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A Comprehensive Study of Generalized Lambert, Generalized Stieltjes, and Stieltjes–Poisson Transforms

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Abstract: In this paper, we explore the properties of the generalized Lambert transform, the L-transform, the generalized Stieltjes transform, and the Stieltjes–Poisson transform within the framework of Lebesgue spaces. We establish Parseval-type relations for each transform, providing a comprehensive analysis of their behaviour and mathematical characteristics.

Keywords: generalized Lambert transform; L-transform; generalized Stieltjes transform; Stieltjes–Poisson transform; Parseval relations; Lebesgue spaces; asymptotic behaviours

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1. Introduction and Preliminaries

We consider the following generalization of the Lambert transform introduced by [1] as

$$(GLM[f])(x) = \int_0^\infty f(t)\frac{xt}{e^{xt}-a}dt,\ x>0,\ a\in\mathbb{R},\ |a|\le 1, \tag{1}$$

given that $f\in\Omega$, where Ω represents the set of functions f that are continuous on $(0,\infty)$ and meet the order estimates outlined in [1]:

$$f(t)=\begin{cases} O(t^{\xi}), & \text{as } t\to 0^+,\\ O(t^{\delta}), & \text{as } t\to +\infty,\end{cases}$$

where $\Re(\xi)>-2$. The parameter δ appearing in the order estimates has no restrictions. For $|a|\le 1,\ x\in(0,\infty),\ t\in(0,\infty)$ one has

$$\frac{xt}{e^{xt}-a}\le\frac{xt}{e^{xt}-1}\le 1.$$

Moreover, for $a\neq 1$ one has

$$\lim_{x\to 0^+}\frac{xt}{e^{xt}-a}=0,$$

and

$$\lim_{x\to+\infty}\frac{xt}{e^{xt}-a}=\lim_{x\to+\infty}\frac{t}{te^{xt}}=0.$$

Notice that the kernel of transform (1) adheres to the relation

$$\frac{xt}{e^{xt}-a}=xt\sum_{k=1}^{\infty}a^{k-1}e^{-kxt},\quad |a|\le 1,\ x\in(0,\infty),\ t\in(0,\infty),$$

as a sum of a geometric series.

Widder [2] introduced the Lambert transform of a suitable function f as

$$F(x) = \int_0^\infty f(t) \frac{1}{e^{xt} - 1} dt, \ x > 0.$$

Widder examined the convergence characteristics and derived an inversion formula for this transform. Subsequently, Goldberg [3] investigated transforms featuring kernels expressed as $\sum_{k=1}^{\infty} a_k e^{-kxt}$, exploring a broad range of sequences of real numbers $\{a_k\}$. An L-transform is an integral transform of the form

$$(L[f])(x) = \int_0^\infty f(t) \sum_{k=1}^{\infty} a_k e^{-kxt} dt, \ x > 0. \tag{2}$$

Negrín [4] obtained an inversion formula for this transform over distributions of compact support on $(0, \infty)$, which is connected with the Post–Widder inversion formula for the distributional Laplace transform [5] (p. 243).

Various authors have investigated different aspects of the Lambert transform and inversion formulae. Among the notable findings, Miller [6] examined the convergence properties of the Lambert transform, crucial for developing inversion formulae, and introduced summability techniques for power series utilizing the Lambert transform [7]. Widder [2] derived an inversion formula in terms of a limit of derivatives involving the Möbius function. Several inversion formulae related to Widder's formula were explored in [8,9]. Raina and Srivastava [1] introduced a generalized Lambert transform linked with the generalized Riemann zeta function, with its inversion formula detailed in [10]. Goldberg [3] introduced a more general kernel for the Lambert transform, employing it to derive inversion formulae for transforms as Stieltjes [11] and Fourier cosine [12]. Raina and Nahar [13] proposed a Lambert transform generalization associated with a class of functions related to the Hurwitz zeta function. Ferreira and López [14] derived asymptotic expansions of the Lambert transform for both large and small values of the variable. Maan et al. [15] explored the generalized Lambert transform over Lebesgue spaces and extended their work to Boehmian spaces. Recently, González and Negrín [16] derived a Post–Widder-type inversion formula applicable to a Widder–Lambert-type integral transform. Additionally, we establish an L^p inversion formula, where $1 < p \leq 2$, for this transform, utilizing the Mellin transform. The inversion formulae obtained by the authors are used to obtain an interesting approach to the famous Riemann hypothesis by means of the Salem equivalence.

For $f \in L^1((0, \infty), t^{-\rho} dt), \ \rho > 0$, the generalized Stieltjes transform of f is defined by

$$(GST[f])(x) = \int_0^\infty \frac{f(t)}{(x+t)^\rho} dt, \ x > 0, \ \rho > 0. \tag{3}$$

T. S. Stieltjes introduced the Stieltjes transform in connection with the semi-infinite interval moment problem [17]. Since then, it has become a valuable tool in various fields such as continuous fractions, probability, and signal processing. While the classical Stieltjes transform has been extended to spaces of generalized functions by multiple authors (referenced in [18]), the exploration of the generalized Stieltjes transform's characteristics has been undertaken in works by different authors [19,20]. Its application is not confined to classical functions; it has also been applied to transforms of distributions [21]. The expansion of the Fourier transform to generalized functions has proven instrumental in studying partial differential equations. The theory and applications of integral transforms of generalized functions have witnessed active research over the last two decades. Despite numerous advancements, it is challenging to encompass all types of such transforms in one survey article. This study aims to analyse recent developments in the generalized Stieltjes transform. Recent studies have extensively explored its properties over weighted Lebesgue spaces and distributions of compact support [22].

Let $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$. For $f \in L^1((0,\infty), t^{\beta-\alpha}dt)$, the Stieltjes–Poisson transform of f is defined by

$$(SPT[f])(x) = \int_0^\infty f(t)\frac{t^\beta}{x^\alpha + t^\alpha}dt,\ x > 0. \tag{4}$$

For the case when $\alpha = 1$ and $\beta = 0$, the resulting transformation is the Stieltjes transform explored by Widder [23] and Goldberg [11], among other researchers. Also, for the case when $\alpha = 2$ and $\beta = 1$, the resulting transformation is known as the Poisson transform, studied in [11,24–26] (among others).

For $\gamma \in \mathbb{R}$ consider the vector space $\mathcal{B}_\gamma$ consisting of all complex-valued measurable functions f on $(0,\infty)$ such that $t^\gamma f(t) \in L^\infty((0,\infty))$.

