0$; *cf.* Example 1 below. As the adjoint $a_\theta(x,D)^*$ of Ching's operator does not leave $\mathcal{S}$ invariant, as can be seen explicitly, e.g., from the proof of Lemma 3.1 in [16], the usual extension to $\mathcal{S}'$ by duality is not possible for $\mathrm{OP}(S^d_{1,1})$.

In general the pathologies of type 1, 1-operators are without doubt reflecting that, most interestingly, this operator class has important applications to non-linear problems:

This was first described around 1980 by Meyer [5,6], who discovered that a composition operator $u \mapsto F \circ u = F(u)$ with $F \in C^\infty$, $F(0) = 0$, can be decomposed in its action on functions $u \in \bigcup_{s>n/p} H^s_p(\mathbb{R}^n)$, by means of a specific u-dependent type 1, 1 symbol $a_u(x,\eta) \in S^0_{1,1}$, as

$$F(u(x)) = a_u(x,D)u(x). \tag{3}$$

He also showed that $a_u(x,D)$ *extends* to a bounded operator on H^t_r for $t>0$, so the fact that $u \mapsto F(u)$ sends H^s_p into itself can be seen from (3) by taking $t=s$ and $r=p$—indeed, this proof method is particularly elegant for non-integer $s > n/p$. It was carried over rigorously to the present type 1, 1-framework in [16] (Section 9), with continuity of $u \mapsto F \circ u$ as a corollary. Some applications of (3) were explained by Taylor [15] (Chapter 3).

Secondly, it was shown in [6] that type 1,1-operators play a main role in the paradifferential calculus of Bony [7] and the microlocal inversion of nonlinear partial differential equations of the form

$$G(x,(D_x^\alpha u(x))_{|\alpha|\le m})=0. \tag{4}$$

This was explicated by Hörmander, who devoted Chapter 10 of [14] to the subject. The resulting set-up was used, e.g., by Hérau [23] in a study of hypoellipticity of (4). Moreover, it was used for propagation of singularities in [14] (Chapter 11), with special emphasis on non-linear hyperbolic equations. Recently paradifferential operators, and thus type 1,1-operators, were also exploited for non-linear Schrödinger operators in constructions of solutions, parametrices and propagation of singularities in global wave front sets; *cf.* works of, e.g., Tataru [30], Delort [31], Nicola and Rodino [32].

Thirdly, both type 1,1-theory as such and Bony's paradifferential techniques played a crucial role in the author's work on semi-linear elliptic boundary problems [25].

Because of the relative novelty of this application, a sketch is given using a typical example. In a bounded C^∞-region $\Omega\subset\mathbb{R}^n$ with normal derivatives $\gamma_j u=(\vec{n}\cdot\nabla)^j u$ at the boundary $\partial\Omega$, and $\Delta:=\partial_{x_1}^2+\cdots+\partial_{x_n}^2$, let $u(x)$ solve the perturbed ℓ-harmonic Dirichlét problem

$$(-\Delta)^\ell u+u^2=f\quad\text{in }\Omega,\qquad \gamma_j u=\varphi_j\quad\text{on }\partial\Omega\text{ for }0\le j<\ell. \tag{5}$$

Without u^2, the linear problem has a well-known solution $u_0=R_\ell f+K_0\varphi_0+\cdots+K_{\ell-1}\varphi_{l-1}$, with operators belonging to the pseudo-differential boundary operator class of Boutet de Monvel [33]. For the non-linear problem in (5), the parametrix construction of [25] yields the solution formula

$$u=P_u^{(N)}(R_\ell f+K_0\varphi_0+\cdots+K_{\ell-1}\varphi_{\ell-1})+(R_\ell L_u)^N u, \tag{6}$$

where the parametrix $P_u^{(N)}$ is the linear map given by the finite Neumann series

$$P_u^{(N)}=I+R_\ell L_u+\cdots+(R_\ell L_u)^{N-1} \tag{7}$$

in terms of the *exact* paralinearisation L_u of u^2 with the sign convention $-L_u(u)=u^2$; *cf.* [25].

One merit of (6) is to show why u's regularity is *unchanged* by the non-linear term u^2: each parametrix $P_u^{(N)}$ is of order 0, hence does not change Sobolev regularity when applied to u_0; while in (6) the remainder $(R_\ell L_u)^N u$ will be in $C^k(\overline{\Omega})$ for every fixed k if N is taken large enough. Indeed, $R_\ell L_u$ has a fixed negative order if u is given with just the weak a priori regularity necessary to make sense of the boundary condition and make u^2 defined and a priori more regular than $(-\Delta)^\ell u$.

Type 1,1-operators are important for the fact that (6) easily implies that extra regularity properties of f in subregions $\Xi\Subset\Omega$ carry over to u; e.g., if $f|_\Xi$ is C^∞ so is $u|_\Xi$. Indeed, such implications boil down to the fact that the exact paralinearisation L_u *factors* through an operator A_u of type 1,1, that is, if r_Ω denotes restriction to Ω and ℓ_Ω is a linear extension to $\mathbb{R}^n\setminus\Omega$,

$$L_u=r_\Omega A_u\ell_\Omega,\qquad A_u\in\operatorname{OP}(S^\infty_{1,1}). \tag{8}$$

Now, by inserting (8) into (6)–(7) for a large N and using cut-off functions supported in Ξ in a well-known way, *cf.* [25] (Theorem 7.8), the regularity of u locally in Ξ is at once improved to the extent permitted by the data f by using the *pseudo-local* property of A_u:

$$\operatorname{sing\,supp} Au\subset\operatorname{sing\,supp} u\quad\text{for}\quad u\in D(A). \tag{9}$$

However, the pseudo-local property of general type 1,1-operators was only proved recently in [16], inspired by the application below (8). Yet, pseudo-locality was anticipated more than three decades ago by Parenti and Rodino [4], who gave an inspiring but incomplete indication, as they relied on the future to bring a specific meaning to $a(x,D)u$ for $u\in\mathcal{S}'\setminus C_0^\infty$.

A rigorous definition of general type $1,1$ operators was first given in [16]. In a way the definition abandons Fourier analysis (temporarily) and mimicks the theory of unbounded operators in Hilbert spaces. This is by viewing a type $1,1$-operator as a densely defined, unbounded operator $a(x,D)\colon \mathcal{S}' \to \mathcal{D}'$ between the two topological vector spaces $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$; thus the graph of $a(x,D)$ may be closed or unclosed in $\mathcal{S}' \times \mathcal{D}'$, *etc*. Indeed, it was proposed in [16] to stipulate that $u \in \mathcal{S}'$ belongs to the domain $D(a(x,D))$ of $a(x,D)$ and to set

$$a(x,D)u := \lim_{m\to\infty} (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\mathrm{i}\,x\cdot\eta} \psi(2^{-m}D_x) a(x,\eta) \psi(2^{-m}\eta) \hat{u}(\eta)\, d\eta \tag{10}$$

whenever this limit does exist in $\mathcal{D}'(\mathbb{R}^n)$ for every $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi = 1$ in a neighbourhood of the origin, and does not depend on such ψ.

In passing it is noted that, beyond the definition, operator theory is also felt in the rules of calculus, since as shown in Proposition 3 below the well-known commutator identity is replaced for type $1,1$-operators by an operator theoretical *inclusion*,

$$a(x,D)D_j + [D_j, a(x,D)] \subset D_j a(x,D). \tag{11}$$

The unconventional definition in (10), by *vanishing frequency modulation*, is a rewriting of the usual one, which is suitable for the present general symbols: clearly (10) gives back the integral in (2) if $u \in \mathcal{S}$. In case $a \in S^d_{1,0}$ this identification extends further to $u \in \mathcal{S}'$ by duality and the calculus of classical pseudo-differential operators. Note that the above integral should be interpreted as the operator $\operatorname{OP}(\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta))$ in $\operatorname{OP}(S^{-\infty})$ acting on u.

Clearly (10) is reminiscent of oscillatory integrals, now with the addition that $u \in D(a(x,D))$ when the regularisation yields a limit independent of the integration factor. Of course it is not a conventional integration factor that is used here, but rather the Fourier multiplier $\psi(2^{-m}D_x)$ that modifies the frequencies of $a(\cdot,\eta)$. While the necessity of this modification was amply elucidated in [16], it is moreover beneficial because the use of $\psi(2^{-m}D_x)$ gives easy access to Littlewood-Paley analysis of $a(x,D)$.

The definition (10) was investigated in [16] from several other perspectives, of which some will be needed below. But mentioned briefly (10) was proved to be maximal among the definitions of $A = a(x,D)$ that gives back the usual operators in $\operatorname{OP}(S^{-\infty})$ and is stable under the limit in (10); A is always defined on $\mathcal{F}^{-1}\mathcal{E}'$; it is pseudo-local but does change wavefront sets in certain cases (even if A is defined on $\bigcup H^s$); and A transports supports via the distribution kernel, *i.e.*, $\operatorname{supp} Au \subset \operatorname{supp} K \circ \operatorname{supp} u$ when $u \in D(A) \bigcap \mathcal{E}'$, with a similar *spectral* support rule for $\operatorname{supp} \hat{u}$; *cf.* (24) below and Appendix B.

For the Weyl calculus, Hörmander [12] noted that type $1,1$-operators do not fit well, as Ching's operator can have discontinuous Weyl-symbol. Conversely Boulkhemair [34,35] showed that the Weyl operator $\iint e^{\mathrm{i}(x-y)\cdot\eta} a(\frac{x+y}{2},\eta) u(y)\, dy\, d\eta/(2\pi)^n$ may give peculiar properties by insertion of $a(x,\eta)$ from $S^d_{1,1}$. E.g., already for Ching's symbol with $d = 0$, the real or imaginary part gives a Weyl operator that is unbounded on H^s for *every* $s \in \mathbb{R}$.

For more remarks on the subject's historic development the reader may refer to Section 2; consult the review in the introduction of [16].

1.2. Outline of Results

The purpose of this paper is to continue the foundational study in [16] and support the definition in (10) with further consequences.

First of all this concerns the hitherto untreated question: under which conditions is a given type $1,1$-operator $a(x,D)$ an everywhere defined and continuous map

$$a(x,D)\colon \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) \quad ? \tag{12}$$

For this it is shown in Proposition 13 and Theorem 23 below to be sufficient that $a(x,\eta)$ fulfils Hörmander's *twisted diagonal condition*, *i.e.*, the partially Fourier transformed symbol

$$\hat{a}(\xi,\eta) = \mathcal{F}_{x\to\xi}a(x,\eta) \tag{13}$$

should vanish in a conical neighbourhood of a non-compact part of the twisted diagonal $\mathcal{T}$ given by $\xi+\eta=0$ in $\mathbb{R}^n\times\mathbb{R}^n$. More precisely this means that for some $B\geq 1$

$$\hat{a}(\xi,\eta)\neq 0 \quad \text{only if} \quad |\xi+\eta|+1\geq |\eta|/B. \tag{14}$$

It should perhaps be noted that it is natural to consider $\hat{a}(\xi,\eta)$, as it is related (*cf.* [16] (Proposition 4.2)) both to the kernel K of $a(x,D)$ and to the kernel $\mathcal{K}$ of $\mathcal{F}^{-1}a(x,D)\mathcal{F}$,

$$(2\pi)^n\mathcal{K}(\xi,\eta) = \hat{a}(\xi-\eta,\eta) = \mathcal{F}_{(x,y)\to(\xi,\eta)}K(x,-y). \tag{15}$$

More generally the $\mathcal{S}'$-continuity (12) is obtained in Theorems 16 and 25 below for the $a(x,\eta)$ in $S^d_{1,1}$ that merely satisfy Hörmander's twisted diagonal condition of *order* σ for all $\sigma\in\mathbb{R}$. These are the symbols which for some $c_{\alpha,\sigma}$ and $0<\varepsilon<1$ fulfil

$$\sup_{x\in\mathbb{R}^n,\, R>0} R^{|\alpha|-d}\Big(\int_{R\leq|\eta|\leq 2R}|D^\alpha_\eta a_{\chi,\varepsilon}(x,\eta)|^2\,\frac{d\eta}{R^n}\Big)^{1/2}\leq c_{\alpha,\sigma}\varepsilon^{\sigma+n/2-|\alpha|}. \tag{16}$$

In this asymptotic formula $\hat{a}_{\chi,\varepsilon}$ denotes a specific localisation of $\hat{a}(x,\eta)$ to the conical neighbourhood $|\xi+\eta|+1\leq 2\varepsilon|\eta|$ of the twisted diagonal $\mathcal{T}$.

Details on the cut-off function χ in (16) are recalled in Section 2.3, in connection with an account of Hörmander's fundamental result that validity of (16) for all $\sigma\in\mathbb{R}$ is equivalent to extendability of $a(x,D)$ to a bounded map $H^{s+d}\to H^s$ for all $s\in\mathbb{R}$, as well as equivalent to the adjoint $a(x,D)^*$ being of type $1,1$.

Of course these results of Hörmander make it natural to expect that the above two conditions (namely (14) and (16) for all σ) are sufficient for the $\mathcal{S}'$-continuity in (12), but this has not been addressed explicitly in the literature before. As mentioned they are verified in Theorem 23, respectively in Theorem 16 by duality and in Theorem 25 by exploiting (16) directly.

In the realm of smooth functions the situation is fundamentally different. Here there is a commutative diagram for *every* type $1,1$-operator $a(x,D)$:

$$\begin{array}{ccccccc} \mathcal{S} & \longrightarrow & \mathcal{S}+\mathcal{F}^{-1}\mathcal{E}' & \longrightarrow & \mathcal{O}_M & \longrightarrow & C^\infty\bigcap\mathcal{S}' \\ {\scriptstyle a(x,D)}\big\downarrow & & {\scriptstyle a(x,D)}\big\downarrow & & \big\downarrow{\scriptstyle a(x,D)} & & \big\downarrow{\scriptstyle a(x,D)} \\ \mathcal{S} & \longrightarrow & \mathcal{O}_M & \longrightarrow & \mathcal{O}_M & \longrightarrow & C^\infty \end{array} \tag{17}$$

The first column is just the integral (2); the second an extension from [16,36]. Column three is an improvement given below of the early contribution of Bourdaud [11] that $a(x,D)$ extends to a map $\mathcal{O}_M\to\mathcal{D}'$, whereby $\mathcal{O}_M$ denotes Schwartz' space of slowly increasing smooth functions.

However, the fourth column restates the full result that a type $1,1$-operator is always defined on the *maximal* space of smooth functions $C^\infty\bigcap\mathcal{S}'$. More precisely, according to Theorem 4 below, it restricts to a strongly continuous map

$$a(x,D)\colon C^\infty(\mathbb{R}^n)\bigcap\mathcal{S}'(\mathbb{R}^n)\to C^\infty(\mathbb{R}^n). \tag{18}$$

It is noteworthy that this holds without any of the conditions (14) and (16). Another point is that, since $C^\infty\not\subset\mathcal{S}'$, it was necessary to ask for a limit in the topology of $\mathcal{D}'$ in (10).

Perhaps it could seem surprising that the described results on (12) and (18) have not been established in their full generality before. However, it should be emphasised that these properties are valid for the operator defined in (10), so they go much beyond the mere extendability discussed by Meyer [6], Bourdaud [10], Hörmander [12–14], Torres [19], Stein [3].

The definition in (10) is also useful because it easily adapts to Littlewood-Paley analysis of type 1,1-operators. Here the systematic point of departure is the well-known paradifferential splitting based on dyadic coronas (*cf.* Section 5 for details), as used by, e.g., Bony [7], Yamazaki [37], Marschall [20]:

$$a(x,D) = a_\psi^{(1)}(x,D) + a_\psi^{(2)}(x,D) + a_\psi^{(3)}(x,D). \tag{19}$$

Since the 1980's splittings like (19) have been used in microlocal analysis of (4) as well as in numerous proofs of continuity of $a(x,D)$ in Sobolev spaces H_p^s and Hölder-Zygmund spaces C_*^s, or the more general Besov and Lizorkin-Triebel scales $B_{p,q}^s$ and $F_{p,q}^s$. For type 1,1-operators (19) was used by Bourdaud [8–10], Marschall [20], Runst [38], and the author in [16,17], and in [36,39] where the Lizorkin-Triebel spaces $F_{p,1}^s$ were shown to be optimal substitutes for the Sobolev spaces H_p^s at the borderline $s = d$ for the domains of operators in $\mathrm{OP}(S_{1,1}^d)$.

It is known that the decomposition (19) follows from the bilinear way ψ enters (10), and that one finds at once the three infinite series in (120)–(122) below, which define the $a_\psi^{(j)}(x,D)$. But it is a main point of Sections 5 and 6 to verify that *each* of these series gives an operator $a_\psi^{(j)}(x,D)$ also belonging to $\mathrm{OP}(S_{1,1}^d)$; which is non-trivial because of the modulation function ψ in (10).

As general properties of the type 1,1-operators $a_\psi^{(1)}(x,D)$ and $a_\psi^{(3)}(x,D)$, they are shown here to satisfy the twisted diagonal condition (14), so (19) can be seen as a main source of such operators. Consequently these terms are harmless as they are defined on $\mathcal{S}'$ because of (12) ff.

Therefore, it is the so-called symmetric term $a_\psi^{(2)}(x,D)$ which may cause $a(x,D)u$ to be undefined, as was previously known, e.g., for functions u in a Sobolev space; *cf.* [36]. This delicate situation is clarified in Theorem 24 with a natural identification of type 1,1-domains, namely

$$D(a(x,D)) = D(a_\psi^{(2)}(x,D)). \tag{20}$$

This might seem obvious at first glance, but really is without meaning before the $a_\psi^{(2)}$-series has been shown to define a type 1,1-operator. Hence (20) is a corollary to the cumbersome book-keeping needed for this identification of $a_\psi^{(2)}(x,D)$. In fact, the real meaning of (20) is that both domains consist of the $u \in \mathcal{S}'$ for which the $a_\psi^{(2)}$-series converges; *cf.* Theorem 24.

In comparison, convergence of the series for $a_\psi^{(1)}(x,D)u$ and $a_\psi^{(3)}(x,D)u$ is in Theorem 22 verified explicitly for all $u \in \mathcal{S}'$, $a \in S_{1,1}^\infty$, and these operators are proved to be of type 1,1. Thus (19) is an identity among type 1,1-operators. It was exploited for estimates of arbitrary $a \in S_{1,1}^d$ in, e.g., Sobolev spaces H_p^s and Hölder–Zygmund spaces C_*^s in [17], by giving full proofs (*i.e.*, the first based on (10)) of the boundedness for all $s > 0$, $1 < p < \infty$,

$$a(x,D)\colon H_p^{s+d}(\mathbb{R}^n) \to H_p^s(\mathbb{R}^n), \qquad a(x,D)\colon C_*^{s+d}(\mathbb{R}^n) \to C_*^s(\mathbb{R}^n). \tag{21}$$

This was generalised in [17] to all $s \in \mathbb{R}$ when a fulfills the twisted diagonal condition of order σ in (16) for all $\sigma \in \mathbb{R}$. This sufficient condition extends results for $p = 2$ of Hörmander [12,13] to $1 < p < \infty$, $s \in \mathbb{R}$. The special case $s = 0 = d$ was considered recently in [26].

