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Article

Convergence Aspects for Generalizations of q-Hypergeometric Functions

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Abstract: In an earlier paper, we found transformation and summation formulas for 43 q-hypergeometric functions of $2n$ variables. The aim of the present article is to find convergence regions and a few conjectures of convergence regions for these functions based on a vector version of the Nova q-addition. These convergence regions are given in a purely formal way, extending the results of Karlsson (1976). The Γ_q-function and the q-binomial coefficients, which are used in the proofs, are adjusted accordingly. Furthermore, limits and special cases for the new functions, e.g., q-Lauricella functions and q-Horn functions, are pointed out.

Keywords: q-Stirling formula; even number of variables; Nova q-addition; inequality

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1. Introduction

The standard work for multiple hypergeometric functions is [1], written by Karlsson and Srivastava. In a preprint from 1976 [2], Per Karlsson found that the restriction of multiple hypergeometric functions to an even number of variables gives a large amount of symmetry and, thus, gives clearly defined convergence regions, integral representations, transformations and reducible cases.

Based on [2] and an earlier paper [3], the aim of the present study is to present convergence regions for q-functions of $2n$ variables. Our philosophy is that $2n$ is an even number, such that the missing generalizations, or some of them at least, could be discovered by consideration of suitable hypergeometric functions, which depend on an even number of variables. We do expect such functions to possess more complicated parameter systems than the four q-Lauricella functions; on the other hand, they should not be so complicated that a practical notation becomes impossible; a reasonably high symmetry will be required. We made the decision that this study should comprise the 43 functions defined in Definitions 10 and 11. Loosely speaking, we may describe them as certain q-hypergeometric functions of $2n$ variables having parameters associated with $1, 2, n$ or $2n$ variables (but not any two, nor any n); for $n = 1$, they reduce to Appell, Horn, Humbert and simpler q-hypergeometric functions.

The formal q-integral representations work best for q-Appell- and q-Lauricella functions and will be given in a subsequent article. We make a brief repetition of these q-Appell- and q-Lauricella functions, with corresponding convergence regions; our 43 functions are natural generalizations, as well as their convergence regions. For brevity, the definitions will be given in table form, as well as their special cases. To make the proofs in the current article, we use vector versions of the Γ_q-function, the q-binomial coefficients and the q-Stirling formula. A couple of lemmas are necessary for these proofs.

The new definitions are given summarily in tables (which occupy far less space than the corresponding sequences of equations); certain results are merely suggested; detailed proofs are given only in certain cases; and conditions of validity are mostly given in introductory remarks, not together with each result. Since methods and results are natural generalizations of those known from the classical theory of (multiple) hypergeometric functions, the concentrated exposition is believed to be acceptable. The usefulness of further investigations is not to be excluded.

This paper is organized as follows: In this section, we give the basic definitions of the first q-functions, together with the Nova q-addition. In Section 2, we define the 43 q-functions of $2n$ variables, together with their limits and special cases. In Section 3, some lemmas are stated and proven. In Section 4, we derive convergence regions of the new functions by the q-Stirling formula.

Definition 1. *Let the q-shifted factorial be defined by:*

$$\langle a;q\rangle_N \equiv \begin{cases} 1, & N=0 \\ \prod_{m=0}^{N-1}(1-q^{a+m}) & N=1,2,\ldots \end{cases} \tag{1}$$

We can write vectors of q-shifted factorials in two ways: Like Exton

$$\langle (g);\vec{q}\rangle_{\vec{k}} \equiv \prod_{j=1}^{n}\langle g_j;q_j\rangle_{k_j} \tag{2}$$

or:

$$\langle a_1,\ldots,a_n;q\rangle_k \equiv \prod_{j=1}^{n}\langle a_j;q\rangle_k \tag{3}$$

A special notation that is sometimes used is:

$$\langle (\alpha);\vec{q}\rangle_{\vec{i}+\vec{j}^-} \equiv \langle \alpha_1;q_1\rangle_{i_1+j_n}\prod_{k=2}^{n}\langle \alpha_k;q_k\rangle_{i_k+j_{k-1}} \tag{4}$$

We have the following inequalities, q-analogues of [2] (p. 11):

$$\langle 1;q\rangle_{|i|} \geq \langle (1);q\rangle_{\vec{i}} \tag{5}$$

$$\langle(1);q\rangle_{i\vec{+}j} \geq \langle(1);q\rangle_{\vec{i}}\langle(1);q\rangle_{\vec{j}} \tag{6}$$

$$\binom{|i+j|}{|i|}_q \geq \binom{i\vec{+}j}{\vec{i}}_q \tag{7}$$

These inequalities motivate the following:

Definition 2. *The q-binomial coefficients are defined in the usual way, and furthermore, we have the following extension:*

$$\binom{|i+j|}{\vec{i}}_q \equiv \frac{\langle 1;q\rangle_{|i+j|}}{\langle 1;q\rangle_{\vec{i}}\langle 1;q\rangle_{\vec{j}}} \tag{8}$$

The q-multinomial coefficient is defined by:

$$\binom{n}{k_1,k_2,\ldots,k_m}_q \equiv \frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_{k_1}\langle 1;q\rangle_{k_2}\ldots\langle 1;q\rangle_{k_m}} \tag{9}$$

where $k_1+k_2+\ldots+k_m=n$. If $\vec{m}$ and $\vec{k}$ are two arbitrary vectors of positive integers with l elements, their q-binomial coefficient is defined as:

$$\binom{\vec{m}}{\vec{k}}_{\vec{q}} \equiv \prod_{j=1}^{l}\binom{m_j}{k_j}_{q_j} \tag{10}$$

We now come to the definition that forms the basis for most of the convergence regions given in this article.

Definition 3. *The notation $\sum_{\vec{m}}$ denotes a multiple summation with the indices $m_1,\ldots,m_n$ running over all non-negative integer values.*

Given an integer k, the formula:

$$m_0+m_1+\ldots+m_j=k \tag{11}$$

determines a set $J_{m_0,\ldots,m_j}\in\mathbb{N}^{j+1}$.

Then, if $f(x)$ is the formal power series $\sum_{l=0}^{\infty}a_lx^l$, its k'-th NWA-power is given by:

$$(\oplus_{q,l=0}^{\infty}a_lx^l)^k \equiv (a_0\oplus_q a_1x\oplus_q\ldots)^k \equiv \sum_{|\vec{m}|=k}\prod_{m_l\in J_{m_0,\ldots,m_j}}(a_lx^l)^{m_l}\binom{k}{\vec{m}}_q \tag{12}$$

The following important function is used in all convergence proofs.

Definition 4. *The q-gamma function is given by:*

$$\Gamma_q(z) \equiv \frac{\langle 1;q\rangle_\infty}{\langle z;q\rangle_\infty}(1-q)^{1-z},\ 0<|q|<1 \tag{13}$$

To save space, the following notation for quotients of Γ_q functions will often be used. If the function values are vectors, we mean the corresponding products of q-gamma functions.

$$\Gamma_q\left[\begin{array}{c}a_1,\ldots,a_p\\ b_1,\ldots,b_r\end{array}\right] \equiv \frac{\Gamma_q(a_1)\ldots\Gamma_q(a_p)}{\Gamma_q(b_1)\ldots\Gamma_q(b_r)} \tag{14}$$

$$\Gamma_{\vec{q}}\left[\begin{array}{c}\vec{a_1},\ldots,\vec{a_p}\\ \vec{b_1},\ldots,\vec{b_r}\end{array}\right]\equiv\frac{\Gamma_q(\vec{a}_1)\ldots\Gamma_q(\vec{a}_p)}{\Gamma_q(\vec{b}_1)\ldots\Gamma_q(\vec{b}_r)} \tag{15}$$

where we have used the notation:

$$\Gamma_q(\vec{a})\equiv\prod_{k=1}^{n}\Gamma_q(a_k) \tag{16}$$

Definition 5. *Let a and b be any elements with commutative multiplication. Then, the NWA q-addition is given by:*

$$(a\oplus_q b)^n\equiv\sum_{k=0}^{n}\binom{n}{k}_q a^k b^{n-k},\ n=0,1,2,\ldots \tag{17}$$

In many cases (see [4–6]), the convergence condition can be stated $(x\oplus_q y)^n<1$.

Definition 6. *The statement:*

$$(x\oplus_q y)^n<1,\ n>N_0,\ n\in\mathbb{N} \tag{18}$$

is denoted by $x\oplus_q y<1$ and similarly for a finite number of letters.

Definition 7. *The four q-Appell functions [5] are given by:*

$$\Phi_1(a;b,b';c|q;x_1,x_2)\equiv\sum_{m_1,m_2=0}^{\infty}\frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}}x_1^{m_1}x_2^{m_2}$$
$$\max(|x_1|,|x_2|)<1 \tag{19}$$

$$\Phi_2(a;b,b';c,c'|q;x_1,x_2)\equiv\sum_{m_1,m_2=0}^{\infty}\frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}}x_1^{m_1}x_2^{m_2}$$
$$|x_1|\oplus_q|x_2|<1 \tag{20}$$

$$\Phi_3(a,a';b,b';c|q;x_1,x_2)\equiv\sum_{m_1,m_2=0}^{\infty}\frac{\langle a;q\rangle_{m_1}\langle a';q\rangle_{m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}}x_1^{m_1}x_2^{m_2}$$
$$\max(|x_1|,|x_2|)<1 \tag{21}$$

$$\Phi_4(a;b;c,c'|q;x_1,x_2)\equiv\sum_{m_1,m_2=0}^{\infty}\frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1+m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}}x_1^{m_1}x_2^{m_2}$$
$$|\sqrt{x_1}|\oplus_q|\sqrt{x_2}|<1 \tag{22}$$

Definition 8. *The q-Lauricella functions [4] are given by:*

$$\Phi_{\mathrm{A}}^{(n)}(a,\vec{b};\vec{c}|q;\vec{x})\equiv\sum_{\vec{m}}\frac{\langle a;q\rangle_m\langle\vec{b};q\rangle_{\vec{m}}\vec{x}^{\vec{m}}}{\langle\vec{c},\vec{1};q\rangle_{\vec{m}}},\ |x_1|\oplus_q\ldots\oplus_q|x_n|<1 \tag{23}$$

$$\Phi_{\mathrm{B}}^{(n)}(\vec{a},\vec{b};c|q;\vec{x})\equiv\sum_{\vec{m}}\frac{\langle\vec{a},\vec{b};q\rangle_{\vec{m}}\vec{x}^{\vec{m}}}{\langle c;q\rangle_m\langle\vec{1};q\rangle_{\vec{m}}},\ \max(|x_1|,\cdots,|x_n|)<1 \tag{24}$$

$$\Phi_{\mathrm{C}}^{(n)}(a,b;\vec{c}|q;\vec{x})\equiv\sum_{\vec{m}}\frac{\langle a,b;q\rangle_m\vec{x}^{\vec{m}}}{\langle\vec{c},\vec{1};q\rangle_{\vec{m}}},\ \sqrt{x_1}\oplus_q\ldots\oplus_q\sqrt{x_n}<1 \tag{25}$$

$$\Phi_{\mathrm{D}}^{(n)}(a,b_1,\ldots,b_n;c|q;x_1,\ldots,x_n)\equiv \sum_{\vec{m}}\frac{\langle a;q\rangle_m\prod_{j=1}^{n}\langle b_j;q\rangle_{m_j}x_j^{m_j}}{\langle c;q\rangle_m\prod_{j=1}^{n}\langle 1;q\rangle_{m_j}},\ \max(|x|,\ldots,|x_n|)<1 \tag{26}$$

