



Article

Operator Smith Algorithm for Coupled Stein Equations from Jump Control Systems

Bo Yu , Ning Dong ^{*}  and Baiquan Hu

School of Science, Hunan University of Technology, Zhuzhou 412007, China; yubo@hut.edu.cn (B.Y.); m23070100015@stu.hut.edu.cn (B.H.)

* Correspondence: dongning@hut.edu.cn

Abstract: Consider a class of coupled Stein equations arising from jump control systems. An operator Smith algorithm is proposed for calculating the solution of the system. Convergence of the algorithm is established under certain conditions. For large-scale systems, the operator Smith algorithm is extended to a low-rank structured format, and the error of the algorithm is analyzed. Numerical experiments demonstrate that the operator Smith iteration outperforms existing linearly convergent iterative methods in terms of computation time and accuracy. The low-rank structured iterative format is highly effective in approximating the solutions of large-scale structured problems.

Keywords: coupled Stein equations; operator Smith algorithm; jump control system; low rank; large-scale problems

MSC: 65F45; 65F10

1. Introduction

Consider the discrete-time jump control systems given by

$$x_{j+1} = A(t_i)x_j + B(t_i)u_j, \quad y_j = C(t_i)x_j, \quad i, j = 1, \dots, m,$$

where $A(t_i) \in \mathbb{R}^{N \times N}$, $B(t_i) \in \mathbb{R}^{N \times l_b}$, and $C(t_i) \in \mathbb{R}^{l_c \times N}$ with $l_b, l_c \ll N$. Here, N represents the scale of the jump control system. Efficient control in the analysis and design of jump systems involves associating the observability Gramian $W_{oi} = \sum_{k=0}^{\infty} (A(t_i)^{\top})^k C(t_i)^{\top} C(t_i) A(t_i)^k$ and controllability Gramian $W_{ci} = \sum_{k=0}^{\infty} A(t_i)^k B(t_i) B(t_i)^{\top} (A(t_i)^{\top})^k$ [1], which are solutions of the corresponding coupled discrete-time Stein equations (CDSEs):

$$\mathcal{S}c_i(X) = X_i - Q_i - A_i^{\top} E_i(X) A_i = 0. \quad (1)$$

Here, for $i = 1, \dots, m$, $A_i \in \mathbb{R}^{N \times N}$ is the input matrix, $Q_i \in \mathbb{R}^{N \times N}$ is symmetric and positive semi-definite, and $E_i(X) = \sum_{j=1}^m p_{ij} X_j \in \mathbb{R}^{N \times N}$ with probability values p_{ij} satisfying $\sum_{j=1}^m p_{ij} = 1$. Numerous methods, ranging from classical to state-of-the-art, have been developed over the past decades to address the single Stein equation (i.e., $m = 1$ in (1)), particularly for special matrix structures. For example, Betser et al. investigated solutions tailored to cases where coefficient matrices are in companion forms [2]. Hueso et al. devised a systolic algorithm for the triangular Stein equation [3]. Li et al. introduced an iterative method for handling large-scale (the term “large-scale” refers to the scale N of the corresponding equations being large) Stein and Lyapunov equations with low-ranked structures [4]. Fan et al. discussed generalized Lyapunov and Stein equations, deriving connections from rational Riccati equations [5]. Yu et al. scrutinized large-scale Stein equations featuring high-ranked structures [6].

For CDSEs (1), the parallel iteration [7], essentially a stationary iteration for linear systems, is a commonly used method to compute the desired solution. This approach



Citation: Yu, B.; Dong, N.; Hu, B. Operator Smith Algorithm for Coupled Stein Equations from Jump Control Systems. *Axioms* **2024**, *13*, 249. <https://doi.org/10.3390/axioms13040249>

Academic Editor: Janis Bajars

Received: 9 March 2024

Revised: 3 April 2024

Accepted: 5 April 2024

Published: 10 April 2024



Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

has been extended to an implicit sequential format [8], leveraging the latest information from obtained solutions to accelerate the iteration of the left part. The gradient-based iterative algorithm, introduced for solving CDSEs, explicitly determines the optimal step size to achieve the maximum convergence rate [9]. By utilizing positive operator theory, two iterative algorithms were established for solving CDSEs [10], later extended to Itô stochastic systems. The continuous-time Lyapunov equations can be transformed into CDSEs via the Cayley transformation [11], although determining the optimal Cayley parameter remains a challenge. For other related iterative methods of continuous-time Lyapunov equations, consult [12–16] and the corresponding references.

CDSEs also arise from solving sub-problems of coupled discrete-time Riccati equations from optimized control systems. The difference method [17] and CM method [18] were proposed to tackle the sub-problems of CDSEs. Ivanov [19] developed two Stein iterations that also exploit the latest information of previously derived solutions for acceleration. Notably, the iteration schemes provided in [8] are almost identical to these two Stein iterations [19], essentially corresponding to the Gauss–Jacobi and the Gauss–Seidel iterations applied to coupled matrix equations. Successive over-relaxation (SOR) iterations and their variants for CDSEs were explored in [20,21], though determining the optimal SOR parameter remains challenging. One limitation of the aforementioned methods is that they are linearly convergent, and the potential structures (such as the low rank and sparseness) of the matrices are not fully exploited. To enhance the convergence rate of iterative methods, the Smith method was employed to solve the single Stein equation [22] and extended to structured large-scale problems [4,11]. This method offers the advantage of being parameter-free and converging quadratically to the desired symmetric solution. A similar idea was extended to address some stochastic matrix equations in [5]. In this paper, we adapt the Smith method to an operator version to compute the solution of CDSEs and subsequently construct the corresponding low-ranked format for large-scale problems. The main contributions encompass the following significant aspects:

- We introduce the operator defined as

$$(\mathcal{F})_i(X) = A_i^\top E_i \left(A^\top E (\dots A^\top E(X) A) \dots A \right) A_i. \quad (2)$$

This operator formulation enables us to adapt the Smith iteration [4,11,22] to an operator version, denoted as the operator Smith algorithm (OSA). By doing so, the iteration maintains quadratic convergence for computing the symmetric solution of CDSEs (1). Our numerical experiments demonstrate that the OSA outperforms existing linearly convergent iterations in terms of both the CPU time and accuracy.

- To address large-scale problems, we structure the OSA in a low-ranked format with twice truncation and compression (TC). One TC applies to the factor in the constructed operator (2), while the other TC applies to the factor in the approximated solution. This approach effectively reduces the column dimensions of the low-rank factors in symmetric solutions.
- We redesign the residual computation to suit large-scale computations. We incorporate practical examples from industries [23] to validate the feasibility and effectiveness of the presented low-ranked OSA. This not only demonstrates its practical utility but also lays the groundwork for exploring various large-scale structured problems.

This paper is structured as follows. Section 2 outlines the iterative scheme of the OSA for CDSEs (1), along with a convergence analysis. Comparative results on small-scale problems highlight the superior performance of the OSA compared to other linearly convergent iterations. Section 3 delves into the development of the low-ranked OSA, providing details on truncation and compression techniques, residual computations, as well as complexity and error analysis. In Section 4, we present numerical experiments from industrial applications to illustrate the effectiveness of the introduced low-ranked OSA in real-world scenarios.

Throughout this paper, I_m (or simply I) is the $m \times m$ identity matrix. For a matrix $A \in \mathbb{R}^{N \times N}$, $\rho(A)$ is the spectral radius of A . Unless stated otherwise, the norm $\|\cdot\|$ is the ∞ -norm of a matrix. For matrices A and $B \in \mathbb{R}^{N \times N}$, the direct sum $A \oplus B$ means the block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. For symmetric matrices A and $B \in \mathbb{R}^{N \times N}$, we say $A > B$ ($A \geq B$) if $A - B$ is a positive definite (semi-definite) matrix.

2. Operator Smith Iteration for CDSEs

2.1. Iteration Scheme

The operator Smith algorithm (OSA) represents a generalization of the Smith iteration applied to a single matrix equation [4].

For each $i = 1, \dots, m$, the operator \mathcal{F} in (2) at k -th iteration is defined as

$$(\mathcal{F})_{i,k}(\cdot) = A_i^\top E_i \left(\overbrace{A^\top E(\dots A^\top E(\cdot) A)}^{2^k-1} \dots A \right) A_i := A_i^\top E_i \left((A^\top E)^{2^k-1}(\cdot) A^{2^k-1} \right) A_i. \quad (3)$$

With the initial $X_{i,0} = Q_i$, the OSA for CDSEs (1) is given by

$$X_{i,k+1} = X_{i,k} + (\mathcal{F})_{i,k}(X_{i,k}), \quad k = 0, 1, 2, \dots, \quad (4)$$

where $X_{i,k}$ represents the k -th iteration satisfying $E_i(X_{i,k}) = \sum_{s=1}^m p_{i,s} X_{s,k}$.

Remark 1. By the definition of the operator $(\mathcal{F})_{i,k}$, it is not difficult to see that $(\mathcal{F})_{i,k+1}$ doubles the former operator $(\mathcal{F})_{i,k}$ for $k = 1, 2, \dots$. Specifically, the operator $(\mathcal{F})_{i,k+1}$ acting on a matrix is equivalent to applying the former operator $(\mathcal{F})_{i,k}$ twice on that matrix. To illustrate, let us consider $m = 2$ as an example. For $Q = (Q_1, Q_2)$, the operator $(\mathcal{F})_{i,0}$ on Q (i.e., $k = 0$ in (3)) is

$$(\mathcal{F})_{i,0}(Q) = A_i^\top E_i(Q) A_i = A_i^\top (p_{i1} Q_1 + p_{i2} Q_2) A_i.$$

Similarly, the operator $(\mathcal{F})_{i,1}$ on Q (i.e., $k = 1$ in (3)) takes the form

$$\begin{aligned} (\mathcal{F})_{i,1}(Q) &= A_i^\top E_i(A^\top E(Q) A) A_i \\ &= A_i^\top (p_{i1} A_1^\top E_1(Q) A_1 + p_{i2} A_2^\top E_2(Q) A_2) A_i \\ &= A_i^\top (p_{i1} A_1^\top (p_{11} Q_1 + p_{12} Q_2) A_1 + p_{i2} A_2^\top (p_{21} Q_1 + p_{22} Q_2) A_2) A_i. \end{aligned}$$

Thus, the effect of $(\mathcal{F})_{i,1}(Q)$ is equivalent to $(\mathcal{F})_{i,0}((\mathcal{F})_{i,0}(Q))$, demonstrating that $(\mathcal{F})_{i,1}$ effectively doubles $(\mathcal{F})_{i,0}$. This doubling property extends the concept of Smith iteration for a single equation [4,11]. In this sense, (4) is referred to as the OSA.