The present results on $a_\psi^{(j)}(x,D)$ are of course natural, but they do rely on two techniques introduced rather recently in works of the author. One ingredient is a *pointwise* estimate

$$|a(x,D)u(x)| \le c\, u^*(x), \qquad x \in \mathbb{R}^n, \tag{22}$$

cf. Section 3 and [29], in terms of the Peetre-Fefferman-Stein maximal function

$$u^*(x) = \sup_{y \in \mathbb{R}^n} \frac{|u(x-y)|}{(1+R|y|)^N}, \quad \text{when} \quad \operatorname{supp} \hat{u} \subset \overline{B}(0,R). \tag{23}$$

Although $u \mapsto u^*$ is non-linear, it is useful for convergence of series: e.g., in H^s_p since it is L_p-bounded, and as shown here also in $\mathcal{S}'$ since it has polynomial bounds $u^*(x) \le c(1+R|x|)^N$.

The second ingredient is the *Spectral Support Rule* from [16]; *cf.* also [36,39]. It provides control of $\operatorname{supp} \mathcal{F}(a(x,D)u)$ in terms of the supports of $\hat{u}$ and $\mathcal{K}(\xi,\eta)$ in (15),

$$\operatorname{supp} \mathcal{F}(a(x,D)u) \subset \overline{\operatorname{supp} \mathcal{K} \circ \operatorname{supp} \mathcal{F}u} = \{\, \xi + \eta \mid (\xi,\eta) \in \operatorname{supp} \hat{a},\ \eta \in \operatorname{supp} \hat{u} \,\}^{-}. \tag{24}$$

The simple case in which $u \in \mathcal{S}$ was covered by Metivier [18] (Proposition 4.2.8). A review of (24) is given in Appendix B, including an equally easy proof for arbitrary $\hat{u} \in \mathcal{E}'$ and $a \in S^d_{1,1}$.

A main purpose of (24) is to avoid a cumbersome approximation by elementary symbols. These were introduced by Coifman and Meyer [40] to reduce the task of bounding the support of $\mathcal{F}(a(x,D)u)$: indeed, elementary symbols have the form $a(x,\eta) = \sum m_j(x)\Phi_j(\eta)$ for multipliers $m_j \in L_\infty$ and a Littlewood-Paley partition $1 = \sum \Phi_j$, so clearly $(2\pi)^n \mathcal{F}a(x,D)u = \sum \hat{m}_j * (\Phi_j \hat{u})$ is a finite sum when $\hat{u} \in \mathcal{E}'$; whence the rule for convolutions yields (24) for such symbols.

However, approximation by elementary symbols is not just technically redundant because of (24), it would also be particularly cumbersome to use for a type 1,1-symbol, as (10) would then have to be replaced by a double-limit procedure. Moreover, in the proof of (19), as well as in the L_p-theory based on it in [17], (24) also yields a significant simplification.

Remark 1. The Spectral Support Rule (24) shows clearly that Hörmander's twisted diagonal condition (14) ensures that $a(x,D)$ cannot change (large) frequencies in $\operatorname{supp} \hat{u}$ to 0. In fact, the support condition in (14) implies that ξ cannot be close to $-\eta$ when $(\xi,\eta) \in \operatorname{supp} \hat{a}$, which by (24) means that $\eta \in \operatorname{supp} \hat{u}$ will be changed by $a(x,D)$ to the frequency $\xi + \eta \ne 0$.

1.3. Contents

Notation is settled in Section 2 along with basics on operators of type 1,1 and the C^∞-results in (17) ff. In Section 3 some pointwise estimates are recalled from [29] and then extended to a version for frequency modulated operators. Section 4 gives a precise analysis of the self-adjoint part of $S^d_{1,1}$, relying on the results and methods from Hörmander's lecture notes [14] (Chapter 9); with consequences derived from the present operator definition. Littlewood-Paley analysis of type 1,1-operators is developed in Section 5. In Section 6 the operators resulting from the paradifferential splitting (19) is further analysed, especially concerning their continuity on $\mathcal{S}'(\mathbb{R}^n)$ and the domain relation (20). Section 7 contains a few final remarks.

2. Preliminaries on Type 1, 1-Operators

Notation and notions from Schwartz' distribution theory, such as the spaces C_0^∞, $\mathcal{S}$, C^∞ of smooth functions and their duals $\mathcal{D}'$, $\mathcal{S}'$, $\mathcal{E}'$ of distributions, and the Fourier transformation $\mathcal{F}$, will be as in Hörmander's book [41] with these exceptions: $\langle\, u,\ \varphi \,\rangle$ denotes the value of a distribution u on a test function φ. The Sobolev space of order $s \in \mathbb{R}$ based on L_p is written H^s_p, and $H^s = H^s_2$. The space $\mathcal{O}_M(\mathbb{R}^n)$ consists of the slowly increasing $f \in C^\infty(\mathbb{R}^n)$, *i.e.*, the f that for each multiindex α and some $N > 0$ fulfils $|D^\alpha f(x)| \le c(1+|x|)^N$.

As usual $t_+ = \max(0,t)$ is the positive part of $t \in \mathbb{R}$ whilst $[t]$ denotes the greatest integer $\le t$. In general, c will denote positive constants, specific to the place of occurrence.

2.1. The General Definition of Type 1, 1-Operators

For type 1, 1-operators the reader may consult [16] for an overview of previous results. The present paper is partly a continuation of [16,36,39], but it suffices to recall just a few facts.

By standard quantization, each operator $a(x,D)$ is defined on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ by

$$a(x,D)u = \mathrm{OP}(a)u(x) = (2\pi)^{-n} \int e^{\mathrm{i}\,x\cdot\eta} a(x,\eta)\mathcal{F}u(\eta)\,d\eta, \qquad u \in \mathcal{S}(\mathbb{R}^n). \tag{25}$$

Hereby its symbol $a(x,\eta)$ is required to be in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, of order $d \in \mathbb{R}$ and type 1, 1, which means that for all multiindices α, $\beta \in \mathbb{N}_0^n$ it fulfils (1), or more precisely has finite seminorms:

$$p_{\alpha,\beta}(a) := \sup_{x,\eta\in\mathbb{R}^n} (1+|\eta|)^{-(d-|\alpha|+|\beta|)} |D_\eta^\alpha D_x^\beta a(x,\eta)| < \infty. \tag{26}$$

The Fréchet space of such symbols is denoted by $S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$, or just $S^d_{1,1}$ for brevity, while as usual $S^{-\infty} = \bigcap_d S^d_{1,1}$. Basic estimates yield that the bilinear map $(a,u) \mapsto a(x,D)u$ is continuous

$$S^d_{1,1} \times \mathcal{S} \to \mathcal{S}. \tag{27}$$

The distribution kernel $K(x,y) = \mathcal{F}^{-1}_{\eta\to z} a(x,\eta)\big|_{z=x-y}$ is well known to be C^∞ for $x \neq y$ also in the type 1, 1 context; *cf.* [16] (Lemma 4.3). It fulfils $\langle\, a(x,D)u,\ \varphi\,\rangle = \langle\, K,\ \varphi \otimes u\,\rangle$ for all u, $\varphi \in \mathcal{S}$.

For arbitrary $u \in \mathcal{S}' \setminus \mathcal{S}$ it is a delicate question whether or not $a(x,D)u$ is defined. The general definition of type 1, 1-operators in [16] uses a symbol modification, exploited throughout below, namely $b(x,\eta) = \psi(2^{-m}D_x)a(x,\eta)$, or more precisely

$$\hat{b}(\xi,\eta) := \mathcal{F}_{x\to\xi} b(x,\eta) = \psi(2^{-m}\xi)\hat{a}(\xi,\eta). \tag{28}$$

Definition 1. If a symbol $a(x,\eta)$ is in $S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$ while $\psi \in C_0^\infty(\mathbb{R}^n)$ is an arbitrary cut-off function equal to 1 in a neighbourhood of the origin, let

$$a_\psi(x,D)u := \lim_{m\to\infty} \mathrm{OP}(\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta))u. \tag{29}$$

When for each such ψ the limit $a_\psi(x,D)u$ exists in $\mathcal{D}'(\mathbb{R}^n)$ and moreover is independent of ψ, then u belongs to the domain $D(a(x,D))$ by definition and

$$a(x,D)u = a_\psi(x,D)u. \tag{30}$$

This way $a(x,D)$ is a linear map $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ with dense domain, as by (27) it contains $\mathcal{S}(\mathbb{R}^n)$. (Use of $D(\cdot)$ for the domain should not be confounded with derivatives, such as D^α or $a(x,D)$.)

This was called definition by *vanishing frequency modulation* in [16], because the removal of high frequencies in x and η achieved by $\psi(2^{-m}D_x)$ and $\psi(2^{-m}\eta)$ disappears for $m \to \infty$. Note that the action on u is well defined for each m in (29) as the modified symbol is in $S^{-\infty}$. Occasionally the function ψ will be referred to as a *modulation function.*

The frequency modulated operator $\mathrm{OP}(\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta))$ has, by the comparison made in [16] (Proposition 5.11), its kernel $K_m(x,y)$ conveniently given as a convolution, up to conjugation by the involution $M\colon (x,y) \mapsto (x, x-y)$,

$$K_m(x,y) = 4^{mn}(\mathcal{F}^{-1}\psi(2^m\cdot) \otimes \mathcal{F}^{-1}\psi(2^m\cdot)) * (K \circ M)(x, x-y). \tag{31}$$

Remark 2. It is used below that when $\varphi, \chi \in C_0^\infty(\mathbb{R}^n)$ are such that $\chi \equiv 1$ on a neighbourhood of $\operatorname{supp}\varphi$, then since $\operatorname{supp}\varphi \otimes (1-\chi)$ is disjoint from the diagonal and bounded in the x-direction, there is convergence in the topology of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$:

$$\varphi(x)(1-\chi(y))K_m(x,y) \xrightarrow[m\to\infty]{} \varphi(x)(1-\chi(y))K(x,y). \tag{32}$$

However, this requires verification because the commutator of the convolution (31) and pointwise multiplication by $\varphi \otimes (1-\chi)$ is a nontrivial pseudo-differential, hence non-local operator. A proof of (32) based on the Regular Convergence Lemma can be found in [16] (Proposition 6.3).

In general the calculus of type 1,1-operators is delicate, *cf.* [12–14], but the following result from [17] is just an exercise (*cf.* the proof there). It is restated here for convenience.

Proposition 2. *When $a(x,\eta)$ is in $S^{d_1}_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and a symbol with constant coefficients $b(\eta)$ belongs to $S^{d_2}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, then $c(x,\eta) := a(x,\eta)b(\eta)$ is in $S^{d_1+d_2}_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and*

$$c(x,D)u = a(x,D)b(D)u. \tag{33}$$

In particular $D(c(x,D)) = D(a(x,D)b(D))$, so the two sides are simultaneously defined.

This result applies especially to differential operators, say $b(D) = D_j$ for simplicity. But as a minor novelty, the classical commutator identity needs an atypical substitute:

Proposition 3. *For $a \in S^d_{1,1}$ the commutator*

$$[D_j, a(x,D)] = D_j a(x,D) - a(x,D)D_j \tag{34}$$

equals $\operatorname{OP}(D_{x_j}a(x,\eta))$ on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, whilst in $\mathcal{S}'(\mathbb{R}^n)$ there is an inclusion in the operator theoretical sense,

$$a(x,D)D_j + [D_j, a(x,D)] \subset D_j a(x,D). \tag{35}$$

The commutator symbol $D_{x_j}a(x,\eta)$ is in $S^{d+|\beta|}_{1,1}$.

Proof. By classical calculations, any modulation function ψ gives the following formula for $u \in \mathcal{S}$, hence for all $u \in \mathcal{S}'$ as the symbols are in $S^{-\infty}$,

$$\begin{aligned} &\operatorname{OP}(\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta)\eta_j)u + \operatorname{OP}(\psi(2^{-m}D_x)D_{x_j}a(x,\eta)\psi(2^{-m}\eta))u \\ &\qquad = D_j \operatorname{OP}(\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta))u. \end{aligned} \tag{36}$$

When both terms on the left have ψ-independent limits for $m \to \infty$, so has the right-hand side. As the first term then is $a(x,D)D_ju$, *cf.* Proposition 2, this entails that the common domain $D(a(x,D)D_j) \bigcap D([D_j, a(x,D)])$ is contained in that of $D_ja(x,D)$, with the same actions. $\square$

The inclusion (35) is strict in some cases, for the domains are not always invariant under differentiation. This is a well-known consequence of the classical counterexamples, which are recalled below for the reader's convenience:

Example 1. The classical example of a symbol of type 1,1 results from an auxiliary function $A \in C_0^\infty(\mathbb{R}^n)$, say with $\operatorname{supp} A \subset \{\, \eta \mid \frac{3}{4} \le |\eta| \le \frac{5}{4} \,\}$, and a fixed vector $\theta \in \mathbb{R}^n$,

$$a_\theta(x,\eta) = \sum_{j=0}^{\infty} 2^{jd} e^{-\mathrm{i}\, 2^j x\cdot\theta} A(2^{-j}\eta). \tag{37}$$

Here $a_\theta \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, since the terms are disjointly supported, and it clearly belongs to $S^d_{1,1}$.

These symbols were used both by Ching [2] and Bourdaud [10] to show L_2-unboundedness for $d = 0$, $|\theta| = 1$. Refining this, Hörmander [12] established that continuity $H^s \to \mathcal{D}'$ with $s > -r$ holds if and only if θ is a zero of A of order $r \in \mathbb{N}_0$. [16] gave an extension to $d \in \mathbb{R}$, $\theta \neq 0$.

The non-preservation of wavefront sets discovered by Parenti and Rodino [4] also relied on $a_\theta(x, \eta)$. Their ideas were extended to all $n \geq 1$, $d \in \mathbb{R}$ in [16] (Section 3.2) and refined by applying $a_{2\theta}(x, D)$ to a product $v(x) f(x \cdot \theta)$, where $v \in \mathcal{F}^{-1}C_0^\infty$ is an analytic function controlling the spectrum, whilst the highly oscillating f is Weierstrass' nowhere differentiable function for orders $d \in]0, 1]$, in a complex version with its wavefront set along a half-line. (Nowhere differentiability was shown with a small microlocalisation argument, explored in [42].)

Moreover, $a_\theta(x, D)$ is unclosable in $\mathcal{S}'$ when A is supported in a small ball around θ, as shown in [16] (Lemma 3.2). Hence Definition 1 cannot in general be simplified to a closure of the graph in $\mathcal{S}' \times \mathcal{D}'$.

As a basic result, it was shown in [16] (Section 4) that the C^∞-subspace $\mathcal{S}(\mathbb{R}^n) + \mathcal{F}^{-1}\mathcal{E}'(\mathbb{R}^n)$ always is contained in the domain of $a(x, D)$ and that

$$a(x, D) \colon \mathcal{S}(\mathbb{R}^n) + \mathcal{F}^{-1}\mathcal{E}'(\mathbb{R}^n) \to \mathcal{O}_M(\mathbb{R}^n). \tag{38}$$

In fact, if $u = v + v'$ is any splitting with $v \in \mathcal{S}$ and $v' \in \mathcal{F}^{-1}\mathcal{E}'$, then

$$a(x, D)u = a(x, D)v + \mathrm{OP}(a(1 \otimes \chi))v', \tag{39}$$

whereby $a(1 \otimes \chi)(x, \eta) = a(x, \eta)\chi(\eta)$ and $\chi \in C_0^\infty(\mathbb{R}^n)$ is arbitrarily chosen so that $\chi = 1$ holds in a neighbourhood of $\operatorname{supp} \mathcal{F}v' \Subset \mathbb{R}^n$. Here $a(x, \eta)\chi(\eta)$ is in $S^{-\infty}$ so that $\mathrm{OP}(a(1 \otimes \chi))$ is defined on $\mathcal{S}'$. Hence $a(x, D)(\mathcal{F}^{-1}\mathcal{E}') \subset \mathcal{O}_M(\mathbb{R}^n)$.

It is a virtue of (38) that $a(x, D)$ is compatible with for example $\mathrm{OP}(S^\infty_{1,0})$; *cf.* [16] for other compatibility questions. Therefore, some well-known facts extend to type $1, 1$-operators:

Example 2. Each $a(x, D)$ of type $1, 1$ is defined on all polynomials and

$$a(x, D)(x^\alpha) = D^\alpha_\eta(e^{\mathrm{i}\, x \cdot \eta} a(x, \eta)) \big|_{\eta = 0}. \tag{40}$$

In fact, $f(x) = x^\alpha$ has $\hat{f}(\eta) = (2\pi)^n(-D_\eta)^\alpha \delta_0(\eta)$ with support $\{0\}$, so it is seen for $v = 0$ in (39) that $a(x, D)f(x) = \langle \hat{f}, (2\pi)^{-n} e^{\mathrm{i}\langle x, \cdot \rangle} a(x, \cdot)\chi(\cdot) \rangle$ where $\chi = 1$ around 0; thence (40).

Example 3. Also when $A = a(x, D)$ is of type $1, 1$, one can recover its symbol from the formula

$$a(x, \xi) = e^{-\mathrm{i}\, x \cdot \xi} A(e^{\mathrm{i}\, x \cdot \xi}). \tag{41}$$

Here $\mathcal{F}e^{\mathrm{i}\langle \cdot, \xi \rangle} = (2\pi)^n \delta_\xi(\eta)$ has compact support, so again it follows from (38) that (via a suppressed cut-off) one has $A(e^{\mathrm{i}\langle \cdot, \xi \rangle}) = \langle \delta_\xi, e^{\mathrm{i}\langle x, \cdot \rangle} a(x, \cdot) \rangle = e^{\mathrm{i}\, x \cdot \xi} a(x, \xi)$.

2.2. General Smooth Functions

To go beyond the smooth functions in (38), it is shown in this subsection how one can extend a remark by Bourdaud [11] on singular integral operators, which shows that every type $1, 1$ symbol $a(x, \eta)$ of order $d = 0$ induces a map $\tilde{A} \colon \mathcal{O}_M \to \mathcal{D}'$.

Indeed, Bourdaud defined $\tilde{A}f$ for $f \in \mathcal{O}_M(\mathbb{R}^n)$ as the distribution that on $\varphi \in C_0^\infty(\mathbb{R}^n)$ is given by the following, using the distribution kernel K of $a(x,D)$ and an auxiliary function $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on a neighbourhood of supp φ,

$$\langle \tilde{A}f, \varphi \rangle = \langle a(x,D)(\chi f), \varphi \rangle + \iint K(x,y)(1-\chi(y))f(y)\varphi(x)\,dy\,dx. \tag{42}$$

However, to free the discussion from the slow growth in $\mathcal{O}_M$, one may restate this in terms of the tensor product $1 \otimes f$ in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ acting on $(\varphi \otimes (1-\chi))K \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, *i.e.*,

$$\langle \tilde{A}f, \varphi \rangle = \langle a(x,D)(\chi f), \varphi \rangle + \langle 1 \otimes f, (\varphi \otimes (1-\chi))K \rangle, \tag{43}$$

One advantage here is that both terms obviously make sense as long as f is smooth and temperate, *i.e.*, for every $f \in C^\infty(\mathbb{R}^n) \bigcap \mathcal{S}'(\mathbb{R}^n)$.