A norm $\|\cdot\|_\gamma$ on $\mathcal{B}_\gamma$ is given by

$$\|f\|_\gamma = \|t^\gamma f(t)\|_{L^\infty((0,\infty))}.$$

With this norm, the map

$$T_\gamma : \mathcal{B}_\gamma \to L^\infty((0,\infty))$$

where for any $f \in \mathcal{B}_\gamma$,

$$(T_\gamma f)(t) = t^\gamma f(t),\ t \in (0,\infty)$$

is an isometric isomorphism from $\mathcal{B}_\gamma$ to $L^\infty((0,\infty))$. Thus, since $L^\infty((0,\infty))$ is complete, then the space $\mathcal{B}_\gamma$ becomes a Banach space. Observe that $\mathcal{B}_0 = L^\infty((0,\infty))$.

Integral transforms are useful tools because they can convert difficult problems in one domain into simpler problems in another domain, often leading to easier mathematical manipulation and solution. They can be used to solve differential equations, integral equations, signal processing problems, image processing problems, and many other mathematical and physical problems [1,2,5,8,9,13,14,17–22]. The selection of the suitable integral transform hinges on the characteristics of the problem and the desired properties of the transformed function. Each transform offers distinct advantages and drawbacks, and the choice thereof depends on the specific requirements of the problem and the properties of the original function.

Studying integral transforms from a mathematical analysis perspective offers insight into their fundamental properties and theoretical foundations, essential for rigorous mathematical understanding. This approach provides a deeper comprehension beyond mere application, fostering innovation and advancement in mathematical theory. In this self-contained work, a comprehensive study of the generalized Lambert transform, the L-transform, the generalized Stieltjes transform, and the Stieltjes–Poisson transform are presented from a functional analytic point of view, emphasizing their mathematical properties.

The paper is structured into six sections. Section 2 analyses the generalized Lambert transform (1) and the L-transform (2), Section 3 analyses the generalized Stieltjes transform (3), and Section 4 analyses the Stieltjes–Poisson transform (4). Section 2 analyses the generalized Lambert transform (1) and the L-transform (2) across spaces $\mathcal{B}_\gamma$, resulting in a Parseval-type relation. Section 3 conducts an analysis of the generalized Stieltjes transform (3) across spaces $\mathcal{B}_\gamma$, also yielding a Parseval-type relation. Section 4 presents a systematic analysis of the Stieltjes–Poisson transform (4). Section 5 focuses on deriving asymptotic behaviours of the generalized Lambert transform, the L-transform, the generalized Stieltjes transform, and the Stieltjes–Poisson transform. Finally, Section 6 gives concluding notes.

2. The Generalized Lambert Transform and the L-Transform over $\mathcal{B}_\gamma$

In this section, we investigate the generalized Lambert transform (1) and the L-transform (2) over the spaces $\mathcal{B}_\gamma$. This exploration yields a Parseval-type relation as a significant outcome.

2.1. The Generalized Lambert Transform over $\mathcal{B}_\gamma$

From (1),

$$(GLM[f])(x) = \int_0^\infty f(t)\frac{xt}{e^{xt}-a}dt,\ x>0,\ a\in\mathbb{R},\ |a|\le 1 \tag{5}$$

for a suitable complex-valued function f on $(0,\infty)$.

Note that

$$\frac{xt^{1-\gamma}}{e^{xt}-a} = xt^{1-\gamma}\sum_{k=1}^{\infty}a^{k-1}e^{-kxt},\ x>0,\ t>0,\ |a|\le 1.$$

Also, for $n\in\mathbb{N}$,

$$\sum_{k=1}^{n}xt^{1-\gamma}|a|^{k-1}e^{-kxt} \le \sum_{k=1}^{\infty}xt^{1-\gamma}|a|^{k-1}e^{-kxt} = \frac{xt^{1-\gamma}}{e^{xt}-|a|},$$

which is integrable on $(0,\infty)$ for $\gamma<1$.

Then, by using the dominated convergence theorem one has

$$\begin{aligned}\int_0^\infty \frac{xt^{1-\gamma}}{e^{xt}-a}dt &= \int_0^\infty \lim_{n\to\infty}\sum_{k=1}^{n}xt^{1-\gamma}a^{k-1}e^{-kxt}dt\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\int_0^\infty xt^{1-\gamma}a^{k-1}e^{-kxt}dt\\ &= \sum_{k=1}^{\infty}\int_0^\infty xt^{1-\gamma}a^{k-1}e^{-kxt}dt.\end{aligned} \tag{6}$$

Observe that

$$\begin{aligned}\int_0^\infty xt^{1-\gamma}e^{-kxt}dt &= \int_0^\infty x\left(\frac{u}{kx}\right)^{1-\gamma}e^{-u}\frac{1}{kx}du,\ \text{(putting } u=kxt\text{)}\\ &= \frac{1}{x^{1-\gamma}k^{2-\gamma}}\int_0^\infty u^{1-\gamma}e^{-u}du\\ &= \frac{\Gamma(2-\gamma)}{x^{1-\gamma}k^{2-\gamma}},\ \gamma<1.\end{aligned} \tag{7}$$

Inserting (7) into (6) one obtains

$$\int_0^\infty \frac{xt^{1-\gamma}}{e^{xt}-a}dt = \frac{\Gamma(2-\gamma)}{x^{1-\gamma}}\sum_{k=1}^{\infty}\frac{a^{k-1}}{k^{2-\gamma}},\ |a|\le 1,\ \gamma<1.$$

Clearly, $\sum_{k=1}^{\infty}\frac{a^{k-1}}{k^{2-\gamma}}$ converges for $|a|\le 1$ since $\left|\frac{a^{k-1}}{k^{2-\gamma}}\right| \le \frac{1}{k^{2-\gamma}}$ and $\sum_{k=1}^{\infty}\frac{1}{k^{2-\gamma}} = \zeta(2-\gamma)$, which converges for $\gamma<1$, with $\zeta(\cdot)$ being the Riemann zeta function.