The following proposition indicates the concrete form of $X_{i,k}$, ($i = 1, \dots, m$) in the OSA (4):

Proposition 1. Let $E_i(Q) = \sum_{s=1}^m p_{i,s} Q_s$. The k -th iteration $X_{i,k}$ generated by (4) admits a representation

$$X_{i,k} = Q_i + A_i^\top E_i \left(\sum_{j=0}^{2^k-2} (A^\top E)^j(Q) A^j \right) A_i \quad (5)$$

for $i = 1, \dots, m$.

Proof. We prove (5) by induction. It is obvious from the OSA that $X_{i,1} = X_{i,0} + \mathcal{F}_{i,0}(Q) = Q_i + A_i^\top E_i(Q) A_i$. Then, (5) holds for $k = 1$.

Assume that (5) holds for $k = l$. Then, one has

$$\begin{aligned}
X_{i,l+1} &= X_{i,l} + \mathcal{F}_{i,l}(X_{\cdot,l}) \\
&= X_{i,l} + A_i^\top E_i ((A^\top E)^{2^l-1} (X_{\cdot,l}) A^{2^l-1}) A_i \\
&= X_{i,l} + A_i^\top E_i \left\{ (A^\top E)^{2^l-1} \left[\sum_{s=1}^m p_{\cdot,s} \left(Q_s \right. \right. \right. \\
&\quad \left. \left. \left. + A_s^\top E_s \left(\sum_{j=0}^{2^l-2} (A^\top E)^j (Q) A^j \right) A_s \right) \right] A^{2^l-1} \right\} A_i \\
&= X_{i,l} + A_i^\top E_i \left\{ (A^\top E)^{2^l-1} \left[Q + \sum_{j=0}^{2^l-2} (A^\top E)^j (Q) A^{j+1} \right] A^{2^l-1} \right\} A_i \\
&= X_{i,l} + A_i^\top E_i \left(\sum_{j=2^l-1}^{2^{l+1}-2} (A^\top E)^j (Q) A^j \right) A_i \\
&= Q_i + A_i^\top E_i \left(\sum_{j=0}^{2^{l+1}-2} (A^\top E)^j (Q) A^j \right) A_i,
\end{aligned}$$

indicating (5) is true for $k = l + 1$. \square

To obtain the convergence of the OSA, we further assume that all A_i for $i = 1, \dots, m$ are d -stable, i.e.,

$$\rho(A_i) < 1.$$

The following theorem concludes the convergence of the OSA:

Theorem 1. Let $\rho := \max_{i=1}^m \rho(A_i)$ and $p := \max_{i,j=1}^m p_{ij}$ such that $2p\rho^2 < 1$. Then, the sequence $\{X_{i,k}\}$ generated by (4) is convergent to the solution

$$X_{i,\infty} = Q_i + A_i^\top E_i \left(\sum_{j=0}^{\infty} (A^\top E)^j (Q) A^j \right) A_i \quad (6)$$

of the CDSEs when A_i is d -stable. Moreover, one has

$$\|X_{i,k} - X_{i,\infty}\| \leq \frac{(2p\rho^2)^{2^k}}{1 - 2p\rho^2} \|Q\|,$$

where $\|Q\| := \max_{i=1}^m \|Q_i\|$ for $i = 1, \dots, m$.

Proof. It follows from Proposition 1 that the solution of CDSEs has the form of (6) when the assumption holds. Subtracting (5) from (6) and taking the norm, one then has

$$\begin{aligned}
\|X_{i,k} - X_{i,\infty}\| &= \|A_i^\top E_i \left(\sum_{j=2^k-1}^{\infty} (A^\top E)^j (Q) A^j \right) A_i\| \\
&\leq (2p\rho^2)^{2^k} \|Q\| (1 + 2p\rho^2 + (2p\rho^2)^2 \dots) \\
&= \frac{(2p\rho^2)^{2^k}}{1 - 2p\rho^2} \|Q\|.
\end{aligned}$$

\square

Remark 2. It is evident from Theorem 1 that the OSA admits the quadratic convergence rate when $2p\rho^2 < 1$. This highlights its superiority over the prevailing linearly convergent iterations [8,19,21,24] both on accuracy and CPU time, as elaborated in the next subsection.

2.2. Examples

In this subsection, we present several examples that highlight the superior performance of the OSA compared to linearly convergent iterations [8,19,21,24]. Notably, the iteration method outlined in [8] is identical to the one in [19]. Additionally, other linearly convergent iterations exhibit similar numerical behaviors. Therefore, for accuracy and CPU time comparisons, we select the iteration method from [8,19], referred to as “FIX” in this

subsection. It is important to note that the discrete-time Lyapunov equation in “FIX” was solved using the built-in function “dlyap” in Matlab 2019a [25].

Example 1. This example is from a slight modification of the all-pass SISO system [26], where the controllability and observability Gramians are quasi-inverse to each other, i.e., $W_{ci}W_{oi} = \sigma_i I$ for some $\sigma_i > 0$. This property indicates that the system has a single Hankel singular value of multiplicity equal to the system’s order. The derived system matrices are as follows:

$$A_1 = 0.4[(I + G_1)^{-1}\bar{A}_1], \quad A_2 = 0.5[(I + G_2)^{-1}\bar{A}_2], \quad Q_1 = L_1^Q(L_1^Q)^\top, \quad Q_2 = L_2^Q(L_2^Q)^\top,$$

where G_1 and G_2 are matrices with zero elements except for the last row of $0.1g_1$ and $0.3g_2$, respectively (both g_1 and g_2 are random row vectors with elements from the interval $(0, 1)$); \bar{A}_1 and \bar{A}_2 are both tri-diagonal matrix $\text{tridiag}(-1, 0, 1)$ but with $\bar{A}_1(1, 1) = -0.5$ and $\bar{A}_2(1, 1) = -0.8$, respectively; $(L_1^Q)^\top = [1, 0, \dots, 0, 1]$; and $(L_2^Q)^\top = [0, 1, 0, \dots, 0, 1, 0]$. We consider $m = 2$ and select the probability matrix $\Pi = \begin{pmatrix} 0.26 & 0.74 \\ 0.53 & 0.47 \end{pmatrix}$.

We evaluate the OSA and FIX for dimensions $N = 400$ and $N = 800$ and present their numerical behaviors in Table 1. Here, δt_k and t_k record the CPU time of the current iteration and accumulated iterations, respectively. The Rel_Res column exhibits the relative residual of CDSEs in each iteration. From Table 1, it is evident that the OSA achieves equation residuals of $O(10^{-16})$ within five iterations for different dimensions. The CPU time required for the OSA is significantly less than that required for FIX. Conversely, FIX maintains equation residuals at the level of $O(10^{-13})$ even after 11 iterations. The symbol “*” in the table indicates that, despite resuming the iteration, it can not further reduce the equation residuals to terminate the FIX.

Table 1. History of CPU time and residual for OSA and FIX in Example 1.

	It.	δt_k	t_k	Rel_Res	δt_k	t_k	Rel_Res
			$N = 400$			$N = 800$	
OSA	1	0.045	0.045	1.38×10^{-1}	0.156	0.156	2.50×10^{-1}
	2	0.070	0.115	1.04×10^{-2}	0.275	0.431	2.00×10^{-2}
	3	0.139	0.254	8.59×10^{-5}	0.434	0.865	1.88×10^{-4}
	4	0.173	0.427	6.15×10^{-9}	0.705	1.571	1.59×10^{-8}
	5	0.284	0.711	2.66×10^{-16}	1.244	2.814	2.47×10^{-16}
FIX	1	0.495	0.495	4.10×10^{-1}	2.276	2.276	7.77×10^{-1}
	2	0.483	0.979	1.26×10^{-1}	2.021	4.297	2.51×10^{-1}
	3	0.471	1.450	4.68×10^{-3}	2.049	6.346	9.85×10^{-3}
	4	0.452	1.902	1.65×10^{-4}	2.078	8.424	3.66×10^{-4}
	5	0.479	2.381	6.00×10^{-6}	2.051	10.503	1.40×10^{-5}
	6	0.606	2.986	2.27×10^{-7}	2.051	12.553	5.58×10^{-7}
	7	0.630	3.617	8.87×10^{-9}	2.031	14.584	2.42×10^{-8}
	8	0.450	4.067	4.02×10^{-10}	2.055	16.639	1.21×10^{-9}
	9	0.462	4.529	1.90×10^{-11}	2.053	18.693	6.05×10^{-11}
	10	0.470	4.998	9.11×10^{-13}	2.046	20.738	3.05×10^{-12}
	11	0.474	5.472	$1.19 \times 10^{-13} *$	2.076	22.814	$2.05 \times 10^{-13} *$

To visualize the residuals of each equation (here, $m = 2$) for different iteration methods, we plot the history of the equation residuals in Figure 1. Here, “S-Resi” and “F-Resi” ($i = 1, 2$) represent the residuals of equations obtained by the OSA and FIX, respectively. The figure illustrates that the OSA has quadratic convergence. Interestingly, although FIX converges rapidly for solving the second equation, it maintains linear convergence for solving the first equation, resulting in an overall linear convergence for FIX.

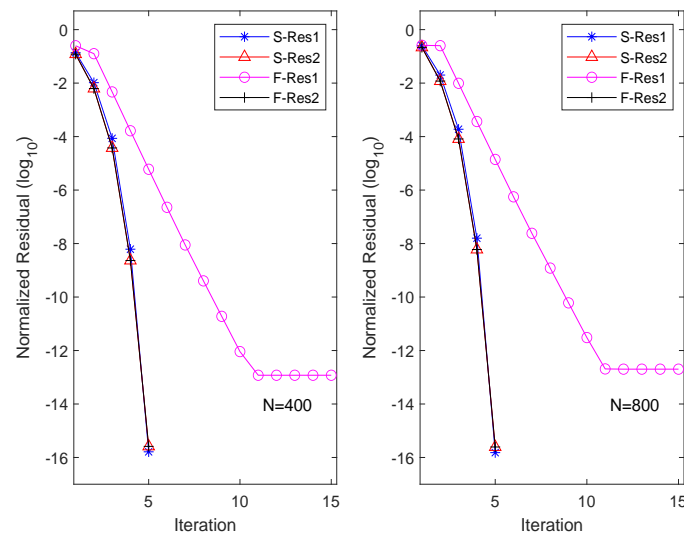


Figure 1. Residual history in each equation for OSA and FIX.