Definition 9. *We also put*

$$\mathrm{F}_2(\vec{a},\vec{a'};c|q;\vec{x},\vec{y})\equiv\sum_{\vec{i},\vec{j}}\frac{\langle\vec{a};q\rangle_{\vec{i}}\langle\vec{a'};q\rangle_{\vec{j}}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}}{\langle c;q\rangle_{|i|+|j|}\langle\vec{1};q\rangle_{\vec{i}}\langle\vec{1};q\rangle_{\vec{j}}} \tag{27}$$

$$\Psi_2(a;\vec{c},\vec{c'}|q;\vec{x},\vec{y})\equiv\sum_{\vec{i},\vec{j}}\frac{\langle a;q\rangle_{|i|+|j|}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}}{\langle\vec{c},\vec{1};q\rangle_{\vec{i}}\langle\vec{c'},\vec{1};q\rangle_{\vec{j}}} \tag{28}$$

We will use the following q-Stirling formula [7]:

$$\Gamma_q(z)\sim\{z\}_q^{\,z-\frac{1}{2}} \tag{29}$$

2. 43 q-Functions of $2n$ Variables

The following functions were defined in [3]; with the exception of *I,J,L,M*, capital italics denote sums, e.g.,

$$A\equiv\sum_{j=1}^{n}a_j \tag{30}$$

Definition 10. *Power series in $2n$ variables are written:*

$$\mathrm{F}(\vec{x},\vec{y})=\sum_{\vec{i},\vec{j}}\frac{\Psi(\vec{i},\vec{j})\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}}{\langle 1;q\rangle_{\vec{i}}\langle 1;q\rangle_{\vec{j}}} \tag{31}$$

If there is a q-factor, we denote it by explicitly giving the exponent as function of $\vec{i},\vec{j}$, separated by a semicolon. Example:

$$\Phi\mathrm{A}_4(a,b;\nu,\sigma|q;\vec{x},\vec{y}||q^{2\binom{\vec{j}}{2}+\vec{j}\vec{\sigma}}) \tag{32}$$

denotes the function with generic name $\Phi\mathrm{A}_4$ and q-factor $q^{2\binom{\vec{j}}{2}+\vec{j}\vec{\sigma}}$.

We give the following two lists of q-hypergeometric functions of $2n$ variables; as before, it is enough to give only $\Psi(\vec{i},\vec{j})$ according to Equation (31). The letters and numbers in the notations are a mix of the notations for Appell and Lauricella functions. We can sometimes switch between vectors and scalars in the definitions. We can permute the indices in a vector $\vec{x}$. We then have $x_{n+1}=x_1$. This means that the vector index is computed modulo n. To translate between [2] and the present paper, we notice that vectors in [2] (p. 5) are not written. In this paper, we often denote these by $\vec{a}$. The product [2] $(b_1)_i\cdots(b_k)_i$ in [2] (p. 5 (9)) here corresponds to $\langle(g);\vec{q}\rangle_{\vec{k}}$.

Definition 11.

Function	$\Psi(\vec{i},\vec{j})$
$\Phi_1(a;b,b';c\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|+\|j\|}\langle b;q\rangle_{\|i\|}\langle b';q\rangle_{\|j\|}}{\langle c;q\rangle_{\|i\|+\|j\|}}$
$\Phi_2(a;b,b';c,c'\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|+\|j\|}\langle b;q\rangle_{\|i\|}\langle b';q\rangle_{\|j\|}}{\langle c;q\rangle_{\|i\|}\langle c';q\rangle_{\|j\|}}$
$\Phi_3(a,a';b,b';c\|q;\vec{x},\vec{y})$	$\frac{\langle a,b;q\rangle_{\|i\|}\langle a',b';q\rangle_{\|j\|}}{\langle c;q\rangle_{\|i\|+\|j\|}}$
$\Phi_4(a,b;c,c'\|q;\vec{x},\vec{y})$	$\frac{\langle a,b;q\rangle_{\|i\|+\|j\|}}{\langle c;q\rangle_{\|i\|}\langle c';q\rangle_{\|j\|}}$
$\Psi_2(a;\vec{c},\vec{c}'\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|+\|j\|}}{\langle c;q\rangle_{\vec{i}}\langle c';q\rangle_{\vec{j}}}$
$\Phi\mathrm{A}(a;\vec{b},\vec{b}';\vec{c},\vec{c}'\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|+\|j\|}\langle b;q\rangle_{\vec{i}}\langle b';q\rangle_{\vec{j}}}{\langle c;q\rangle_{\vec{i}}\langle c';q\rangle_{\vec{j}}}$
$\Phi\mathrm{B}(\vec{a},\vec{a}',\vec{b},\vec{b}';c\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\vec{i}}\langle a';q\rangle_{\vec{j}}\langle b;q\rangle_{\vec{i}}\langle b';q\rangle_{\vec{j}}}{\langle c;q\rangle_{\|i\|+\|j\|}}$
$\Phi\mathrm{C}(a,b;\vec{c},\vec{c}'\|q;\vec{x},\vec{y})$	$\frac{\langle a,b;q\rangle_{\|i\|+\|j\|}}{\langle c;q\rangle_{\vec{i}\langle c';q\rangle_{\vec{j}}}}$
$\Phi\mathrm{D}(a;\vec{b},\vec{b}';c\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|+\|j\|}\langle b;q\rangle_{\vec{i}}\langle b';q\rangle_{\vec{j}}}{\langle c;q\rangle_{\|i\|+\|j\|}}$
$\Phi\mathrm{A}_1(a,a';\vec{b};\vec{c}\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|}\langle a';q\rangle_{\|j\|}\langle b;q\rangle_{i\vec{+}j}}{\langle c;q\rangle_{i\vec{+}j}}$
$\Phi\mathrm{A}_{\overline{1}}(a;\vec{b},\vec{b}';\vec{c}\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|+\|j\|}\langle b;q\rangle_{\vec{i}}\langle b';q\rangle_{\vec{j}}}{\langle c;q\rangle_{i\vec{+}j}}$
$\Phi\mathrm{A}_2(a,a';\vec{b};\vec{c},\vec{c}'\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|}\langle a';q\rangle_{\|j\|}\langle b;q\rangle_{i\vec{+}j}}{\langle c;q\rangle_{\vec{i}}\langle c';q\rangle_{\vec{j}}}$
$\Phi\mathrm{A}_3(a,a';\vec{b},\vec{b}';\vec{c}\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|}\langle a';q\rangle_{\|j\|}\langle b;q\rangle_{\vec{i}}\langle b';q\rangle_{\vec{j}}}{\langle c;q\rangle_{i\vec{+}j}}$
$\Phi\mathrm{A}_4(a;\vec{b};\vec{c},\vec{c}'\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|+\|j\|}\langle b;q\rangle_{i\vec{+}j}}{\langle c;q\rangle_{\vec{i}}\langle c';q\rangle_{\vec{j}}}$
$\Phi\mathrm{B}_1(\vec{a},\vec{b},\vec{b}';c\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{i\vec{+}j}\langle b;q\rangle_{\vec{i}}\langle b';q\rangle_{\vec{j}}}{\langle c;q\rangle_{\|i\|+\|j\|}}$
$\Phi\mathrm{B}_2(\vec{a},\vec{b},\vec{b}';c,c'\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{i\vec{+}j}\langle b;q\rangle_{\vec{i}}\langle b';q\rangle_{\vec{j}}}{\langle c;q\rangle_{\|i\|}\langle c';q\rangle_{\|j\|}}$
$\Phi\mathrm{B}_4(\vec{a},\vec{b};c,c'\|q;\vec{x},\vec{y})$	$\frac{\langle a,b;q\rangle_{i\vec{+}j}}{\langle c;q\rangle_{\|i\|}\langle c';q\rangle_{\|j\|}}$
$\Phi\mathrm{C}_1(a,b,b';\vec{c}\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|+\|j\|}\langle b;q\rangle_{\|i\|}\langle b';q\rangle_{\|j\|}}{\langle c;q\rangle_{i\vec{+}j}}$
$\Phi\mathrm{C}_2(a,b,b';\vec{c},\vec{c}'\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|+\|j\|}\langle b;q\rangle_{\|i\|}\langle b';q\rangle_{\|j\|}}{\langle c;q\rangle_{\vec{i}}\langle c';q\rangle_{\vec{j}}}$
$\Phi\mathrm{C}_3(a,a';b,b';\vec{c}\|q;\vec{x},\vec{y})$	$\frac{\langle a,b;q\rangle_{\|i\|}\langle a',b';q\rangle_{\|j\|}}{\langle c;q\rangle_{i\vec{+}j}}$
$\Phi\mathrm{D}_1(a,a';\vec{b};c\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|}\langle a';q\rangle_{\|j\|}\langle b;q\rangle_{i\vec{+}j}}{\langle c;q\rangle_{\|i\|+\|j\|}}$
$\Phi\mathrm{D}_2(a,a';\vec{b};c,c'\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|}\langle a';q\rangle_{\|j\|}\langle b;q\rangle_{i\vec{+}j}}{\langle c;q\rangle_{\|i\|}\langle c';q\rangle_{\|j\|}}$
$\Phi\mathrm{D}_{\overline{2}}(a;\vec{b},\vec{b}';c,c'\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|+\|j\|}\langle b;q\rangle_{\vec{i}}\langle b';q\rangle_{\vec{j}}}{\langle c;q\rangle_{\|i\|}\langle c';q\rangle_{\|j\|}}$
$\Phi\mathrm{D}_3(a,a';\vec{b},\vec{b}';c\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|}\langle a';q\rangle_{\|j\|}\langle b;q\rangle_{\vec{i}}\langle b';q\rangle_{\vec{j}}}{\langle c;q\rangle_{\|i\|+\|j\|}}$
$\Phi\mathrm{D}_4(a;\vec{b};c,c'\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|+\|j\|}\langle b;q\rangle_{i\vec{+}j}}{\langle c;q\rangle_{\|i\|}\langle c';q\rangle_{\|j\|}}$

Definition 12. *When all parameters to the left of* $|$ *are vectors, we can also let* q *be a vector.*