Moreover, for φ with support in the interior $\mathcal{C}^\circ$ of a compact set $\mathcal{C} \subset \mathbb{R}^n$ and $\chi = 1$ on a neighbourhood of $\mathcal{C}$, the right-hand side of (43) gives the same value for any $\tilde{\chi} \in C_0^\infty$ equal to 1 around $\mathcal{C}$, for in the difference of the right-hand sides equals 0 since $\langle a(x,D)((\chi - \tilde{\chi})f), \varphi \rangle$ is seen from the kernel relation to equal $-\langle 1 \otimes f, (\varphi(\tilde{\chi} - \chi))K \rangle$.

Crude estimates of (43) now show that $\tilde{A}f$ yields a distribution in $\mathcal{D}'(\mathcal{C}^\circ)$, and the above χ-independence implies that it coincides in $\mathcal{D}'(\mathcal{C}^\circ \bigcap \mathcal{C}_1^\circ)$ with the distribution defined from another compact set $\mathcal{C}_1$. Since $\mathbb{R}^n = \bigcup \mathcal{C}^\circ$, the *recollement de morceaux* theorem yields that a distribution $\tilde{A}f \in \mathcal{D}'(\mathbb{R}^n)$ is defined by (43).

There is also a more explicit formula for $\tilde{A}f$: when $\tilde{\varphi} \in C_0^\infty$ is chosen so that $\tilde{\varphi} \equiv 1$ around $\mathcal{C}$ while supp $\tilde{\varphi}$ has a neighbourhood where $\chi = 1$, then $\varphi = \tilde{\varphi}\varphi$ in (43) gives, for $x \in \mathcal{C}^\circ$,

$$\tilde{A}f(x) = a(x,D)(\chi f)(x) + \langle f, (\tilde{\varphi}(x)(1-\chi(\cdot)))K(x,\cdot) \rangle. \tag{44}$$

Now $\tilde{A}f \in C^\infty$ follows, for the first term is in $\mathcal{S}$, and the second coincides in $\mathcal{C}^\circ$ with a function in $\mathcal{S}$, as a corollary to the construction of $g \otimes f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ for $f, g \in \mathcal{S}'(\mathbb{R}^n)$.

Post festum, it is seen in (43) that when $f \to 0$ in both C^∞ and in $\mathcal{S}'$, then $\chi f \to 0$ in $\mathcal{S}$ while $1 \otimes f \to 0$ in $\mathcal{S}'$. Therefore, $\tilde{A}f \to 0$ in $\mathcal{D}'$, which is a basic continuity property of $\tilde{A}$.

By setting $\tilde{A}$ in relation to Definition 1, the above gives the new result that $a(x,D)$ always is a map defined on the *maximal* set of smooth functions, *i.e.*, on $C^\infty \bigcap \mathcal{S}'$:

Theorem 4. *Every $a(x,D) \in \mathrm{OP}(S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n))$ restricts to a map*

$$a(x,D)\colon C^\infty(\mathbb{R}^n) \bigcap \mathcal{S}'(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n), \tag{45}$$

which locally is given by formula (44). The map (45) is continuous when $C^\infty(\mathbb{R}^n)$ has the usual Fréchet space structure and $\mathcal{S}'(\mathbb{R}^n)$ has the strong dual topology.

The intersection $C^\infty \bigcap \mathcal{S}'$ is topologised by enlarging the set of seminorms on C^∞ by those on $\mathcal{S}'$. Here the latter have the form $f \mapsto \sup_{\psi \in \mathcal{B}} |\langle f, \psi \rangle|$ for an arbitrary bounded set $\mathcal{B} \subset \mathcal{S}$.

Proof. Let $A_m = \mathrm{OP}(\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta))$ so that $a(x,D)u = \lim_m A_m u$ when u belongs to $D(a(x,D))$. With $f \in C^\infty \bigcap \mathcal{S}'$ and φ, χ as above, this is the case for $u = \chi f \in C_0^\infty$.

Exploiting the convergence in Remark 2 in (43), it is seen that

$$\langle \tilde{A}f, \varphi \rangle = \lim_m \langle A_m(\chi f), \varphi \rangle + \lim_m \iint K_m(x,y)(1-\chi(y))f(y)\varphi(x)\,dy\,dx. \tag{46}$$

Here the integral equals $\langle A_m(f-\chi f), \varphi \rangle$ by the kernel relation, for $A_m \in \mathrm{OP}(S^{-\infty})$ and f may as an element of $\mathcal{S}'$ be approached from C_0^∞. So (46) yields

$$\langle \tilde{A}f, \varphi \rangle = \lim_m \langle A_m(\chi f), \varphi \rangle + \lim_m \langle A_m(f-\chi f), \varphi \rangle = \lim_m \langle A_m f, \varphi \rangle. \tag{47}$$

Thus $A_m f \to \tilde{A}f$, which by (43) is independent of ψ. Hence $\tilde{A} \subset a(x,D)$ as desired.

That $a(x,D)(C^\infty \cap \mathcal{S}')$ is contained in C^∞ now follows from the remarks to (44).

When $f \to 0$ in C^∞, then clearly $D^\alpha a(x,D)(f\chi) \to 0$ in $\mathcal{S}$, hence uniformly on $\mathbb{R}^n$. It is also straightforward to see that $(\tilde{\varphi}(x)(1-\chi(\cdot)))K(x,\cdot)$ stays in a bounded set in $\mathcal{S}(\mathbb{R}^n)$ as x runs through $\mathcal{C}$. Therefore, when $f \to 0$ also in the *strong* dual topology on $\mathcal{S}'$, then the second term in (44) tends to 0 uniformly with respect to $x \in \mathcal{C}$. As x-derivatives may fall on $K(x,\cdot)$, the same argument gives that $\sup_{\mathcal{C}} |D^\alpha \tilde{A}f| \to 0$. Hence $f \to 0$ in $C^\infty \bigcap \mathcal{S}'$ implies $D^\alpha \tilde{A}f \to 0$ in $C^\infty(\mathbb{R}^n)$, which gives the stated continuity property. □

Remark 3. It is hardly a drawback that continuity in Theorem 4 holds for the strong dual topology on $\mathcal{S}'$, as for *sequences* weak and strong convergence are equivalent (a well-known consequence of the fact that $\mathcal{S}$ is a Montel space).

In view of Theorem 4, the difficulties for type 1,1-operators do not stem from growth at infinity for C^∞-functions. Obviously the codomain C^∞ is not contained in $\mathcal{S}'$, but this is not just made possible by the use of $\mathcal{D}'$ in Definition 1, it is indeed decisive for the above construction.

In the proof above, the fact that $\tilde{A}f \in C^\infty$ also follows from the pseudo-local property of $a(x,D)$; *cf.* [16] (Theorem 6.4). The direct argument above is rather short, though. In addition to the smoothness, the properties of $a(x,D)f$ can be further sharpened by slow growth of f:

Corollary 5. *Every type 1, 1-operator $a(x,D)$ leaves $\mathcal{O}_M(\mathbb{R}^n)$ invariant.*

Proof. If $f \in \mathcal{O}_M$, then it follows that $(1+|x|)^{-2N} D^\alpha \tilde{A}f(x)$ is bounded for sufficiently large N, since in the second contribution to (44) clearly $(1+|y|)^{-2N} f(y)$ is in L_1 for large N: the resulting factor $(1+|y|)^{2N}$ may be absorbed by K, using that $r = \operatorname{dist}(\operatorname{supp}\tilde{\varphi}, \operatorname{supp}(1-\chi)) > 0$, since for $x \in \operatorname{supp}\tilde{\varphi}$, $y \in \operatorname{supp}(1-\chi)$,

$$\begin{aligned}(1+|y|)^{2N}|D_x^\alpha K(x,y)| &\le (1+|x|)^{2N}\max(1,1/r)^{2N}(r+|x-y|)^{2N}|D_x^\alpha K(x,y)| \\ &\le c(1+|x|)^{2N}\sup_{x\in\mathbb{R}^n}\int |(4\Delta_\eta)^N\big((\eta+D_x)^\alpha a(x,\eta)\big)|\,d\eta,\end{aligned} \tag{48}$$

where the supremum is finite for $2N > d+|\alpha|+n$ (by induction $(\eta+D_x)^\alpha \colon S^d_{1,1} \to S^{d+|\alpha|}_{1,1}$). Moreover, $c = \max(1,1/r)^{2N}/(2\pi)^n$ can be chosen uniformly for $x \in \mathbb{R}^n$ as it suffices to have (44) with $0 \le \chi \le 1$ and $\mathcal{C} = \overline{B}(0,j)$, $\operatorname{supp}\tilde{\varphi} = \overline{B}(0,j+1)$ and $\chi^{-1}(\{1\}) = \overline{B}(0,j+2)$ for an arbitrarily large $j \in \mathbb{N}$, which yields $r=1$, $c \le 1$. Thus $(1+|x|)^{-2N}|D^\alpha \tilde{A}f(x)|$ is less than $s_{\alpha,N}\int_{\mathbb{R}^n}(1+|y|)^{-2N}|f(y)|\,dy$ for all $x \in \mathbb{R}^n$, $s_{\alpha,N}$ as the supremum in (48). Hence $\tilde{A}f \in \mathcal{O}_M$. □

Example 4. The space $C^\infty(\mathbb{R}^n) \bigcap \mathcal{S}'(\mathbb{R}^n)$ clearly contains functions of non-slow growth, e.g.,

$$f(x) = e^{x_1+\dots+x_n}\cos(e^{x_1+\dots+x_n}). \tag{49}$$

Recall that $f \in \mathcal{S}'$ because $f = \mathrm{i}\,D_1 g$ for $g(x) = \sin(e^{x_1+\dots+x_n})$, which is in $L_\infty \subset \mathcal{S}'$. But $g \notin \mathcal{O}_M$, so already for $a(x,D) = \mathrm{i}\,D_1$ the space $\mathcal{O}_M$ cannot contain the range in Theorem 4.

Remark 4. Prior to the $T1$-theorem, David and Journé explained in [28] how a few properties of the distribution kernel of a continuous map $T\colon C_0^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ makes $T(1)$ well defined modulo constants; in particular if $T \in \mathrm{OP}(S^0_{1,1})$. Bourdaud [11] used this construction for $\tilde{A}$, so by Theorem 4 this extension of $T \in \mathrm{OP}(S^0_{1,1})$ from [28] is contained in Definition 1.

2.3. Conditions along the Twisted Diagonal

As the first explicit condition formulated for the symbol of a type 1,1-operator, Hörmander [12] proved that $a(x,D)$ has an extension by continuity

$$H^{s+d} \to H^s \quad \text{for every } s \in \mathbb{R} \tag{50}$$

whenever $a \in S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ fulfils the *twisted diagonal condition*: for some $B \geq 1$

$$\hat{a}(\xi,\eta) = 0 \quad \text{where} \quad B(1+|\xi+\eta|) < |\eta|. \tag{51}$$

In detail this means that the partially Fourier transformed symbol $\hat{a}(\xi,\eta) := \mathcal{F}_{x\to\xi} a(x,\eta)$ is trivial in a conical neighbourhood of a non-compact part of the twisted diagonal

$$\mathcal{T} = \{\, (\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \mid \xi+\eta = 0 \,\}. \tag{52}$$

Localisations to certain conical neighbourhoods of $\mathcal{T}$ were also used by Hörmander [12–14] as

$$\hat{a}_{\chi,\varepsilon}(\xi,\eta) = \hat{a}(\xi,\eta)\chi(\xi+\eta,\varepsilon\eta), \tag{53}$$

whereby the cut-of function $\chi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is chosen to satisfy

$$\chi(t\xi,t\eta) = \chi(\xi,\eta) \quad \text{for} \quad t \geq 1,\ |\eta| \geq 2 \tag{54}$$

$$\operatorname{supp}\chi \subset \{\, (\xi,\eta) \mid 1 \leq |\eta|,\ |\xi| \leq |\eta| \,\} \tag{55}$$

$$\chi = 1 \quad \text{in} \quad \{\, (\xi,\eta) \mid 2 \leq |\eta|,\ 2|\xi| \leq |\eta| \,\}. \tag{56}$$

Using this, Hörmander introduced and analysed a milder condition than the strict vanishing in (51). Namely, for some $\sigma \in \mathbb{R}$ the symbol should satisfy an estimate, for all multiindices α and $0 < \varepsilon < 1$,

$$N_{\chi,\varepsilon,\alpha}(a) := \sup_{\substack{R>0,\\ x\in\mathbb{R}^n}} R^{-d}\Big(\int_{R\leq|\eta|\leq 2R} |R^{|\alpha|} D^\alpha_\eta a_{\chi,\varepsilon}(x,\eta)|^2 \,\frac{d\eta}{R^n}\Big)^{\frac{1}{2}} \leq c_{\alpha,\sigma}\varepsilon^{\sigma+n/2-|\alpha|}. \tag{57}$$

This is an asymptotic formula for $\varepsilon \to 0$. It always holds for $\sigma = 0$, *cf.* [14] (Lemma 9.3.2):

Lemma 6. *When $a \in S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and $0 < \varepsilon \leq 1$, then $a_{\chi,\varepsilon} \in C^\infty$ and*

$$|D^\alpha_\eta D^\beta_x a_{\chi,\varepsilon}(x,\eta)| \leq C_{\alpha,\beta}(a)\varepsilon^{-|\alpha|}(1+|\eta|)^{d-|\alpha|+|\beta|} \tag{58}$$

$$\Big(\int_{R\leq|\eta|\leq 2R} |D^\alpha_\eta a_{\chi,\varepsilon}(x,\eta)|^2 \,d\eta\Big)^{1/2} \leq C_\alpha R^d(\varepsilon R)^{n/2-|\alpha|}. \tag{59}$$

The map $a \mapsto a_{\chi,\varepsilon}$ is continuous in $S^d_{1,1}$.

The last remark on continuity has been inserted here for later reference. It is easily verified by observing in the proof of [14] (Lemma 9.3.2) (to which the reader is referred) that the constant $C_{\alpha,\beta}(a)$ is a continuous seminorm in $S^d_{1,1}$.

In case $\sigma > 0$ there is a faster convergence to 0 in (57). In [13] this was proved to imply that $a(x, D)$ is bounded as a densely defined map

$$H^{s+d}(\mathbb{R}^n) \to H^s(\mathbb{R}^n) \quad \text{for} \quad s > -\sigma. \tag{60}$$

The reader may consult [14] (Theorem 9.3.5) for this (whilst [14] (Theorem 9.3.7) gives four pages of proof of necessity of $s \geq -\sup \sigma$, with supremum over all σ satisfying (57)).

Consequently, if $\hat{a}(\xi, \eta)$ is so small along $\mathcal{T}$ that (57) holds for all $\sigma \in \mathbb{R}$, there is boundedness $H^{s+d} \to H^s$ for all $s \in \mathbb{R}$. e.g., this is the case when (51) holds, for since

$$\operatorname{supp} \hat{a}_{\chi,\varepsilon} \subset \{ (\xi, \eta) \mid 1 + |\xi + \eta| \leq 2\varepsilon|\eta| \}, \tag{61}$$

clearly $a_{\chi,\varepsilon} \equiv 0$ for $2\varepsilon < 1/B$ then.

Example 5. For the present paper it is useful to exploit Ching's symbol (37) to show the existence of symbols fulfilling (57) for a given σ, at least for $\sigma \in \mathbb{N}$. To do so one may fix $|\theta| = 1$ and take some $A(\eta)$ in $C_0^\infty(\{ \eta \mid \frac{3}{4} < |\eta| < \frac{5}{4} \})$ with a zero of order σ at θ, so that Taylor's formula gives $|A(\eta)| \leq c|\eta - \theta|^\sigma$ in a neighbourhood of θ:

Indeed, as $\hat{a}(x, \eta) = (2\pi)^n \sum_{j=0}^\infty 2^{jd} \delta(\xi + 2^j\theta) A(2^{-j}\eta)$, clearly

$$a_{\theta,\chi,\varepsilon}(x, \eta) = \sum_{j=0}^\infty 2^{jd} e^{-\mathrm{i}\, x \cdot 2^j\theta} \chi(\eta - 2^j\theta, \varepsilon\eta) A(2^{-j}\eta). \tag{62}$$

Because $[R, 2R]$ is contained in $[\frac{3}{4}2^{j-1}, \frac{3}{2}2^{j-1}] \cup [\frac{3}{4}2^j, \frac{3}{2}2^j]$ for some $j \in \mathbb{Z}$, it suffices to estimate the integral in (57) only for $R = 3 \cdot 2^{j-2}$ with $j \geq 1$. Then it involves only the jth term, *i.e.*,

$$\int_{R \leq |\eta| \leq 2R} |a_{\theta,\chi,\varepsilon}(x, \eta)|^2 \, d\eta = \int_{R \leq |\eta| \leq 2R} R^{2d} |A(\eta/R)|^2 |\chi(\eta - R\theta, \varepsilon\eta)|^2 \, d\eta. \tag{63}$$

By the choice of χ, the integrand is 0 unless $|\eta - R\theta| \leq \varepsilon|\eta| \leq 2\varepsilon R$ and $1 \leq \varepsilon R$, so for small ε,

$$\int_{R \leq |\eta| \leq 2R} |a_{\theta,\chi,\varepsilon}(x, \eta)|^2 \, d\eta \leq \|\chi\|_\infty^2 R^{n+2d} \int_{|\zeta - \theta| \leq 2\varepsilon} c|\zeta - \theta|^{2\sigma} \, d\zeta \leq c' \varepsilon^{2\sigma+n} R^{n+2d}. \tag{64}$$

Applying $(RD_\eta)^\alpha$ before integration, $(RD_\eta)^\gamma$ may fall on $A(\eta/R)$, which lowers the degree and yields (at most) $\varepsilon^{n/2+\sigma-|\gamma|}$. In the factor $(RD_\eta)^{\alpha-\gamma}\chi(\eta - R\theta, \varepsilon\eta)$ the homogeneity of degree $-|\alpha - \gamma|$ applies for $\varepsilon R \geq 2$ and yields a bound in terms of finite suprema over $B(\theta, 2) \times B(0, 2)$, hence is $\mathcal{O}(1)$; else $\varepsilon R < 2$ so the factor is $\mathcal{O}(R^{|\alpha-\gamma|}) = \mathcal{O}(\varepsilon^{|\gamma|-|\alpha|})$ when non-zero, as both entries are in norm less than 4 then. Altogether this verifies (57).

A lower bound of (63) by $c\varepsilon^{2\sigma+n} R^{n+2d}$ is similar (*cf.* [14] (Example 9.3.3) for $\sigma = 0 = d$) when $|A(\eta)| \geq c_0|\eta - \theta|^\sigma$, which is obtained by taking A as a localisation of $|\eta - \theta|^\sigma$ for even σ (so $A \in C^\infty$). This implies that (57) does not hold for larger values of σ for this $a_\theta(x, \eta)$.

3. Pointwise Estimates

A crucial technique in this paper will be to estimate $|a(x, D)u(x)|$ at an arbitrary point $x \in \mathbb{R}^n$. Some of recent results on this by the author [29] are recalled here and further elaborated in Section 3.2 with an estimate of frequency modulated operators.