We further consider modified system matrices

$$A_1 = 0.48[(I + G_1)^{-1}\bar{A}_1], \quad A_2 = 0.98[(I + G_2)^{-1}\bar{A}_2],$$

where G_1 and G_2 are matrices with zero elements except for the last row of $0.6g_1$ and $0.8g_2$, respectively. In this case, the spectral radii of matrices A_1 and A_2 are 0.96 and 0.95, respectively. We rerun the OSA and FIX algorithm, and the obtained results are recorded in Figure 2. From the plot, it can be observed that for $N = 400$, the OSA requires nine iterations to achieve equation residuals at the $O(10^{-15})$ level, with a total time of 10.17 s. Conversely, FIX, after consuming 51.31 s over 90 iterations, only achieves equation residuals at the scale of $O(10^{-13})$. Further numerical experiments demonstrate that even by increasing the number of iterations, FIX fails to further reduce the residual level. Similar numerical results are obtained for the scale of $N = 800$.

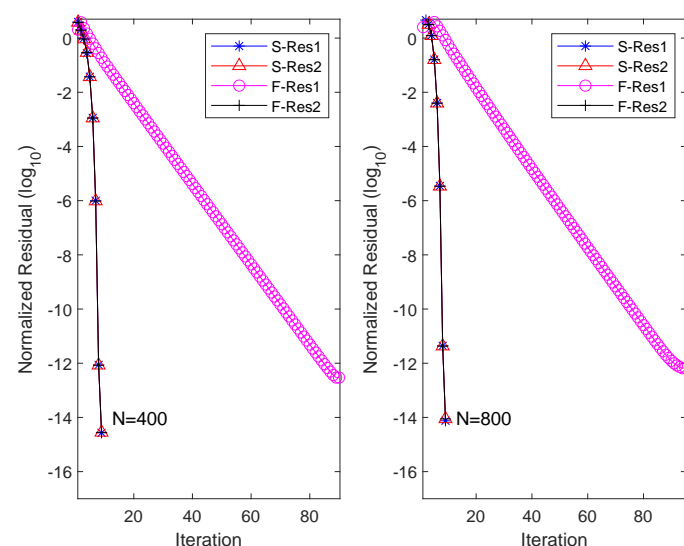


Figure 2. Residual history in each equation for OSA and FIX when $\rho(A_1) = 0.96$ and $\rho(A_2) = 0.95$.

Example 2. Consider a slight modification of a chemical reaction by a convection reaction partial differential equation on the unit square [27], given by

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial y^2} + \frac{\partial^2 x}{\partial z^2} + 20 \frac{\partial x}{\partial z} - 180x + f(y, z)x(t),$$

where x is a function of time (t), vertical position (v), and horizontal position (z). The boundaries of interest in this problem lie on a square with opposite corners at $(0, 0)$ and $(1, 1)$. The function $x(t, v, z)$ is zero on these boundaries. This PDE is discretized using centered difference approximations on a grid of $n_v \times n_z$ points. The dimension N of A is the product of the state space dimension $n_v n_z$, resulting in a sparsity pattern of A as

$$A = \begin{pmatrix} -734 & 171 & \overbrace{0 \ \cdots \ 0}^7 & 196 \\ -9 & -734 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 196 \\ 0 & & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 196 & \ddots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ & \ddots & & \ddots & \ddots & \ddots & \ddots & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & 171 \\ & & & 196 & 0 & \cdots & 0 & -9 & -734 \end{pmatrix}.$$

Here we take

$$A_1 = 10^{-3} \times \xi_1 A, \quad A_2 = 10^{-3} \times \xi_2 A, \quad Q_1 = L_1^Q (L_1^Q)^\top, \quad Q_2 = L_2^Q (L_2^Q)^\top,$$

where

$$(L_1^Q)^\top = [\overbrace{1, \dots, 1}^7, 0, \dots, 0, \overbrace{1, \dots, 1}^7], \quad (L_2^Q)^\top = [\overbrace{0, \dots, 0}^7, \overbrace{1, \dots, 1}^7, 0, \dots, 0, \overbrace{1, \dots, 1}^7, \overbrace{0, \dots, 0}^7].$$

The parameter $m = 2$ and the probability matrix $\Pi = \begin{pmatrix} 0.244 & 0.756 \\ 0.342 & 0.658 \end{pmatrix}$.

Similarly, we applied the OSA and FIX iterations to CDSEs with dimensions $N = 350$ and $N = 700$, and the results are presented in Table 2. It is evident that the OSA can achieve equation residuals of $O(10^{-14})$ within five iterations, with the required CPU time approximately one-ninth of that needed for the FIX iterations. However, FIX only attains equation residuals of $O(10^{-13})$ after 10 iterations. We also depict the residuals of the two different iteration methods for their respective m equations (here $m = 2$) in Figure 3. The figure illustrates that the OSA exhibits quadratic convergence, while FIX demonstrates only linear convergence.

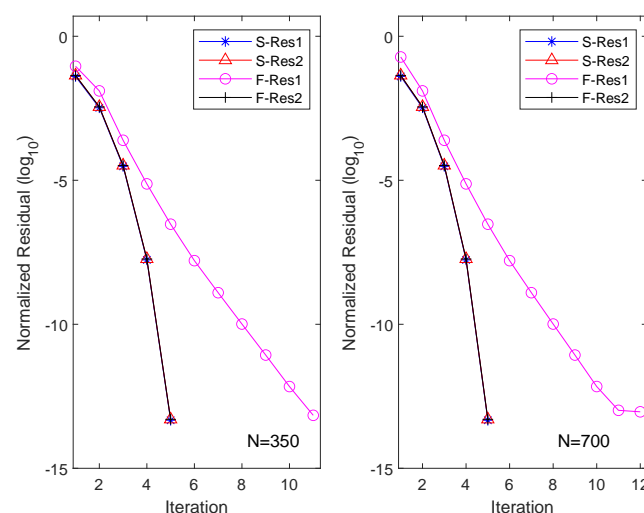


Figure 3. Residual history in each equation for OSA and FIX in Example 2.

Table 2. History of CPU time and residual for OSA and FIX in Example 2.

	It.	δt_k	t_k	Rel_Res	δt_k	t_k	Rel_Res
		$N = 350$			$N = 700$		
OSA	1	0.001	0.001	4.37×10^{-2}	0.003	0.003	4.37×10^{-2}
	2	0.005	0.06	3.54×10^{-3}	0.003	0.006	3.54×10^{-3}
	3	0.009	0.015	3.27×10^{-5}	0.015	0.865	3.27×10^{-5}
	4	0.032	0.047	1.85×10^{-8}	0.036	0.051	1.85×10^{-8}
	5	0.235	0.281	5.01×10^{-14}	0.331	0.382	5.01×10^{-14}
FIX	1	0.291	0.291	1.91×10^{-1}	1.252	1.252	1.91×10^{-1}
	2	0.231	0.523	1.27×10^{-2}	1.246	2.499	1.27×10^{-2}
	3	0.234	0.757	2.45×10^{-4}	1.294	3.793	2.45×10^{-4}
	4	0.240	0.997	7.50×10^{-6}	1.231	5.024	7.50×10^{-6}
	5	0.227	1.224	2.99×10^{-7}	1.242	6.266	2.99×10^{-7}
	6	0.254	1.478	1.62×10^{-8}	1.215	7.482	1.62×10^{-8}
	7	0.240	1.718	1.25×10^{-9}	1.302	8.784	1.25×10^{-9}
	8	0.228	1.946	1.02×10^{-10}	1.255	10.039	1.02×10^{-10}
	9	0.231	2.177	8.55×10^{-12}	1.243	11.281	8.55×10^{-12}
	10	0.244	2.421	6.89×10^{-13}	1.231	12.513	6.90×10^{-13}

3. Structured Algorithm for Large-Scale Problems

In numerous practical scenarios [23], matrix A_i is often sparse, and Q_i is typically of a low-ranked structure. Therefore, in this section, we adapt the OSA to a low-ranked format, well-suited for large-scale computations.

3.1. Structured Iteration Scheme

Given the initial matrices $X_{i,0} = Q_i = L_i^Q (L_i^Q)^\top$ for $i = 1, \dots, m$, we show that the iteration (4) can be organized as the following format:

$$X_{i,k} = L_{i,k}^Q K_{i,k}^Q (L_{i,k}^Q)^\top, \quad \mathcal{F}_{i,k}(X_{i,k}) = L_{i,k}^{F_{2k}} K_{i,k}^{F_{2k}} (L_{i,k}^{F_{2k}})^\top, \quad k = 0, 1, 2, \dots, \quad (7)$$

where $L_{i,k}^{F_{2k}}, K_{i,k}^{F_{2k}}, L_{i,k}^Q$, and $K_{i,k}^Q$ are of forms in the following proposition.