Thus, for $f \in \mathcal{B}_\gamma,\ |a| \le 1,\ \gamma < 1,\ x > 0$, one has

$$\begin{aligned} |(GLM[f])(x)| &\le \int_0^\infty |f(t)| \frac{xt}{e^{xt}-a} dt \\ &= \int_0^\infty t^\gamma |f(t)| \frac{xt^{1-\gamma}}{e^{xt}-a} dt \\ &\le \operatorname*{ess\,sup}_{t\in(0,\infty)} \{t^\gamma |f(t)|\} \int_0^\infty \frac{xt^{1-\gamma}}{e^{xt}-a} dt \\ &= \|f\|_\gamma \frac{\Gamma(2-\gamma)}{x^{1-\gamma}} \sum_{k=1}^\infty \frac{a^{k-1}}{k^{2-\gamma}} \\ &\le \|f\|_\gamma \frac{\Gamma(2-\gamma)}{x^{1-\gamma}} \zeta(2-\gamma). \end{aligned} \tag{8}$$

From this estimate one obtains, for $0 < q < \infty,\ |a| \le 1,\ \gamma < 1$, and for w being a measurable function on $(0,\infty)$ such that $w > 0$ almost everywhere on $(0,\infty)$, $f \in \mathcal{B}_\gamma$

$$\int_0^\infty |(GLM[f])(x)|^q w(x) dx \le \|f\|_\gamma^q \Gamma(2-\gamma)^q \zeta(2-\gamma)^q \int_0^\infty x^{-q(1-\gamma)} w(x) dx.$$

These results are summarized in the next proposition.

Proposition 1. *Assume w is a measurable function on $(0,\infty)$ such that $w > 0$ almost everywhere on $(0,\infty)$ and $0 < q < \infty,\ |a| \le 1,\ \gamma < 1$. If $\int_0^\infty x^{-q(1-\gamma)} w(x) dx < \infty$, then the generalized Lambert transform (GLM) given by (5) satisfying*

$$GLM : \mathcal{B}_\gamma \to L^q((0,\infty), w(x)dx)$$

is a bounded linear operator.

Example 1. *Examples of weights w being $\gamma < 1$ are*

(i) $w(x) = e^{rx}$, $x \in (0,\infty)$, *valid for $r < 0$ and $0 < q < \frac{1}{1-\gamma}$.*

(ii) $w(x) = (1+x)^r$, $x \in (0,\infty)$, *valid for $r < q(1-\gamma) - 1$ and $0 < q < \frac{1}{1-\gamma}$.*

Remark 1. *For the case $\gamma = 0$ this proposition agrees with Proposition 3.1 in* [15].

The next result exhibits a Parseval relation for the generalized Lambert transform given by (5).

Theorem 1. *If $f \in \mathcal{B}_\gamma$, $g \in L^1((0,\infty), \frac{dx}{x^{1-\gamma}})$, where $\gamma < 1$, then the following Parseval relation holds:*

$$\int_0^\infty (GLM[f])(x) g(x) dx = \int_0^\infty f(x) (GLM[g])(x) dx.$$

Proof. Applying Fubini's theorem in the following, we obtain

$$\begin{aligned} \int_0^\infty (GLM[f])(x) g(x) dx &= \int_0^\infty \int_0^\infty f(t) \frac{xt}{e^{xt}-a} dt g(x) dx \\ &= \int_0^\infty \int_0^\infty g(x) \frac{xt}{e^{xt}-a} dx f(t) dt \\ &= \int_0^\infty f(t) (GLM[g])(t) dt. \end{aligned}$$

The Fubini theorem holds here because from (8) one has

$$\int_0^\infty |(GLM[f])(x)||g(x)|dx \leq \|f\|_\gamma \Gamma(2-\gamma)\zeta(2-\gamma)\int_0^\infty \frac{|g(x)|}{x^{1-\gamma}}dx < \infty.$$

Observe that $GLM[g]$ exists for $g \in L^1((0,\infty), \frac{dx}{x^{1-\gamma}})$.
In fact,

$$\int_0^\infty g(x)\frac{xt}{e^{xt}-a}dx = \int_0^\infty \frac{g(x)}{x^{1-\gamma}}\frac{x^{2-\gamma}t}{e^{xt}-a}dx.$$

Now, for $\gamma < 1$,

$$\frac{x^{2-\gamma}t}{e^{xt}-a} \to 0, \text{ as } x \to 0^+,$$

and

$$\frac{x^{2-\gamma}t}{e^{xt}-a} \to 0, \text{ as } x \to +\infty.$$

Then, for each $t \in (0,\infty)$,

$$\begin{aligned}\int_0^\infty |g(x)|\frac{xt}{e^{xt}-a}dx \quad &= \int_0^\infty \frac{|g(x)|}{x^{1-\gamma}}\frac{x^{2-\gamma}t}{e^{xt}-a}dx \\ &\leq N_t\int_0^\infty \frac{|g(x)|}{x^{1-\gamma}}dx < \infty,\end{aligned}$$

where $N_t > 0$ satisfies $\frac{x^{2-\gamma}t}{e^{xt}-a} \leq N_t$ for all $x \in (0,\infty)$. □

Remark 2. *For the case $\gamma = 0$ this theorem agrees with Theorem 3.3 in* [15].

Observe that for $\gamma < 1$, $f \in \mathcal{B}_\gamma$, $x \in (0,\infty)$,

$$\left|x^{1-\gamma}(GLM[f])(x)\right| \leq \|f\|_\gamma\Gamma(2-\gamma)\zeta(2-\gamma),$$

and so

$$\|GLM[f]\|_{1-\gamma} \leq \|f\|_\gamma\Gamma(2-\gamma)\zeta(2-\gamma).$$

Therefore, one has the next result.

Proposition 2. *For $\gamma < 1$, and with $\mathcal{B}_\gamma$ being defined as in Section 1, the generalized Lambert transform (GLM) given by* (5) *satisfying*

$$GLM : \mathcal{B}_\gamma \to \mathcal{B}_{1-\gamma}$$

is a bounded linear operator.

2.2. The L-Transform over $\mathcal{B}_\gamma$

From (2),

$$(L[f])(x) = \int_0^\infty f(t)\sum_{k=1}^\infty a_k e^{-kxt}dt, \ x > 0, \text{ for some } a_k, \ k \in \mathbb{N}, \tag{9}$$

and for a suitable complex-valued function f on $(0,\infty)$, we obtain formula (3) of [3].

For this case, when the sequence $\{a_k\}_{k=1}^{\infty}$ is bounded consisting of non-negative numbers with $a_1 > 0$ and $\{b_n\}_{n=1}^{\infty}$ being the (unique) sequence such that

$$\sum_{d|m} a_d b_{m|d} = \begin{cases} 1, & m = 1, \\ 0, & m = 2, 3, \cdots, \end{cases} \tag{10}$$

with the summation running over all the divisors d of m and $\{b_n\}_{n=1}^{\infty}$ also being bounded, then an inversion formula via Post–Widder for (2) is obtained for some class of functions f (Theorem 1 of [11]).