3.1. The Factorisation Inequality

First of all, by [29] (Theorem 4.1), when $\operatorname{supp}\hat{u} \subset \overline{B}(0,R)$, the action on u by the operator $a(x,D)$ can be *separated* from u at the cost of an estimate, which is the *factorisation inequality*

$$|a(x,D)u(x)| \le F_a(N,R;x)u^*(N,R;x). \tag{65}$$

Hereby $u^*(x) = u^*(N,R;x)$ denotes the maximal function of Peetre-Fefferman-Stein type,

$$u^*(N,R;x) = \sup_{y\in\mathbb{R}^n} \frac{|u(x-y)|}{(1+R|y|)^N} = \sup_{y\in\mathbb{R}^n} \frac{|u(y)|}{(1+R|x-y|)^N}. \tag{66}$$

The parameter N is often chosen to satisfy $N \ge \operatorname{order}\hat{u}$.

The a-factor F_a, also called the symbol factor, only depends on u in a vague way, *i.e.*, only through the N and R in (66). It is related to the distribution kernel of $a(x,D)$ as

$$F_a(N,R;x) = \int_{\mathbb{R}^n} (1+R|y|)^N |\mathcal{F}^{-1}_{\eta\to y}(a(x,\eta)\chi(\eta))|\,dy, \tag{67}$$

where $\chi \in C_0^\infty(\mathbb{R}^n)$ should equal 1 in a neighbourhood of $\operatorname{supp}\hat{u}$, or of $\bigcup_x \operatorname{supp} a(x,\cdot)\hat{u}(\cdot)$.

In (65) both factors are easily controlled. For one thing the non-linear map $u \mapsto u^*$ has long been known to have bounds with respect to the L_p-norm; *cf.* [29] (Theorem 2.6) for an elementary proof. But in the present paper it is more important that $u^*(x)$ is polynomially bounded thus: $|u(y)| \le c(1+|y|)^N \le c(1+R|y-x|)^N(1+|x|)^N$ holds according to the Paley-Wiener-Schwartz Theorem if $N \ge \operatorname{order}\hat{u}$ and $R \ge 1$, which by (66) implies

$$u^*(N,R;x) \le c(1+|x|)^N, \qquad x \in \mathbb{R}^n. \tag{68}$$

Here it is first recalled that every $u \in \mathcal{S}'$ has finite order since, for $\psi \in \mathcal{S}$,

$$|\langle u, \psi \rangle| \le c p_N(\psi), \tag{69}$$
$$p_N(\psi) = \sup\{\, (1+|x|)^N |D^\alpha u(x)| \mid x \in \mathbb{R}^n,\ |\alpha| \le N \,\}. \tag{70}$$

Indeed, since $(1+|x|)^N$ is finite on $\operatorname{supp}\psi$ for $\psi \in C_0^\infty$, u is of order N. To avoid a discussion of the converse, it will throughout be convenient to call the least integer N fulfilling (69) the *temperate* order of u, written $N = \operatorname{order}_{\mathcal{S}'}(u)$.

Returning to (68), when the compact spectrum of u results from Fourier multiplication, then the below $\mathcal{O}(2^{kN})$-information on the constant will be used repeatedly in the present paper.

Lemma 7. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$ be arbitrary and $N \ge \operatorname{order}_{\mathcal{S}'}(\hat{u})$. When $\psi \in C_0^\infty(\mathbb{R}^n)$ has support in $\overline{B}(0,R)$, then $w = \psi(2^{-k}D)u$ fulfils*

$$w^*(N,R2^k;x) \le C2^{kN}(1+|x|)^N, \qquad k \in \mathbb{N}_0, \tag{71}$$

for a constant C independent of k.

Proof. As $\psi(2^{-k}D)u(x) = \langle \hat{u}, \psi(2^{-k}\cdot)e^{\mathrm{i}\langle x,\cdot\rangle}(2\pi)^{-n} \rangle$, continuity of $\hat{u}\colon \mathcal{S} \to \mathbb{C}$ yields

$$|w(x)| \le c \sup\{\, (1+|\xi|)^N |D^\alpha_\xi(\psi(2^{-k}\xi)e^{\mathrm{i}\langle x,\xi\rangle})| \mid \xi \in \mathbb{R}^n,\ |\alpha| \le N \,\}. \tag{72}$$

Since $(1+|\xi|)^N|D^\alpha\psi(2^{-k}\xi)| \le c'2^{k(N-|\alpha|)}$, Leibniz' rule gives that $|w(x)| \le c''2^{kN}(1+|x|)^N$. Proceeding as before the lemma, the claim follows with $C = c''\max(1,R^{-N})$. $\square$

Secondly, for the a-factor in (67) one has $F_a \in C(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ and an estimate highly reminiscent of the Mihlin-Hörmander conditions for Fourier multipliers:

Theorem 8. *Assume $a(x,\eta)$ is in $S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and let $F_a(N,R;x)$ be given by (67) for parameters $R, N > 0$, with the auxiliary function taken as $\chi = \psi(R^{-1}\cdot)$ for $\psi \in C_0^\infty(\mathbb{R}^n)$ equalling 1 in a set with non-empty interior. Then one has for all $x \in \mathbb{R}^n$ that*

$$0 \le F_a(x) \le c_{n,N} \sum_{|\alpha| \le [N+\frac{n}{2}]+1} \Big(\int_{R\,\operatorname{supp}\psi} |R^{|\alpha|} D^\alpha_\eta a(x,\eta)|^2 \,\frac{d\eta}{R^n} \Big)^{1/2}. \tag{73}$$

For the elementary proof the reader may consult [29]; *cf.* Theorem 4.1 and Section 6 there. A further analysis of how F_a depends on $a(x,\eta)$ and R is a special case of [29] (Corollary 4.3):

Corollary 9. *Assume $a \in S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and let N, R and ψ be as in Theorem 8. When $R \ge 1$ there is a seminorm p on $S^d_{1,1}$ and a constant $c > 0$, that depends only on n, N and ψ, such that*

$$0 \le F_a(x) \le c_1 p(a) R^{\max(d,[N+n/2]+1)} \quad \textit{for all} \quad x \in \mathbb{R}^n. \tag{74}$$

If $\psi(\eta) \ne 0$ only holds in a corona $0 < \theta_0 \le |\eta| \le \Theta_0$, and $\psi(\eta) = 1$ holds for $\theta_1 \le |\eta| \le \Theta_1$, then

$$0 \le F_a(x) \le c_0 p(a) R^d \quad \textit{for all} \quad x \in \mathbb{R}^n, \tag{75}$$

whereby $c_0 = c_1 \max(1, \theta_0^{d-N-[n/2]-1}, \theta_0^d)$.

The above asymptotics is $\mathcal{O}(R^d)$ for $R \to \infty$ if d is large. This can be improved when $a(x,\eta)$ is modified by removing the low frequencies in the x-variable (*cf.* the $a^{(3)}$-term in Section 5 below). In fact, with a second spectral quantity $Q > 0$, the following is contained in [29] (Corollary 4.4):

Corollary 10. *When $a_Q(x,\eta) = \varphi(Q^{-1}D_x)a(x,\eta)$ for some $a \in S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi = 0$ in a neighbourhood of $\xi = 0$, then there is a seminorm p on $S^d_{1,1}$ and constants c_M, depending only on M, n, N, ψ and φ, such that*

$$0 \le F_{a_Q}(N,R;x) \le c_M p(a) Q^{-M} R^{\max(d+M,[N+n/2]+1)} \quad \textit{for} \quad M,Q,R > 0. \tag{76}$$

Here $d+M$ can replace $\max(d+M,[N+n/2]+1)$ when the auxiliary function ψ in F_{a_Q} fulfils the corona condition in Corollary 9.

Remark 5. By the proofs in [29], the seminorms in Corollaries 9 and 10 may be chosen in the same way for all d, namely $p(a) = \sum_{|\alpha| \le [N+n/2]+1} p_{\alpha,0}(a)$; *cf.* (26).

3.2. Estimates of Frequency Modulated Operators

The results in the previous section easily give the following, which later in Sections 5 and 6 will be used repeatedly.

Proposition 11. *For $a(x,\eta)$ in $S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$, $v \in \mathcal{S}'(\mathbb{R}^n)$ and arbitrary $\Phi, \Psi \in C_0^\infty(\mathbb{R}^n)$, for which Ψ is constant in a neighbourhood of the origin and with its support in $\overline{B}(0,R)$ for some $R \ge 1$, there is for $k \in \mathbb{N}_0$ and $N \ge \operatorname{order}_{\mathcal{S}'}(\mathcal{F}v)$, cf. (70), a polynomial bound*

$$\big|\operatorname{OP}\big(\Phi(2^{-k}D_x)a(x,\eta)\Psi(2^{-k}\eta)\big)v(x)\big| \le C(k)(1+|x|)^N, \tag{77}$$

whereby

$$C(k) = \begin{cases} c2^{k(N+d)_+} & \text{for } N+d \neq 0, \\ ck & \text{for } N+d = 0. \end{cases} \tag{78}$$

For $0 \notin \operatorname{supp}\Psi$ *this may be sharpened to* $C(k) = c2^{k(N+d)}$ *for all values of* $N+d$.

Proof. In this proof it is convenient to let $a^k(x,\eta) = \Phi(2^{-k}D_x)a(x,\eta)$ and $v^k = \Psi(2^{-k}D)v$. Then the factorisation inequality (65) gives

$$|a^k(x,D)v^k(x)| \leq F_{a^k}(N, R2^k; x) \cdot (v^k)^*(N, R2^k; x). \tag{79}$$

Since $N \geq \operatorname{order}_{\mathcal{S}'}(\hat{v})$, Lemma 7 gives $(v^k)^*(N, R2^k; x) \leq C2^{kN}(1+|x|)^N$, $x \in \mathbb{R}^n$.

In case $0 \notin \operatorname{supp}\Psi$, the auxiliary function $\chi = \psi(\cdot/(R2^k))$ used in F_{a^k}, *cf.* Theorem 8, can be so chosen that it fulfils the corona condition in Corollary 9; e.g., it is possible to have $\Theta_1 = 1$ and $\theta_1 = r/R$ when $\Psi \equiv 0$ on $B(0,r)$. Since Remark 5 implies $p(a^k) \leq p(a) \int |\mathcal{F}^{-1}\Phi(y)|\, dy$,

$$0 \leq F_{a^k}(N, R2^k; x) \leq c_0 \|\mathcal{F}^{-1}\Phi\|_1 p(a) R^d 2^{kd}. \tag{80}$$

When combined with the above, this inequality yields the claim in case $0 \notin \Psi$.

In the general case one has $v^k = v_k + v_{k-1} + \cdots + v_1 + v^0$, whereby v_j denotes the difference $v^j - v^{j-1} = \Psi(2^{-j}D)v - \Psi(2^{-j+1}D)v$. Via (65) this gives the starting point

$$|a^k(x,D)v^k(x)| \leq |a^k(x,D)v^0(x)| + \sum_{j=1}^{k} F_{a^k}(N, R2^j; x) v_j^*(N, R2^j; x). \tag{81}$$

As $\tilde{\Psi} = \Psi - \Psi(2\cdot)$ does not have 0 in its support, the above shows that for $j = 1, \ldots, k$ one has $F_{a^k}(N, R2^j; x) \leq c_0 \|\mathcal{F}^{-1}\Phi\|_1 p(a) R^d 2^{jd}$. Lemma 7 yields polynomial bounds of v_j^*, say with a constant C', so the sum on the right-hand side of (81) is estimated, for $d+N \neq 0$, by

$$\sum_{j=1}^{k} c_0 C' R^d p(a) 2^{j(N+d)} (1+|x|)^N \leq \frac{c_0 C' R^d}{2^{|d+N|}-1} p(a) (1+|x|)^N 2^{(k+1)(N+d)_+}. \tag{82}$$

In case $d+N = 0$ the k bounds are equal.

The remainder in (81) fulfils $|a^k(x,D)v^0(x)| \leq c_1 R^{N'}(1+|x|)^N$ for a large N'; *cf.* the first part of Corollary 9 and Lemma 7. Altogether $|a^k(x,D)v^k(x)| \leq C(k)(1+|x|)^N$. □

4. Adjoints of Type 1, 1-Operators

For classical pseudo-differential operators $a(x,D)\colon \mathcal{S} \to \mathcal{S}'$ it is well known that the adjoint $a(x,D)^*\colon \mathcal{S} \to \mathcal{S}'$ has symbol $a^*(x,\eta) = e^{\mathrm{i} D_x \cdot D_\eta} \overline{a(x,\eta)}$, and that $a \mapsto a^*$ sends, e.g., $S^d_{1,0}$ into itself.

4.1. The Basic Lemma

In order to show that the twisted diagonal condition (51) also implies continuity $a(x,D)\colon \mathcal{S}' \to \mathcal{S}'$, a basic result on the adjoint symbols is recalled from [12,14] (Lemma 9.4.1):

Lemma 12. *When* $a(x,\eta)$ *is in* $S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ *and for some* $B \geq 1$ *satisfies the twisted diagonal condition* (51), *then the adjoint symbol* $b(x,\eta) = e^{\mathrm{i} D_x \cdot D_\eta} \overline{a(x,\eta)}$ *is in* $S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ *in this case and*

$$\hat{b}(\xi,\eta) = 0 \quad \text{when} \quad |\xi + \eta| > B(|\eta|+1). \tag{83}$$

Moreover,

$$|D_\eta^\alpha D_x^\beta b(x,\eta)| \le C_{\alpha\beta}(a) B(1+B^{d-|\alpha|+|\beta|})(1+|\eta|)^{d-|\alpha|+|\beta|}, \tag{84}$$

for certain seminorms $C_{\alpha\beta}$ that are continuous on $S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and do not depend on B.

The twisted diagonal condition (51) implies that $a^*(x,D) = b(x,D)$ is a map $\mathcal{S} \to \mathcal{S}$, as it is of type $1,1$ by Lemma 12, so then $a(x,D)$ has the continuous linear *extension* $b(x,D)^* \colon \mathcal{S}' \to \mathcal{S}'$. It is natural to expect that this coincides with the definition of $a(x,D)$ by vanishing frequency modulation:

Proposition 13. *If $a(x,\eta) \in S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ fulfils (51), then $a(x,D)$ is a continuous linear map $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ that equals the adjoint of $b(x,D) \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, when $b(x,\eta)$ is the adjoint symbol as in Lemma 12.*

Proof. When $\psi \in C_0^\infty(\mathbb{R}^n)$ is such that $\psi = 1$ in a neighbourhood of the origin, a simple convolution estimate (*cf.* [16]) (Lemma 2.1) gives that in the topology of $S^{d+1}_{1,1}$,

$$\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta) \to a(x,\eta) \quad \text{for} \quad m \to \infty. \tag{85}$$

Since the supports of the partially Fourier transformed symbols

$$\psi(2^{-m}\xi)\mathcal{F}_{x\to\xi}a(\xi,\eta)\psi(2^{-m}\eta), \quad m \in \mathbb{N}, \tag{86}$$

are contained in $\operatorname{supp} \mathcal{F}_{x\to\xi}a(\xi,\eta)$, clearly this sequence also fulfils (51) for the same B. As the passage to adjoint symbols by (84) is continuous from the metric subspace of $S^d_{1,1}$ fulfilling (51) to $S^{d+1}_{1,1}$, one therefore has that

$$b_m(x,\eta) := e^{\mathrm{i}\,D_x\cdot D_\eta}(\overline{\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta)}) \xrightarrow[m\to\infty]{} e^{\mathrm{i}\,D_x\cdot D_\eta}\overline{a(x,\eta)} =: b(x,\eta). \tag{87}$$

Combining this with the fact that $b(x,D)$ as an operator on the Schwartz space depends continuously on the symbol, one has for $u \in \mathcal{S}'(\mathbb{R}^n)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}(\, b(x,D)^*u \mid \varphi\,) &= (\, u \mid \lim_{m\to\infty} \operatorname{OP}(b_m(x,\eta))\varphi\,) \\ &= \lim_{m\to\infty} (\operatorname{OP}(\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta))u \mid \varphi\,).\end{aligned} \tag{88}$$

As the left-hand side is independent of ψ the limit in (29) is so, hence the definition of $a(x,D)$ gives that every $u \in \mathcal{S}'(\mathbb{R}^n)$ is in $D(a(x,D))$ and $a(x,D)u = b(x,D)^*u$ as claimed. □

The mere extendability to $\mathcal{S}'$ under the twisted diagonal condition (51) could have been observed already in [12,14], but the above result seems to be the first sufficient condition for a type $1,1$-operator to be *defined* on the entire $\mathcal{S}'(\mathbb{R}^n)$.

4.2. The Self-Adjoint Subclass $\widetilde{S}^d_{1,1}$

Proposition 13 shows that (51) suffices for $D(a(x,D)) = \mathcal{S}'$. But (51) is too strong to be necessary; a vanishing to infinite order along $\mathcal{T}$ should suffice.

In this section, the purpose is to prove that $a(x,D) \colon \mathcal{S}' \to \mathcal{S}'$ is continuous if more generally the twisted diagonal condition of order σ, that is (57), holds for all $\sigma \in \mathbb{R}$.

This will supplement Hörmander's investigation in [12–14], from where the main ingredients are recalled. Using (53) and $\mathcal{F}_{x\to\xi}$ one has in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$,

$$a(x,\eta) = (a(x,\eta) - a_{\chi,1}(x,\eta)) + \sum_{\nu=0}^{\infty} (a_{\chi,2^{-\nu}}(x,\eta) - a_{\chi,2^{-\nu-1}}(x,\eta)). \tag{89}$$

Here the first term $a(x,\eta) - a_{\chi,1}(x,\eta)$ fulfils (51) for $B = 1$, so Proposition 13 applies to it. Introducing $e_\varepsilon(x, D)$ like in [14] (Section 9.3) as

$$\hat{e}_\varepsilon(x,\eta) = \hat{a}_{\chi,\varepsilon}(\xi,\eta) - \hat{a}_{\chi,\varepsilon/2}(\xi,\eta) = (\chi(\xi+\eta,\varepsilon\eta) - \chi(\xi+\eta,\varepsilon\eta/2))\hat{a}(x,\eta), \tag{90}$$

it is useful to infer from the choice of χ that

$$\operatorname{supp}\hat{e}_\varepsilon \subset \{\, (\xi,\eta) \mid \tfrac{\varepsilon}{4}|\eta| \le \max(1,|\xi+\eta|) \le \varepsilon|\eta| \,\}. \tag{91}$$

In particular this yields that $\hat{e}_\varepsilon = 0$ when $1 + |\xi+\eta| < |\eta|\varepsilon/4$, so e_ε fulfils (51) for $B = 4/\varepsilon$. Hence the terms $e_{2^{-\nu}}$ in (89) do so for $B = 2^{\nu+2}$.