Function	$\Psi(\vec{i},\vec{j})$
$\Phi\mathrm{a}(a;\vec{b};\vec{c}\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|+\|j\|}\langle b;q\rangle_{i+\vec{j}-}}{\langle c;q\rangle_{i\vec{+}j}}$
$\Phi\mathrm{a}_1(a,a';\vec{b};\vec{c}\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|}\langle a';q\rangle_{\|j\|}\langle b;q\rangle_{i+\vec{j}-}}{\langle c;q\rangle_{i\vec{+}j}}$
$\Phi\mathrm{b}(\vec{a},\vec{b};c\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{i\vec{+}j}\langle b;q\rangle_{i+\vec{j}-}}{\langle c;q\rangle_{\|i\|+\|j\|}}$
$\Phi\mathrm{b}_4(\vec{a},\vec{b};c,c'\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{i\vec{+}j}\langle b;q\rangle_{i+\vec{j}-}}{\langle c;q\rangle_{\|i\|}\langle c';q\rangle_{\|j\|}}$
$\Phi\mathrm{g}(\vec{a},\vec{b};\vec{c}\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{i\vec{+}j}\langle b;q\rangle_{i+\vec{j}-}}{\langle c;q\rangle_{i\vec{+}j}}$
$\Phi\mathrm{g}_1(\vec{a},\vec{a}',\vec{b};\vec{c}\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\vec{i}}\langle a';q\rangle_{\vec{j}}\langle b;q\rangle_{i+\vec{j}-}}{\langle c;q\rangle_{i\vec{+}j}}$
$\Phi\mathrm{g}_4(\vec{a},\vec{b};\vec{c},\vec{c}'\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{i\vec{+}j}\langle b;q\rangle_{i+\vec{j}-}}{\langle c;q\rangle_{\vec{i}}\langle c';q\rangle_{\vec{j}}}$
$\Phi\overline{\mathrm{g}}(\vec{a},\vec{b};\vec{c}\|q;\vec{x},\vec{y})$	$\frac{\langle a,b;q\rangle_{i+\vec{j}-}}{\langle c;q\rangle_{i\vec{+}j}}$
$\mathrm{MA}(a,a';\vec{c}\|q;\vec{x},\vec{y})$	$\frac{\langle a;q\rangle_{\|i\|}\langle a';q\rangle_{\|j\|}}{\langle c;q\rangle_{i\vec{+}j}}$
$\mathrm{MB}(\vec{b};c,c'\|q;\vec{x},\vec{y})$	$\frac{\langle b;q\rangle_{i\vec{+}j}}{\langle c;q\rangle_{\|i\|}\langle c';q\rangle_{\|j\|}}$
$\mathrm{Ma}(\vec{b};\vec{c}\|q;\vec{x},\vec{y})$	$\frac{\langle b;q\rangle_{i+\vec{j}-}}{\langle c;q\rangle_{i\vec{+}j}}$
$\mathrm{G}_1(a;b;b'\|q;\vec{x},\vec{y})$	$\langle a;q\rangle_{\|i\|+\|j\|}\langle b;q\rangle_{\|j\|-\|i\|}\langle b';q\rangle_{\|i\|-\|j\|}$
$\mathrm{G}_2(a,a';b,b'\|q;\vec{x},\vec{y})$	$\langle a;q\rangle_{\|i\|}\langle a';q\rangle_{\|j\|}\langle b;q\rangle_{\|j\|-\|i\|}\langle b';q\rangle_{\|i\|-\|j\|}$
$\mathrm{G}A_1(a;\vec{b},\vec{b}'\|q;\vec{x},\vec{y})$	$\langle a;q\rangle_{\|i\|+\|j\|}\langle b;q\rangle_{j\vec{-}i}\langle b';q\rangle_{i\vec{-}j}$
$\mathrm{G}A_2(a,a';\vec{b},\vec{b}'\|q;\vec{x},\vec{y})$	$\langle a;q\rangle_{\|i\|}\langle a';q\rangle_{\|j\|}\langle b;q\rangle_{j\vec{-}i}\langle b';q\rangle_{i\vec{-}j}$
$\mathrm{GD}_1(a,a';\vec{b}\|q;\vec{x},\vec{y})$	$\langle a;q\rangle_{\|j\|-\|i\|}\langle a';q\rangle_{\|i\|-\|j\|}\langle b;q\rangle_{i\vec{+}j}$
$\mathrm{GD}_2(a,a';\vec{b},\vec{b}'\|q;\vec{x},\vec{y})$	$\langle a;q\rangle_{\|j\|-\|i\|}\langle a';q\rangle_{\|i\|-\|j\|}\langle b;q\rangle_{\vec{i}}\langle b';q\rangle_{\vec{j}}$
$\mathrm{Gg}_1(\vec{a},\vec{b},\vec{b}'\|q;\vec{x},\vec{y})$	$\langle a;q\rangle_{i+\vec{j}-}\langle b;q\rangle_{j\vec{-}i}\langle b';q\rangle_{i\vec{-}j}$

2.1. Elementary Special and Limiting Cases

The following rather obvious relations could easily be derived.

1. For $x = 0$, or $y = 0$, the above functions reduce to q-Lauricella or q-Humbert functions or to a product:

$\Phi\mathrm{A}_1$, $\Phi\mathrm{A}_{\overline{1}}$, $\Phi\mathrm{A}_2$, $\Phi\mathrm{A}_3$, $\Phi\mathrm{A}_4$, Φa, Φa_1, GA_1, GA_2	Φ_{A}	Φ_g, Φ_{g_1} Φ_{g_4}, $\Phi_{\overline{g}}$	$\prod\ {}_2\Phi_1$
$\Phi\mathrm{C}_1$, $\Phi\mathrm{C}_2$, $\Phi\mathrm{C}_3$	$\Phi\mathrm{C}$	$\mathrm{G}g_1$	$\prod\ {}_2\Phi_1$
$\Phi\mathrm{B}_1$, $\Phi\mathrm{B}_2$, $\Phi\mathrm{B}_4$, Φb, Φb_4	Φ_{B}	MA	Ψ_2
$\Phi\mathrm{D}_1$, $\Phi\mathrm{D}_2$, $\Phi\mathrm{D}_{\overline{2}}$, $\Phi\mathrm{D}_3$, $\Phi\mathrm{D}_4$, GD_1, GD_2	Φ_{D}	MB Ma	F_2 $\prod\ {}_1\Phi_1$

2. When a numerator parameter is equal to zero, the result is unity or a simpler function. For example, the functions $\Phi\mathrm{A}_1$, $\Phi\mathrm{A}_2$,
$\Phi\mathrm{A}_3$, Φa_1, GA_2 reduce to a $\Phi\mathrm{A}$ for $a = 0$.

3. For $n = 1$, the functions reduce to q-Appell, q-Humbert or q-Horn functions, or to a q-hypergeometric function with argument $x \oplus_q y$ according to the following table:

$\Phi\mathrm{A}_1,\ \Phi\mathrm{A}_{\overline{1}},\ \Phi\mathrm{B}_1,$ $\Phi\mathrm{C}_1,\ \Phi\mathrm{D}_1,$ $\Phi a_1,\ \Phi g_1$	Φ_1	$\Phi\mathrm{A}_4,\ \Phi\mathrm{B}_4,\ \Phi\mathrm{D}_4,$ $\Phi b_4,\ \Phi g_4$	Φ_4	$\mathrm{GA}_1,\ \mathrm{GD}_1,\ \mathrm{G}g_1$	G_1
$\Phi\mathrm{A}_2,\ \Phi\mathrm{B}_2,\ \Phi\mathrm{C}_2,$ $\Phi\mathrm{D}_2,\ \Phi\mathrm{D}_{\overline{2}}$	Φ_2	$\Phi a,\ \Phi b,$ $\Phi g,\ \Phi\overline{g}$	${}_2\Phi_1$	$\Phi\mathrm{A}_3,\ \Phi\mathrm{C}_3,\ \Phi\mathrm{D}_3$	Φ_3
$\mathrm{GA}_2,\ \mathrm{GD}_2$	G_2	MA; MB	$\mathrm{F}_2;\ \Psi_2$	Ma	${}_1\Phi_1$

4. For $n = 2$, the series in the same boxes are identical. The K- and D-series are q-analogues of series introduced by Exton [8–10].

$\Phi\mathrm{A}_{\overline{1}},\ \Phi\mathrm{D}_{\overline{2}},\ \mathrm{K}_{12}$	$\Phi\mathrm{A}_1,\ \Phi\mathrm{D}_2,\ \Phi g$	$\Phi\mathrm{A}_3,\ \Phi\mathrm{B}_2,\ \Phi g_1$	$\Phi\mathrm{A}_2,\ \Phi g_4$
$\Phi\mathrm{A}_4,\ \Phi\mathrm{C}_2,\ \mathrm{K}_5$	$\Phi\mathrm{B}_1,\ \Phi\mathrm{D}_3,\ \mathrm{K}_{20}$	$\Phi\mathrm{B}_4,\ \Phi\mathrm{C}_3$	$\Phi\mathrm{A}_1,\ \Phi b_4$
$\Phi\mathrm{D}_1,\ \Phi b,\ \mathrm{K}_{16}$	MA, MB, Ma	$\mathrm{GD}_1, \mathrm{D}_1$	$\mathrm{GD}_2, \mathrm{D}_5$
$\Phi\mathrm{C}_1,\ \Phi\mathrm{D}_4,\ \Phi a,\ \mathrm{K}_3$			

5. In the following cases, a reduction to a product of inverse q-shifted factorials takes place:

Function	$\Phi\mathrm{A}_1$	$\Phi\mathrm{D}_2$	$\Phi\mathrm{D},\ \Phi g$	$\mathrm{GA}_1,\ \mathrm{GA}_2,\ \mathrm{G}g_1$	$\mathrm{GD}_1, \mathrm{GD}_2$
with	$b = c$	$a = c,\ a' = c'$	$a = c$	$b + b' = 1$	$a + a' = 1$

3. A Couple of Lemmas

Lemma 13. *A q-analogue of [2] (p. 11): if the general term in a power series with summation indices $\vec{i}, \vec{j}$ is multiplied by a factor $f(\vec{i}, \vec{j})$, which satisfies:*

$$\alpha(1 + |\{i\}_q| + |\{j\}_q|)^\xi < |f(\vec{i}, \vec{j})| < \beta(1 + |\{i\}_q| + |\{j\}_q|)^\eta \tag{33}$$

where $\alpha, \beta \in (0, \infty)$ and $\xi, \eta \in \mathbb{R}$, then the region of convergence is unaltered.

Lemma 14. *A q-analogue of [2] (p. 11): The region of convergence for a multiple q-hypergeometric series is independent of the parameters provided that they do not take exceptional values, i.e., $0, -1, -2, \cdots$, in general.*

Proof. An alteration of a parameter value is equivalent to the insertion of a factor in the general term that has the form $f(\vec{i}, \vec{j}) = \dfrac{\langle \alpha; q\rangle_{\vec{l}}}{\langle \beta; q\rangle_{\vec{l}}}$, where $\vec{l}$ denotes the relevant linear combination of q-shifted factorials. Since exceptional values are excluded, f does not take the values $0, \infty$; and the q-Stirling formula implies:

$$\lim_{|l| \to \infty} |f| = O(|\{l\}_q|^{\mathrm{Re}(\alpha - \beta)}) \tag{34}$$

The statement in the lemma now follows by Lemma 13. □

Lemma 15. *[2] (p. 13): Assume that all $a_i > 0$. Then, the sum:*

$$\sum_{I = i} \vec{a}^{\vec{i}} \tag{35}$$

is equivalent to $(\max\{a_m\})^i$ for convergence purposes.

Proof. The inequalities:

$$(\max\{a_m\})^i < \sum_{I=i} \vec{a}^{\vec{i}} < (i+1)^{n-1}(\max\{a_m\})^i \tag{36}$$

are obvious. By Lemma 13, the factor $(i+1)^{n-1}$ is unimportant. □

Theorem 16. *[2](p. 13): For a positive sequence $\{F(i)\}_{j=1}^{n}$, we have the following inequalities:*

$$(i+1)^{-n+1}\left[\sum_{I=i} F(i)\right]^2 < \sum_{I=i} F(i)^2 < \left[\sum_{I=i} F(i)\right]^2 \tag{37}$$

Thus, a sum of squares is equivalent to the square of the sum for convergence purposes; we denote this equivalence by $\sim$.