Proposition 2. Let $L_i^Q \in \mathbb{R}^{N \times l_i}$ in $X_{i,0}$ be the initial factor with $l_i \ll N$. The sequences $\mathcal{F}_{i,k}(X_{i,k})$ and $X_{i,k}$ generated by (4) are factorized as in (7), where

$$L_{i,k}^{F_{2k}} = A_i^\top [L_{1,k}^{F_{2k-1}}, \dots, L_{m,k}^{F_{2k-1}}], \quad K_{i,k}^{F_{2k}} = p_{i,1} K_{1,k}^{F_{2k-1}} \oplus \dots \oplus p_{i,m} K_{m,k}^{F_{2k-1}}, \quad (8)$$

$$L_{i,k}^Q = [L_{i,k-1}^Q, L_{i,k-1}^{F_{2k-1}}], \quad K_{i,k}^Q = K_{i,k-1}^Q \oplus K_{i,k-1}^{F_{2k-1}} \quad (9)$$

and

$$L_{i,k}^{F_1} = A_i^\top [L_{1,k}^Q, \dots, L_{m,k}^Q], \quad K_{i,k}^{F_1} = p_{i,1} K_{1,k}^Q \oplus \dots \oplus p_{i,m} K_{m,k}^Q, \quad K_{i,0}^Q = I, \quad L_{i,0}^Q = L_i^Q.$$

Proof. Given the initial matrix $X_{i,0} = Q_i = L_i^Q (L_i^Q)^\top = L_{i,0}^Q K_{i,0}^Q (L_{i,0}^Q)^\top$, it then follows from (3) that

$$\mathcal{F}_{i,0}(X_{i,0}) = A_i^\top E_i(X_{i,0}) A_i = A_i^\top \left(\sum_{s=1}^m p_{i,s} Q_s \right) A_i = L_{i,0}^{F_1} K_{i,0}^{F_1} (L_{i,0}^{F_1})^\top.$$

So (7) holds for $k = 0$. Assume (7) is true for $k = l$. It follows from the iteration scheme (4) that

$$X_{i,l+1} = L_{i,l}^Q K_{i,l}^Q (L_{i,l}^Q)^\top + L_{i,l}^{F_{2l}} K_{i,l}^{F_{2l}} (L_{i,l}^{F_{2l}})^\top = L_{i,l+1}^Q K_{i,l+1}^Q (L_{i,l+1}^Q)^\top.$$

Recalling (3) again, it follows

$$\begin{aligned}
\mathcal{F}_{i,l+1}(X_{:,l+1}) &= A_i^\top E_i \left((A^\top E)^{2^{l+1}-1} (X_{:,l+1}) A^{2^{l+1}-1} \right) A_i \\
&= A_i^\top E_i \left((A^\top E)^{2^{l+1}-2} \cdot A^\top E (X_{:,l+1}) A \cdot A^{2^{l+1}-2} \right) A_i \\
&= A_i^\top E_i \left((A^\top E)^{2^{l+1}-2} \left(\sum_{s=1}^m p_{:,s} A_s^\top [L_{1,l+1}^Q, \dots, L_{m,l+1}^Q] [p_{s,1} K_{1,l+1}^Q \oplus \dots \right. \right. \\
&\quad \left. \left. \oplus p_{s,m} K_{1,l+1}^Q] [L_{1,l+1}^Q, \dots, L_{m,l+1}^Q]^\top A_s \right) A^{2^{l+1}-2} \right) A_i \\
&= A_i^\top E_i \left((A^\top E)^{2^{l+1}-2} \left(\sum_{s=1}^m p_{:,s} L_{s,l+1}^{F_1} K_{s,l+1}^{F_1} (L_{s,l+1}^{F_1})^\top \right) A^{2^{l+1}-2} \right) A_i \\
&= A_i^\top E_i \left((A^\top E)^{2^{l+1}-3} \left(\sum_{s=1}^m p_{:,s} A_s^\top [L_{1,l+1}^{F_1}, \dots, L_{m,l+1}^{F_1}] [p_{s,1} K_{1,l+1}^{F_1} \oplus \dots \right. \right. \\
&\quad \left. \left. \oplus p_{s,m} K_{1,l+1}^{F_1}] [L_{1,l+1}^{F_1}, \dots, L_{m,l+1}^{F_1}]^\top A_s \right) A^{2^{l+1}-3} \right) A_i \\
&= A_i^\top E_i \left((A^\top E)^{2^{l+1}-3} \left(\sum_{s=1}^m p_{:,s} L_{s,l+1}^{F_2} K_{s,l+1}^{F_2} (L_{s,l+1}^{F_2})^\top \right) A^{2^{l+1}-3} \right) A_i \\
&= \dots \\
&= A_i^\top E_i \left(\sum_{s=1}^m p_{:,s} L_{s,l+1}^{F_{2^{l+1}-1}} K_{s,l+1}^{F_{2^{l+1}-1}} (L_{s,l+1}^{F_{2^{l+1}-1}})^\top \right) A_i \\
&= L_{i,l+1}^{F_{2^{l+1}}} K_{i,l+1}^{F_{2^{l+1}}} (L_{i,l+1}^{F_{2^{l+1}}})^\top.
\end{aligned}$$

Then (7) holds true for $k = l + 1$. The proof is complete. \square

3.2. Truncation and Compression

It is evident that the columns of $L_{i,k}^{F_{2^k}}$ at the k -th iteration will scale approximately as $O(m^{2^{k-1}} l_i)$, where l_i is the initial column number of the factor L_i^Q . Consequently, we implement the truncation and compression (TC) to reduce the column number of low-rank factors [11,28]. Notably, our algorithm employs the TC technique twice within one iteration: once for $L_{i,k}^{F_{2^k}}$ and once for $L_{i,k}^Q$.

For simplicity in notation, we omit the subscript i for low-rank factors. Recalling (8) and (9), we perform TC on $L_k^{F_{2^k}}$ and L_k^Q using QR decompositions with column pivoting. Then, one has

$$\begin{aligned}
L_k^{F_{2^k}} P_k^{F_{2^k}} &= [Q_k^{F_{2^k}} \tilde{Q}_k^{F_{2^k}}] \begin{bmatrix} U_1^{F_{2^k}} & U_2^{F_{2^k}} \\ 0 & \tilde{U}_k^{F_{2^k}} \end{bmatrix}, \quad \|\tilde{U}_k^{F_{2^k}}\| < u_0^f \tau, \\
L_k^Q P_k^Q &= [Q_k^Q \tilde{Q}_k^Q] \begin{bmatrix} U_{k,1}^Q & U_{k,2}^Q \\ 0 & \tilde{U}_k^Q \end{bmatrix}, \quad \|\tilde{U}_k^Q\| < u_0^q \tau,
\end{aligned} \tag{10}$$

where $P_k^{F_{2^k}}$ and P_k^Q are permutation matrices ensuring that the diagonal elements of the decomposed block triangular matrices decrease in absolute value. Additionally, u_0^f and u_0^q represent constants, and τ is some small tolerance controlling TC. Let $m_k^{f_{2^k}}$ and m_k^q denote the respective column numbers of $L_k^{F_{2^k}}$ and L_k^Q , bounded above by a given m_{\max} . Then, their ranks satisfy

$$r_k^{f_{2^k}} := \text{rank}(L_k^{F_{2^k}}) \leq m_k^{f_{2^k}} \leq m_{\max} \quad \text{and} \quad r_k^q := \text{rank}(L_k^Q) \leq m_k^q \leq m_{\max}$$

with $m_{\max} \ll N$. The truncated factors are still denoted as

$$L_k^{F_{2k}} P_{F_{2k}} = Q_{F_{2k}} [U_1^{F_{2k}} \ U_2^{F_{2k}}] := Q_{F_{2k}} U_{F_{2k}}, \quad (11)$$

$$L_k^Q P_k^Q = Q_k^Q [U_{k,1}^Q \ U_{k,2}^Q] := Q_k^Q U_k^Q, \quad (12)$$

respectively. The compressed kernels are denoted as

$$K_k^{F_{2k}} := U_{F_{2k}}^{F_{2k}} P_{F_{2k}}^{F_{2k}} K_k^{F_{2k}} (U_{F_{2k}}^{F_{2k}} P_{F_{2k}}^{F_{2k}})^\top, \quad (13)$$

$$K_k^Q := U_k^Q P_k^Q K_k^Q (U_k^Q P_k^Q)^\top, \quad (14)$$

where $K_k^{F_{2k}}$ and K_k^Q represent the abbreviated kernels in (8) and (9) without subscript i , respectively.

3.3. Computation of Residuals

Given the initial matrix $X_{i,0} = Q_i = L_i^Q (L_i^Q)^\top$, for $i = 1, \dots, m$ the initial residual of the CDSEs is

$$Sc_0(X_{i,0}) = X_{i,0} - A_i^\top E_i(X_{i,0}) A_i - Q_i = L_{i,0}^R K_{i,0}^R (L_{i,0}^R)^\top, \quad (15)$$

where $L_{i,0}^R = A_i^\top [L_{1,0}^Q, \dots, L_{m,0}^Q]$, $K_{i,0}^R = p_{i,1} I \oplus \dots \oplus p_{i,m} I$.

When the k -th iteration $X_{i,k} = L_{i,k}^Q K_{i,k}^Q (L_{i,k}^Q)^\top$ is available, the residual of CDSEs has the decomposition

$$Sc_k(X_{i,k}) = X_{i,k} - A_i^\top E_i(X_{i,k}) A_i - Q_i = L_{i,k}^R K_{i,k}^R (L_{i,k}^R)^\top \quad (16)$$

with

$$L_{i,k}^R = [L_{i,0}^Q, L_{i,k}^Q, L_{i,k}^{F_{2k}}], \quad K_{i,k}^R = -I \oplus K_{i,k}^Q \oplus -K_{i,k}^{F_{2k}}.$$

Similarly, we also impose the truncation and compression on $L_{i,k}^R$, $k \geq 0$, i.e, implementing the QR decomposition with pivoting:

$$L_{i,k}^R P_{i,k}^R = [Q_{i,k}^R \ \tilde{Q}_{i,k}^R] \begin{bmatrix} U_{i,k,1}^R & U_{i,k,2}^R \\ 0 & \tilde{U}_{i,k}^R \end{bmatrix}, \quad \|\tilde{U}_{i,k}^R\| < u_0^r \tau, \quad (17)$$

where $P_{i,k}^R$ is a pivoting matrix and u_0^r is some constant. Let $U_{i,k}^R = [U_{i,k,1}^R, U_{i,k,2}^R]$. The corresponding kernel of residual is also denoted as

$$K_{i,k}^R := U_{i,k}^R P_{i,k}^R K_{i,k}^R (U_{i,k}^R P_{i,k}^R)^\top, \quad (18)$$

and the terminating condition of the whole algorithm is chosen to be

$$\text{Rel_Res} = \max_i \frac{\|K_{i,k}^R\|}{\|K_{i,0}^R\|} \leq \epsilon \quad (19)$$

with ϵ being the tolerance.

3.4. Large-Scale Algorithm and Complexity

The OSA with a low-rank structure equipped with TC is summarized in the following OSA_lr Algorithm 1.