The next result characterises the $a \in S^d_{1,1}$ for which the adjoint symbol a^* is again in $S^d_{1,1}$; *cf.* the below condition (i). As adjoining is an involution, these symbols constitute the class

$$\tilde{S}^d_{1,1} := S^d_{1,1} \cap (S^d_{1,1})^*. \tag{92}$$

Theorem 14. *When $a(x,\eta)$ is a symbol in $S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ the following properties are equivalent:*

(i) *The adjoint symbol $a^*(x,\eta)$ is also in $S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$.*

(ii) *For arbitrary $N > 0$ and α, β there is some constant $C_{\alpha,\beta,N}$ such that*

$$|D^\alpha_\eta D^\beta_x a_{\chi,\varepsilon}(x,\eta)| \le C_{\alpha,\beta,N}\varepsilon^N(1+|\eta|)^{d-|\alpha|+|\beta|} \quad \textit{for} \quad 0 < \varepsilon < 1. \tag{93}$$

(iii) *For all $\sigma \in \mathbb{R}$ there is a constant $c_{\alpha,\sigma}$ such that for $0 < \varepsilon < 1$*

$$\sup_{R>0,\ x\in\mathbb{R}^n} R^{|\alpha|-d}\Big(\int_{R\le|\eta|\le 2R} |D^\alpha_\eta a_{\chi,\varepsilon}(x,\eta)|^2 \,\frac{d\eta}{R^n}\Big)^{1/2} \le c_{\alpha,\sigma}\varepsilon^{\sigma+\frac{n}{2}-|\alpha|}. \tag{94}$$

In the affirmative case $a \in \tilde{S}^d_{1,1}$, cf. (92)*, and a^* fulfils an estimate*

$$|D^\alpha_\eta D^\beta_x a^*(x,\eta)| \le (C_{\alpha,\beta}(a) + C'_{\alpha,\beta,N})(1+|\eta|)^{d-|\alpha|+|\beta|} \tag{95}$$

for a certain continuous seminorm $C_{\alpha,\beta}$ on $S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and some finite sum $C'_{\alpha,\beta,N}$ of constants fulfilling the inequalities in (ii).

It should be observed from (i) that $a(x,\eta)$ fulfils condition (ii) or (iii) if and only if $a^*(x,\eta)$ does so — whereas neither (ii) nor (iii) make this obvious. But (ii) immediately gives the (expected) inclusion $\tilde{S}^d_{1,1} \subset \tilde{S}^{d'}_{1,1}$ for $d' > d$. Condition (iii) is close in spirit to the Mihlin-Hörmander multiplier theorem, and it is useful for the estimates shown later in Section 6.

Remark 6. Conditions (ii), (iii) both hold either for all χ satisfying (57) or for none, for (i) does not depend on χ. It suffices to verify (ii) or (iii) for $0 < \varepsilon < \varepsilon_0$ for some convenient $\varepsilon_0 \in]0,1[$. This is implied by Lemma 6 since every power ε^p is bounded on the interval $[\varepsilon_0, 1]$.

Theorem 14 was undoubtedly known to Hörmander, for he stated the equivalence of (i) and (ii) explicitly in [12] (Theorem 4.2) and [14] (Theorem 9.4.2) and gave brief remarks on (iii) in the latter. Equivalence with continuous extensions $H^{s+d} \to H^s$ for all $s \in \mathbb{R}$ was also shown. However, the exposition there left a considerable burden of verification to the reader.

Moreover, Theorem 14 was used without proof in a main L_p-theorem in [17], and below in Section 4.2.4 a corollary to the proof will follow and decisively enter the first proof of $\mathcal{S}'$-continuity. Hence full details on the main result in Theorem 14 should be in order here:

4.2.1. Equivalence of (ii) and (iii)

That (ii) implies (iii) is seen at once by insertion, taking $\beta = 0$ and $N = \sigma + \frac{n}{2} - |\alpha|$.

Conversely, note first that $|\xi + \eta| \leq \varepsilon|\eta|$ in the spectrum of $a_{\chi,\varepsilon}(\cdot, \eta)$. That is, $|\xi| \leq (1+\varepsilon)|\eta|$ so Bernstein's inequality gives

$$|D_x^\beta D_\eta^\alpha a_{\chi,\varepsilon}(x,\eta)| \leq ((1+\varepsilon)|\eta|)^{|\beta|} \sup_{x\in\mathbb{R}} |D_\eta^\alpha a_{\chi,\varepsilon}(x,\eta)|. \tag{96}$$

Hence $C_{\alpha,\beta,N} = 2^{|\beta|}C_{\alpha,0,N}$ is possible, so it suffices to prove (iii) $\implies$ (ii) only for $\beta = 0$.

For the corona $1 \leq |\zeta| \leq 2$ Sobolev's lemma gives for $f \in C^\infty(\mathbb{R}^n)$,

$$|f(\zeta)| \leq c_1 \Big(\sum_{|\beta|\leq[n/2]+1} \int_{1\leq|\zeta|\leq 2} |D^\beta f(\zeta)|^2 \, d\zeta \Big)^{1/2}. \tag{97}$$

Substituting $D_\eta^\alpha a_{\chi,\varepsilon}(x, R\zeta)$ and $\zeta = \eta/R$, whereby $R \leq |\eta| \leq 2R$, $R > 0$, yields

$$\begin{aligned}
|D_\eta^\alpha a_{\chi,\varepsilon}(x,\eta)| &\leq c_1 \Big(\sum_{|\beta|\leq[n/2]+1} R^{2|\beta|} \int_{R\leq|\eta|\leq 2R} |D_\eta^{\alpha+\beta} a_{\chi,\varepsilon}(x,\eta)|^2 \frac{d\eta}{R^n} \Big)^{1/2} \\
&\leq c_1 \Big(\sum_{|\beta|\leq[n/2]+1} R^{2d-2|\alpha|} C^2_{\alpha+\beta,\sigma} \varepsilon^{2(\sigma+\frac{n}{2}-|\alpha|-|\beta|)} \Big)^{1/2} \\
&\leq c_1 \Big(\sum_{|\beta|\leq[n/2]+1} C^2_{\alpha+\beta,\sigma} \Big)^{1/2} \varepsilon^{\sigma-1-|\alpha|} R^{d-|\alpha|}.
\end{aligned} \tag{98}$$

Here $R^{d-|\alpha|} \leq (1+|\eta|)^{d-|\alpha|}$ for $d \geq |\alpha|$, that leads to (ii) as $\sigma \in \mathbb{R}$ can be arbitrary.

For $|\alpha| > d$ it is first noted that, by the support condition on χ, one has $a_{\chi,\varepsilon}(x,\eta) \neq 0$ only for $2R \geq |\eta| \geq \varepsilon^{-1} > 1$. But $R \geq 1/2$ yields $R^{d-|\alpha|} \leq (\frac{1}{3}(\frac{1}{2} + 2R))^{d-|\alpha|} \leq 6^{|\alpha|-d}(1+|\eta|)^{d-|\alpha|}$, so (ii) follows from the above.

4.2.2. The Implication (ii) $\implies$ (i) and the Estimate

The condition (ii) is exploited for each term in the decomposition (89). Setting $b_\nu(x,\eta) = e^*_{2^{-\nu}}(x,\eta)$ it follows from Lemma 12 that b_ν is in $S^d_{1,1}$ by the remarks after (91), *cf.* (89) ff, and (84) gives

$$|D_\eta^\alpha D_x^\beta b_\nu(x,\eta)| \leq C_{\alpha,\beta}(e_\nu) 2^{\nu+2} (1 + 2^{(\nu+2)(d-|\alpha|+|\beta|)})(1+|\eta|)^{d-|\alpha|+|\beta|}. \tag{99}$$

Now (ii) implies that $C_{\alpha,\beta}(a_{\chi,2^{-\nu}}) \leq C'_{\alpha,\beta,N} 2^{-\nu N}$ for all $N > 0$ (with other constants $C'_{\alpha,\beta,N}$ as the seminorms $C_{\alpha,\beta}$ may contain derivatives of higher order than $|\alpha|$ and $|\beta|$). Hence $C_{\alpha,\beta}(e_{2^{-\nu}}) \leq C'_{\alpha,\beta,N} 2^{1-\nu N}$. It follows from this that $\sum b_\nu$ converges to some b in $S^d_{1,1}$ (in the Fréchet topology of this space), so that $a^*(x,\eta) = b(x,\eta)$ is in $S^d_{1,1}$. More precisely, (84) and the above yields for $N = 2 + (d - |\alpha| + |\beta|)_+$

$$\begin{aligned}
\frac{|D_\eta^\alpha D_x^\beta a^*(x,\eta)|}{(1+|\eta|)^{d-|\alpha|+|\beta|}} &\leq 2^N C_{\alpha,\beta}(a - a_{\chi,1}) + \sum_{\nu=0}^\infty C_{\alpha,\beta}(e_{2^{-\nu}}) 2^{\nu+2} (1 + 2^{(\nu+2)(d-|\alpha|+|\beta|)_+}) \\
&\leq 2^N C_{\alpha,\beta}(a - a_{\chi,1}) + \sum_{\nu=0}^\infty 16 C'_{\alpha,\beta,N} 2^{-\nu(N-1)} 2^{(\nu+2)(d-|\alpha|+|\beta|)_+} \\
&\leq 2^N C_{\alpha,\beta}(a - a_{\chi,1}) + 4^{N+2} C'_{\alpha,\beta,N}.
\end{aligned} \tag{100}$$

Invoking continuity from Lemma 6 in the first term, the last part of the theorem follows.

4.2.3. Verification of (i) $\Longrightarrow$ (ii)

It suffices to derive another decomposition

$$a = A + \sum_{\nu=0}^{\infty} a_\nu, \tag{101}$$

in which $A \in S^{-\infty}$ and each $a_\nu \in S^d_{1,1}$ with $\hat{a}_\nu(\xi,\eta) = 0$ for $2^{\nu+1}|\xi+\eta| < |\xi|$ and seminorms $C_{\alpha,\beta}(a_\nu) = \mathcal{O}(2^{-\nu N})$ for each $N > 0$.

Indeed, when $\chi(\xi+\eta, \varepsilon\eta) \neq 0$ the triangle inequality gives $|\xi+\eta| \leq \varepsilon|\eta| \leq \varepsilon|\xi+\eta| + \varepsilon|\xi|$, whence $|\xi+\eta|(1-\varepsilon)/\varepsilon \leq |\xi|$, so that for one thing

$$\hat{a}_{\chi,\varepsilon}(x,\eta) = \chi(\xi+\eta,\varepsilon\eta)\hat{A}(x,\eta) + \sum_{2^{\nu+1}>(1-\varepsilon)/\varepsilon} \chi(\xi+\eta,\varepsilon\eta)\hat{a}_\nu(x,\eta). \tag{102}$$

Secondly, for each seminorm $C_{\alpha,\beta}$ in $S^d_{1,1}$ one has $C_{\alpha,\beta}(a_{\nu,\chi,\varepsilon}) \leq \varepsilon^{-|\alpha|} C_{\alpha,\beta}(a_\nu)$ by Lemma 6, so by estimating the geometric series by its first term, the above formula entails that

$$C_{\alpha,\beta}(a_{\chi,\varepsilon}) \leq C_{\alpha,\beta}(A_{\chi,\varepsilon}) + \sum_{\varepsilon 2^{\nu+1} > 1-\varepsilon} \frac{C_{N+|\alpha|}}{\varepsilon^{|\alpha|} 2^{\nu(N+|\alpha|)}} \leq C_{\alpha,\beta}(A_{\chi,\varepsilon}) + \frac{c}{\varepsilon^{|\alpha|}} (\frac{2\varepsilon}{1-\varepsilon})^{N+|\alpha|}. \tag{103}$$

This gives the factor ε^N in (ii) for $0 < \varepsilon \leq 1/2$. For $1/2 < \varepsilon < 1$ the series is $\mathcal{O}(\varepsilon^{-|\alpha|})$ because $2^{-\nu} \leq 1 < 2\varepsilon/(1-\varepsilon)$ for all ν. However, $1 \leq (2\varepsilon)^{N+|\alpha|}$ for such ε, so (ii) will follow for all $\varepsilon \in]0,1[$. (It is seen directly that $|A_{\chi,\varepsilon}(x,\eta)| \leq c\varepsilon^N(1+|\eta|)^d$, *etc.*, for only the case $\varepsilon|\eta| \geq 1$ is non-trivial, and then $\varepsilon^{-N} \leq (1+|\eta|)^N$ while $A \in S^{-\infty}$.)

In the deduction of (101) one can use a Littlewood-Paley partition of unity, say $1 = \sum_{\nu=0}^{\infty} \Phi_\nu$ with dilated functions $\Phi_\nu(\eta) = \Phi(2^{-\nu}\eta) \neq 0$ only for $\frac{11}{20}2^\nu \leq |\eta| \leq \frac{13}{10}2^\nu$ if $\nu \geq 1$. Beginning with a trivial split $a^* = A_0 + A_1$ into two terms for which $A_0 \in S^{-\infty}$ and $A_1 \in S^d_{1,1}$ such that $A_1(x,\eta) = 0$ for $|\eta| < 1/2$, this gives

$$\hat{a}^*(\xi,\eta) = \hat{A}_0(\xi,\eta) + \sum_{\nu=0}^{\infty} \Phi_\nu(\xi/|\eta|)\hat{A}_1(\xi,\eta). \tag{104}$$

This yields the desired $a_\nu(x,\eta)$ by taking the adjoint of $\mathcal{F}^{-1}_{\xi\to x}(\Phi_\nu(\frac{\xi}{|\eta|})\hat{A}_1(\xi,\eta))$, that is, of the symbol $\int |2^\nu\eta|^n \check{\Phi}(|2^\nu\eta|y) A_1(x-y,\eta)\,dy$. Indeed, it follows directly from [12] (Proposition 3.3) (where the proof uses Taylor expansion and vanishing moments of $\check{\Phi}$ for $\nu \geq 1$) that a^*_ν belongs to $S^d_{1,1}$ with $(2^{N\nu}a^*_\nu)_{\nu\in\mathbb{N}}$ bounded in $S^d_{1,1}$ for all $N > 0$. Therefore (104) gives (101) by inverse Fourier transformation. Moreover, $\hat{a}^*_\nu(\xi,\eta)$ is for $\nu \geq 1$ is supported by the region

$$\tfrac{11}{20}2^\nu|\eta| \leq |\xi| \leq \tfrac{13}{20}2^\nu|\eta|, \tag{105}$$

where a fortiori $1 + |\xi+\eta| \geq |\xi| - |\eta| \geq (\frac{11}{20}2^\nu - 1)|\eta| \geq \frac{1}{10}|\eta|$, so it is clear that

$$\hat{a}^*_\nu(\xi,\eta) = 0 \quad \text{if } 10(|\xi+\eta|+1) < |\eta|. \tag{106}$$

According to Lemma 12 this implies that $a_\nu = a^{**}_\nu$ is also in $S^d_{1,1}$ and that, because of the above boundedness in $S^d_{1,1}$, there is a constant c independent of ν such that

$$\begin{aligned} |D^\alpha_\eta D^\beta_x a_\nu(x,\eta)| &\leq C_{\alpha,\beta}(a^*_\nu) 10(1 + 10^{d-|\alpha|+|\beta|})(1+|\eta|)^{d-|\alpha|+|\beta|} \\ &\leq c2^{-N\nu}(1+|\eta|)^{d-|\alpha|+|\beta|}. \end{aligned} \tag{107}$$

Therefore, the a_ν tend rapidly to 0, which completes the proof of Theorem 14.

4.2.4. Consequences for the Self-Adjoint Subclass

One can set Theorem 14 in relation to the definition by vanishing frequency modulation, simply by elaborating on the above proof:

Corollary 15. *On $\widetilde{S}^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ the adjoint operation is stable with respect to vanishing frequency modulation in the sense that, when $a\in\widetilde{S}^d_{1,1}$, $\psi\in C_0^\infty(\mathbb{R}^n)$ with $\psi=1$ around 0, then*

$$\big(\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta)\big)^* \xrightarrow[m\to\infty]{} a(x,\eta)^* \tag{108}$$

holds in the topology of $S^{d+1}_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$.

Proof. For brevity $b_m(x,\eta)=\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta)$ denotes the symbol that is frequency modulated in both variables. The proof consists in insertion of $a(x,\eta)-b_m(x,\eta)$ into (100), where the first sum tends to 0 for $m\to\infty$ by majorised convergence.

Note that for each $\nu\geq 0$ in the first sum of (100) one must control $C_{\alpha,\beta}(e^m_{2^{-\nu}})$ for $m\to\infty$ and

$$\hat{e}^m_{2^{-\nu}}(\xi,\eta)=(\chi(\xi+\eta,2^{-\nu}\eta)-\chi(\xi+\eta,2^{-\nu-1}\eta))(1-\psi(2^{-m}\xi)\psi(2^{-m}\eta))\hat{a}(\xi,\eta). \tag{109}$$

To do so, a convolution estimate first gives $p_{\alpha,\beta}(b_m)\leq c\sum_{\gamma\leq\alpha}p_{\gamma,\beta}(a)$, whence $(b_m)_{m\in\mathbb{N}}$ is bounded in $S^d_{1,1}$. Similar arguments yield that $b_m\to a$ in $S^{d+1}_{1,1}$ for $m\to\infty$; *cf.* [16] (Lemma 2.1). Moreover, for each $\nu\geq 0$, every seminorm $p_{\alpha,\beta}$ now on $S^{d+1}_{1,1}$, gives

$$p_{\alpha,\beta}(e^m_{2^{-\nu}})\leq p_{\alpha,\beta}((a-b_m)_{\chi,2^{-\nu}})+p_{\alpha,\beta}((a-b_m)_{\chi,2^{-\nu-1}}). \tag{110}$$

Here both terms on the right-hand side tend to 0 for $m\to\infty$, in view of the continuity of $a\mapsto a_{\chi,\varepsilon}$ on $S^{d+1}_{1,1}$; *cf.* Lemma 6. Hence $C_{\alpha,\beta}(e^m_{2^{-\nu}})\to 0$ for $m\to\infty$.

It therefore suffices to replace d by $d+1$ in (100) and majorise. However, $a\mapsto a_{\chi,\varepsilon}$ commutes with $a\mapsto b_m$ as maps in $\mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^n)$, so since $a\in\widetilde{S}^{d+1}_{1,1}$, it follows from (ii) that

$$p_{\alpha,\beta}((a-b_m)_{\chi,\varepsilon})\leq p_{\alpha,\beta}(a_{\chi,\varepsilon})+c\sum_{\gamma\leq\alpha}p_{\gamma,\beta}(a_{\chi,\varepsilon})\leq(1+c)(\sum_{\gamma\leq\alpha}C_{\gamma,\beta,N})\varepsilon^N\leq C'_{\alpha,\beta,N}\varepsilon^N. \tag{111}$$

Using this in the previous inequality, $C_{\alpha,\beta}(e^m_{2^{-\nu}})\leq C2^{-\nu N}$ is obtained for C independent of $m\in\mathbb{N}$. Now it follows from (100) that $b_m(x,\eta)^*\to a(x,\eta)^*$ in $S^{d+1}_{1,1}$ as desired. □

Thus prepared, the proof of Proposition 13 can now be repeated from (87) onwards, which immediately gives the first main result of the paper:

Theorem 16. *When a symbol $a(x,\eta)$ of type $1,1$ belongs to the class $\tilde{S}^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$, as characterised in Theorem 14, then $a(x,D)$ is everywhere defined and continuous*

$$a(x,D)\colon\mathcal{S}'(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n) \tag{112}$$

It equals the adjoint of $\operatorname{OP}(e^{\mathrm{i}\,D_x\cdot D_\eta}\bar{a}(x,\eta))\colon\mathcal{S}\to\mathcal{S}$.