One observes that for the class of functions $\mathcal{B}_\gamma$, $\gamma < 0$, Theorem 1 of [11] holds, and thus, for these $\{a_k\}_{k=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ we obtain an inversion formula for (2) over $\mathcal{B}_\gamma$, $\gamma < 0$. The result is summarized as follows:

Theorem 2 (Inversion Formula)**.** *Let $\{a_k\}$ be a bounded sequence of non-negative numbers with $a_1 > 0$. Let $\{b_n\}_{n=1}^{\infty}$ be the (unique) sequence such that*

$$\sum_{d|m} a_d b_{m|d} = \begin{cases} 1, & m = 1, \\ 0, & m = 2, 3, \cdots, \end{cases}$$

the summation running over all the divisors d of m. If the sequence $\{b_n\}_{n=1}^{\infty}$ is also bounded and with $f \in \mathcal{B}_\gamma$, $\gamma < 0$, then

$$\lim_{p\to\infty} \frac{(-1)^p}{p!}\left(\frac{p}{t}\right)^{p+1} \sum_{n=1}^{\infty} b_n n^p (L[f])^{(p)}\left(\frac{np}{t}\right) = f(t)$$

almost everywhere on $(0, \infty)$, where $L[f]$ denotes the L-transform (2) of f and $H^{(p)}(z)$ denotes the conventional p-th derivative of H with respect to its argument z.

As a consequence of this inversion formula one obtains the next result.

Corollary 1 (Injectivity)**.** *Assuming the hypothesis of Theorem 2 and supposing that f, $g \in \mathcal{B}_\gamma$, $\gamma < 0$, with $L[f] = L[g]$, then $f = g$ almost everywhere on $(0, \infty)$.*

For $f \in \mathcal{B}_\gamma$, $\gamma < 0$, one has some interesting relations between the L-transform (2) and the real Laplace transform $\mathcal{L}$ of f given by

$$(\mathcal{L}[f])(x) = \int_0^\infty f(t)e^{-xt}dt,\ x > 0.$$

In fact, by means of Theorem 6.1 for the case $r = 1$ in [3], and since the corresponding $\alpha(t) = \int_0^t f(u)du$, $t > 0$, satisfies

$$\begin{aligned} |\alpha(t)| \le \int_0^t |f(u)|du &\le N \cdot \int_0^t u^{-\gamma}du \\ &= \frac{N}{1-\gamma} \cdot t^{1-\gamma},\ t > 0,\ \text{since } \gamma < 0, \end{aligned}$$

where $N > 0$ satisfies $|f(t)| \le \frac{N}{t^\gamma}$ almost everywhere on $(0, \infty)$, with $f \in \mathcal{B}_\gamma$, one obtains

$$\int_0^1 \frac{|\alpha(t)|}{t^2}dt \le \frac{N}{1-\gamma}\int_0^1 \frac{t^{1-\gamma}}{t^2}dt < \infty,\ \text{for } \gamma < 0.$$

So condition 5 of Theorem 6.1 of [3] for the case $r = 1$ is fulfilled, and then,

$$(L[f])(x) = \sum_{k=1}^{\infty} a_k(\mathcal{L}[f])(kx),\ x > 0,\ f \in \mathcal{B}_\gamma,\ \gamma < 0. \tag{11}$$

Also, from Theorem 6.4 of [3] for the case $r = 1$, and since

$$\int_0^1 \frac{|\alpha(t)\log t|}{t^2}dt \leq \frac{N}{1-\gamma}\int_0^1 \frac{t^{1-\gamma}|\log t|}{t^2}dt = -\frac{N}{1-\gamma}\int_0^1 t^{-1-\gamma}\log t\,dt < \infty, \text{ for } \gamma < 0.$$

So, condition 5 of Theorem 6.4 of [3] for the case $r = 1$ is fulfilled, and then,

$$(\mathcal{L}[f])(x) = \sum_{n=1}^{\infty} b_n(L[f])(nx),\ x > 0,\ f \in \mathcal{B}_\gamma,\ \gamma < 0. \tag{12}$$

Also, for the case $a = 0$ of the generalized Lambert transform (1), one has

$$(\mathcal{L}[f])(x) = \frac{1}{x}\cdot\left(GLM\left[\frac{f(t)}{t}\right]\right)(x),\ x > 0,\ f \in \mathcal{B}_\gamma,\ \gamma < 1, \tag{13}$$

and

$$(GLM[f])(x) = x\cdot(\mathcal{L}[tf(t)])(x),\ x > 0,\ f \in \mathcal{B}_\gamma,\ \gamma < 0. \tag{14}$$

Then, using Formulae (11) and (13) the following relation is obtained:

$$(L[f])(x) = \frac{1}{x}\cdot\sum_{k=1}^{\infty}\frac{a_k}{k}\left(GLM\left[\frac{f(t)}{t}\right]\right)(kx),\ x > 0,\ f \in \mathcal{B}_\gamma,\ \gamma < 0.$$

Also, using Formulae (12) and (14) the following relation is obtained:

$$(GLM[f])(x) = x\cdot\sum_{n=1}^{\infty} b_n(L[tf(t)])(nx),\ x > 0,\ f \in \mathcal{B}_\gamma,\ \gamma < -1.$$

There follows two examples of pairs of sequences satisfying (10).

Example 2. *If $a_k = 1,\ k = 1,2,\cdots$, it is well known that $b_n = \mu_n$, where $\{\mu_n\}_{n=1}^{\infty}$ are the Möbius numbers, defined as $\mu = 1,\ \mu_n = (-1)^s$ if n is the product of s distinct primes, $\mu_n = 0$ if n is divisible by a square. For this example, (10) reads*

$$\sum_{d|m}\mu_{m|d} = \begin{cases} 1, & m = 1, \\ 0, & m = 2,3,\cdots. \end{cases}$$

Note that for this example the L-transform (2) becomes

$$\begin{aligned}(L[f])(x) &= \int_0^{\infty} f(t)\sum_{k=1}^{\infty} e^{-kxt}dt \\ &= \int_0^{\infty} f(t)\frac{1}{e^{xt}-1}dt,\ x > 0.\end{aligned}$$

Example 3. *If $a_{2k-1} = 1,\ k = 1,2,\cdots,\ a_{2k} = 0,\ k = 1,2,\cdots$, then $b_{2k-1} = \mu_{2k-1}$, $n = 1,2,\cdots, b_{2n} = 0,\ n = 1,2,\cdots$, [3] (p. 556, Example 2).*