To show the computational complexity of the algorithm OSA_lr, we assume that all matrices A_i for $i = 1, \dots, m$ are sufficiently sparse. This allows us to consider the cost of both the product $A_i B$ and solving the equation $A_i X = B$, which are both within the range of cN floating-point operations (flops), where B is an $N \times m_b$ matrix with $m_b \ll N$ and c is a constant. Additionally, the number of truncated columns of $L_{i,k}^{F_{2k-1}}$ and $L_{i,k}^Q$ for all

$i = 1, \dots, m$ are denoted as m_k^f and m_k^q , respectively. The flops and memory of the k -th iteration are summarized in Table 3 below.

Algorithm 1: Algorithm OSA_lr. Solve large-scale CDSEs with low-ranked Q_i

Inputs: Sparse matrices A_i , low-rank factors L_i^Q for $i = 1, \dots, m$, probability matrix $\Pi \in \mathbb{R}^{m \times m}$, truncation tolerance τ , upper bound m_{\max} and the iteration tolerance ϵ .
Outputs: Low-ranked matrix L_i^X and the kernel matrix K_i^X with the solution $X_i \approx L_i^X K_i^X (L_i^X)^\top$.
1. Set $L_{i,0}^Q = L_i^Q$ and $K_{i,0}^Q = I_i$ for $i = 1, \dots, m$.
2. For $k = 1, \dots$, until convergence, do
3. Compute $K_{i,k}^{F_{2k}}$ and $L_{i,k}^{F_{2k}}$ as in (8).
4. Truncate and compress $L_{i,k}^{F_{2k}}$ as in (10) with accuracy $u_0^f \tau$.
5. Construct compressed low-ranked factor $L_{i,k}^{F_{2k}}$ and kernel $K_{i,k}^{F_{2k}}$ as in (11) and (13).
6. Compute $K_{i,k}^Q$ and $L_{i,k}^Q$ as in (9).
7. Truncate and compress $L_{i,k}^Q$ as in (10) with accuracy $u_0^q \tau$.
8. Construct compressed low-ranked factor $L_{i,k}^Q$ and kernel $K_{i,k}^Q$ as in (12) and (14).
9. Evaluate the relative residual Rel_Res in (19).
11. If $\text{Rel_Res} < \epsilon$, break, End.
12. $k := k + 1$;
14. End (For)
18. Output $K_i^X := K_{i,k}^Q$, $L_i^X := L_{i,k}^Q$.

Table 3. Complexity and memory at k -th iteration in algorithm OSA_lr.

Items	Flops	Memory
$L_{i,k}^{F_{2k-1}}$	$cm_{k-1}^q 2^{k-1} (m^{2^{k-1}} + m)N$	$m^{2^{k-1}} m_{k-1}^q N$
$K_{i,k}^{F_{2k-1}}$	$m(m_{k-1}^q)^2 (1 + m^{2k}) (k + 1) / 2$	$(m^{2^{k-1}} m_{k-1}^q)^2$
$L_{i,k}^{F_{2k-1}}$ QR *	$2(m^{2^{k-1}} m_{k-1}^q)^2 (N - m^{2^{k-1}} m_{k-1}^q / 3)$	$(m_k^f)^2$
Compressed $K_{i,k}^{F_{2k-1}}$	$4m_k^f (m^{2^{k-1}} m_{k-1}^q)^2$	$(m_k^f)^2$
$L_{i,k}^Q$	—	$(m_{k-1}^q + m_k^f)N$
$K_{i,k}^Q$	—	$(m_{k-1}^q + m_k^f)^2$
$L_{i,k}^Q$ QR *	$2(m_k^f m_{k-1}^q)^2 (N - m_k^f m_{k-1}^q / 3)$	$(m_k^q)^2$
Compressed $K_{i,k}^Q$	$4m_k^q (m_k^f + m_{k-1}^q)^2$	$(m_k^q)^2$

* Householder QR decomposition is used [29].

3.5. Error Analysis

In this subsection, we will conduct the error analysis of OSA_lr. For $i = 1, \dots, m$, let

$$\delta A_i = A_i - \hat{A}_i, \quad \delta X_{i,k} = X_{i,k} - \hat{X}_{i,k} \quad (20)$$

be the errors yielded by roundoff or iteration. Here, A_i and $X_{i,k}$ are true matrices, while \hat{A}_i and $\hat{X}_{i,k}$ are the practical iteration matrices. The following lemma indicates the error propagation of the operator.

Lemma 1. Given errors $\delta X_{i,k}$ and δA_i as in (20), the error of the operator is

$$\|\delta \mathcal{F}_{i,l}(X_k)\| \leq (mp)^{2^l} (\alpha^{2^{l+1}} \delta X_k + 2^{l+1} q_k \alpha^{2^{l+1}-1} \delta A) \quad \text{for } l \leq k,$$

where $\alpha = \max_i \{\|A_i\|\}$, $q_k = \max_i \{\|X_{i,k}\|\}$, $\delta A = \max_i \{\|\delta A_i\|\}$, $\delta X_k = \max_i \{\|\delta X_{i,k}\|\}$, and $p = \max_{i,j} \{p_{i,j}\}$.

Proof. By merely retaining one order error of δA_i and $\delta X_{i,k}$, it follows from the definition of $\mathcal{F}_{i,l}(X_k)$ in (3) that the practical operator is

$$\begin{aligned} \widehat{\mathcal{F}}_{i,l}(X_k) &= (A_i + \delta A_i)^\top \left\{ \overbrace{\sum_{j=1}^m p_{j,\cdot} (A_j + \delta A_j)^\top \dots \left\{ \sum_{s=1}^m p_{s,\cdot} (A_s + \delta A_s)^\top \left[\sum_{k=1}^m p_{k,\cdot} (X_{i,k} + \delta X_{i,k}) \right] \right\}}^{2^l-1} \right. \\ &\quad \left. \overbrace{(A_s + \delta A_s) \dots (A_j + \delta A_j)}^{2^l-1} \right\} (A_i + \delta A_i) \\ &\leq (A_i + \delta A_i)^\top \left\{ \overbrace{\sum_{j=1}^m p_{j,\cdot} (A_j + \delta A_j)^\top \dots \left\{ \sum_{t=1}^m p_{t,\cdot} (A_t + \delta A_t)^\top \left[\mathcal{F}_{i,0}(X_k) + (mp)^2 (A_s^\top \delta X_{i,k} A_s \right. \right. \right.}^{2^l-2} \\ &\quad \left. \left. \left. + (\delta A_s)^\top X_{i,k} A_s + A_s^\top X_{i,k} \delta A_s \right] (A_t + \delta A_t) \right\} \dots (A_j + \delta A_j) \right\} (A_i + \delta A_i) \end{aligned} \quad (21)$$

$$\begin{aligned} &\leq (A_i + \delta A_i)^\top \left\{ \overbrace{\sum_{j=1}^m p_{j,\cdot} (A_j + \delta A_j)^\top \dots \left\{ \sum_{u=1}^m p_{u,\cdot} (A_u + \delta A_u)^\top \left[\mathcal{F}_{i,1}(X_k) + (mp)^3 ((A_s A_t)^\top \delta X_{i,k} A_s A_t \right. \right. \right.}^{2^l-3} \\ &\quad \left. \left. \left. + (\delta A_s A_t)^\top X_{i,k} A_s A_t + (A_s A_t)^\top X_{i,k} \delta A_s A_t + (A_s \delta A_t)^\top X_{i,k} A_s A_t \right. \right. \right. \\ &\quad \left. \left. \left. + (\delta A_s A_t)^\top X_{i,k} A_s A_t \right] (A_u + \delta A_u) \right\} \dots (A_j + \delta A_j) \right\} (A_i + \delta A_i) \\ &\leq \dots \end{aligned} \quad (22)$$

Note that error items in $\left[\cdot \right]$ in (21) and (22) have corresponding upper bounds $(mp)^2 (\alpha^2 \delta X_k + 2q_k \alpha \delta A)$ and $(mp)^3 (\alpha^4 \delta X_k + 4q_k \alpha^3 \delta A)$, respectively. After multiplying the left by $(A_i + \delta A_i)^\top$ and the right by $A_i + \delta A_i$ in the outermost layer, the upper bounds of the errors are $(mp)^2 (\alpha^4 \delta X_k + 4q_k \alpha^3 \delta A)$ and $(mp)^3 (\alpha^6 \delta X_k + 6q_k \alpha^5 \delta A)$, respectively. Then, by the induction, it is not difficult to see that the final error is

$$(mp)^{2^l} (\alpha^{2^{l+1}} \delta X_k + 2^{l+1} q_k \alpha^{2^{l+1}-1} \delta A).$$

□

We have the following error bound at the $k+1$ -th iteration.

Theorem 2. Given errors δX_k and δA_i as in (20), the error at the $k+1$ -th iteration has the bound

$$\delta X_{k+1} \leq (1 + (mp\alpha^2))^{2^k} \delta X_k + 2^{k+1} (mp)^{2^k} \alpha^{2^{k+1}-1} q_k \delta A + O(\tau), \quad (23)$$

where m , p , α , and q_k are defined in Lemma 1 and τ is the error of TC described in Section 3.2.

Proof. It follows from the iteration format (4) that the error at the $k+1$ -th iteration has the upper bound

$$\|\delta X_{i,k+1}\| \leq \|\delta X_{i,k}\| + \|\delta \mathcal{F}_{i,k}(X_k)\| + O(\tau),$$

where $O(\tau)$ represents the truncation and compression error on $\mathcal{F}_{i,k}(X_k)$ and $X_{i,k}$. By taking $l = k$ in Lemma 1, one has

$$\|\delta X_{i,k+1}\| \leq \delta X_k + (mp)^{2^k} \alpha^{2^{k+1}} \delta X_k + 2^{k+1} (mp)^{2^k} q_k \alpha^{2^{k+1}-1} \delta A + O(\tau)$$

and (23) holds true. □

4. Numerical Examples

In this section, we illustrate the effectiveness of OSA_lr in computing symmetric solutions to large-scale CDSEs (1) through practical examples [23,26,27,30–33]. The algorithm OSA_lr was coded by MATLAB 2019a on a 64-bit PC running Windows 10. The computer is equipped with a 3.0 GHz Intel Core i5 processor with six cores and six threads, 32 GB RAM, and a machine unit roundoff value of $\text{eps} = 2.22 \times 10^{-16}$. The maximum allowed column number of the low-ranked factors in OSA_lr is bounded by $m_{\max} = 1000$, and the tolerance for the TC of columns is set to $\tau = 10^{-16}$. In our experiments, we also attempted using $N \cdot \text{eps}$ as the TC tolerance for τ but found it had no impact on the computation accuracy. The residuals of the equations are calculated in (19) with a termination tolerance of $\epsilon = 10^{-13}$. It is noteworthy that we no longer compare with the linearly convergent iterative methods in Section 2, as the computational complexity of those algorithms per iteration is $O(N^3)$.