Like for Proposition 13, there seems to be no previous attempts in the literature to obtain this clarification (Theorem 16 was stated without proof in [17]). However, it seems to be open whether (112) conversely implies that $a\in\tilde{S}^d_{1,1}$.

5. Dyadic Corona Decompositions

This section adopts Littlewood-Paley techniques to provide a passage to auxiliary operators $a^{(j)}(x,D)$, $j=1,2,3$, which may be easily analysed with the pointwise estimates of Section 3.

5.1. The Paradifferential Splitting

Recalling the definition of type $1,1$-operators in (29) and (30), it is noted that to each modulation function ψ, *i.e.*, $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi = 1$ in a neighbourhood of 0, there exist $R > r > 0$ with $R \geq 1$ satisfying

$$\psi(\xi) = 1 \quad \text{for} \quad |\xi| \leq r; \qquad \psi(\xi) = 0 \quad \text{for} \quad |\xi| \geq R. \tag{113}$$

For fixed ψ it is convenient to take an integer $h \geq 2$ so large that $2R < r2^h$.

To obtain a Littlewood-Paley decomposition from ψ, set $\varphi = \psi - \psi(2\cdot)$. Then a dilation of this function is supported in a corona,

$$\operatorname{supp} \varphi(2^{-k}\cdot) \subset \{\, \xi \mid r2^{k-1} \leq |\xi| \leq R2^k \,\}, \qquad \text{for } k \geq 1. \tag{114}$$

The identity $1 = \psi(x) + \sum_{k=1}^\infty \varphi(2^{-k}\xi)$ follows by letting $m \to \infty$ in the telescopic sum,

$$\psi(2^{-m}\xi) = \psi(\xi) + \varphi(\xi/2) + \cdots + \varphi(\xi/2^m). \tag{115}$$

Using this, functions $u(x)$ and symbols $a(x,\eta)$ will be localised to frequencies $|\eta| \approx 2^j$ as

$$u_j = \varphi(2^{-j}D)u, \qquad a_j(x,\eta) = \varphi(2^{-j}D_x)a(x,\eta). \tag{116}$$

Localisation to balls given by $|\eta| \leq R2^j$ are written with upper indices,

$$u^j = \psi(2^{-j}D)u, \qquad a^j(x,\eta) = \psi(2^{-j}D_x)a(x,\eta). \tag{117}$$

In addition $u_0 = u^0$ and $a_0 = a^0$; as an *index convention* they are all taken $\equiv 0$ for $j < 0$. (To avoid having two different meanings of sub- and superscripts, the dilations $\psi(2^{-j}\cdot)$ are written as such, with the corresponding Fourier multiplier as $\psi(2^{-j}D)$, and similarly for φ). Note that the corresponding operators are $a^k(x,D) = \operatorname{OP}(\psi(2^{-k}D_x)a(x,\eta))$, *etc.*

Inserting the relation (115) twice in (29), bilinearity gives

$$\operatorname{OP}(\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta))u = \sum_{j,k=0}^{m} a_j(x,D)u_k. \tag{118}$$

Of course the sum may be split in three groups having $j \leq k-h$, $|j-k| < h$ and $k \leq j-h$. For $m \to \infty$ this yields the well-known paradifferential decomposition

$$a_\psi(x,D)u = a_\psi^{(1)}(x,D)u + a_\psi^{(2)}(x,D)u + a_\psi^{(3)}(x,D)u, \tag{119}$$

whenever a and u fit together such that the three series below converge in $\mathcal{D}'(\mathbb{R}^n)$:

$$a_\psi^{(1)}(x,D)u = \sum_{k=h}^\infty \sum_{j\leq k-h} a_j(x,D)u_k = \sum_{k=h}^\infty a^{k-h}(x,D)u_k \tag{120}$$

$$\begin{aligned} a_\psi^{(2)}(x,D)u = \sum_{k=0}^\infty \big(& a_{k-h+1}(x,D)u_k + \cdots + a_{k-1}(x,D)u_k + a_k(x,D)u_k \\ &+ a_k(x,D)u_{k-1} + \cdots + a_k(x,D)u_{k-h+1}\big) \end{aligned} \tag{121}$$

$$a_\psi^{(3)}(x,D)u = \sum_{j=h}^\infty \sum_{k\leq j-h} a_j(x,D)u_k = \sum_{j=h}^\infty a_j(x,D)u^{j-h}. \tag{122}$$

Note the shorthand $a^{k-h}(x,D)$ for $\sum_{j\le k-h} a_j(x,D) = \mathrm{OP}(\psi(2^{h-k}D_x)a(x,\eta))$, *etc.* Using this and the index convention, the so-called symmetric term in (121) has the brief form

$$a^{(2)}_\psi(x,D)u = \sum_{k=0}^{\infty}((a^k - a^{k-h})(x,D)u_k + a_k(x,D)(u^{k-1} - u^{k-h})). \tag{123}$$

In the following the subscript ψ is usually dropped because this auxiliary function will be fixed (ψ was left out already in a_j and a^j; *cf.* (116) and (117)). Note also that the above $a^{(j)}(x,D)$ for now is just a convenient notation for the infinite series. The full justification of this operator notation will first result from Theorems 22–24 below.

Remark 7. It was tacitly used in (118) and (120)–(122) that one has

$$a_j(x,D)u_k = \mathrm{OP}(a_j(x,\eta)\varphi(2^{-k}\eta))u. \tag{124}$$

This is because, with $\chi \in C_0^\infty$ equalling 1 on $\operatorname{supp}\varphi(2^{-k}\cdot)$, both sides are equal to

$$\mathrm{OP}(a_j(x,\eta)\chi(\eta))u_k. \tag{125}$$

Indeed, while this is trivial for the right-hand side of (124), where the symbol is in $S^{-\infty}$, it is for the type 1,1-operator on the left-hand side of (124) a fact that follows at once from (38). Thus the inclusion $\mathcal{F}^{-1}\mathcal{E}' \subset D(a(x,D))$ in (38) is crucial for the simple formulae in the present paper. Analogously Definition 1 may be rewritten briefly as $a(x,D)u = \lim_m a^m(x,D)u^m$.

The importance of the decomposition in (120)–(122) lies in the fact that the summands have localised spectra, e.g., there is a dyadic corona property:

Proposition 17. *If $a \in S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$, and r, R are chosen as in (113) for each auxiliary function ψ, then every $h \in \mathbb{N}$ such that $2R < r2^h$ gives*

$$\operatorname{supp}\mathcal{F}(a^{k-h}(x,D)u_k) \subset \{\, \xi \mid R_h 2^k \le |\xi| \le \frac{5R}{4}2^k \,\} \tag{126}$$

$$\operatorname{supp}\mathcal{F}(a_k(x,D)u^{k-h}) \subset \{\, \xi \mid R_h 2^k \le |\xi| \le \frac{5R}{4}2^k \,\}, \tag{127}$$

whereby $R_h = \frac{r}{2} - R2^{-h} > 0$.

Proof. By (114) and the Spectral Support Rule, *cf.* the last part of Theorem 27,

$$\operatorname{supp}\mathcal{F}(a^{k-h}(x,D)u_k) \subset \{\, \xi + \eta \mid (\xi,\eta) \in \operatorname{supp}(\psi_{h-k}\otimes 1)\hat{a},\ r2^{k-1} \le |\eta| \le R2^k \,\}. \tag{128}$$

So by the triangle inequality every $\zeta = \xi + \eta$ in the support fulfils, as $h \ge 2$,

$$r2^{k-1} - R2^{k-h} \le |\zeta| \le R2^{k-h} + R2^k \le \tfrac{5}{4}R2^k. \tag{129}$$

This shows (126) and (127) follows analogously. □

To achieve simpler constants one could take h so large that $4R \le r2^h$, which instead of R_h would allow $r/4$ (and $9R/8$). But the present choice of h is preferred in order to reduce the number of terms in $a^{(2)}(x,D)u$.

In comparison the terms in $a^{(2)}(x,D)u$ only satisfy a dyadic ball condition. Previously this was observed, e.g., for functions $u \in \bigcup H^s$ in [36], as was the fact that when the twisted diagonal condition (51) holds, then the situation improves for large k. This is true for arbitrary u:

Proposition 18. *When $a \in S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$, $u \in \mathcal{S}'(\mathbb{R}^n)$, and r, R are chosen as in (113) for each auxiliary function ψ, then every $h \in \mathbb{N}$ such that $2R < r2^h$ gives*

$$\begin{gathered}\operatorname{supp}\mathcal{F}\big(a_k(x,D)(u^{k-1}-u^{k-h})\big)\bigcup\operatorname{supp}\mathcal{F}\big((a^k-a^{k-h})(x,D)u_k\big)\\ \subset\{\,\xi\in\mathbb{R}^n \mid |\xi|\le 2R2^k\,\}\end{gathered} \tag{130}$$

If (51) holds for some $B \geq 1$, then the support is contained in the annulus

$$\{\,\xi \mid \frac{r}{2^{h+1}B}2^k \le |\xi| \le 2R2^k\,\} \quad \textit{for all } k \ge h+1+\log_2(\frac{B}{r}). \tag{131}$$

Proof. As in Proposition 17, $\operatorname{supp}\mathcal{F}a_k(x,D)(u^{k-1}-u^{k-h})$ is seen to be a subset of

$$\{\,\xi+\eta \mid (\xi,\eta)\in\operatorname{supp}(\varphi_k\otimes 1)\hat{a},\ r2^{k-h}\le|\eta|\le R2^{k-1}\,\}. \tag{132}$$

Thence any ζ in the support fulfils $|\zeta| \le R2^k + R2^{k-1} = (3R/2)2^k$. If (51) holds, then one has $B(1+|\xi+\eta|) \ge |\eta|$ on $\operatorname{supp}\mathcal{F}_{x\to\xi}a$, so for all k larger than the given limit

$$|\zeta| \ge \tfrac{1}{B}|\eta| - 1 \ge \tfrac{1}{B}r2^{k-h} - 1 \ge (\tfrac{r}{2^hB} - 2^{-k})2^k \ge \tfrac{r}{2^{h+1}B}2^k. \tag{133}$$

The term $(a^k - a^{k-h})(x,D)u_k$ is analogous but will cause $3R/2$ to be replaced by $2R$. □

Remark 8. The inclusions in Propositions 17 and 18 have been a main reason for the introduction of the paradifferential splitting (119) in the 1980's, but they were then only derived for elementary symbols; *cf.* [9,10,37]. With the Spectral Support Rule, *cf.* Theorem 27, this restriction is redundant; *cf.* also the remarks to (24) in the introduction.

5.2. Polynomial Bounds

In the treatment of $a^{(1)}(x,D)u$ and $a^{(3)}(x,D)u$ in (120) and (122) one may conveniently commence by observing that, according to Proposition 17, the terms in these series fulfil condition (A1) in Lemma 26 for $\theta_0 = \theta_1 = 1$.

So to deduce their convergence from Lemma 26, it remains to obtain the polynomial bounds in (A2). For this it is natural to use the efficacy of the pointwise estimates in Section 3:

Proposition 19. *If $a(x,\eta)$ is in $S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and $N \ge \operatorname{order}_{\mathcal{S}'}(\mathcal{F}u)$ fulfils $d+N \ne 0$, then*

$$|a^{k-h}(x,D)u_k(x)| \le c2^{k(N+d)}(1+|x|)^N, \tag{134}$$

$$|a_k(x,D)u^{k-h}(x)| \le c2^{k(N+d)_+}(1+|x|)^N, \tag{135}$$

$$|(a^k-a^{k-h})(x,D)u_k(x)| \le c2^{k(N+d)}(1+|x|)^N, \tag{136}$$

$$|a_k(x,D)(u^{k-1}-u^{k-h})(x)| \le c2^{k(N+d)}(1+|x|)^N. \tag{137}$$

Proof. The second inequality follows by taking the two cut-off functions in Proposition 11 as $\Phi = \varphi$ and $\Psi = \psi(2^{-h}\cdot)$. The first claim is seen by interchanging their roles, *i.e.*, for $\Phi = \psi(2^{-h}\cdot)$ and $\Psi = \varphi$; the latter is 0 around the origin so $N+d$ is obtained without the positive part.

Clearly similar estimates hold for the terms in $a^{(2)}(x,D)u$. e.g., taking $\psi - \psi(2^{-h}\cdot)$ and φ, respectively, as the cut-off functions in Proposition 11, one finds for $k \ge h$ the estimate in (136). Note that the positive part can be avoided for $0 \le k < h$ by using a sufficiently large constant. □

The difference in the above estimates appears because u_k in (134) has spectrum in a corona. However, one should not confound this with spectral inclusions like (A1) that one might obtain after application of $a^{k-h}(x,D)$, for these are irrelevant for the pointwise estimates here.

5.3. Induced Paradifferential Operators

Although (120)–(122) yield a well-known splitting, the operator notation $a^{(j)}(x,D)$ requires justification in case of type 1, 1-operators.

Departing from the right hand sides of (120)–(122) one is via (124) led directly to the symbols

$$a^{(1)}(x,\eta) = \sum_{k=h}^{\infty} a^{k-h}(x,\eta)\varphi(2^{-k}\eta) \tag{138}$$

$$a^{(3)}(x,\eta) = \sum_{j=h}^{\infty} a_j(x,\eta)\psi(2^{-(j-h)}\eta). \tag{139}$$

In addition, letting $\delta_{k\geq h}$ stand for 1 when $k \geq h$ and for 0 in case $k < h$,

$$\begin{aligned} a^{(2)}(x,\eta) = \sum_{k=1}^{\infty} \big((a^k(x,\eta) - a^{k-h}(x,\eta))\varphi(2^{-k}\eta) \\ + a_k(x,\eta)(\psi(2^{-(k-1)}\eta) - \psi(2^{-(k-h)}\eta)\delta_{k\geq h})\big) + a^0(x,\eta)\psi(\eta) \end{aligned} \tag{140}$$

These three series converge in the Fréchet space $S^{d+1}_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$, for the sums are locally finite. Therefore, it is clear that

$$a(x,\eta) = a^{(1)}(x,\eta) + a^{(2)}(x,\eta) + a^{(3)}(x,\eta), \tag{141}$$

where some of the partially Fourier transformed symbols have conical supports,

$$\hat{a}^{(1)}(\xi,\eta) \neq 0 \implies |\xi| \leq \tfrac{2R}{r2^h}|\eta|, \qquad \hat{a}^{(3)}(\xi,\eta) \neq 0 \implies |\eta| \leq \tfrac{2R}{r2^h}|\xi|. \tag{142}$$

This well-known fact follows from the supports of ψ and φ. But a sharper exploitation gives

Proposition 20. *For each $a \in S^d_{1,1}$ and every modulation function $\psi \in C_0^\infty(\mathbb{R}^n)$ in (113), the associated symbols $a^{(1)}_\psi(x,\eta)$ and $a^{(3)}_\psi(x,\eta)$ fulfil the twisted diagonal condition (51).*

Proof. When $\hat{a}^{(3)}(\xi,\eta) \neq 0$ it follows from (142), which in particular yields $|\eta| < |\xi|$, that

$$|\xi+\eta| \geq |\xi| - |\eta| \geq |\xi|(1 - \tfrac{2R}{r2^h}) > |\eta|(1 - \tfrac{2R}{r2^h}). \tag{143}$$

Therefore, $\hat{a}^{(3)}(\xi,\eta) = 0$ whenever $B_1|\xi+\eta| < |\eta|$ holds for $B_1 = (1 - \frac{2R}{r2^h})^{-1}$; a fortiori (51) is fulfilled with $B = B_1 > 1$. The case of $a^{(1)}$ is a little simpler. □

To elucidate the role of the twisted diagonal, note that the lower bound in Proposition 17 reappears by using $|\xi| \geq r2^{k-1}$ in the middle of (143).

Anyhow, it is a natural programme to verify that $u \in \mathcal{S}'$ belongs to the domain of the operator $a^{(j)}(x,D)$ precisely when the previously introduced series denoted $a^{(j)}(x,D)u$ converges; *cf.* (120)–(122). In view of the definition by vanishing frequency modulation in (29) ff, this will necessarily be lengthy because a second modulation function Ψ has to be introduced.

To indicate the details for $a^{(1)}(x,\eta)$, let $\psi, \Psi \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 around the origin, and let ψ be used as the fixed modulation function entering $a^{(1)}(x,D) = a^{(1)}_\psi(x,D)$ in (120); and set $\varphi = \psi - \psi(2\cdot)$. The numbers r, R and h are then chosen in relation to ψ as in (113).

Moreover, Ψ is used for the frequency modulation made when Definition 1 is applied to $a^{(1)}_\psi(x,D)$. This gives the following identity in $S^d_{1,1}$, where prime indicates a finite sum,

$$\Psi(2^{-m}D_x)a^{(1)}(x,\eta)\Psi(2^{-m}\eta) = \textstyle\sum_{k=h}^{m+\mu} a^{k-h}(x,\eta)\varphi(2^{-k}\eta) + \sum'_k \Psi(2^{-m}D_x)a^{k-h}(x,\eta)\varphi(2^{-k}\eta)\Psi(2^{-m}\eta). \tag{144}$$

Indeed, if $\lambda, \Lambda > 0$ fulfil that $\Psi(\eta) = 1$ for $|\eta| \leq \lambda$ while $\Psi = 0$ for $|\eta| \geq \Lambda$, the support of $\varphi(2^{-k}\eta)$ in (138) lies by (114) in one of the 'harmless' level sets $\Psi(2^{-m}\eta) = 1$ or $\Psi(2^{-m}\eta) = 0$ when, respectively,

$$R2^k \leq \lambda 2^m \quad \text{or} \quad r2^{k-1} \geq \Lambda 2^m. \tag{145}$$

That is, supp $\varphi(2^{-k}\cdot)$ is contained in these level sets unless k fulfils

$$m + \log_2(\lambda/R) < k < m + 1 + \log_2(\Lambda/r). \tag{146}$$

Therefore, the primed sum has at most $1 + \log_2 \frac{R\Lambda}{r\lambda}$ terms, independently of the parameter m; in addition $\Psi(2^{-m}\eta)$ and $\Psi(2^{-m}D_x)$ disappear from the other terms, as stated in (144).