Proof. The first inequality is proven in the following way by using the relation between the arithmetic and geometric mean values. The second inequality is obvious.

$$\begin{aligned}\left[\sum_{I=i} F(i)\right]^2 &= \sum_{I=i} F(i) \sum_{J=j} F(j) \leq \frac{1}{2}\sum_{I=i}\sum_{J=j}(F(i)^2 + F(j)^2)\\ &= \sum_{I=i}\sum_{J=i} F(i)^2 < (i+1)^{n-1}\sum_{I=i} F(i)^2\end{aligned} \tag{38}$$

□

Lemma 17. *The following inequality holds for all sequences $\{x_m\}_{m=1}^{n}$:*

$$\sum_{|r|=m}\left(\binom{m}{\vec{r}}_q\right)^2 |\vec{x}^{\vec{r}}| < \left(\sqrt{|x_1|} \oplus_q \ldots \oplus_q \sqrt{|x_n|}\right)^{2m} \tag{39}$$

4. Convergence Regions

In this final section, we will give convergence regions (and guesses of convergence regions) for most of the 43 functions from Section 2. Some of these convergence regions will be vector versions of the previously given convergence regions for q-Appell- [5] and q-Lauricella-functions [4]; they all use formula Equation (12). The special case $q = 1$ corresponds to the regions given in [2] (p. 10). For each convergence region, we give the corresponding functions and give the proof for one of the functions.

In a few cases, the convergence region is slightly different, and we point this out. We then give guesses of convergence regions that are between two regions; the smaller region is then always the region from [2]. We cannot prove these exceptional cases, and they remain conjectures.

Convergence regions for general multiple q-functions of this kind involving the Nova q-addition have not been given before. The rest of the section consists of theorems (and conjectures) followed by proofs.

Theorem 18. *Let $n \geq 2$. The convergence region for the functions:*

$$\Phi\mathrm{B}_1,\ \Phi\mathrm{D}_1,\ \Phi\mathrm{D}_3,\ \mathrm{GD}_2, \Phi b \tag{40}$$

is indicated.

$$\forall m : |x_m| < 1, |y_m| < 1 \tag{41}$$

Proof. The proof is for the function $\Phi\mathrm{B}_1$. The coefficient of $\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}$ is equal to:

$$A_{i,\vec{j}} \equiv \Gamma_q \left[\begin{array}{c} a+\vec{i}+j, b\vec{+}i, b'\vec{+}j, c, \vec{1}, \vec{1} \\ \vec{a}, \vec{b}, \vec{b'}, c+i+j, 1\vec{+}i, 1\vec{+}j \end{array} \right] \tag{42}$$

According to the $q-$Stirling formula, $\lim_{i,\vec{j}\to\vec{\infty}}$:

$$A_{i,\vec{j}} \sim \Gamma_q \left[\begin{array}{c} c \\ \vec{a}, \vec{b}, \vec{b'} \end{array} \right] \lim_{i,\vec{j}\to\vec{\infty}} \{i\vec{+}j\}_q^{a-1} \{\vec{i}\}_q^{b-1} \{\vec{j}\}_q^{b'-1} \{i+j\}_q^{1-c} \\ \frac{1}{\binom{|i+j|}{i\vec{+}j}_q} \tag{43}$$

The maximum values of the real parts of $\vec{a}, \vec{b}, \vec{b'}, c$ are $\alpha, \beta, \beta', \gamma$, and N is a number, such that:

$$N > \left| \Gamma_q \left[\begin{array}{c} c \\ \vec{a}, \vec{b}, \vec{b'} \end{array} \right] \right| \tag{44}$$

For $\vec{i,j}$ big enough, we have:

$$|A_{i,\vec{j}} \vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| < N \left| \frac{1}{\binom{|i+j|}{i\vec{+}j}_q} \right| \{i+j\}_q^{\alpha-1} \{i\}_q^{\beta-1} \{j\}_q^{\beta'-1} \{i+j\}_q^{1-\gamma} |\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| \tag{45}$$

If ϵ_1 denotes a positive number bigger than the greatest of $\beta-1$ and $\beta'-1$ and ϵ_1' is a sufficiently big number, we have:

$$\{i\}_q^{\beta-1}\{j\}_q^{\beta'-1} < \{i\}_q^{\epsilon_1}\{j\}_q^{\epsilon_1} < \frac{\{i+j\}_q^{2\epsilon_1'}}{4^{\epsilon_1}} \tag{46}$$

Therefore:

$$\sum_{i,\vec{j}} |A_{i,\vec{j}} \vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| < \frac{N}{4^{\epsilon_1}} \sum_{i,\vec{j}} \left| \frac{1}{\binom{|i+j|}{i\vec{+}j}_q} \{i+j\}_q^{2\epsilon_1'+\alpha-\gamma} \vec{x}^{\vec{i}}\vec{y}^{\vec{j}} \right| = \frac{N}{4^{\epsilon_1}} \\ \sum_{\vec{k}} \left| \frac{1}{\binom{|k|}{\vec{k}}_q} \prod_m \sum_{i_m+j_m=k_m} \{i+j\}_q^{2\epsilon_1'+\alpha-\gamma} \vec{x}^{\vec{i}}\vec{y}^{\vec{j}} \right| \tag{47}$$

and the series converges $\forall m : |x_m| < 1, |y_m| < 1$ and $\Phi\mathrm{B}_1$ converges in the same region. $\square$

Theorem 19. *The convergence region for the functions:*

$$\Phi\mathrm{B}_2,\ \Phi\mathrm{D}_2,\ \mathrm{GD}_1 \tag{48}$$

is indicated.

$$\forall m : |x_m| \oplus_q |y_m| < 1 \tag{49}$$

Proof. The proof is for the function $\Phi\mathrm{B}_2$. The coefficient of $\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}$ is equal to:

$$A_{i,\vec{j}} \equiv \Gamma_q \left[\begin{array}{c} a+\vec{i}+j, b\vec{+}i, b'\vec{+}j, c, c', \vec{1}, \vec{1} \\ \vec{a}, \vec{b}, \vec{b'}, c+i, c'+j, 1\vec{+}i, 1\vec{+}j \end{array} \right] \tag{50}$$

According to the $q-$Stirling formula, $\lim_{\vec{i,j}\to\vec{\infty}}$:

$$A_{\vec{i,j}} \sim \Gamma_q \left[\begin{array}{c} c, c' \\ \vec{a}, \vec{b}, \vec{b'} \end{array} \right] \lim_{\vec{i,j}\to\vec{\infty}} \{\vec{i}\}_q^{b-1} \{\vec{j}\}_q^{b'-1} \{i \vec{+} j\}_q^{a-1} \{i\}_q^{1-c} \{j\}_q^{1-c'} \frac{\langle (1); q\rangle_{i\vec{+}j}}{\langle 1; q\rangle_{|i|} \langle 1; q\rangle_{|j|}} \tag{51}$$

The maximum values of the real parts of $\vec{a}, \vec{b}, \vec{b'}, c, c'$ are $\alpha, \beta, \beta', \gamma, \gamma'$, and N is a number, such that:

$$N > \left| \Gamma_q \left[\begin{array}{c} c, c' \\ \vec{a}, \vec{b}, \vec{b'} \end{array} \right] \right| \tag{52}$$

For $\vec{i,j}$ big enough, we have:

$$|A_{\vec{i,j}} \vec{x}^{\vec{i}} \vec{y}^{\vec{j}}| < N \left| \frac{\langle (1); q\rangle_{i\vec{+}j}}{\langle 1; q\rangle_{|i|} \langle 1; q\rangle_{|j|}} \right| \{i\}_q^{\beta-1} \{j\}_q^{\beta'-1} \{i+j\}_q^{\alpha-1} \{i\}_q^{1-\gamma} \{j\}_q^{1-\gamma'} |\vec{x}^{\vec{i}} \vec{y}^{\vec{j}}| \tag{53}$$

If:

1. ϵ_1 denotes a positive number bigger than the greatest of $\beta - 1$ and $\beta' - 1$ and ϵ'_1 is a sufficiently big number
2. ϵ_2 denotes a positive number bigger than the greatest of $1-\gamma$ and $1-\gamma'$ and ϵ'_2 denotes a sufficiently large number,

we have:

$$\{i\}_q^{\beta-1} \{j\}_q^{\beta'-1} < \{i\}_q^{\epsilon_1} \{j\}_q^{\epsilon_1} < \frac{\{i+j\}_q^{2\epsilon'_1}}{4^{\epsilon_1}} \tag{54}$$

$$\{i\}_q^{1-\gamma} \{j\}_q^{1-\gamma'} < \{i\}_q^{\epsilon_2} \{j\}_q^{\epsilon_2} < \frac{\{i+j\}_q^{2\epsilon'_2}}{4^{\epsilon_2}} \tag{55}$$

Therefore:

$$\begin{aligned} \sum_{\vec{i,j}} |A_{\vec{i,j}} \vec{x}^{\vec{i}} \vec{y}^{\vec{j}}| &< \frac{N}{4^{\epsilon_1+\epsilon_2}} \sum_{\vec{i,j}} \left| \frac{\langle (1); q\rangle_{i\vec{+}j}}{\langle 1; q\rangle_{|i|} \langle 1; q\rangle_{|j|}} \{i+j\}_q^{2\epsilon'_1+2\epsilon'_2+\alpha-1} \vec{x}^{\vec{i}} \vec{y}^{\vec{j}} \right| \\ &\overset{\text{by}(5)}{\leq} \frac{N}{4^{\epsilon_1+\epsilon_2}} \prod_m \sum_{i_m, j_m} \left| \frac{\langle (1); q\rangle_{i_m+j_m}}{\langle 1; q\rangle_{i_m} \langle 1; q\rangle_{j_m}} \{i+j\}_q^{2\epsilon'_1+2\epsilon'_2+\alpha-1} \vec{x}^{\vec{i}} \vec{y}^{\vec{j}} \right| \end{aligned} \tag{56}$$

and the series converges for $\oplus_{q,m=1}^n |x_m| \oplus_q |y_m| < 1$ and $\Phi\mathrm{B}_2$ converges in the same region. $\square$

Theorem 20. *Let $n \geq 2$. The convergence region for the functions:*

$$\Phi g, \ \mathrm{G}g_1, \ \Phi g_1 \tag{57}$$

is indicated.

$$\forall m > 1 : |x_m| \oplus_q |y_{m-1}| < 1 \tag{58}$$

Proof. The proof is for the function Φg_1. The coefficient of $\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}$ is equal to:

$$A_{\vec{i,j}} \equiv \Gamma_q \left[\begin{array}{c} a \vec{+} i, a' \vec{+} j, b + \vec{i} + j^-, \vec{c}, \vec{1}, \vec{1} \\ \vec{a}, \vec{a'}, \vec{b}, c + \vec{i} + j, 1 \vec{+} i, 1 \vec{+} j \end{array} \right] \tag{59}$$

According to the $q-$Stirling formula, $\lim_{\vec{i,j} \to \vec{\infty}}$:

$$A_{\vec{i,j}} \sim \Gamma_q \left[\begin{array}{c} \vec{c} \\ \vec{a}, \vec{a'}, \vec{b} \end{array} \right] \lim_{\vec{i,j} \to \vec{\infty}} \{\vec{i}\}_q^{a-1} \{\vec{j}\}_q^{a'-1} \{i \vec{+} j\}_q^{b-c} \frac{\langle (1); q \rangle_{i+\vec{j}^-}}{\langle (1); q \rangle_{i\vec{+}j}} \tag{60}$$