Example 3. We still employ the modification of the all-pass SISO system [26] in Section 2, but here we take $N = 12,000$. We list the calculated results of OSA_lr in Table 4, where the columns δ_k and t_k record the CPU time for each iteration and for cumulative iterations, respectively. The Resi and Rel_Resi ($i = 1, 2$) columns provide the absolute residual $\|K_{i,k}^R\|$ and relative residual $\|K_{i,k}^R\| / \|K_{i,0}^R\|$ computed by OSA_lr at each iteration, respectively. The $m_k^{Q_i}$ ($i = 1, 2$) columns indicate the column number of the low-ranked factor $L_{i,k}^Q$.

From the table, it is evident that OSA_lr achieves the prescribed equation residual level within five iterations, and the residual history demonstrates the quadratic convergence of the algorithm. The column count $m_k^{Q_i}$ for the low-ranked factor $L_{i,k}^Q$ expands at a rate greater than twice with each iteration, resulting in an exponential increase in the CPU time. Particularly, significant growth in the CPU time occurs during the third and fourth iterations. In numerical experiments, we observed that this substantial increase in the CPU time primarily lies in the residual computation step, specifically in Step 9 of OSA_lr. Hence, further investigation on the efficient evaluation of the equation residual is a crucial consideration for large-scale computations. We also plot the residual history of OSA_lr in Figure 7 to show its performance, where R-Res_i ($i = 1, 2$) denotes the relative residual of the i -th equation.

Table 4. CPU time and residual in Example 3.

It.	δt_k	t_k	Res1	Rel_Res1	Res2	Rel_Res2	$m_k^{Q_1}$	$m_k^{Q_2}$
1	0.012	0.012	2.06×10^0	2.06×10^0	3.06×10^0	3.06×10^0	3	3
2	0.135	0.147	1.35×10^{-1}	1.35×10^{-1}	6.15×10^{-2}	6.15×10^{-2}	9	9
3	0.659	0.806	1.44×10^{-3}	1.44×10^{-3}	6.44×10^{-4}	6.44×10^{-4}	21	21
4	35.765	35.571	1.80×10^{-7}	1.80×10^{-7}	7.51×10^{-8}	7.51×10^{-8}	46	46
5	488.519	525.090	4.42×10^{-14}	4.42×10^{-14}	2.72×10^{-14}	2.72×10^{-14}	109	109

Example 4. We continue to examine CDSEs from Example 2 [27,30], but with larger scales $N = 21,000, 28,000$, and $35,000$. The derived results of OSA_lr are presented in Table 5.

The symbols δ_k , t_k , Resi, Rel_Resi, and $m_k^{Q_i}$ ($i = 1, 2$) are defined similarly to those in Example 3. In all experiments, the equation residuals (in log 10) reached the predetermined residual level by the sixth iteration. For equations of different dimensions, the Resi ($i = 1, 2$) columns indicate that the algorithm OSA_lr is of nearly quadratic convergence, except for the final two iterations. The $m_k^{Q_i}$ column reveals that, in the fifth and sixth iterations, the column number of the factor $L_{i,k}^Q$ increased by nearly five times and six times, respectively. This resulted in a substantial increase in the CPU time during the last two iterations. A further detailed analysis indicated that this increased time primarily came from the com-

putation of equation residuals in the final two steps. The performance of OSA_lr on the residual history with $N = 35,000$ is plotted in Figure 7, where R-Res_i ($i = 1, 2$) denotes the relative residual of the i -th equation.

Table 5. CPU time and residual in Example 4.

It.	δt_k	t_k	Res1	Rel_Res1	Res2	Rel_Res2	m_k^{Q1}	m_k^{Q2}
$N = 21,000$								
1	0.005	0.005	1.51×10^1	1.51×10^1	1.55×10^1	1.55×10^1	3	3
2	0.010	0.015	4.73×10^{-2}	4.73×10^{-2}	2.70×10^{-2}	2.70×10^{-2}	7	7
3	0.019	0.034	6.10×10^{-4}	6.10×10^{-4}	3.83×10^{-4}	3.83×10^{-4}	15	15
4	0.053	0.088	6.30×10^{-7}	6.30×10^{-7}	4.04×10^{-7}	4.04×10^{-7}	31	31
5	1.354	1.441	3.67×10^{-12}	3.67×10^{-12}	2.37×10^{-12}	2.37×10^{-12}	157	157
6	235.119	235.119	1.93×10^{-14}	1.93×10^{-14}	7.13×10^{-15}	7.13×10^{-15}	908	908
$N = 28,000$								
1	0.006	0.006	1.51×10^1	1.51×10^1	1.55×10^1	1.55×10^1	3	3
2	0.017	0.023	4.73×10^{-2}	4.73×10^{-2}	2.70×10^{-2}	2.70×10^{-2}	7	7
3	0.029	0.052	6.10×10^{-4}	6.10×10^{-4}	3.83×10^{-4}	3.83×10^{-4}	15	15
4	0.078	0.130	6.30×10^{-7}	6.30×10^{-7}	4.04×10^{-7}	4.04×10^{-7}	31	31
5	1.373	1.503	4.09×10^{-12}	4.09×10^{-12}	2.86×10^{-12}	2.86×10^{-12}	147	147
6	239.690	241.193	1.59×10^{-14}	1.59×10^{-14}	8.68×10^{-15}	8.68×10^{-15}	908	908
$N = 35,000$								
1	0.008	0.008	1.51×10^1	1.51×10^1	1.55×10^1	1.55×10^1	3	3
2	0.025	0.034	4.73×10^{-2}	4.73×10^{-2}	2.70×10^{-2}	2.70×10^{-2}	7	7
3	0.043	0.077	6.10×10^{-4}	6.10×10^{-4}	3.83×10^{-4}	3.83×10^{-4}	15	15
4	0.098	0.175	6.30×10^{-7}	6.30×10^{-7}	4.04×10^{-7}	4.04×10^{-7}	31	31
5	1.617	1.793	4.76×10^{-12}	4.76×10^{-12}	3.46×10^{-12}	3.46×10^{-12}	161	161
6	248.356	250.148	1.97×10^{-14}	1.97×10^{-14}	8.62×10^{-15}	8.62×10^{-15}	908	908

Example 5. Consider the thermal convective flow control systems in [23,30,31]. These problems involve a flow region with a prescribed velocity profile, incorporating convective transport. Achieving solution accuracy with upwind finite element schemes typically requires a considerable number of elements for a physically meaningful simulation. In the illustrated scenario (see the left side of Figure 4), a 3D model of a chip is subjected to forced convection, utilizing tetrahedral element type SOLID70 as described by [34]. Both the Dirichlet boundary conditions and initial conditions are set to 0.

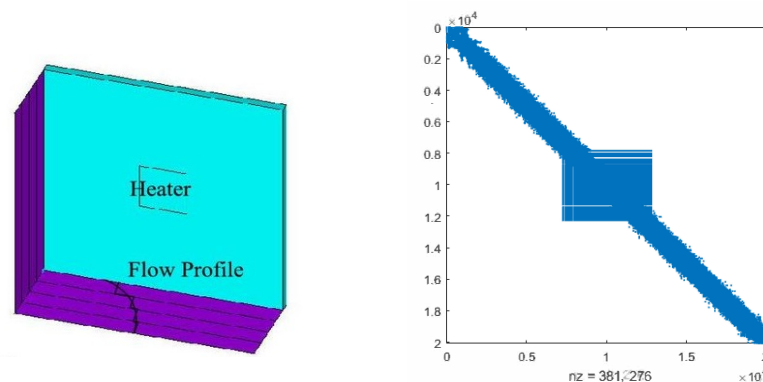


Figure 4. The 3D model of the chip and the discretized matrix A_1 .

We consider the case that the fluid speed is zero and the discretization matrices are symmetric. The system matrices are

$$A_1 = (I + r_1 BB^\top)^{-1}A, \quad A_2 = (I + r_2 BB^\top)^{-1}A, \quad Q_1 = Q_2 = C^\top C,$$

where r_1 and r_2 are random numbers from $(0, 1)$ and matrices $A \in \mathbb{R}^{N \times N}$, $B \in \mathbb{R}^{N \times 1}$, and $C \in \mathbb{R}^{1 \times N}$ ($N = 20,082$) can be found at [23]. Since A_1 and A_1 have almost the same sparse structure, we only plot A_1 in the right of Figure 4, where the non-zero elements attain the scale of 381,276. In the numerical experiments, we set $m = 2$ and use the probability matrix $\Pi = \begin{pmatrix} 0.631 & 0.369 \\ 0.143 & 0.857 \end{pmatrix}$. We ran OSA_lr for 10 different r_1 and r_2 and recorded the averaged CPU time, residual of CDSEs, and column dimension of the low-ranked factor in Table 6.

Table 6. CPU time and residual in Example 5.

It.	δt_k	t_k	Res1	Rel_Res1	Res2	Rel_Res2	$m_k^{Q_1}$	$m_k^{Q_2}$
1	0.017	0.017	2.06×10^{-1}	2.22×10^0	6.13×10^{-1}	1.05×10^0	10	10
2	0.083	0.100	9.73×10^{-6}	5.03×10^{-5}	9.83×10^{-6}	1.69×10^{-5}	22	22
3	0.665	0.766	2.74×10^{-10}	1.47×10^{-9}	2.75×10^{-10}	4.72×10^{-10}	46	46
4	1073.56	1074.34	2.48×10^{-17}	1.33×10^{-16}	2.49×10^{-16}	4.28×10^{-16}	112	112

The computational results in Table 6 reveal that OSA_lr requires only four iterations to achieve the predetermined equation residual accuracy. Moreover, the Resi and Rel_Resi columns indicate that OSA_lr exhibits quadratic convergence. The $m_k^{Q_i}$ column demonstrates that the column count of the low-rank factor $L_k^{Q_i}$ approximately doubles in the first three iterations but experiences a close to three-fold increase in the final iteration. In terms of the CPU time for each iteration, the time required for the final iteration is significantly greater than the sum of the preceding three. A detailed analysis indicates that the primary reason for this phenomenon is similar to the previous examples, wherein the computation of equation residuals in the algorithm accounts for the majority of the time. The performance of OSA_lr on the residual history is plotted in Figure 7, where R-Res_i ($i = 1, 2$) denotes the relative residual of the i -th equation.