Note that for this example the L-transform (2) becomes

$$\begin{aligned}(L[f])(x) &= \int_0^\infty f(t)\sum_{k=1}^\infty e^{-(2k-1)xt}dt \\ &= \int_0^\infty f(t)\frac{1}{e^{xt}-e^{-xt}}dt \\ &= \int_0^\infty f(t)\frac{cosech(xt)}{2}dt,\ x>0.\end{aligned}$$

Arguing as the preceding subsection, with $a_k \le M$, for some $M > 0$, for all $k \in \mathbb{N}$ and using the dominated convergence theorem one arrives at the estimate

$$|(L[f])(x)| \le \|f\|_\gamma \cdot M \cdot \frac{\Gamma(1-\gamma)}{x^{1-\gamma}}\zeta(1-\gamma),\ x>0,\ f\in\mathcal{B}_\gamma,\ \gamma<0. \tag{15}$$

In fact, for $\gamma < 0$, and using the dominated convergence theorem, one has that

$$\int_0^\infty \frac{t^{-\gamma}}{e^{xt}-1}dt = \frac{\Gamma(1-\gamma)}{x^{1-\gamma}}\sum_{k=1}^\infty \frac{1}{k^{1-\gamma}} = \frac{\Gamma(1-\gamma)}{x^{1-\gamma}}\zeta(1-\gamma). \tag{16}$$

Thus, for $f \in \mathcal{B}_\gamma,\ \gamma < 0,\ x > 0$ one obtains

$$\begin{aligned}|(L[f])(x)| &\le \int_0^\infty |f(t)|\sum_{k=1}^\infty a_k e^{-kxt}dt \\ &\le M\cdot\int_0^\infty t^\gamma|f(t)|t^{-\gamma}\sum_{k=1}^\infty e^{-kxt}dt \\ &\le M\cdot \operatorname*{ess\,sup}_{t\in(0,\infty)}\{t^\gamma|f(t)|\}\int_0^\infty \frac{t^{-\gamma}}{e^{xt}-1}dt \\ &= M\cdot\|f\|_\gamma\cdot\frac{\Gamma(1-\gamma)}{x^{1-\gamma}}\zeta(1-\gamma)\quad(\text{by means of (16)}),\end{aligned}$$

where $a_k \le M$, for all $k \in \mathbb{N}$.

From the estimate (15) one obtains for $0 < q < \infty,\ \gamma < 0$ and for w being a measurable function on $(0, \infty)$ such that $w > 0$ almost everywhere on $(0, \infty)$, $f \in \mathcal{B}_\gamma$

$$\int_0^\infty |(L[f])(x)|^q w(x)dx \le \|f\|_\gamma^q \Gamma(1-\gamma)^q\zeta(1-\gamma)^q\int_0^\infty x^{-q(1-\gamma)}w(x)dx.$$

These results are summarized in the next proposition.

Proposition 3. *Assume w is a measurable function on $(0, \infty)$ such that $w > 0$ almost everywhere on $(0, \infty)$ and $0 < q < \infty,\ \gamma < 0$. If $\int_0^\infty x^{-q(1-\gamma)}w(x)dx < \infty$, then for the L-transform (2) one has that*

$$L : \mathcal{B}_\gamma \to L^q((0,\infty), w(x)dx)$$

is a bounded linear operator.

Example 4. *Examples of weights w are the same as those considered in Example 1 of the previous subsection, replacing $\gamma < 1$ by $\gamma < 0$.*

The next result exhibits a Parseval relation for the L-transform.

Theorem 3. *If $f \in \mathcal{B}_\gamma$, $g \in L^1((0,\infty), \frac{dx}{x^{1-\gamma}})$, where $\gamma < 0$, then the following Parseval relation holds:*

$$\int_0^\infty (L[f])(x)g(x)dx = \int_0^\infty f(x)(L[g])(x)dx.$$

Proof. The proof is similar to that given in Theorem 2.4 using estimate (15) instead of (8). □

Observe that for $\gamma < 0$, $f \in \mathcal{B}_\gamma$, $x \in (0,\infty)$,

$$\left| x^{1-\gamma}(L[f])(x) \right| \leq \|f\|_\gamma \cdot M \cdot \Gamma(1-\gamma)\zeta(1-\gamma),$$

and so

$$\|L[f]\|_{1-\gamma} \leq \|f\|_\gamma \Gamma(1-\gamma)\zeta(1-\gamma).$$

Therefore, one has the next result.

Proposition 4. *For $\gamma < 0$, and with $\mathcal{B}_\gamma$ being defined as in Section 1, the L-transform given by (9) satisfying*

$$L : \mathcal{B}_\gamma \to \mathcal{B}_{1-\gamma}$$

is a bounded linear operator.

3. The Generalized Stieltjes Transform over $\mathcal{B}_\gamma$

From (3),

$$(GST[f])(x) = \int_0^\infty \frac{f(t)}{(x+t)^\rho} dt, \; x > 0, \; \rho > 0 \tag{17}$$

for a suitable complex-valued function f on $(0,\infty)$.

For $x > 0$,

$$|(GST[f])(x)| \leq \int_0^\infty |f(t)| t^\gamma \frac{t^{-\gamma}}{(x+t)^\rho} dt \leq \|f\|_\gamma \int_0^\infty \frac{t^{-\gamma}}{(x+t)^\rho} dt,$$

where $f \in \mathcal{B}_\gamma$ and $1 - \rho < \gamma < 1$.

Observe that for $1 - \rho < \gamma < 1$,

$$\begin{aligned} \int_0^\infty \frac{t^{-\gamma}}{(x+t)^\rho} dt \qquad &= \frac{1}{x^\rho} \int_0^\infty \frac{t^{-\gamma}}{(1+\frac{t}{x})^\rho} dt \\ &= \frac{1}{x^\rho} \int_0^\infty \frac{v^{-\gamma}}{(1+v)^\rho} x dv, \; \left(\text{putting } v = \frac{t}{x}\right) \\ &= \frac{1}{x^{\rho+\gamma-1}} B(1-\gamma, \rho+\gamma-1), \end{aligned}$$

where $B(\cdot,\cdot)$ denotes the beta function.