The maximum values of the real parts of $\vec{a}, \vec{a'}, \vec{b}, \vec{c}$ are $\alpha, \alpha', \beta, \gamma$, and N is a number, such that:

$$N > \left| \Gamma_q \left[\begin{array}{c} \vec{c} \\ \vec{a}, \vec{a'}, \vec{b} \end{array} \right] \right| \tag{61}$$

For $\vec{i,j}$ big enough, we have:

$$|A_{\vec{i,j}} \vec{x}^{\vec{i}} \vec{y}^{\vec{j}}| < N \left| \frac{\langle (1); q \rangle_{i+\vec{j}^-}}{\langle (1); q \rangle_{i\vec{+}j}} \right| \{i\}_q^{\alpha-1} \{j\}_q^{\alpha'-1} \{i+j\}_q^{\beta-\gamma} |\vec{x}^{\vec{i}} \vec{y}^{\vec{j}}| \tag{62}$$

If ϵ_1 denotes a positive number bigger than the greatest of $\alpha - 1$ and $\alpha' - 1$ and ϵ_1' denotes a sufficiently large number, we have:

$$\{i\}_q^{\alpha-1} \{j\}_q^{\alpha'-1} < \{i\}_q^{\epsilon_1} \{j\}_q^{\epsilon_1} < \frac{\{i+j\}_q^{2\epsilon_1'}}{4^{\epsilon_1}} \tag{63}$$

Therefore:

$$\sum_{\vec{i,j}} |A_{\vec{i,j}} \vec{x}^{\vec{i}} \vec{y}^{\vec{j}}| < \frac{N}{4^{\epsilon_1}} \sum_{\vec{i,j}} \left| \frac{\langle (1); q \rangle_{i+\vec{j}^-}}{\langle (1); q \rangle_{i\vec{+}j}} \{i+j\}_q^{2\epsilon_1' + \beta - \gamma} \vec{x}^{\vec{i}} \vec{y}^{\vec{j}} \right| \overset{\text{by}(6)}{\leq} \frac{N}{4^{\epsilon_1}} \prod_m \sum_{i_m, j_{m-1}} \left| \frac{\langle 1; q \rangle_{i_m + j_{m-1}}}{\langle 1; q \rangle_{i_m} \langle 1; q \rangle_{j_{m-1}}} \{i+j\}_q^{2\epsilon_1' + \beta - \gamma} x_m{}^{i_m} y_{m-1}{}^{j_{m-1}} \right| \tag{64}$$

and the series converges for $|x_m| \oplus_q |y_{m-1}| < 1$ and $\Phi\mathrm{G}_1$ converges in the same region. $\square$

Theorem 21. *The convergence region for the function* $\Phi\mathrm{B}_4$ *is indicated.*

$$\forall m : \sqrt{x_m} \oplus_q \sqrt{y_m} < 1 \tag{65}$$

Proof. The coefficient of $\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}$ is equal to:

$$A_{\vec{i,j}} \equiv \Gamma_q \left[\begin{array}{c} a + \vec{i} + j, b + \vec{i} + j, c, c', \vec{1}, \vec{1} \\ \vec{a}, \vec{b}, c + i, c' + j, 1 \vec{+} i, 1 \vec{+} j \end{array} \right] \tag{66}$$

According to the $q-$Stirling formula, $\lim_{\vec{i,j} \to \vec{\infty}}$:

$$A_{\vec{i,j}} \sim \Gamma_q \left[\begin{array}{c} c, c', \\ \vec{a}, \vec{b} \end{array} \right] \lim_{\vec{i,j} \to \vec{\infty}} \{i \vec{+} j\}_q^{a+b-2} \{i\}_q^{1-c} \{j\}_q^{1-c'} \frac{\langle (1); q \rangle_{i\vec{+}j}^2}{\langle 1; q \rangle_{|i|} \langle 1; q \rangle_{|j|} \langle (1); q \rangle_{\vec{i}} \langle (1); q \rangle_{\vec{j}}} \tag{67}$$

The maximum values of the real parts of $\vec{a}, \vec{b}, c, c'$ are $\alpha, \beta, \gamma, \gamma'$, and N is a number, such that:

$$N > \left| \Gamma_q \left[\begin{array}{c} c, c' \\ \vec{a}, \vec{b} \end{array} \right] \right| \tag{68}$$

For $\vec{i,j}$ big enough, we have:

$$\begin{aligned} |A_{\vec{i,j}} \vec{x}^{\vec{i}} \vec{y}^{\vec{j}}| < N & \left| \frac{\langle (1); q \rangle^2_{i \vec{+} j}}{\langle 1; q \rangle_{|i|} \langle 1; q \rangle_{|j|} \langle (1); q \rangle_{\vec{i}} \langle (1); q \rangle_{\vec{j}}} \right| \\ & \{ i \vec{+} j \}_q^{\alpha+\beta-2} \{ i \}_q^{1-\gamma} \{ j \}_q^{1-\gamma'} |\vec{x}^{\vec{i}} \vec{y}^{\vec{j}}| \end{aligned} \tag{69}$$

If ϵ_1 denotes a positive number bigger than the greatest of $1-\gamma$ and $1-\gamma'$ and ϵ'_1 is a sufficiently big number, we have:

$$\{ i \}_q^{1-\gamma} \{ j \}_q^{1-\gamma'} < \{ i \}_q^{\epsilon_1} \{ j \}_q^{\epsilon_1} < \frac{\{ i+j \}_q^{2\epsilon'_1}}{4^{\epsilon_1}} \tag{70}$$

Therefore:

$$\begin{aligned} \sum_{\vec{i,j}} |A_{\vec{i,j}} \vec{x}^{\vec{i}} \vec{y}^{\vec{j}}| &< \frac{N}{4^{\epsilon_1}} \sum_{\vec{i,j}} \left| \frac{\langle (1); q \rangle^2_{i \vec{+} j}}{\langle 1; q \rangle_{|i|} \langle 1; q \rangle_{|j|} \langle (1); q \rangle_{\vec{i}} \langle (1); q \rangle_{\vec{j}}} \{ i+j \}_q^{2\epsilon'_1+\alpha+\beta-2} \vec{x}^{\vec{i}} \vec{y}^{\vec{j}} \right| \\ &\overset{\text{by}(5)}{\leq} \frac{N}{4^{\epsilon_1}} \sum_{\vec{k}} \prod_m \sum_{i_m+j_m=k_m} \left| \frac{\langle 1; q \rangle^2_{i_m+j_m}}{\langle 1; q \rangle^2_{i_m} \langle 1; q \rangle^2_{j_m}} \{ i+j \}_q^{2\epsilon'_1+\alpha+\beta-2} {x_m}^{i_m} {y_m}^{j_m} \right| \\ &\overset{\text{by}(39)}{<} \frac{N}{4^{\epsilon_1}} \sum_{\vec{k}} \prod_m \{ k_m \}_q^{2\epsilon'_1+\alpha+\beta-2} (\sqrt{x_m} \oplus_q \sqrt{y_m})^{2k_m} \end{aligned} \tag{71}$$

and the series converges $\forall m : \sqrt{x_m} \oplus_q \sqrt{y_m} < 1$ and $\Phi\mathrm{B}_4$ converges in the same region. $\square$

Theorem 22. *Let $n \geq 2$. The convergence region for the function $\Phi\overline{g}$ is indicated.*

$$\forall m > 1 : \sqrt{x_m} \oplus_q \sqrt{y_{m-1}} < 1 \tag{72}$$

Proof. The coefficient of $\vec{x}^{\vec{i}} \vec{y}^{\vec{j}}$ is equal to:

$$A_{\vec{i,j}} \equiv \Gamma_q \left[\begin{array}{c} a + \vec{i} + j_-, b + \vec{i} + j_-, \vec{c}, \vec{1}, \vec{1} \\ \vec{a}, \vec{b}, c + \vec{i} + j, 1 \vec{+} i, 1 \vec{+} j \end{array} \right] \tag{73}$$

According to the $q-$Stirling formula, $\lim_{\vec{i,j} \to \vec{\infty}}$:

$$A_{\vec{i,j}} \sim \Gamma_q \left[\begin{array}{c} \vec{c} \\ \vec{a}, \vec{b} \end{array} \right] \lim_{\vec{i,j} \to \vec{\infty}} \{ i \vec{+} j \}_q^{a+b-c-1} \frac{\langle (1); q \rangle^2_{i \vec{+} j_-}}{\langle (1); q \rangle_{i \vec{+} j} \langle (1); q \rangle_{\vec{i}} \langle (1); q \rangle_{\vec{j}}} \tag{74}$$

The maximum values of the real parts of $\vec{a}, \vec{b}, \vec{c}$ are α, β, γ, and N is a number, such that:

$$N > \left| \Gamma_q \left[\begin{array}{c} \vec{c} \\ \vec{a}, \vec{b} \end{array} \right] \right| \tag{75}$$

For $\vec{i,j}$ big enough, we have:

$$|A_{\vec{i,j}}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| < N\left|\frac{\langle(1);q\rangle^2_{\vec{i+j_-}}}{\langle(1);q\rangle_{\vec{i+j}}\langle(1);q\rangle_{\vec{i}}\langle(1);q\rangle_{\vec{j}}}\right|\{i+j\}_q^{\alpha+\beta-\gamma-1}|\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| \tag{76}$$

Therefore, we have:

$$\begin{aligned}\sum_{\vec{i,j}}|A_{\vec{i,j}}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| &< N\sum_{\vec{i,j}}\left|\frac{\langle(1);q\rangle^2_{\vec{i+j_-}}}{\langle(1);q\rangle_{\vec{i+j}}\langle(1);q\rangle_{\vec{i}}\langle(1);q\rangle_{\vec{j}}}\{i+j\}_q^{\alpha+\beta-\gamma-1}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}\right| \\ &\overset{\text{by}(6)}{\leq} N\sum_{\vec{k}}\prod_m\sum_{i_m+j_m=k_m}\left|\frac{\langle 1;q\rangle^2_{i_m+j_{m-1}}}{\langle 1;q\rangle^2_{i_m}\langle 1;q\rangle^2_{j_{m-1}}}\{i+j\}_q^{\alpha+\beta-\gamma-1}{x_m}^{i_m}{y_{m-1}}^{j_{m-1}}\right| \\ &\overset{\text{by}(39)}{<} N\sum_{\vec{k}}\prod_m\{k_m\}_q^{\alpha+\beta-\gamma-1}(\sqrt{x_m}\oplus_q\sqrt{y_{m-1}})^{2k_m}\end{aligned} \tag{77}$$

and the series converges for $\forall m>1:\sqrt{x_m}\oplus_q\sqrt{y_{m-1}}<1$, and $\Phi\overline{g}$ converges in the same region. □

Theorem 23. *The convergence region for the function* $\Phi\mathrm{D}_{\overline{2}}$ *is indicated.*

$$\max_m|x_m|\oplus_q\max_m|y_m|<1 \tag{78}$$

Proof. The coefficient of $\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}$ is equal to:

$$A_{\vec{i,j}}\equiv\Gamma_q\left[\begin{matrix}a+i+j,b\vec{+}i,b'\vec{+}j,c,c',\vec{1},\vec{1}\\ a,\vec{b},\vec{b'},c+i,c'+j,1\vec{+}i,1\vec{+}j\end{matrix}\right] \tag{79}$$