Example 6. Consider the structurally vertical stand model from machinery control systems, depicted on the left side of Figure 5, representing a segment of a machine tool [30]. In this structural component, a set of guide rails is situated on one of its surfaces. Throughout the machining process, a tool slide traverses various positions along these rails [32]. The model was created and meshed using ANSYS. For spatial discretization, the finite element method with linear Lagrange elements was employed and implemented in FEniCS.

The derived system matrices are

$$A_1 = -0.01[(I + r_1 BB^\top)^{-1}A], \quad A_2 = -0.015[(I + r_2 BB^\top)^{-1}A], \\ Q_1 = L_1^Q (L_1^Q)^\top, \quad Q_2 = L_2^Q (L_2^Q)^\top,$$

where r_1 and r_2 are random numbers from $(0, 1)$ and L_1^Q and $L_2^Q \in \mathbb{R}^{N \times 1}$ are vectors with all elements being zeros except five ones located in rows (3341, 6743, 8932, 11,324, and 16,563) and rows (1046, 2436, 6467, 8423, and 12,574), respectively. As the sparse structures of A_1 and A_2 are analogous, the right of Figure 5 only exhibits the structure of A_1 , which contains 602,653 non-zero elements. Matrices $A \in \mathbb{R}^{16,626 \times 16,626}$ and $B \in \mathbb{R}^{16,626 \times 1}$ can be found at [23]. In this example, we set $m = 2$ and use the probability matrix $\Pi = \begin{pmatrix} 0.564 & 0.436 \\ 0.785 & 0.215 \end{pmatrix}$.

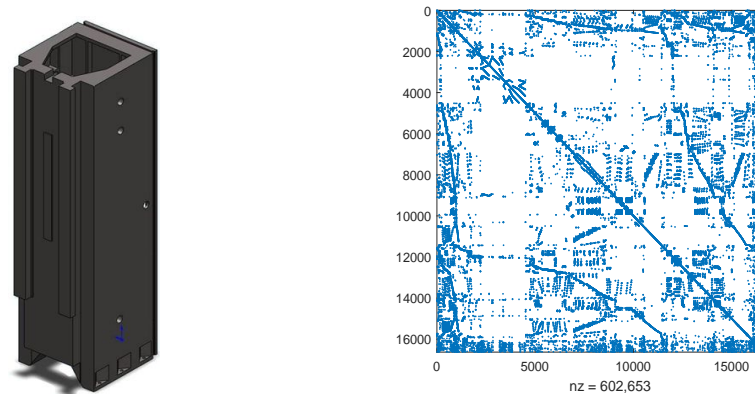


Figure 5. The vertical stand model and the discretized matrix A_1 .

We utilized OSA_lr to solve the CDSEs, and the computed results are presented in Table 7. It can be observed from the table that OSA_lr terminates after four iterations, achieving a high-precision solution, where the equation residuals reach a level of $O(10^{-15})$ to $O(10^{-16})$. The iteration history in the Resi and Rel_Resi columns illustrates the quadratic convergence of OSA_lr. The $m_k^{Q_i}$ column indicates that in the second, third, and fourth iterations, the column count of the low-rank factor $L_k^{Q_i}$ approximately doubles, triples, and quadruples, respectively. This demonstrates that the truncation and compression techniques have a limited impact on reducing the column count of $L_k^{Q_i}$ in this scenario. Similarly, δt_k reveals that the CPU time for the final iteration is significantly greater than the sum of the preceding three, primarily due to the algorithm spending substantial time computing equation residuals in the last iteration.

Table 7. CPU time and residual in Example 6.

It.	δt_k	t_k	Res1	Rel_Res1	Res2	Rel_Res2	$m_k^{Q_1}$	$m_k^{Q_2}$
1	0.023	0.023	5.01×10^0	5.01×10^0	5.01×10^0	5.01×10^0	3	3
2	0.091	0.114	1.78×10^{-6}	1.78×10^{-6}	3.77×10^{-6}	3.77×10^{-6}	7	7
3	0.787	0.901	2.25×10^{-10}	2.25×10^{-11}	4.52×10^{-11}	4.52×10^{-11}	25	25
4	164.941	165.842	5.59×10^{-15}	5.59×10^{-15}	5.96×10^{-16}	5.96×10^{-16}	119	119

Example 7. Consider a semi-discretized heat transfer problem aimed at optimizing the cooling of steel profiles in control systems, as discussed in works by [23,30]. The order of the models varies due to the application of different refinements to the computational mesh. For the discretization process, the ALBERTA-1.2 fem-toolbox [33] and the linear Lagrange elements are utilized. The initial mesh (depicted on the left in Figure 6) is generated using MATLAB's pde tool.

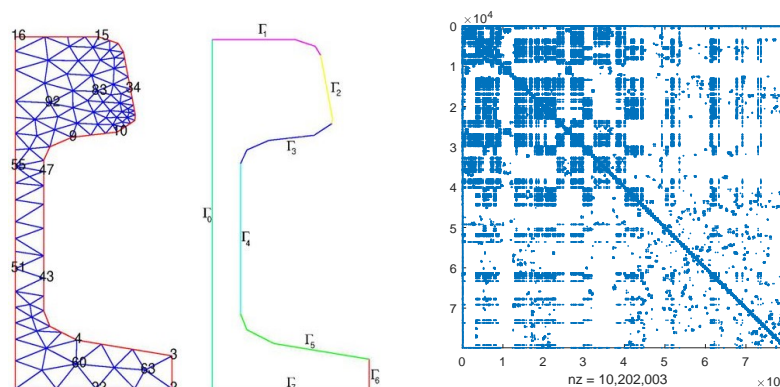


Figure 6. The initial mesh for cooling of steel and the discretized matrix A_1 and A_2 .

We slightly modify the model matrices as follows:

$$\begin{aligned} A_1 &= r_1[(I + BB^\top)^{-1}A], & A_2 &= r_2[(I + BB^\top)^{-1}A], \\ Q_1 &= L_1^Q(L_1^Q)^\top, & Q_2 &= L_2^Q(L_2^Q)^\top, \end{aligned}$$

where $r_1 = 0.009$ and $r_2 = 0.008$ for $N = 20,209$ and $r_1 = 0.0015$ and $r_2 = 0.0012$ for $N = 79,841$. In this experiment, we take $L_1^Q = r_3 C'$ and $L_2^Q = r_4 C'$, with r_3, r_4 being a random number in $(0, 1)$. Matrices $A \in \mathbb{R}^{N \times N}$, $B \in \mathbb{R}^{N \times 1}$, and $C \in \mathbb{R}^{1 \times N}$ can be found at [23]. The sparse structures of matrices A_1 and A_2 are nearly identical, and for illustration purposes, we only display the structure of A_1 with $N = 79,841$ on the right side of Figure 6. The probability matrix is defined as

$$\Pi = \begin{pmatrix} 0.713 & 0.287 \\ 0.584 & 0.416 \end{pmatrix}.$$

We employed OSA_lr to solve CDSEs under two different dimensions, and the computed results are presented in Table 8. It is evident from the table that OSA_lr terminates after achieving the predetermined equation residual levels for various dimensions. The Rel_Resi column indicates a significant decrease in the relative equation residuals to the level of $O(10^{-8})$ by the second iteration, enabling the algorithm to obtain a high-precision solution in only four iterations. The two columns of $m_k^{Q_i}$ demonstrate that, for different dimensions, the column count of the low-rank factor $L_k^{Q_i}$ increases by a factor of two, indicating that truncation and compression techniques effectively constrain the growth of the column count of $L_k^{Q_i}$ in this scenario. Similarly, the δt_k column reveals that in the final iteration, due to the computation of equation residuals, OSA_lr consumes considerably more CPU time than the sum of the preceding three iterations. The performances of OSA_lr on the residual history with $N = 20,209, 79,841$ are plotted in Figure 7, where R-Res- i ($i = 1, 2$) denotes the relative residual of the i -th equation.

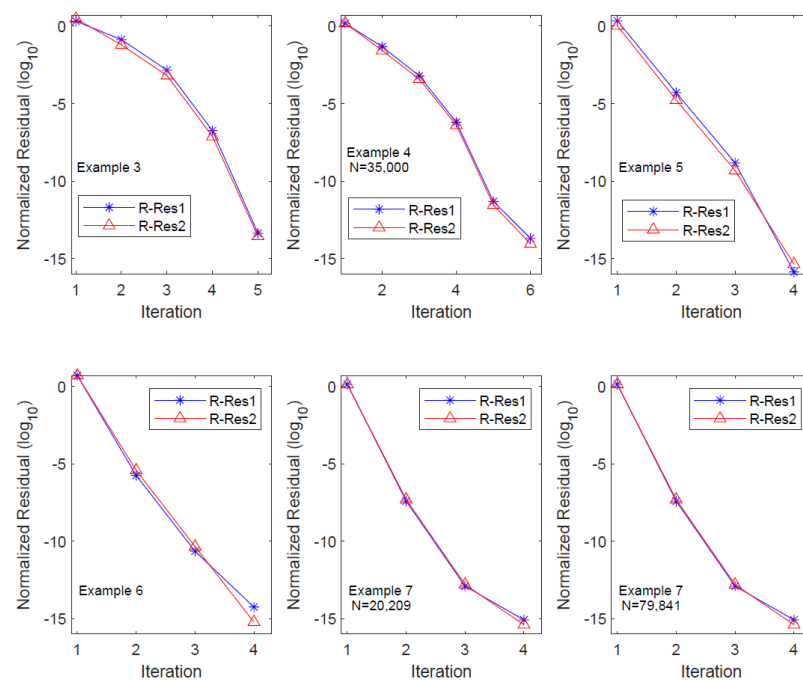


Figure 7. Relative residual histories for OSA_lr in Examples 3–7.

Table 8. CPU time and residual in Example 7.