Thus, for $f \in \mathcal{B}_\gamma$, $1 - \rho < \gamma < 1$, $x > 0$,

$$|(GST[f])(x)| \leq \|f\|_\gamma \frac{1}{x^{\rho+\gamma-1}} B(1-\gamma, \rho+\gamma-1). \tag{18}$$

From this estimate one obtains for $0 < q < \infty$, $1 - \rho < \gamma < 1$ and for w being a measurable function on $(0,\infty)$ such that $w > 0$ almost everywhere on $(0,\infty)$, $f \in \mathcal{B}_\gamma$,

$$\int_0^\infty |(GST[f])(x)|^q w(x) dx \leq \|f\|_\gamma^q \{B(1-\gamma, \rho+\gamma-1)\}^q \int_0^\infty x^{-q(\rho+\gamma-1)} w(x) dx.$$

These results are summarized in the next proposition.

Proposition 5. *Assume w is a measurable function on $(0, \infty)$ such that $w > 0$ almost everywhere on $(0, \infty)$ and $0 < q < \infty$, $1 - \rho < \gamma < 1$. If $\int_0^\infty x^{-q(\rho+\gamma-1)} w(x)dx < \infty$, then for the generalized Stieltjes transform GST one has that*

$$GST : \mathcal{B}_\gamma \to L^q((0, \infty), w(x)dx)$$

is a bounded linear operator.

Example 5. *Examples of weights w, with $1 - \rho < \gamma < 1$, are:*

(i) $w(x) = e^{rx}$, $x \in (0, \infty)$, *valid for $r < 0$ and $0 < q < \frac{1}{\rho+\gamma-1}$.*

(ii) $w(x) = (1 + x)^r$, $x \in (0, \infty)$, *valid for $r < q(\rho + \gamma - 1) - 1$ and $0 < q < \frac{1}{\rho+\gamma-1}$.*

The next result exhibits a Parseval relation for the generalized Stieltjes transform.

Theorem 4. *If $f \in \mathcal{B}_\gamma$, $g \in L^1((0, \infty), \frac{dx}{x^{\rho+\gamma-1}})$, where $1 - \rho < \gamma < 1$, then the following Parseval relation holds:*

$$\int_0^\infty (GST[f])(x)g(x)dx = \int_0^\infty f(x)(GST[g])(x)dx.$$

Proof. The proof is similar to that given in Theorem 1 using estimate (18) instead of (8). □

Now, for $k \in \mathbb{N}$ one has

$$D_t^k \left\{ \frac{1}{(x+t)^\rho} \right\} = \frac{(-1)^k (\rho)_k}{(x+t)^{\rho+k}},$$

where $(\rho)_k = \rho(\rho + 1) \cdots (\rho + k - 1)$.

Thus, for $f \in C_c^k((0, \infty))$, the space of compactly supported functions on $(0, \infty)$ which are k-times differentiable functions with continuity, one obtains

$$\begin{aligned} \left(GST[D^k f]\right)(x) &= \int_0^\infty \left(D^k f\right)(t) \frac{1}{(x+t)^\rho} dt \\ &= (-1)^k \int_0^\infty f(t) D_t^k \left\{ \frac{1}{(x+t)^\rho} \right\} dt \\ &= (\rho)_k \int_0^\infty f(t) \frac{1}{(x+t)^{\rho+k}} dt, \ x > 0. \end{aligned}$$

So, from Theorem 4, the next result follows.

Theorem 5. *If $f \in C_c^k((0, \infty))$, $k \in \mathbb{N}$, $1 - \rho < \gamma < 1$, and $g \in L^1((0, \infty), \frac{dx}{x^{\rho+\gamma-1}})$, then*

$$\int_0^\infty (GST_k[f])(x)g(x)dx = \frac{1}{(\rho)_k} \int_0^\infty \left(D^k f\right)(x)(GST[g])(x)dx,$$

where $(GST_k[f])(x) = \int_0^\infty \frac{f(t)}{(x+t)^{\rho+k}} dt$, $x > 0$.

Observe that for $1 - \rho < \gamma < 1$, $f \in \mathcal{B}_\gamma$, $x \in (0, \infty)$, and from estimate (18), one obtains

$$x^{\rho+\gamma-1} |(GST[f])(x)| \leq \|f\|_\gamma B(1 - \gamma, \rho + \gamma - 1),$$

and so,

$$\|GST[f]\|_{\rho+\gamma-1} \leq \|f\|_{\gamma} B(1-\gamma, \rho+\gamma-1).$$

Therefore, one has the next result.

Proposition 6. *For $1-\rho<\gamma<1$, and with $\mathcal{B}_{\gamma}$ being as defined in Section 1, the generalized Stieltjes transform GST given by (17) satisfying*

$$GST : \mathcal{B}_{\gamma} \to \mathcal{B}_{\rho+\gamma-1}$$

is a bounded linear operator.

4. The Stieltjes–Poisson Transform over $\mathcal{B}_{\gamma}$

From (4),

$$(SPT[f])(x) = \int_0^{\infty} f(t) \frac{t^{\beta}}{x^{\alpha}+t^{\alpha}} dt, \ \alpha, \beta \in \mathbb{R}, \ \alpha \neq 0, \ x>0, \tag{19}$$

for a suitable complex-valued function f on $(0,\infty)$.

Note that for $x>0$,

$$\begin{aligned} |(SPT[f])(x)| \quad & \leq \int_0^{\infty} |f(t)| t^{\gamma} \frac{t^{\beta-\gamma}}{x^{\alpha}+t^{\alpha}} dt \\ & \leq \|f\|_{\gamma} \int_0^{\infty} \frac{t^{\beta-\gamma}}{x^{\alpha}+t^{\alpha}} dt, \end{aligned}$$

where $f \in \mathcal{B}_{\gamma}$, $0<\frac{\beta-\gamma+1}{\alpha}<1$.

Observe that

$$\int_0^{\infty} \frac{t^{\beta-\gamma}}{x^{\alpha}+t^{\alpha}} dt = \frac{1}{x^{\alpha-\beta+\gamma-1}} \frac{1}{|\alpha|} B\left(\frac{\beta-\gamma+1}{\alpha}, \frac{\alpha-\beta+\gamma-1}{\alpha}\right),$$

where $B(\cdot,\cdot)$ denotes the beta function and $0<\frac{\beta-\gamma+1}{\alpha}<1$.