According to the $q-$Stirling formula, $\lim_{\vec{i,j}\to\vec{\infty}}$:

$$A_{\vec{i,j}}\sim\Gamma_q\left[\begin{matrix}c,c'\\ a,\vec{b},\vec{b'}\end{matrix}\right]\lim_{\vec{i,j}\to\vec{\infty}}\vec{\{i\}}_q^{b-1}\vec{\{j\}}_q^{b'-1}\{i+j\}_q^{a-1}\{i\}_q^{1-c}\{j\}_q^{1-c'} \binom{|i+j|}{|i|}_q \tag{80}$$

The maximum values of the real parts of $a,\vec{b},\vec{b'},c,c'$ are $\alpha,\beta,\beta',\gamma,\gamma'$, and N is a number, such that:

$$N>\left|\Gamma_q\left[\begin{matrix}c,c'\\ a,\vec{b},\vec{b'}\end{matrix}\right]\right| \tag{81}$$

For $\vec{i,j}$ big enough, we have:

$$|A_{\vec{i,j}}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}|<N\left|\binom{|i+j|}{|i|}_q\right|\{i\}_q^{\beta-1}\{j\}_q^{\beta'-1}\{i+j\}_q^{\alpha-1}\{i\}_q^{1-\gamma}\{j\}_q^{1-\gamma'}|\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| \tag{82}$$

If:

1. ϵ_1 denotes a positive number bigger than the greatest of $\beta-1$ and $\beta'-1$ and ϵ_1' is a sufficiently big number

2. ϵ_2 denotes a positive number bigger than the greatest of $1-\gamma$ and $1-\gamma'$ and ϵ_2' denotes a sufficiently large number,

we have:

$$\{i\}_q^{\beta-1}\{j\}_q^{\beta'-1} < \{i\}_q^{\epsilon_1}\{j\}_q^{\epsilon_1} < \frac{\{i+j\}_q^{2\epsilon_1'}}{4^{\epsilon_1}} \tag{83}$$

$$\{i\}_q^{1-\gamma}\{j\}_q^{1-\gamma'} < \{i\}_q^{\epsilon_2}\{j\}_q^{\epsilon_2} < \frac{\{i+j\}_q^{2\epsilon_2'}}{4^{\epsilon_2}} \tag{84}$$

Therefore:

$$\begin{aligned}\sum_{\vec{i},\vec{j}} |A_{\vec{i},\vec{j}}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| &< \frac{N}{4^{\epsilon_1+\epsilon_2}} \sum_{\vec{i},\vec{j}} \left| \binom{|i+j|}{|i|}_q \{i+j\}_q^{2\epsilon_1'+2\epsilon_2'+\alpha-1}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}} \right| = \frac{N}{4^{\epsilon_1+\epsilon_2}} \\ \sum_{\vec{i},\vec{j}} &\left| \binom{i \vec{+} j}{\vec{i}}_q \{i+j\}_q^{2\epsilon_1'+2\epsilon_2'+\alpha-1} \sum_{I=i} |\vec{x}^{\vec{i}}| \sum_{J=j} |\vec{y}^{\vec{j}}| \right| \overset{\text{by lemma (15)}}{\sim} \frac{N}{4^{\epsilon_1+\epsilon_2}} \\ \sum_{\vec{i},\vec{j}} &\left| \binom{i \vec{+} j}{\vec{i}}_q \right| \{i+j\}_q^{2\epsilon_1'+2\epsilon_2'+\alpha-1} \mathrm{max}|x_m|^i \mathrm{max}|y_m|^j\end{aligned} \tag{85}$$

and the series converges for $\max_m |x_m| \oplus_q \max_m |y_m| < 1$, and $\Phi\mathrm{D}_{\overline{2}}$ converges in the same region. □

Theorem 24. *The convergence region for the functions:*

$$\Phi\mathrm{A}_1,\ \Phi\mathrm{A}_3,\ \mathrm{GA}_2 \tag{86}$$

is indicated.

$$\oplus_{q,m=1}^{n} |x_m| < 1\ \wedge \oplus_{q,m=1}^{n} |y_m| < 1 \tag{87}$$

Proof. The proof is for the function $\Phi\mathrm{A}_3$. The coefficient of $\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}$ is equal to:

$$A_{\vec{i},\vec{j}} \equiv \Gamma_q \left[\begin{matrix} \vec{c}, a+i, a'+j, b \vec{+} i, b' \vec{+} j, \vec{1}, \vec{1} \\ c+\vec{i}+j, a, a', \vec{b}, \vec{b'}, 1 \vec{+} i, 1 \vec{+} j \end{matrix} \right] \tag{88}$$

According to the $q-$Stirling formula, $\lim_{\vec{i},\vec{j}\to\vec{\infty}}$:

$$\begin{aligned}A_{\vec{i},\vec{j}} \sim \Gamma_q \left[\begin{matrix} \vec{c} \\ a, a', \vec{b}, \vec{b'} \end{matrix} \right] \lim_{\vec{i},\vec{j}\to\vec{\infty}} \{\vec{i}\}_q^{b-1} \{\vec{j}\}_q^{b'-1} \{i \vec{+} j\}_q^{1-c} \{i\}_q^{a-1}\{j\}_q^{a'-1} \\ \frac{\langle 1;q\rangle_{|i|}\langle 1;q\rangle_{|j|}}{\langle (1);q\rangle_{i\vec{+}j}}\end{aligned} \tag{89}$$

The maximum values of the real parts of $a, a', \vec{b}, \vec{b'}, \vec{c}$ are $\alpha, \alpha', \beta, \beta', \gamma$, and N is a number, such that:

$$N > \left| \Gamma_q \left[\begin{matrix} \vec{c} \\ a, a', \vec{b}, \vec{b'} \end{matrix} \right] \right| \tag{90}$$

For $\vec{i,j}$ big enough, we have:

$$|A_{\vec{i},\vec{j}}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| < N \left| \frac{\langle 1;q\rangle_{|i|}\langle 1;q\rangle_{|j|}}{\langle (1);q\rangle_{i\vec{+}j}} \right| \{i\}_q^{\beta-1}\{j\}_q^{\beta'-1}\{i \vec{+} j\}_q^{1-\gamma}\{i\}_q^{\alpha-1}\{j\}_q^{\alpha'-1}|\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| \tag{91}$$

If:

1. ϵ_1 denotes a positive number bigger than the greatest of $\beta-1$ and $\beta'-1$ and ϵ_1' is a sufficiently big number
2. ϵ_2 denotes a positive number bigger than the greatest of $\alpha-1$ and $\alpha'-1$ and ϵ_2' denotes a sufficiently large number,

we have:

$$\{i\}_q^{\beta-1}\{j\}_q^{\beta'-1} < \{i\}_q^{\epsilon_1}\{j\}_q^{\epsilon_1}\frac{\{i+j\}_q^{2\epsilon_1'}}{4^{\epsilon_1}} \tag{92}$$

$$\{i\}_q^{\alpha-1}\{j\}_q^{\alpha'-1} < \{i\}_q^{\epsilon_2}\{j\}_q^{\epsilon_2} < \frac{\{i+j\}_q^{2\epsilon_2'}}{4^{\epsilon_2}} \tag{93}$$

Therefore:

$$\begin{aligned}\sum_{\vec{i},\vec{j}}|A_{\vec{i},\vec{j}}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| &< \frac{N}{4^{\epsilon_1+\epsilon_2}}\sum_{\vec{i},\vec{j}}\left|\frac{\langle 1;q\rangle_{|i|}\langle 1;q\rangle_{|j|}}{\langle(1);q\rangle_{i\vec{+}j}}\{i+j\}_q^{2\epsilon_1'+2\epsilon_2'+1-\gamma}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}\right|\\ &\overset{\text{by}(6)}{\leq} \frac{N}{4^{\epsilon_1+\epsilon_2}}\sum_{\vec{i},\vec{j}}\left|\binom{|i|}{\vec{i}}_q\binom{|j|}{\vec{j}}_q\{i+j\}_q^{2\epsilon_1'+2\epsilon_2'+1-\gamma}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}\right|\end{aligned} \tag{94}$$

and the series converges for $\oplus_{q,m=1}^n|x_m| < 1$, $\oplus_{q,m=1}^n|y_m| < 1$ and $\Phi\mathrm{A}_3$ converges in the same region. □

Theorem 25. *The convergence region for the function $\Phi\mathrm{C}_3$ is governed by:*

$$\oplus_{q,m=1}^n|\sqrt{x_m}| < 1 \wedge \oplus_{q,m=1}^n|\sqrt{y_m}| < 1 \tag{95}$$

Proof. The coefficient of $\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}$ is equal to:

$$A_{\vec{i},\vec{j}} \equiv \Gamma_q\left[\begin{array}{c} a+i, b+i, a'+j, b'+j, \vec{c}, \vec{1}, \vec{1}\\ a, a', b, b', c+\vec{i}+j, 1\vec{+}i, 1\vec{+}j\end{array}\right] \tag{96}$$

According to the $q-$Stirling formula, $\lim_{\vec{i},\vec{j}\to\vec{\infty}}$:

$$\begin{aligned}A_{\vec{i},\vec{j}} \sim \Gamma_q\left[\begin{array}{c}\vec{c}\\ a, a', b, b'\end{array}\right]\lim_{\vec{i},\vec{j}\to\vec{\infty}}\{i\}_q^{a+b-2}\{j\}_q^{a'+b'-2}\{i\vec{+}j\}_q^{1-c}\\ \binom{|i|}{\vec{i}}_q\binom{|j|}{\vec{j}}_q\frac{\langle 1;q\rangle_{|i|}\langle 1;q\rangle_{|j|}}{\langle(1);q\rangle_{i\vec{+}j}}\end{aligned} \tag{97}$$

The maximum values of the real parts of $a, a', b, b', \vec{c}$ are $\alpha, \alpha', \beta, \beta', \gamma$, and N is a number, such that:

$$N > \left|\Gamma_q\left[\begin{array}{c}\vec{c}\\ a, a', b, b'\end{array}\right]\right| \tag{98}$$

For $\vec{i,j}$ big enough, we have:

$$\begin{aligned}|A_{\vec{i},\vec{j}}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| < N\left|\binom{|i|}{\vec{i}}_q\binom{|j|}{\vec{j}}_q\frac{\langle 1;q\rangle_{|i|}\langle 1;q\rangle_{|j|}}{\langle(1);q\rangle_{i\vec{+}j}}\right|\\ \{i\}_q^{\alpha+\beta-2}\{j\}_q^{\alpha'+\beta'-2}\{i+j\}_q^{1-\gamma}|\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}|\end{aligned} \tag{99}$$

If ϵ_1 denotes a positive number bigger than the greatest of $\alpha+\beta-2$ and $\alpha'+\beta'-2$ and ϵ_1' is a sufficiently big number, we have:

$$\{i\}_q^{\alpha+\beta-2}\{j\}_q^{\alpha'+\beta'-2} < \{i\}_q^{\epsilon_1}\{j\}_q^{\epsilon_1} < \frac{\{i+j\}_q^{2\epsilon_1'}}{4^{\epsilon_1}} \tag{100}$$