Consequently, with $\mu = [\log_2(\lambda/R)]$ and $k = m + l$, for $l \in \mathbb{Z}$, one has for $u \in \mathcal{S}'(\mathbb{R}^n)$ that

$$\begin{aligned} \operatorname{OP}(\Psi(2^{-m}D_x)a^{(1)}(x,\eta)\Psi(2^{-m}\eta))u &= \sum_{k=h}^{m+\mu} a^{k-h}(x,D)u_k \\ &+ \sideset{}{'}\sum_{\mu<l<1+\log_2(\Lambda/r)} \operatorname{OP}(\Psi(2^{-m}D_x)\psi(2^{h-l-m}D_x)a(x,\eta)\varphi(2^{-m-l}\eta)\Psi(2^{-m}\eta))u. \end{aligned} \tag{147}$$

A similar reasoning applies to $a^{(3)}(x,\eta)$. The main difference is that the possible inclusion of supp $\varphi(2^{-j}\cdot)$, into the level sets where $\Psi(2^{-m}\cdot)$ equals 1 or 0, in this case applies to the symbol $\Psi(2^{-m}D_x)a_j(x,\eta) = \mathcal{F}^{-1}_{\xi\to x}(\Psi(2^{-m}\xi)\varphi(2^{-j}\xi)\hat{a}(\xi,\eta))$. Therefore, one has for the same μ,

$$\begin{aligned} \operatorname{OP}(\Psi(2^{-m}D_x)a^{(3)}(x,\eta)\Psi(2^{-m}\eta))u &= \sum_{j=h}^{m+\mu} a_j(x,D)u^{j-h} \\ &+ \sideset{}{'}\sum_{\mu<l<1+\log_2(\Lambda/r)} \operatorname{OP}(\Psi(2^{-m}D_x)\varphi(2^{-l-m}D_x)a(x,\eta)\psi(2^{h-m-l}\eta)\Psi(2^{-m}\eta))u. \end{aligned} \tag{148}$$

Treating $a^{(2)}_\psi(x,D)$ analogously, it is not difficult to see that once again the central issue is whether supp $\varphi(2^{-k}\cdot)$ is contained in the set where $\Psi(2^{-m}\cdot) = 1$ or $= 0$. So when $m \geq h$ for simplicity, one has for the same μ, and with primed sums over the same integers l as above,

$$\begin{aligned} \operatorname{OP}(\Psi(2^{-m}D_x)a^{(2)}(x,\eta)\Psi(2^{-m}\eta))u &= \sum_{k=0}^{m+\mu} \left((a^k - a^{k-h})(x,D)u_k + a_k(x,D)(u^{k-1} - u^{k-h})\right) \\ &+ \sum{}' \operatorname{OP}(\Psi(2^{-m}D_x)(a^{m+l}(x,\eta) - a^{m+l-h}(x,\eta))\varphi(2^{-m-l}\eta)\Psi(2^{-m}\eta))u \\ &+ \sum{}' \operatorname{OP}(\Psi(2^{-m}D_x)a_{m+l}(x,\eta)(\psi(2^{1-m-l}\eta) - \psi(2^{h-m-l}\eta))\Psi(2^{-m}\eta))u. \end{aligned} \tag{149}$$

The programme introduced after Proposition 20 is now completed by letting $m \to \infty$ in (147)–(149) and observing that the infinite series in (120)–(122) reappear in this way. Of course, this relies on the fact that the remainders in the primed sums over l can be safely ignored:

Proposition 21. *When $a(x,\eta)$ is given in $S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and Ψ, $\psi \in C_0^\infty(\mathbb{R}^n)$ equal 1 in neighbourhoods of the origin, then it holds for every $u \in \mathcal{S}'(\mathbb{R}^n)$ that each term (with l fixed) in the primed sums in (147) and (148) tends to 0 in $\mathcal{S}'(\mathbb{R}^n)$ for $m \to \infty$.*

This is valid for (149) too, if $a(x,\eta)$ in addition fulfils the twisted diagonal condition (51).

Proof. To show that each remainder term tends to 0 for $m \to \infty$ and fixed l, it suffices to verify (A1) and (A2) in view of Remark 13.

For $a^{(1)}_\psi(x,D)$, note that by repeating the proof of Proposition 17 (ignoring Ψ) each remainder in (147) has ξ in its spectrum only when $(R_h 2^l)2^m \leq |\xi| \leq \frac{5\cdot 2^l}{4}R2^m$.

Moreover, each remainder term is $\leq c2^{k(N+d)}(1+|x|)^N$ for $N \geq \operatorname{order}_{\mathcal{S}'}(\hat{u})$ according to Proposition 11, for with the cut-off functions $\Psi\psi(2^{h-l}\cdot)$ and $\varphi(2^{-l}\cdot)\Psi$ the latter is 0 around the origin. So a crude estimate by $c2^{k(N+d_+)}(1+|x|)^{N+d_+}$ shows that (A2) is fulfilled.

Similarly for the primed sum in (148), where $\psi(2^{h-l}\cdot)\Psi$ is 1 around the origin; which again results in the bound $c2^{k(N+d_+)}(1+|x|)^{N+d_+}$ for $N \neq -d$.

The procedure also works for (149), for (A1) is verified as in Proposition 18, *cf.* (131), because the extra spectral localisations provided by $\Psi(2^{-m}\cdot)$ cannot increase the spectra. For the pointwise estimates one may now use, e.g., $\Psi\varphi(2^{-l}\cdot)$ and $(\psi(2^{1-l}\cdot) - \psi(2^{h-l}\cdot))\Psi$ as the cut-off functions in the last part of (149). This yields the proof of Proposition 21. □

An extension of the proposition's remainder analysis to general $a_\psi^{(2)}(x,D)$ without a condition on the behaviour along the twisted diagonal does not seem feasible. But such results will follow in Section 6 from a much deeper investigation of $a(x,D)$ itself; *cf.* Theorem 24.

Remark 9. The type 1,1-operator $a^{(1)}(x,D)$ induced by (138) is a *paradifferential* operator in the sense of Bony [7], as well as in Hörmander's framework of residue classes in [14] (Chapter 10). The latter follows from (142), but will not be pursued here. $a^{(2)}(x,D)$ and $a^{(3)}(x,D)$ are also called paradifferential operators, following Yamazaki [37]. The decomposition (119)–(122) can be traced back to Kumano-go and Nagase, who used a variant of $a^{(1)}(x,\eta)$ to smooth non-regular symbols, *cf.* [43] (Theorem 1.1). It was exploited in continuity analysis of pseudo-differential operators in, e.g., [20,24,36,37].

Remark 10. For pointwise multiplication decompositions analogous to (119) were used implicitly by Peetre [44], Triebel [45]; and more explicitly in the paraproducts of Bony [7]. Moreover, for $a = a(x)$ Definition 1 reduces to the product $\pi(a,u)$ introduced formally in [46] as

$$\pi(a,u) = \lim_{m\to\infty} a^m \cdot u^m. \tag{150}$$

This was analysed in [46], including continuity properties deduced from (119), that essentially is a splitting of the generalised pointwise product $\pi(\cdot,\cdot)$ into paraproducts. Partial associativity, *i.e.*, $f\pi(a,u) = \pi(fu,a) = \pi(a,fu)$ for $f \in C^\infty$, was first obtained with the refined methods developed later in [16] (Theorem 6.7), though.

6. Action on Temperate Distributions

6.1. Littlewood-Paley Analysis of Type 1, 1-Operators

First the full set of conclusions is drawn for the operators $a^{(j)}(x,D)$, $j = 1,2,3$ studied in Section 5.3. Of course none of them have anomalies if $a(x,\eta)$ fulfils (51):

Theorem 22. *When $a(x,\eta)$ is a symbol in $S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ for some $d\in\mathbb{R}$ and $\psi \in C_0^\infty(\mathbb{R}^n)$ equals 1 around the origin, then the associated type 1,1-operators $a_\psi^{(1)}(x,D)$ and $a_\psi^{(3)}(x,D)$ are everywhere defined continuous linear maps*

$$a_\psi^{(1)}(x,D),\ a_\psi^{(3)}(x,D)\colon \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n), \tag{151}$$

that are given by formulae (120) and (122), where the infinite series converge rapidly in $\mathcal{S}'(\mathbb{R}^n)$ for every $u \in \mathcal{S}'(\mathbb{R}^n)$. The adjoints are also in $\operatorname{OP}(S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n))$.

If furthermore $a(x,\eta)$ fulfils the twisted diagonal condition (51), these conclusions are valid verbatim for the operator $a_\psi^{(2)}(x,D)$ given by the series in (121).

Proof. As the symbols $a^{(1)}(x,\eta)$ and $a^{(3)}(x,\eta)$ both belong to $S^d_{1,1}$ and fulfil (51) by Proposition 20, the corresponding operators are defined and continuous on $\mathcal{S}'$ by Proposition 13, with $a^{(1)}(x,D)^*$ and $a^{(3)}(x,D)^*$ both of type 1,1.

Since $\operatorname{supp} \mathcal{F}_{x\to\xi} a^{(2)} \subset \operatorname{supp} \mathcal{F}_{x\to\xi} a$ it follows that $a^{(2)}(x,D)$ satisfies (51), when $a(x,\eta)$ does so. Hence the preceding argument also applies to $a^{(2)}(x,D)$, so that it is continuous on $\mathcal{S}'$ with its adjoint being of type $1,1$.

Moreover, the series $\sum_{k=0}^{\infty} a^{k-h}(x,D)u_k$ in (120) converges rapidly in $\mathcal{S}'$ for every $u \in \mathcal{S}'$. This follows from 1° of Lemma 26, for the terms fulfil (A1) and (A2) by Proposition 17, *cf.* (126), and Proposition 19, respectively. (The latter gives a bound by $2^{k(N+d_+)}(1+|x|)^{N+d_+}$.) Now the distribution $\sum_{k=0}^{\infty} a^{k-h}(x,D)u_k$ equals $a^{(1)}(x,D)u$ because of formula (147), since the primed sum there goes to 0 for $m \to \infty$, as shown in Proposition 21.

Similarly Lemma 26 yields convergence of the series (122) for $a^{(3)}(x,D)u$ when $u \in \mathcal{S}'$. By Propositions 18 and 19, convergence of the $a^{(2)}$-series in (123) also follows from Lemma 26. The series identify with the operators in view of the remark made prior to Proposition 21. □

It should be emphasized that duality methods and pointwise estimates contribute in two different ways in Theorem 22: once the symbol $a^{(1)}(x,\eta)$ has been introduced, continuity on $\mathcal{S}'(\mathbb{R}^n)$ of the associated type $1,1$-operator $a^{(1)}(x,D)$ is obtained by duality through Proposition 13. However, the pointwise estimates in Section 3 yield (vanishing of the remainder terms, hence) the identification of $a^{(1)}(x,D)u$ with the series in (120). Furthermore, the pointwise estimates also give an explicit proof of the fact that $a^{(1)}(x,D)$ is defined on the entire $\mathcal{S}'(\mathbb{R}^n)$, for the right-hand side of (120) does not depend on the modulation function Ψ. Similar remarks apply to $a^{(3)}(x,D)$. Thus duality methods and pointwise estimates together lead to a deeper analysis of type $1,1$-operators.

Remark 11. Theorem 22 and its proof generalise a result of Coifman and Meyer [47] (Chapter 15) in three ways. They stated Lemma 26 for $\theta_0 = \theta_1 = 1$ and derived a corresponding fact for paramultiplication, though only with a treatment of the first and third term.

Going back to the given $a(x,D)$, one derives from Theorem 22 and (119) the following

Theorem 23. *When $a \in S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ fulfils the twisted diagonal condition (51), then the associated type 1, 1-operator $a(x,D)$ defined by vanishing frequency modulation is an everywhere defined continuous linear map*

$$a(x,D)\colon \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n), \tag{152}$$

with its adjoint $a(x,D)^*$ also in $\mathrm{OP}(S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n))$. The operator fulfils

$$a(x,D)u = a^{(1)}_\psi(x,D)u + a^{(2)}_\psi(x,D)u + a^{(3)}_\psi(x,D)u \tag{153}$$

for every $\psi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 in a neighbourhood of the origin, and the series in (120)–(122) converge rapidly in $\mathcal{S}'(\mathbb{R}^n)$ for every $u \in \mathcal{S}'(\mathbb{R}^n)$.

To extend the discussion to general $a(x,D)$ without vanishing along the twisted diagonal, note that Theorem 22 at least shows that $a^{(1)}(x,D)u$ and $a^{(3)}(x,D)u$ are always defined and that (120) and (122) are operator identities.

It remains to justify the operator notation $a^{(2)}(x,D)$ in (121) and to give its precise relation to $a(x,D)$ itself. The point of departure is of course the symbol splitting (141); the corresponding type $1,1$-operators are still denoted by $a^{(j)}(x,D)$. However, to avoid ambiguity the series in (120)–(122) will now be temporarily written as $A^{(j)}_\psi u$, whence (119) amounts to

$$a(x,D)u = A^{(1)}_\psi u + A^{(2)}_\psi u + A^{(3)}_\psi u \quad \text{for } u \in \mathcal{S}'. \tag{154}$$

Here the left-hand side exists if and only if the series $A_\psi^{(2)}u$ converges, as $A_\psi^{(1)}u$, $A_\psi^{(3)}u$ always converge by Theorem 22. This strongly indicates that (155) below is true. In fact, this main result of the analysis is obtained by frequency modulation:

Theorem 24. *When $a(x,\eta) \in S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and $a^{(2)}(x,\eta)$ denotes the type 1, 1-symbol in (140), derived from the paradifferential decomposition (119), then*

$$D(a^{(2)}(x,D)) = D(a(x,D)) \tag{155}$$

and $u \in \mathcal{S}'(\mathbb{R}^n)$ belongs to these domains if and only if the series (121), or equivalently (123), converges in $\mathcal{D}'(\mathbb{R}^n)$ — in which case (also) formulae (121) and (123) are operator identities.

Proof. A variant of (154) follows at once from (141), using a second modulation function Ψ and the brief notation from Remark 7,

$$a^m(x,D)u^m = \sum_{l=1,2,3} \mathrm{OP}(\Psi(2^{-m}D_x)a^{(l)}(x,\eta)\Psi(2^{-m}\eta))u. \tag{156}$$

Here the terms with $l = 1$ and $l = 3$ always have Ψ-independent limits for $m \to \infty$ according to Theorem 22, so it is clear from the definition by vanishing frequency modulation that $u \in D^{(2)}(a(x,D))$ is equivalent to $u \in D(a(x,D))$, hence to convergence of $A_\Psi^{(2)}u$, *cf.* (154) ff.

As for the last claim, whenever $u \in D(a(x,D))$, then passage to the limit ($m \to \infty$) in the above equation yields the following, when (147), (149) and (148) are applied, now with the remainders in the primed sums there denoted by $R_m^{(1)}u$, $R_m^{(2)}u$, $R_m^{(3)}u$ for brevity:

$$\begin{aligned} a(x,D)u &= \lim_{m\to\infty} \sum_{l=1,2,3} \mathrm{OP}(\Psi(2^{-m}D_x)a^{(l)}(x,\eta)\Psi(2^{-m}\eta))u \\ &= A_\psi^{(1)}u + A_\psi^{(2)}u + A_\psi^{(3)}u + 0 + \lim_{m\to\infty} R_m^{(2)}u + 0. \end{aligned} \tag{157}$$

Note that convergence of $R_m^{(2)}$ follows from that of the other six terms; *cf.* Proposition 21. Compared to (154) this yields $\lim_m R_m^{(2)} = 0$, which via (149) gives that $a^{(2)}(x,D)u = A_\psi^{(2)}u$. □

6.2. The Twisted Diagonal Condition of Arbitrary Order

When $a(x,\eta)$ is in the self-adjoint subclass $\widetilde{S}^d_{1,1}$, then it follows Theorem 16 that the domains in (155) equal $\mathcal{S}'$.

However, it is interesting to give an explicit proof that the domains in (155) equal $\mathcal{S}'$ whenever $a \in \widetilde{S}^d_{1,1}$. This can be done in a natural way by extending the proof of Theorem 24, where the special estimates in (57) enter the convergence proof for $a^{(2)}(x,D)u$ directly, because they are rather close to the symbol factors from the factorisation inequalities in Section 3. The full generality with $\theta_0 < \theta_1$ in the corona criterion Lemma 26 is also needed now.

Theorem 25. *Suppose $a(x,\eta) \in \widetilde{S}^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$, i.e., $a(x,\eta)$ fulfils one of the three equivalent conditions in Theorem 14. Then the conclusions of Theorems 23 and 24 remain valid for $a(x,D)$; in particular one has $D(a(x,D)) = \mathcal{S}'(\mathbb{R}^n)$.*

Proof. The continuity on $\mathcal{S}'$ is assured by Theorem 16. For the convergence of the series in the paradifferential splitting, it is convenient to write, in the notation of (57) ff,

$$a(x,\eta) = (a(x,\eta) - a_{\chi,1}(x,\eta)) + a_{\chi,1}(x,\eta), \tag{158}$$

where $a - a_{\chi,1}$ satisfies (51) for $B = 1$, so that Theorem 23 applies to it. As $a_{\chi,1}$ is in $\widetilde{S}^d_{1,1}$ like a and $a - a_{\chi,1}$ (the latter by Proposition 13), one may reduce to the case in which

$$\hat{a}(x,\eta) \neq 0 \implies \max(1, |\xi + \eta|) \leq |\eta|. \tag{159}$$

Continuing under this assumption, it is according to Theorems 22 and 24 enough to show for all $u \in \mathcal{S}'$ that there is convergence of the two contributions to $a^{(2)}(x,D)u$,

$$\sum_{k=0}^{\infty} (a^k - a^{k-h})(x,D)u_k, \qquad \sum_{k=1}^{\infty} a_k(x,D)(u^{k-1} - u^{k-h}). \tag{160}$$

Since the terms here are functions of polynomial growth by Proposition 19, it suffices to improve the estimates there; and to do so for $k \geq h$.

Using Hörmander's localisation to a neighbourhood of $\mathcal{T}$, *cf.* (54)–(56), one arrives at

$$\hat{a}_{k,\chi,\varepsilon}(\xi,\eta) = \hat{a}(\xi,\eta)\varphi(2^{-k}\xi)\chi(\xi+\eta, \varepsilon\eta), \tag{161}$$

This leaves the remainder $b_k(x,\eta) = a_k(x,\eta) - a_{k,\chi,\varepsilon}(x,\eta)$, that applied to the above difference $v_k = u^{k-1} - u^{k-h} = \mathcal{F}^{-1}((\varphi(2^{1-k}\cdot) - \varphi(2^{h-k}\cdot))\hat{u})$ gives

$$a_k(x,D)v_k = a_{k,\chi,\varepsilon}(x,D)v_k + b_k(x,D)v_k. \tag{162}$$

To utilise the pointwise estimates, fix $N \geq \operatorname{order}_{\mathcal{S}'}(\hat{u})$ so that $d + N \neq 0$; and pick $\Psi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 in a neighbourhood of the corona $\frac{r}{R}2^{-1-h} \leq |\eta| \leq 1$ and equal to 0 outside the set with $\frac{r}{R}2^{-2-h} \leq |\eta| \leq 2$. Taking the dilated function $\Psi(\eta/(R2^k))$ as the auxiliary function in the symbol factor, the factorisation inequality (65) and Theorem 8 give

$$|a_{k,\chi,\varepsilon}(x,D)v_k(x)| F_{a_{k,\chi,\varepsilon}}(N, R2^k; x) v_k^*(N, R2^k; x) \tag{163}$$

which is estimated from above by

$$cv_k^*(x) \sum_{|\alpha|=0}^{N+[n/2]+1} \Big(\int_{r2^{k-h-2} \leq |\eta| \leq R2^{k+1}} |(R2^k)^{|\alpha|-n/2} D_\eta^\alpha a_{k,\chi,\varepsilon}(x,\eta)|^2 \, d\eta \Big)^{1/2}. \tag{164}$$

Here the ratio of the limits is $2R/(r2^{-h-2}) > 32$, so with extension to $R2^{k+1-L} \leq |\eta| \leq R2^{k+1}$, there is $L \geq 6$ dyadic coronas. This gives an estimate by $c(R2^k)^d L^{1/2} N_{\chi,\varepsilon,\alpha}(a_k)$. In addition, Minkowski's inequality gives

$$\begin{aligned} N_{\chi,\varepsilon,\alpha}(a_k) &\leq \sup_{\rho>0} \rho^{|\alpha|-d} \int_{\mathbb{R}^n} |2^{kn}\check{\varphi}(2^k y)| \Big(\int_{\rho \leq |\eta| \leq 2\rho} |D_\eta^\alpha a_{\chi,\varepsilon}(x-y,\eta)|^2 \frac{d\eta}{\rho^n} \Big)^{1/2} dy \\ &\leq c N_{\chi,\varepsilon,\alpha}(a). \end{aligned} \tag{165}$$

So it follows from the above that

$$|a_{k,\chi,\varepsilon}(x,D)v_k(x)| \leq cv_k^*(N, R2^k; x) \Big(\sum_{|\alpha| \leq N+[n/2]+1} c_{\alpha,\sigma}\varepsilon^{\sigma+n/2-|\alpha|} \Big) L^{1/2} (R2^k)^d. \tag{166}$$

Using Lemma 7 and taking $\varepsilon = 2^{-k\theta}$, say for $\theta = 1/2$ this gives

$$|a_{k,\chi,2^{-k\theta}}(x,D)v_k(x)| \leq c(1+|x|)^N 2^{-k(\sigma-1-2d-3N)/2}. \tag{167}$$

Choosing $\sigma > 3N + 2d + 1$, the series $\sum_k \langle\, a_{k,\chi,\varepsilon}(x,D)v_k,\, \phi \,\rangle$ converges rapidly for $\phi \in \mathcal{S}$.