It.	δt_k	t_k	Res1	Rel_Res1	Res2	Rel_Res2	$m_k^{Q_1}$	$m_k^{Q_2}$
N = 20,209								
1	0.011	0.011	8.30×10^0	1.34×10^0	6.82×10^0	1.34×10^0	12	12
2	0.052	0.063	2.39×10^{-7}	3.84×10^{-8}	2.48×10^{-7}	4.86×10^{-8}	24	24
3	0.129	0.192	7.72×10^{-13}	1.24×10^{-13}	8.01×10^{-11}	1.57×10^{-13}	48	48
4	489.420	489.612	5.13×10^{-15}	8.25×10^{-16}	2.07×10^{-15}	4.06×10^{-16}	96	96
N = 79,841								
1	0.614	0.614	8.29×10^0	1.33×10^0	6.81×10^0	1.33×10^0	12	12
2	5.140	5.754	1.77×10^{-7}	2.84×10^{-8}	1.06×10^{-7}	2.09×10^{-8}	24	24
3	16.946	22.700	4.74×10^{-12}	7.63×10^{-13}	2.86×10^{-12}	5.60×10^{-13}	48	48
4	874.870	895.570	2.73×10^{-15}	4.39×10^{-16}	3.67×10^{-15}	7.19×10^{-16}	96	96

5. Conclusions

This paper introduces an OSA method for coupled Stein equations in a class of jump systems. The convergence of the algorithm is established. For large-structured problems, the OSA method is extended to a low-rank structured iterative format, and an error propagation analysis of the algorithm is conducted. Numerical experiments, drawn from practical problems [23], indicate that in small-scale computations, the OSA outperforms existing linearly convergent iterative methods in terms of both the CPU time and accuracy. In large-scale computations, OSA_lr efficiently computes high-precision solutions for CDSEs. Nevertheless, the experiments reveal that the time spent on residual computation in the final iteration is relatively high. Therefore, improving the efficiency of the algorithm's termination criteria is a direction for further research in future work, and it is currently under consideration.

Author Contributions: Conceptualization, B.Y.; methodology, B.Y.; software, B.H.; validation, N.D.; formal analysis, N.D. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported in part by the NSF of China (11801163), the NSF of Hunan Province (2021JJ50032, 2023JJ50165, 2024JJ7162), and the Degree & Postgraduate Education Reform Project of Hunan University of Technology and Hunan Province (JGYB23009, 2024JGYB210).

Data Availability Statement: All examples and data can be found in [30].

Conflicts of Interest: The authors declare no conflicts of interest.

References

- Chen, C.-T. *Linear System Theory and Design*, 3rd ed.; Oxford University Press: New York, NY, USA, 1999.
- Betser, A.; Cohen, N.; Zeheb, E. On solving the Lyapunov and Stein equations for a companion matrix. *Syst. Control Lett.* **1995**, *25*, 211–218. [\[CrossRef\]](#)
- Hueso, J.L.; Martínez, G.; Hernández, V. A systolic algorithm for the triangular Stein equation. *J. VLSI Signal Process. Syst. Signal Image Video Technol.* **1993**, *5*, 49–55. [\[CrossRef\]](#)
- Li, T.-X.; Weng, P.C.-Y.; Chu, E.K.-W.; Lin, W.-W. Large-scale Stein and Lyapunov Equations, Smith Method, and Applications. *Numer. Algorithms* **2013**, *63*, 727–752. [\[CrossRef\]](#)
- Fan, H.-Y.; Weng, P.C.-Y.; Chu, K.-W. Numerical solution to generalized Lyapunov/Stein and rational Riccati equations in stochastic control. *Numer. Algorithms* **2016**, *71*, 245–272. [\[CrossRef\]](#)
- Yu, B.; Dong, N.; Tang, Q. Factorized squared Smith method for large-scale Stein equations with high-rank terms. *Automatica* **2023**, *154*, 111057. [\[CrossRef\]](#)
- Borno, I.; Gajic, Z. Parallel algorithm for solving coupled algebraic Lyapunov equations of discrete-time jump linear systems. *Comput. Math. Appl.* **1995**, *30*, 1–4. [\[CrossRef\]](#)
- Wu, A.-G.; Duan, G.-R. New Iterative algorithms for solving coupled Markovian jump Lyapunov equations. *IEEE Trans. Auto. Control* **2015**, *60*, 289–294.
- Zhou, B.; Duan, G.-R.; Li, Z.-Y. Gradient based iterative algorithm for solving coupled matrix equations. *Syst. Control. Lett.* **2009**, *58*, 327–333. [\[CrossRef\]](#)

10. Li, Z.-Y.; Zhou, B.; Lam, J.; Wang, Y. Positive operator based iterative algorithms for solving Lyapunov equations for Itô stochastic systems with Markovian jumps. *Appl. Math. Comput.* **2011**, *217*, 8179–8195. [CrossRef]
11. Yu, B.; Fan, H.-Y.; Chu, E.K.-W. Smith method for projected Lyapunov and Stein equations. *UPB Sci. Bull. Ser. A* **2018**, *80*, 191–204.
12. Sun, H.-J.; Zhang, Y.; Fu, Y.-M. Accelerated smith iterative algorithms for coupled Lyapunov matrix equations. *J. Frankl. Inst.* **2017**, *354*, 6877–6893. [CrossRef]
13. Li, T.-Y.; Gajic, Z. Lyapunov iterations for solving coupled algebraic Riccati equations of Nash differential games and algebraic Riccati equations of zero-sum games. In *New Trends in Dynamic Games and Applications*; Olsder, G.J., Ed.; Annals of the International Society of Dynamic Games; Birkhäuser: Boston, MA, USA, 1995; Volume 3.
14. Wicks, M.; De Carlo, R. Solution of Coupled Lyapunov Equations for the Stabilization of Multimodal Linear Systems. In Proceedings of the 1997 American Control Conference (Cat. No.97CH36041), Albuquerque, NM, USA, 4–6 June 1997; Volume 3, pp. 1709–1713.
15. Ivanov, I.G. An Improved method for solving a system of discrete-time generalized Riccati equations. *J. Numer. Math. Stoch.* **2011**, *3*, 57–70.
16. Qian, Y.-Y.; Pang, W.-J. An implicit sequential algorithm for solving coupled Lyapunov equations of continuous-time Markovian jump systems. *Automatica* **2015**, *60*, 245–250. [CrossRef]
17. Costa, O.L.V.; Aya, J.C.C. Temporal difference methods for the maximal solution of discrete-time coupled algebraic Riccati equations. *J. Optim. Theory Appl.* **2001**, *109*, 289–309. [CrossRef]
18. Costa, O.L.V.; Marques, R.P. Maximal and stabilizing Hermitian solutions for discrete-time coupled algebraic Riccati equations. *Math. Control Signals Syst.* **1999**, *12*, 167–195. [CrossRef]
19. Ivanov, I.G. Stein iterations for the coupled discrete-time Riccati equations. *Nonlinear Anal. Theory Methods Appl.* **2009**, *71*, 6244–6253. [CrossRef]
20. Bai, L.; Zhang, S.; Wang, S.; Wang, K. Improved SOR iterative method for coupled Lyapunov matrix equations. *Afr. Math.* **2021**, *32*, 1457–1463. [CrossRef]
21. Tian, Z.-L.; Xu, T.-Y. An SOR-type algorithm based on IO iteration for solving coupled discrete Markovian jump Lyapunov equations. *Filomat* **2021**, *35*, 3781–3799. [CrossRef]
22. Penzl, T. A cyclic low-rank Smith method for large sparse Lyapunov equations. *SIAM J. Sci. Comput.* **1999**, *21*, 1401–1408. [CrossRef]
23. Korvink, G.; Rudnyi, B. Oberwolfach Benchmark Collection. In *Dimension Reduction of Large-Scale Systems*; Benner, P., Sorensen, D.C., Mehrmann, V., Eds.; Lecture Notes in Computational Science and Engineering; Springer: Berlin/Heidelberg, Germany, 2005; Volume 45.
24. Wang, Q.; Lam, J.; Wei, Y.; Chen, T. Iterative solutions of coupled discrete Markovian jump Lyapunov equations. *Comput. Math. Appl.* **2008**, *55*, 843–850. [CrossRef]
25. Mathworks. *MATLAB User's Guide*; Mathworks: Natick, MA, USA, 2020. Available online: https://www.mathworks.com/help/pdf_doc/matlab/index.html (accessed on 15 March 2024).
26. Ober, R.J. Asymptotically Stable All-Pass Transfer Functions: Canonical Form, Parametrization and Realization. *IFAC Proc. Vol.* **1987**, *20*, 181–185. [CrossRef]
27. Chahlaoui, Y.; Van Dooren, P. Benchmark examples for model reduction of linear time-invariant dynamical systems. In *Dimension Reduction of Large-Scale Systems*; Springer: Berlin/Heidelberg, Germany, 2005; Volume 45, pp. 379–392.
28. Chu, E.K.-W.; Weng, P.C.-Y. Large-scale discrete-time algebraic Riccati equations—Doubling algorithm and error analysis. *J. Comput. Appl. Math.* **2015**, *277*, 115–126. [CrossRef]
29. Higham, N.J. *Functions of Matrices: Theory and Computation*; SIAM: Philadelphia, PA, USA, 2008.
30. Chahlaoui, Y.; Van Dooren, P. A collection of benchmark examples for model reduction of linear time invariant dynamical systems. *Work. Note* **2002**. Available online: <https://eprints.maths.manchester.ac.uk/1040/1/ChahlaouiV02a.pdf> (accessed on 15 March 2024).
31. Moosmann, C.; Greiner, A. Convective thermal flow problems. In *Dimension Reduction of Large-Scale Systems*; Lecture Notes in Computational Science and Engineering; Springer: Berlin/Heidelberg, Germany, 2005; Volume 45, pp. 341–343.
32. Lang, N. *Numerical Methods for Large-Scale Linear Time-Varying Control Systems and Related Differential Matrix Equations*; Logos-Verlag: Berlin, Germany, 2018.
33. Schmidt, A.; Siebert, K. *Design of Adaptive Finite Element Software—The Finite Element Toolbox ALBERTA*; Lecture Notes in Computational Science and Engineering; Springer: Berlin/Heidelberg, Germany, 2005; Volume 42.
34. Harper, C.A. *Electronic Packaging and Interconnection Handbook*; McGraw-Hill: New York, NY, USA, 1997.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.