Therefore:

$$\begin{aligned} &\sum_{\vec{i},\vec{j}} \left|A_{\vec{i},\vec{j}}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}\right| < \frac{N}{4^{\epsilon_1}} \sum_{\vec{i},\vec{j}} \left| \binom{|i|}{\vec{i}}_q \binom{|j|}{\vec{j}}_q \frac{\langle 1;q\rangle_{|i|}\langle 1;q\rangle_{|j|}}{\langle (1);q\rangle_{\vec{i}\widetilde{+}\vec{j}}} \{i+j\}_q^{2\epsilon_1'+1-\gamma} \right| \\ &\left|\vec{x}^{\vec{i}}\right| \left|\vec{y}^{\vec{j}}\right| \overset{\text{by}(6)}{\leq} \frac{N}{4^{\epsilon_1}} \sum_{\vec{i}} \binom{|i|}{\vec{i}}_q^2 |\vec{x}^{\vec{i}}| \sum_{\vec{j}} \binom{|j|}{\vec{j}}_q^2 |\vec{y}^{\vec{j}}| \{i+j\}_q^{2\epsilon_1'+1-\gamma} \\ &\overset{\text{by}(39)}{<} \frac{N}{4^{\epsilon_1}} \sum_{\vec{i}} \binom{|i|}{\vec{i}}_q |\vec{x}^{\vec{i}}| \sum_{\vec{j}} \binom{|j|}{\vec{j}}_q |\vec{y}^{\vec{j}}| \{i+j\}_q^{2\epsilon_1'+1-\gamma} \end{aligned} \tag{101}$$

and we get the estimate for the convergence of $\Phi\mathrm{C}_3$. □

Conjecture 26. *In this conjecture, we have two convergence regions in* $\mathbb{R}^{2n}$*: Equations (102) and (103). We have Equation (102)* $\subset$ *Equation (103).*

$$\sum_m |x_m| + |y_m| < 1 \tag{102}$$

$$\oplus_{q,m=1}^{n} \left(|x_m| \oplus_q |y_m|\right) < 1 \tag{103}$$

The convergence regions for the function GA_1 *are somewhere between Equation (102) and (103).*

Proof. The coefficient of $\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}$ in GA_1 is equal to:

$$A_{\vec{i},\vec{j}} \equiv \Gamma_q \left[\begin{matrix} a+i+j, b+\vec{j}-i, b'+\vec{i}-j, \vec{1}, \vec{1} \\ a, b, b', 1\widetilde{+}i, 1\widetilde{+}j \end{matrix} \right] \tag{104}$$

According to the $q-$Stirling formula, $\lim_{\vec{i},\vec{j}\to\vec{\infty}}$:

$$\begin{aligned} &A_{\vec{i},\vec{j}} \sim \Gamma_q \left[\begin{matrix} \cdot \\ a, b, b' \end{matrix} \right] \lim_{\vec{i},\vec{j}\to\vec{\infty}} \{i+j\}_q^{a-1}\{j-i\}_q^{b-1}\{i-j\}_q^{b'-1} \\ &\frac{\langle 1;q\rangle_{|i+j|}}{\langle (1);q\rangle_{\vec{i}}(1);q\rangle_{\vec{j}}} \end{aligned} \tag{105}$$

The maximum values of the real parts of a, b, b' are α, β, β', and N is a number, such that:

$$N > \left| \Gamma_q \left[\begin{matrix} \cdot \\ a, b, b' \end{matrix} \right] \right| \tag{106}$$

For $\vec{i,j}$ big enough, we have:

$$\begin{aligned} &|A_{\vec{i},\vec{j}}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| < N \left| \frac{\langle 1;q\rangle_{|i+j|}}{\langle (1);q\rangle_{\vec{i}}(1);q\rangle_{\vec{j}}} \right| \\ &\left| \{j-i\}_q^{\beta-1}\{i-j\}_q^{\beta'-1}\{i+j\}_q^{\alpha-1} \right| |\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| \end{aligned} \tag{107}$$

If ϵ_1 denotes a positive number bigger than the greatest of $\beta-1$ and $\beta'-1$ and ϵ'_1 is a sufficiently big number, we have:

$$\{j-i\}_q^{\beta-1}\{i-j\}_q^{\beta'-1}<\{i\}_q^{\epsilon_1}\{j\}_q^{\epsilon_1}<\frac{\{i+j\}_q^{2\epsilon'_1}}{4^{\epsilon_1}} \tag{108}$$

Therefore:

$$\begin{aligned}
&\sum_{\vec{i},\vec{j}}|A_{\vec{i},\vec{j}}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}|<\frac{N}{4^{\epsilon_1}}\sum_{\vec{i},\vec{j}}\left|\frac{\langle 1;q\rangle_{|i+j|}}{\langle (1);q\rangle_{\vec{i}}(1);q\rangle_{\vec{j}}}\{i+j\}_q^{2\epsilon'_1+\alpha-1}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}\right|\\
=&\frac{N}{4^{\epsilon_1}}\sum_{\vec{i},\vec{j}}\left|\binom{i\vec{+}j}{\vec{i}}_q\{i+j\}_q^{2\epsilon'_1+\alpha-1}\left|\sum_{I=i}\left[\binom{|i|}{\vec{i}}_q\sqrt{\vec{x}^{\vec{i}}}\right]^2\sum_{J=j}\left[\binom{|j|}{\vec{j}}_q\sqrt{\vec{y}^{\vec{j}}}\right]^2\right|\right.\\
\overset{\text{by}(37)}{\sim}&\frac{N}{4^{\epsilon_1}}\sum_{\vec{i},\vec{j}}\left|\binom{i\vec{+}j}{\vec{i}}_q\{i+j\}_q^{2\epsilon'_1+\alpha-1}\right|\left[\sum_{I=i}\binom{|i|}{\vec{i}}_q\prod_{m=1}^{n}\sqrt{|x_m|^{i_m}}\right]^2\left[\sum_{J=j}\binom{|j|}{\vec{j}}_q\prod_{m=1}^{n}\sqrt{|y_m|^{j_m}}\right]^2\\
=&\frac{N}{4^{\epsilon_1}}\sum_{\vec{i},\vec{j}}\left|\binom{i\vec{+}j}{\vec{i}}_q\{i+j\}_q^{2\epsilon'_1+\alpha-1}\right|\left[\left[\oplus_{q,m=1}^{n}\sqrt{|x_m|}\right]^{|i|}\right]^2\left[\left[\oplus_{q,m=1}^{n}\sqrt{|y_m|}\right]^{|j|}\right]^2\\
\overset{\text{by}(37)}{\sim}&\frac{N}{4^{\epsilon_1}}\sum_{\vec{i},\vec{j}}\left|\binom{i\vec{+}j}{\vec{i}}_q\{i+j\}_q^{2\epsilon'_1+\alpha-1}\right|\left[\oplus_{q,m=1}^{n}|x_m|\right]^{|i|}\left[\oplus_{q,m=1}^{n}|y_m|\right]^{|j|}
\end{aligned} \tag{109}$$

and we get the estimate for the convergence of GA_1 in the conjecture. □

Conjecture 27. *In this conjecture, we have two convergence regions in* $\mathbb{R}^{2n}$*: Equations (110) and (111). We have Equation (110)* $\subset$ *Equation (111).*

$$(\sum_m|\sqrt{x_m}|)^2+(\sum_m|\sqrt{y_m}|)^2<1 \tag{110}$$

$$\oplus_{q,m=1}^{n}|\sqrt{x_m}|\oplus_q|\sqrt{y_m}|<1 \tag{111}$$

The convergence region for the function $\Phi\mathrm{C}_2$ *is somewhere between Equations (110) and (111).*

Proof. The coefficient of $\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}$ in $\Phi\mathrm{C}_2$. is equal to

$$A_{\vec{i},\vec{j}}\equiv\Gamma_q\left[\begin{array}{c}a+i+j,b+i,b'+j,\vec{c},\vec{c'},\vec{1},\vec{1}\\ a,b,b',c\vec{+}i,c'\vec{+}j,1\vec{+}i,1\vec{+}j\end{array}\right] \tag{112}$$

According to the $q-$Stirling formula, $\lim_{\vec{i},\vec{j}\to\vec{\infty}}$:

$$\begin{aligned}&A_{\vec{i},\vec{j}}\sim\Gamma_q\left[\begin{array}{c}\vec{c},\vec{c'}\\ a,b,b'\end{array}\right]\lim_{\vec{i},\vec{j}\to\vec{\infty}}\{i+j\}_q^{a-1}\{i\}_q^{b-1}\{j\}_q^{b'-1}\{\vec{i}\}_q^{1-c}\{\vec{j}\}_q^{1-c'}\\ &\binom{|i|}{\vec{i}}_q\binom{|j|}{\vec{j}}_q\frac{\langle 1;q\rangle_{|i+j|}}{\langle (1);q\rangle_{\vec{i}}(1);q\rangle_{\vec{j}}}\end{aligned} \tag{113}$$

The maximum values of the real parts of $a,b,b',\vec{c},\vec{c'}$ are $\alpha,\beta,\beta',\gamma,\gamma'$, and N is a number, such that:

$$N>\left|\Gamma_q\left[\begin{array}{c}\vec{c},\vec{c'}\\ a,b,b'\end{array}\right]\right| \tag{114}$$

For $\vec{i,j}$ big enough, we have:

$$|A_{\vec{i,j}}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| < N\left|\binom{|i|}{\vec{i}}_q\binom{|j|}{\vec{j}}_q\frac{\langle 1;q\rangle_{|i+j|}}{\langle(1);q\rangle_{\vec{i}}(1);q\rangle_{\vec{j}}}\right| \\ \{i\}_q^{\beta-1}\{j\}_q^{\beta'-1}\{i+j\}_q^{\alpha-1}\{\vec{i}\}_q^{1-\gamma}\{\vec{j}\}_q^{1-\gamma'}|\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| \tag{115}$$

1. If ϵ_1 denotes a positive number bigger than the greatest of $\beta-1$ and $\beta'-1$ and ϵ_1' is a sufficiently big number, we have:

$$\{i\}_q^{\beta-1}\{j\}_q^{\beta'-1} < \{i\}_q^{\epsilon_1}\{j\}_q^{\epsilon_1} < \frac{\{i+j\}_q^{2\epsilon_1'}}{4^{\epsilon_1}} \tag{116}$$

2. If ϵ_2 denotes a positive number bigger than the greatest of $1-\gamma$ and $1-\gamma'$ and ϵ_2' is a sufficiently big number, we have:

$$\{i\}_q^{1-\gamma}\{j\}_q^{1-\gamma'} < \{i\}_q^{\epsilon_2}\{j\}_q^{\epsilon_2} < \frac{\{i+j\}_q^{2\epsilon_2'}}{4^{\epsilon_2}} \tag{117}$$