In fact,

$$\begin{aligned} \int_0^{\infty} \frac{t^{\beta-\gamma}}{x^{\alpha}+t^{\alpha}} dt \quad & = \frac{1}{x^{\alpha}} \int_0^{\infty} \frac{t^{\beta-\gamma}}{1+\left(\frac{t}{x}\right)^{\alpha}} dt \\ & = \frac{1}{x^{\alpha-\beta+\gamma-1}} \int_0^{\infty} \frac{v^{\beta-\gamma}}{1+v^{\alpha}} dv, \ \left(\text{putting } v=\frac{t}{x}\right) \\ & = \frac{1}{x^{\alpha-\beta+\gamma-1}} \frac{1}{|\alpha|} \int_0^{\infty} y^{\frac{\beta-\gamma+1}{\alpha}-1} (1+y)^{-1} dy, \ \left(\text{putting } y=v^{\alpha}\right) \\ & = \frac{1}{x^{\alpha-\beta+\gamma-1}} \frac{1}{|\alpha|} B\left(\frac{\beta-\gamma+1}{\alpha}, \frac{\alpha-\beta+\gamma-1}{\alpha}\right), \end{aligned}$$

valid for $0<\frac{\beta-\gamma+1}{\alpha}<1$.

Thus, for $f \in \mathcal{B}_{\gamma}$, $0<\frac{\beta-\gamma+1}{\alpha}<1$, $x>0$,

$$|(SPT[f])(x)| \leq \|f\|_{\gamma} \frac{1}{x^{\alpha-\beta+\gamma-1}} \frac{1}{|\alpha|} B\left(\frac{\beta-\gamma+1}{\alpha}, \frac{\alpha-\beta+\gamma-1}{\alpha}\right). \tag{20}$$

From this estimate one obtains for $0 < q < \infty$, $0 < \frac{\beta-\gamma+1}{\alpha} < 1$ and for w being a measurable function on $(0,\infty)$ such that $w > 0$ almost everywhere on $(0,\infty)$, $f \in \mathcal{B}_\gamma$

$$\begin{aligned}\int_0^\infty |(SPT[f])(x)|^q w(x)dx &\leq \|f\|_\gamma^q \frac{1}{|\alpha|^q}\left\{B\left(\frac{\beta-\gamma+1}{\alpha}, \frac{\alpha-\beta+\gamma-1}{\alpha}\right)\right\}^q \\ &\times \int_0^\infty x^{-q(\alpha-\beta+\gamma-1)} w(x)dx.\end{aligned}$$

These results are summarized in the next proposition.

Proposition 7. *Assume w is a measurable function on $(0,\infty)$ such that $w > 0$ almost everywhere on $(0,\infty)$ and $0 < q < \infty$, $0 < \frac{\beta-\gamma+1}{\alpha} < 1$. If $\int_0^\infty x^{-q(\alpha-\beta+\gamma-1)} w(x)dx < \infty$, then for the Stieltjes–Poisson transform (SPT) one has that*

$$SPT : \mathcal{B}_\gamma \to L^q((0,\infty), w(x)dx)$$

is a bounded linear operator.

Example 6. *Examples of weights w, with $0 < \frac{\beta-\gamma+1}{\alpha} < 1$, are:*

(i) *$w(x) = e^{rx}$, $x \in (0,\infty)$, valid for $r < 0$ and $0 < q < \frac{1}{\alpha-\beta+\gamma-1}$ (when $\alpha-\beta+\gamma-1 > 0$), or $r < 0$ and $0 < q < \infty$ (when $\alpha-\beta+\gamma-1 \leq 0$).*

(ii) *$w(x) = (1+x)^r$, $x \in (0,\infty)$, valid for $r < q(\alpha-\beta+\gamma-1) - 1$ and $0 < q < \frac{1}{\alpha-\beta+\gamma-1}$ (when $\alpha-\beta+\gamma-1 > 0$), or $r < q(\alpha-\beta+\gamma-1) - 1$ and $0 < q < \infty$ (when $\alpha-\beta+\gamma-1 \leq 0$).*

The next result exhibits a Parseval relation for the Stieltjes–Poisson transform.

Theorem 6. *If $f \in \mathcal{B}_\gamma$ and $g \in L^1((0,\infty), \frac{dx}{x^{\alpha-\beta+\gamma-1}})$, where $\alpha > 0$ and $\beta+1-\alpha < \gamma < \beta+1$, then the following Parseval relation holds:*

$$\int_0^\infty (SPT[f])(x)g(x)dx = \int_0^\infty f(x)(SPT^*[g])(x)dx,$$

where $(SPT^*[g])(x) = \int_0^\infty g(t)\frac{x^\beta}{x^\alpha+t^\alpha}dt$, $x > 0$.

Proof. Applying Fubini's theorem in the following, we obtain

$$\begin{aligned}\int_0^\infty (SPT[f])(x)g(x)dx &= \int_0^\infty \int_0^\infty f(t)\frac{t^\beta}{x^\alpha+t^\alpha}dt g(x)dx \\ &= \int_0^\infty \int_0^\infty g(x)\frac{t^\beta}{x^\alpha+t^\alpha}dx f(t)dt \\ &= \int_0^\infty f(t)(SPT^*[g])(t)dt.\end{aligned}$$

The Fubini theorem holds here because from (20) one has

$$\begin{aligned}\int_0^\infty |(SPT[f])(x)||g(x)|dx &\leq \|f\|_\gamma \frac{1}{|\alpha|}B\left(\frac{\beta-\gamma+1}{\alpha}, \frac{\alpha-\beta+\gamma-1}{\alpha}\right) \\ &\times \int_0^\infty \frac{|g(x)|}{x^{\alpha-\beta+\gamma-1}}dx < \infty.\end{aligned}$$

Observe that $SPT^*[g]$ exists for $g \in L^1((0,\infty), \frac{dx}{x^{\alpha-\beta+\gamma-1}})$ with $\alpha > 0$ and $\beta+1-\alpha < \gamma < \beta+1$.