To treat $\sum_{k=0}^{\infty} b_k(x,D)v_k$ it is observed that $\hat{a}_{k,\chi,2^{-k\theta}}(x,\eta) = \hat{a}_k(x,\eta)$ holds in the set where $\chi(\xi+\eta, 2^{-k\theta}\eta) = 1$, that is, when $2\max(1,|\xi+\eta|) \le 2^{-k\theta}|\eta|$, so by (159),

$$\operatorname{supp} \hat{b}_k \subset \{\, (\xi,\eta) \mid 2^{-1-k\theta}|\eta| \le \max(1,|\xi+\eta|) \le |\eta| \,\}. \tag{168}$$

This implies by Theorem 27 that $\zeta = \xi+\eta$ is in $\operatorname{supp} \mathcal{F} b_k(x,D)v_k$ only if both

$$|\zeta| \le |\eta| \le R2^k \tag{169}$$

$$\max(1,|\zeta|) \ge 2^{-1-k\theta}|\eta| \ge r2^{k(1-\theta)-h-2}. \tag{170}$$

When k fulfils $2^{k(1-\theta)} > 2^{h+2}/r$, so that the last right-hand side is >1, these inequalities give

$$(r2^{-h-2})2^{k(1-\theta)} \le |\zeta| \le R2^k. \tag{171}$$

This shows that the corona condition (A1) in Lemma 26 is fulfilled for $\theta_0 = 1-\theta = 1/2$ and $\theta_1 = 1$, and the growth condition (A2) is easily checked since both $a_{k,\chi,\varepsilon}(x,D)v_k$ and $a_k(x,D)v_k$ are estimated by $2^{k(N+d_+)}(1+|x|)^{N+d_+}$, as can be seen from (167) and Proposition 11, respectively. Hence $\sum b_k(x,D)v_k$ converges rapidly.

For the series $\sum_{k=0}^{\infty} |\langle (a^k - a^{k-h})(x,D)u_k, \phi\rangle|$ it is not complicated to modify the above. Indeed, the pointwise estimates of the v_k^* are easily carried over to u_k^*, for $R2^k$ was used as the outer spectral radius of v_k; and $r2^{k-h-1}$ may also be used as the inner spectral radius of u_k. In addition the symbol $a^k - a^{k-h}$ can be treated by replacing $\varphi(2^{-k}\xi)$ by $\psi(2^{-k}\xi) - \psi(2^{h-k}\xi)$ in (161) ff., for the use of Minkowski's inequality will now give the factor $\int |\psi - \psi(2^h\cdot)|\,dy$ in the constant. For the remainder

$$\tilde{b}_k(x,D)u_k = (a^k - a^{k-h})(x,D)u_k - (a^k - a^{k-h})_{\chi,\varepsilon}(x,D)u_k \tag{172}$$

one can apply the treatment of $b_k(x,D)v_k$ verbatim. $\square$

Remark 12. The analysis in Theorem 25 is also exploited in the L_p-theory of type 1, 1-operators in [17]. Indeed, the main ideas of the above proof was used in [17] (Section 5.3) to derive certain continuity results in the Lizorkin-Triebel scale $F^s_{p,q}(\mathbb{R}^n)$ for $p < 1$, which (except for a small loss of smoothness) generalise results of Hounie and dos Santos Kapp [26].

7. Final Remarks

In view of the satisfying results on type 1, 1-operators in $\mathcal{S}'(\mathbb{R}^n)$, *cf.* Section 6, and the continuity results in the scales H^s_p, C^s_*, $F^s_{p,q}$ and $B^s_{p,q}$ presented in [17], their somewhat unusual definition by vanishing frequency modulation in Definition 1 should be well motivated.

As an open problem, it remains to characterise the type 1, 1-operators $a(x,D)$ that are everywhere defined and continuous on $\mathcal{S}'(\mathbb{R}^n)$. For this it was shown above to be sufficient that $a(x,\eta)$ is in $\widetilde{S}^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$, and it could of course be conjectured that this is necessary as well.

Similarly, since the works of Bourdaud and Hörmander, *cf.* [9] (Chapter IV), [10,12,13] and also [14], it has remained an open problem to determine

$$\mathbb{B}(L_2(\mathbb{R}^n)) \cap \operatorname{OP}(S^0_{1,1}). \tag{173}$$

Indeed, this set was shown by Bourdaud to contain the self-adjoint subclass $\operatorname{OP}(\widetilde{S}^0_{1,1})$, and this sufficient condition has led some authors to a few unfortunate statements, for example that lack of L_2-boundedness for $\operatorname{OP}(S^0_{1,1})$ is "attributable to the lack of self adjointness". But self-adjointness is not necessary, since already Bourdaud, by modification of Ching's operator (37), gave an example [10] (p. 1069) of an operator $\sigma(x,D)$ in $\mathbb{B}(L_2) \bigcap \operatorname{OP}(S^0_{1,1} \setminus \widetilde{S}^0_{1,1})$; that is, this $\sigma(x,D)^*$ is not of type 1, 1.

However, it could be observed that $N_{\chi,\varepsilon,\alpha}(a_\theta) = \mathcal{O}(\varepsilon^{n/2-|\alpha|})$ by Lemma 6 is valid for Ching's symbol a_θ and that this estimate is sharp for the L_2-unbounded version of $a_\theta(x,D)$, by the last part of Example 5. Therefore, the condition

$$N_{\chi,\varepsilon,\alpha}(a) = o(\varepsilon^{n/2-|\alpha|}) \quad \text{for } \varepsilon \to 0 \tag{174}$$

is conjectured to be necessary for L_2-continuity of a given $a(x,D)$ in $\mathrm{OP}(S^0_{1,1})$.

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Appendix A. Dyadic Corona Criteria

Convergence of a series $\sum_{j=0}^\infty u_j$ of temperate distributions follows if the u_j both fulfil a growth condition and have their spectra in suitable dyadic coronas. This is a special case of Lemma 26, which for $\theta_0 = \theta_1 = 1$ was given by Coifman and Meyer [47] (Chapter 15) without arguments.

Extending the proof given in [48], the refined version in Lemma 26 allows the inner and outer radii of the spectra to grow at different exponential rates $\theta_0 < \theta_1$, even though the number of overlapping spectra increases with j. This is crucial for Theorem 25, so a full proof is given.

Lemma 26. 1° *Let $(u_j)_{j\in\mathbb{N}_0}$ be a sequence in $\mathcal{S}'(\mathbb{R}^n)$ fulfilling that there exist $A > 1$ and $\theta_1 \geq \theta_0 > 0$ such that* $\operatorname{supp} \hat{u}_0 \subset \{\, \xi \mid |\xi| \leq A \,\}$ *while for $j \geq 1$*

$$\operatorname{supp} \hat{u}_j \subset \{\, \xi \mid \tfrac{1}{A} 2^{j\theta_0} \leq |\xi| \leq A 2^{j\theta_1} \,\}, \tag{A1}$$

and that for suitable constants $C \geq 0$, $N \geq 0$,

$$|u_j(x)| \leq C 2^{jN\theta_1} (1+|x|)^N \textit{ for all } j \geq 0. \tag{A2}$$

Then $\sum_{j=0}^\infty u_j$ converges rapidly in $\mathcal{S}'(\mathbb{R}^n)$ to a distribution u, for which $\hat{u}$ is of order N.

2° *For every $u \in \mathcal{S}'(\mathbb{R}^n)$ both (A1) and (A2) are fulfilled for $\theta_0 = \theta_1 = 1$ by the functions $u_0 = \Phi_0(D)u$ and $u_j = \Phi(2^{-j}D)u$ when $\Phi_0, \Phi \in C_0^\infty(\mathbb{R}^n)$ and $0 \notin \operatorname{supp}\Phi$. In particular this is the case for a Littlewood-Paley decomposition $1 = \Phi_0 + \sum_{j=1}^\infty \Phi(2^{-j}\xi)$.*

Proof. In 2° it is clear that Φ is supported in a corona, say $\{\, \xi \mid \frac{1}{A} \leq |\xi| \leq A \,\}$ for a large $A > 0$; hence (A1). (A2) follows from the proof of Lemma 7.

The proof of 1° exploits a well-known construction of an auxiliary function: taking $\psi_0 \in C_0^\infty(\mathbb{R}^n)$ depending on $|\xi|$ alone and so that $0 \leq \psi_0 \leq 1$ with $\psi_0(\xi) = 1$ for $|\xi| \leq 1/(2A)$ while $\psi_0(\xi) = 0$ for $|\xi| \geq 1/A$, then

$$\frac{d}{dt}\psi_0(\frac{\xi}{t}) = \psi(\frac{\xi}{t})\frac{1}{t} \quad \text{for} \quad \psi(\xi) = -\xi \cdot \nabla\psi_0(\xi), \tag{A3}$$

which by integration for $1 \leq t \leq \infty$ gives an uncountable partition of unity

$$1 = \psi_0(\xi) + \int_1^\infty \psi(\frac{\xi}{t})\,\frac{dt}{t}, \quad \xi \in \mathbb{R}^n. \tag{A4}$$

Clearly the support of $\psi(\xi/t)$ is compact and given by $A|\xi| \leq t \leq 2A|\xi|$ when ξ is fixed. For $j \geq 1$ this implies

$$\hat{u}_j = \hat{u}_j\psi_0 + \hat{u}_j \int_1^\infty \psi(\frac{\xi}{t})\,\frac{dt}{t} = \hat{u}_j \int_{2^{j\theta_0}}^{A^2 2^{j\theta_1+1}} \psi(\frac{\xi}{t})\,\frac{dt}{t}. \tag{A5}$$

Defining $\psi_j \in C_0^\infty(\mathbb{R}^n)$ as the last integral here, $\psi_j = 1$ on $\operatorname{supp} \hat{u}_j$; so if $\varphi \in \mathcal{S}$,

$$|\langle u_j, \overline{\varphi} \rangle| \le \big\|(1+|x|^2)^{-\frac{N+n}{2}} u_j\big\|_2 \big\|(1+|x|^2)^{\frac{N+n}{2}} \mathcal{F}^{-1}(\psi_j \hat{\varphi})\big\|_2. \tag{A6}$$

The first norm is $\mathcal{O}(2^{N\theta_1 j})$ by (A2). For the second, note that

$$\operatorname{supp} \psi_j \subset \{\, \xi \in \mathbb{R}^n \mid A^{-1} 2^{j\theta_0 - 1} \le |\xi| \le A 2^{j\theta_1 + 1} \,\} \tag{A7}$$

and $\|D^\alpha \psi_j\|_\infty \le 2^{-j\theta_0|\alpha|} \|D^\alpha \psi\|_\infty / |\alpha|$ for $\alpha \ne 0$ while $\|\psi_j\|_\infty \le \operatorname{diam}(\psi_0(\mathbb{R}^n)) \le 1$ by (A3). In addition the identity $(1+|x|^2)^{N+n} \mathcal{F}^{-1} = \mathcal{F}^{-1}(1-\Delta)^{N+n}$ gives for arbitrary $k > 0$,

$$\begin{aligned} &\|(1+|x|^2)^{N+n} \mathcal{F}^{-1}(\psi_j \hat{\varphi})\|_2 \\ &\qquad \le \sum_{|\alpha|,|\beta| \le N+n} c_{\alpha,\beta} \|D^\alpha \psi_j\|_\infty \|(1+|\xi|)^{k+n/2} D^\beta \hat{\varphi}\|_\infty \Big(\int_{2^{j\theta_0 - 1}/A}^\infty r^{-1-2k}\, dr \Big)^{1/2}. \end{aligned} \tag{A8}$$

Here $\|D^\alpha \psi_j\|_\infty = \mathcal{O}(1)$, so because of the L_2-norm the above is $\mathcal{O}(2^{-jk\theta_0})$ for every $k > 0$.

Hence $\langle u_j, \overline{\varphi} \rangle = \mathcal{O}(2^{j(\theta_1 N - \theta_0 k)})$, so $k > N\theta_1/\theta_0$ yields that $\sum_{j=0}^\infty \langle u_j, \varphi \rangle$ converges. $\square$

Remark 13. The above proof yields that the conjunction of (A1) and (A2) implies $\langle u_j, \varphi \rangle = \mathcal{O}(2^{-jN})$ for all $N > 0$; hence there is *rapid* convergence of $u = \sum_{j=0}^\infty u_j$ in $\mathcal{S}'$ in the sense that $\langle u - \sum_{j<k} u_j, \varphi \rangle = \sum_{j \ge k} \langle u_j, \varphi \rangle = \mathcal{O}(2^{-kN})$ for $N > 0$, $\varphi \in \mathcal{S}$.

Appendix B. The Spectral Support Rule

To control the spectrum of $x \mapsto a(x,D)u$, *i.e.*, the support of $\xi \mapsto \mathcal{F}a(x,D)u$, there is a simple rule which is recalled here for the reader's convenience.

Writing $\mathcal{F}a(x,D)\mathcal{F}^{-1}(\hat{u})$ instead, the question is clearly how the support of $\mathcal{F}u$ is changed by the conjugated operator $\mathcal{F}a(x,D)\mathcal{F}^{-1}$. In terms of its distribution kernel $\mathcal{K}(\xi,\eta)$, *cf.* (15), one should expect the spectrum of $a(x,D)u$ to be contained in

$$\Xi := \operatorname{supp} \mathcal{K} \circ \operatorname{supp} \mathcal{F}u = \{\, \xi \in \mathbb{R}^n \mid \exists \eta \in \operatorname{supp} \hat{u} \colon (\xi,\eta) \in \operatorname{supp} \mathcal{K} \,\}. \tag{B1}$$

For $\operatorname{supp} \mathcal{F}u \Subset \mathbb{R}^n$ this was proved in [36]; but in general the closure $\overline{\Xi}$ should be used instead:

Theorem 27. *Let $a \in S^\infty_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and suppose $u \in D(a(x,D))$ has the property that (29) holds in the topology of $\mathcal{S}'(\mathbb{R}^n)$ for some $\psi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 around the origin. Then*

$$\operatorname{supp} \mathcal{F}(a(x,D)u) \subset \overline{\Xi}, \tag{B2}$$

$$\Xi = \{\, \xi + \eta \mid (\xi,\eta) \in \operatorname{supp} \mathcal{F}_{x \to \xi} a,\ \eta \in \operatorname{supp} \mathcal{F}u \,\}. \tag{B3}$$

When $u \in \mathcal{F}^{-1}\mathcal{E}'(\mathbb{R}^n)$ the $\mathcal{S}'$-convergence holds automatically and Ξ is closed for such u.

The reader is referred to [16] for the deduction of this from the kernel formula. Note that whilst (15) yields (B3), it suffices for (B2) to take any $v \in C_0^\infty(\mathbb{R}^n)$ with support disjoint from Ξ and verify that

$$\langle \mathcal{F}a(x,D)\mathcal{F}^{-1}\hat{u}, v \rangle = \langle \mathcal{K}, v \otimes \hat{u} \rangle = 0. \tag{B4}$$

Here the middle expression makes sense as $\langle (v \otimes \hat{u})\mathcal{K}, 1 \rangle$, as noted in [16], using the remarks to Definition 3.1.1 in [41]. However, the first equality sign is in general not trivial to justify: the limit in Definition 1 is decisive for this.

Remark 14. There is a simple proof of (B2) in the main case that $\hat{u} \in \mathcal{E}'$: If $a \in S^d_{1,0}$ and v is as above, then (B1) yields $\operatorname{dist}(\operatorname{supp}\mathcal{K}, \operatorname{supp}(v \otimes \hat{u})) > 0$ since $\operatorname{supp}\hat{u} \Subset \mathbb{R}^n$. So with $\hat{u}_\varepsilon = \varphi_\varepsilon * \hat{u}$ for some $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\overset{\wedge}{\varphi}(0) = 1$, $\varphi_\varepsilon = \varepsilon^{-n}\varphi(\cdot/\varepsilon)$, all sufficiently small $\varepsilon > 0$ give

$$\operatorname{supp}\mathcal{K} \bigcap \operatorname{supp} v \otimes \hat{u}_\varepsilon = \emptyset. \tag{B5}$$

Therefore, one has, since $\hat{u}_\varepsilon \in C_0^\infty(\mathbb{R}^n)$,

$$\langle\, \mathcal{F}a(x,D)\mathcal{F}^{-1}\hat{u},\, v \,\rangle = \lim_{\varepsilon\to 0}\langle\, \mathcal{F}a(x,D)\mathcal{F}^{-1}\hat{u}_\varepsilon,\, v \,\rangle = \lim_{\varepsilon\to 0}\langle\, \mathcal{K},\, v \otimes \hat{u}_\varepsilon \,\rangle = 0. \tag{B6}$$

For general $b(x,\eta)$ in $S^d_{1,1}$ one may set $a(x,\eta) = b(x,\eta)\chi(\eta)$ for a $\chi \in C_0^\infty$ equal to 1 on an open ball containing $\operatorname{supp}\hat{u}$. Then a is in $S^{-\infty}$ with associated kernel $\mathcal{K}_a(\xi,\eta) = \mathcal{K}_b(\xi,\eta)\chi(\eta)$ because of (15). Moreover, the set Ξ is unchanged by this replacement, so (B6) gives

$$\langle\, \mathcal{F}b(x,D)\mathcal{F}^{-1}\hat{u},\, v \,\rangle = \langle\, \mathcal{F}a(x,D)\mathcal{F}^{-1}\hat{u},\, v \,\rangle = \lim_{\varepsilon\to 0}\langle\, \mathcal{K}_a,\, v \otimes \hat{u}_\varepsilon \,\rangle = 0. \tag{B7}$$

The argument in Remark 14 clearly covers the applications of Theorem 27 in this paper.

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