Therefore:

$$\begin{aligned}\sum_{\vec{i,j}}|A_{\vec{i,j}}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| &< \frac{N}{4^{\epsilon_1+\epsilon_2}}\sum_{\vec{i,j}}\left|\binom{|i|}{\vec{i}}_q\binom{|j|}{\vec{j}}_q\frac{\langle 1;q\rangle_{|i+j|}}{\langle(1);q\rangle_{\vec{i}}(1);q\rangle_{\vec{j}}}\{i+j\}_q^{2\epsilon_1'+\alpha-1}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}\right| \\ &= \frac{N}{4^{\epsilon_1+\epsilon_2}}\sum_{\vec{i,j}}\left|\binom{i\dotplus j}{\vec{i}}_q\{i+j\}_q^{2\epsilon_1'+\alpha-1}\sum_{I=i}\left[\binom{|i|}{\vec{i}}_q\sqrt{\vec{x}^{\vec{i}}}\right]^2\sum_{J=j}\left[\binom{|j|}{\vec{j}}_q\sqrt{\vec{y}^{\vec{j}}}\right]^2\right| \\ &\overset{\text{by}(37)}{\sim} \frac{N}{4^{\epsilon_1+\epsilon_2}}\sum_{\vec{i,j}}\left|\binom{i\dotplus j}{\vec{i}}_q\{i+j\}_q^{2\epsilon_1'+\alpha-1}\right|\left[\sum_{I=i}\binom{|i|}{\vec{i}}_q\prod_{m=1}^n\sqrt{|x_m|^{i_m}}\right]^2\left[\sum_{J=j}\binom{|j|}{\vec{j}}_q\prod_{m=1}^n\sqrt{|y_m|^{j_m}}\right]^2 \\ &= \frac{N}{4^{\epsilon_1+\epsilon_2}}\sum_{\vec{i,j}}\left|\binom{i\dotplus j}{\vec{i}}_q\{i+j\}_q^{2\epsilon_1'+\alpha-1}\right|\left[\left[\oplus_{q,m=1}^n\sqrt{|x_m|}\right]^{|i|}\right]^2\left[\left[\oplus_{q,m=1}^n\sqrt{|y_m|}\right]^{|j|}\right]^2\end{aligned} \tag{118}$$

and we get the approximate estimate for the convergence of $\Phi\mathrm{C}_2$ in the conjecture. □

Conjecture 28. *In this conjecture, we have two convergence regions in $\mathbb{R}^{2n}$: Equations (119) and (120). We have Equation (119) $\subset$ Equations (120).*

$$\sum_m(\sqrt{|x_m} + \sqrt{|y_m}|)^2 < 1 \tag{119}$$

$$\forall m:\sqrt{|x_m|}\oplus_q\sqrt{|y_m|} < 1 \tag{120}$$

The convergence region for the function $\Phi\mathrm{A}_4$ is somewhere between Equations (119) and (120).

Proof. The coefficient of $\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}$ in $\Phi\mathrm{A}_4$. is equal to:

$$A_{\vec{i,j}} \equiv \Gamma_q\begin{bmatrix} a+i+j, b+\vec{i}+j, \vec{c}, \vec{c'}, \vec{1}, \vec{1} \\ a, \vec{b}, c\dotplus i, c'\dotplus j, 1\dotplus i, 1\dotplus j\end{bmatrix} \tag{121}$$

According to the $q-$Stirling formula, $\lim_{i,\vec{j}\to\vec{\infty}}$:

$$A_{i,\vec{j}} \sim \Gamma_q \left[\begin{matrix} \vec{c}, \vec{c'} \\ a, \vec{b} \end{matrix} \right] \lim_{i,\vec{j}\to\vec{\infty}} \{i+j\}_q^{a-1} \{\vec{i}\}_q^{1-c} \{\vec{j}\}_q^{1-c'} \{i \vec{+} j\}_q^{b-1} \binom{|i+j|}{\vec{i}}_q \binom{i \vec{+} j}{\vec{i}}_q \tag{122}$$

The maximum values of the real parts of $a, \vec{b}, \vec{c}, \vec{c'}$ are $\alpha, \vec{\beta}, \gamma, \gamma'$, and N is a number, such that:

$$N > \left| \Gamma_q \left[\begin{matrix} \vec{c}, \vec{c'} \\ a, \vec{b} \end{matrix} \right] \right| \tag{123}$$

For $i, \vec{j}$ big enough, we have:

$$|A_{i,\vec{j}} \vec{x}^{\vec{i}} \vec{y}^{\vec{j}}| < N \{i+j\}_q^{\alpha-1} \{\vec{i}\}_q^{1-\gamma} \{j\}_q^{1-\gamma'} \{i \vec{+} j\}_q^{\beta-1} \binom{|i+j|}{\vec{i}}_q \binom{i \vec{+} j}{\vec{i}}_q |\vec{x}^{\vec{i}} \vec{y}^{\vec{j}}| \tag{124}$$

If ϵ_1 denotes a positive number bigger than the greatest of $1-\gamma$ and $1-\gamma'$ and ϵ_1' is a sufficiently big number, we have:

$$\{i\}_q^{1-\gamma} \{j\}_q^{1-\gamma'} < \{i\}_q^{\epsilon_1} \{j\}_q^{\epsilon_1} < \frac{\{i+j\}_q^{2\epsilon_1'}}{4^{\epsilon_1}} \tag{125}$$

Therefore:

$$\begin{aligned} &\sum_{i,\vec{j}} |A_{i,\vec{j}} \vec{x}^{\vec{i}} \vec{y}^{\vec{j}}| < \frac{N}{4^{\epsilon_1}} \sum_{i,\vec{j}} \left| \binom{|i+j|}{\vec{i}}_q \binom{i \vec{+} j}{\vec{i}}_q \{i+j\}_q^{2\epsilon_1'+\alpha+\beta-2} \vec{x}^{\vec{i}} \vec{y}^{\vec{j}} \right| = \frac{N}{4^{\epsilon_1}} \\ &\sum_{\vec{k}} \binom{|k|}{\vec{k}}_q \prod_{m=1}^{n} \sum_{i_m+j_m=k_m} \left[\binom{i_m+j_m}{i_m}_q \sqrt{|x_m|^{i_m}} \sqrt{|y_m|^{j_m}} \right]^2 \left| \{i+j\}_q^{2\epsilon_1'+\alpha+\beta-2} \right| \\ &\overset{\text{by}(37)}{\sim} \frac{N}{4^{\epsilon_1}} \sum_{\vec{k}} \left| \binom{|k|}{\vec{k}}_q \right| \prod_{m=1}^{n} \left[\sum_{i_m+j_m=k_m} \binom{i_m+j_m}{i_m}_q \sqrt{|x_m|^{i_m}} \sqrt{|y_m|^{j_m}} \right]^2 \\ &\left| \{i+j\}_q^{2\epsilon_1'+\alpha+\beta-2} \right| = \frac{N}{4^{\epsilon_1}} \sum_{\vec{k}} \binom{|k|}{\vec{k}}_q \left| \{k\}_q^{2\epsilon_1'+\alpha+\beta-2} \right| \prod_{m=1}^{n} \left[\left[\sqrt{|x_m|} \oplus_q \sqrt{|y_m|} \right]^{k_m} \right]^2 \end{aligned} \tag{126}$$

and we get the estimate for the convergence of ΦA_4 in the conjecture. □

Conjecture 29. *In this conjecture, we have two convergence regions in $\mathbb{R}^{2n}$: Equations (127) and (128). We have Equation (127) $\subset$ Equation (128).*

$$\sum_m (\max\{|x_m|, |y_m|\}) < 1 \tag{127}$$

$$\oplus_{q,m=1}^{n} \max\{|x_m|, |y_m|\} < 1 \tag{128}$$

The convergence region for the function $\Phi\text{A}_{\bar{1}}$ is somewhere between Equations (127) and (128).

Proof. The coefficient of $\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}$ in $\Phi A_{\bar{1}}$ is equal to:

$$A_{\vec{i},\vec{j}} \equiv \Gamma_q \left[\begin{array}{c} a+i+j, b\vec{+}i, b'\vec{+}j, \vec{c}, \vec{1}, \vec{1} \\ a, \vec{b}, \vec{b'}, c+\vec{i}+j, 1\vec{+}j \end{array} \right] \tag{129}$$

According to the $q-$Stirling formula, $\lim_{\vec{i},\vec{j}\to\vec{\infty}}$:

$$A_{\vec{i},\vec{j}} \sim \Gamma_q \left[\begin{array}{c} \vec{c} \\ a, \vec{b}, \vec{b'} \end{array} \right] \lim_{\vec{i},\vec{j}\to\vec{\infty}} \{\vec{i}\}_q^{b-1} \{\vec{j}\}_q^{b'-1} \{i+j\}_q^{a-1} \{i\vec{+}j\}_q^{1-c} \binom{|i+j|}{i\vec{+}j}_q \tag{130}$$

The maximum values of the real parts of $a, \vec{b}, \vec{b'}, \vec{c}$ are $\alpha, \beta, \beta', \gamma$, and N is a number, such that:

$$N > \left| \Gamma_q \left[\begin{array}{c} \vec{c} \\ a, \vec{b}, \vec{b'} \end{array} \right] \right| \tag{131}$$

For $\vec{i,j}$ big enough, we have:

$$|A_{\vec{i},\vec{j}}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| < N \binom{|i+j|}{i\vec{+}j}_q \{i\}_q^{\beta-1}\{j\}_q^{\beta'-1}\{i+j\}_q^{\alpha-1}\{i\vec{+}j\}_q^{1-\gamma}|\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| \tag{132}$$

If ϵ_1 denotes a positive number bigger than the greatest of $\beta-1$ and $\beta'-1$ and ϵ_1' is a sufficiently big number, we have:

$$\{i\}_q^{\beta-1}\{j\}_q^{\beta'-1} < \{i\}_q^{\epsilon_1}\{j\}_q^{\epsilon_1} < \frac{\{i+j\}_q^{2\epsilon_1'}}{4^{\epsilon_1}} \tag{133}$$

Therefore:

$$\begin{aligned} &\sum_{\vec{i},\vec{j}} |A_{\vec{i},\vec{j}}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}}| < \frac{N}{4^{\epsilon_1}} \sum_{\vec{i},\vec{j}} \left| \binom{|i+j|}{i\vec{+}j}_q \{i+j\}_q^{2\epsilon_1'+\alpha-\gamma}\vec{x}^{\vec{i}}\vec{y}^{\vec{j}} \right| = \frac{N}{4^{\epsilon_1}} \\ &\sum_{\vec{k}} \left| \binom{|k|}{\vec{k}}_q \prod_m \sum_{i_m+j_m=k_m} \{i+j\}_q^{2\epsilon_1'+\alpha-\gamma} {x_m}^{i_m}{y_m}^{j_m} \right| \overset{\text{by lemma (15)}}{\sim} \frac{N}{4^{\epsilon_1}} \\ &\sum_{\vec{k}} \binom{|k|}{\vec{k}}_q \prod_{m=1}^{n} [\max\{|x_m|, |y_m|\}]^{k_m} \{k_m\}_q^{2\epsilon_1'+\alpha-\gamma} \\ &\sim \frac{N}{4^{\epsilon_1}} \left[\oplus_{q,m=1}^{n} \max\{|x_m|, |y_m|\}\right] \end{aligned} \tag{134}$$

and the series converges for $\oplus_{q,m=1}^{n}\max\{|x_m|, |y_m|\} < 1$, and we get the estimate for the convergence of $\Phi A_{\bar{1}}$ in the conjecture. $\square$

5. Conclusions

We have found a new type of convergence region of the vector type. To visualize these regions, one should start with the case $q = 1$ and imagine the q-deformed regions as slightly larger, increasing with the inverse of q. We have not found convergence regions for all functions, and there are other ways to describe the convergence regions in Section 4. There are also ways to describe where the series diverge.

The connection to quantum groups comes from the q-Lie algebras and q-Lie groups, which have been presented at the Group 30 meeting in Ghent 2014. Further articles of this type are in progress.

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Conflicts of Interest

The authors declare no conflict of interest.

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