In fact,

$$\int_0^\infty g(x)\frac{t^\beta}{t^\alpha+x^\alpha}dx = t^\beta\int_0^\infty \frac{g(x)}{x^{\alpha-\beta+\gamma-1}}\frac{x^{\alpha-\beta+\gamma-1}}{t^\alpha+x^\alpha}dx.$$

Now, with $\alpha>0$ and $\beta+1-\alpha<\gamma<\beta+1$,

$$\frac{x^{\alpha-\beta+\gamma-1}}{t^\alpha+x^\alpha}\to 0, \text{ as } x\to 0^+,$$

and

$$\frac{x^{\alpha-\beta+\gamma-1}}{t^\alpha+x^\alpha}\to 0, \text{ as } x\to +\infty.$$

Then, for each $t\in(0,\infty)$,

$$\int_0^\infty |g(x)|\frac{t^\beta}{t^\alpha+x^\alpha}dx \le N_t\int_0^\infty \frac{|g(x)|}{x^{\alpha-\beta+\gamma-1}}dx<\infty,$$

where $N_t>0$ satisfies $t^\beta\cdot\frac{x^{\alpha-\beta+\gamma-1}}{t^\alpha+x^\alpha}\le N_t$ for all $x\in(0,\infty)$. $\square$

Observe that for $0<\frac{\beta-\gamma+1}{\alpha}<1$, $f\in\mathcal{B}_\gamma$, $x\in(0,\infty)$, and from estimate (20), one obtains

$$x^{\alpha-\beta+\gamma-1}|(SPT[f])(x)|\le \|f\|_\gamma\frac{1}{|\alpha|}B\left(\frac{\beta-\gamma+1}{\alpha},\frac{\alpha-\beta+\gamma-1}{\alpha}\right),$$

and so,

$$\|SPT[f]\|_{\alpha-\beta+\gamma-1}\le \|f\|_\gamma\frac{1}{|\alpha|}B\left(\frac{\beta-\gamma+1}{\alpha},\frac{\alpha-\beta+\gamma-1}{\alpha}\right).$$

Therefore, one has the next result.

Proposition 8. *For $0<\frac{\beta-\gamma+1}{\alpha}<1$, and with $\mathcal{B}_\gamma$ being defined as in Section 1, the Stieltjes–Poisson transform (SPT) given by (19) satisfying*

$$SPT:\mathcal{B}_\gamma\to\mathcal{B}_{\alpha-\beta+\gamma-1}$$

is a bounded linear operator.

5. Asymptotic Behaviours over $\mathcal{B}_\gamma$

When considering the generalized Lambert transform and using estimate (8) one has the following.

Proposition 9. *For $f\in\mathcal{B}_\gamma$, $\gamma<1$, then*

$$\lim_{x\to\infty}\{x^r(GLM[f])(x)\}=0, \text{ for any } r<1-\gamma.$$

In particular, for $f\in\mathcal{B}_\gamma$, $\gamma<1$,

$$\lim_{x\to\infty}(GLM[f])(x)=0.$$

Also, for $f\in\mathcal{B}_0=L^\infty((0,\infty))$ it follows that

$$\lim_{x\to\infty}\{x^r(GLM[f])(x)\}=0, \text{ for any } r<1.$$

In particular, for $f \in \mathcal{B}_0 = L^\infty((0,\infty))$,

$$\lim_{x\to\infty} (GLM[f])(x) = 0.$$

When considering the L-transform and using estimate (15) one has

Proposition 10. *For* $f \in \mathcal{B}_\gamma$, $\gamma < 0$, *then*

$$\lim_{x\to\infty} \{x^r (L[f])(x)\} = 0, \text{ for any } r < 1-\gamma.$$

In particular, for $f \in \mathcal{B}_\gamma$, $\gamma < 0$,

$$\lim_{x\to\infty} (L[f])(x) = 0.$$

Remark 3. *The case when* $f \in \mathcal{B}_0 = L^\infty((0,\infty))$ *is not included in Proposition 10 because* $\gamma < 0$.

When considering the generalized Stieltjes transform and using estimate (18), one has

Proposition 11. *For* $f \in \mathcal{B}_\gamma$, $1-\rho < \gamma < 1$, *then*

$$\lim_{x\to\infty} \{x^r (GST[f])(x)\} = 0, \text{ for any } r < \rho + \gamma - 1.$$

In particular, for $f \in \mathcal{B}_\gamma$, $1-\rho < \gamma < 1$,

$$\lim_{x\to\infty} (GST[f])(x) = 0.$$

Also, for $f \in \mathcal{B}_0 = L^\infty((0,\infty))$ *it follows that*

$$\lim_{x\to\infty} \{x^r (GST[f])(x)\} = 0, \text{ for any } r < \rho - 1.$$

In particular, for $f \in L^\infty((0,\infty))$ *and* $\rho > 1$,

$$\lim_{x\to\infty} (GST[f])(x) = 0. \tag{21}$$

Remark 4. *Note that the asymptotic behaviour (21) does not include the Stieltjes transform* ($\rho = 1$).

When considering the Stieltjes–Poisson transform and using estimate (20), one has

Proposition 12. *For* $f \in \mathcal{B}_\gamma$, $0 < \frac{\beta-\gamma+1}{\alpha} < 1$, *then*

$$\lim_{x\to\infty} \{x^r (SPT[f])(x)\} = 0, \text{ for any } r < \alpha - \beta + \gamma - 1. \tag{22}$$

In particular, for $f \in \mathcal{B}_\gamma$, $\alpha > 0$ *and* $\beta + 1 - \alpha < \gamma < \beta + 1$,

$$\lim_{x\to\infty} (SPT[f])(x) = 0. \tag{23}$$

Also, for $f \in \mathcal{B}_0 = L^\infty((0,\infty))$, $0 < \frac{\beta-\gamma+1}{\alpha} < 1$, *it follows that*

$$\lim_{x\to\infty} \{x^r (SPT[f])(x)\} = 0, \text{ for any } r < \alpha - \beta - 1. \tag{24}$$

In particular, for $f \in \mathcal{B}_0 = L^\infty((0,\infty))$, $\alpha > 0$ *and* $-1 < \beta < \alpha - 1$,

$$\lim_{x\to\infty} (SPT[f])(x) = 0. \tag{25}$$

Remark 5. *The asymptotic behaviours* (22) *and* (23) *hold for the Stieltjes transform ($\alpha = 1, \beta = 0$) when $f \in \mathcal{B}_\gamma$, $0 < \gamma < 1$, and for the Poisson transform ($\alpha = 2, \beta = 1$) when $f \in \mathcal{B}_\gamma$, $0 < \gamma < 2$. Note that the asymptotic behaviours* (24) *and* (25) *do not include either the Stieltjes transform or the Poisson transform.*

6. Conclusions

In conclusion, this research paper investigates the properties of the generalized Lambert transform, the L-transform, the generalized Stieltjes, and the Stieltjes–Poisson transform within the context of Lebesgue spaces, specifically focusing on the space $\mathcal{B}_\gamma$. Through our exploration we have successfully established Parseval-type relations and examined the asymptotic behaviour of each transform. The findings not only contribute valuable insights into the characteristics of these transforms but also provide a foundation for further exploration of similar properties for various integral transforms over the space $\mathcal{B}_\gamma$. Our findings lay a foundation for further exploration of similar properties in diverse integral transforms within the space $\mathcal{B}_\gamma$.

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