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On a Stability of Non-Stationary Discrete Schemes with Respect to Interpolation Errors

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Abstract: The aim of this article is to analyze the efficiency and accuracy of finite-difference and finite-element Galerkin schemes for non-stationary hyperbolic and parabolic problems. The main problem solved in this article deals with the construction of accurate and efficient discrete schemes on nonuniform and dynamic grids in time and space. The presented stability and convergence analysis enables improving the existing accuracy estimates. The obtained stability results show explicitly the rate of accumulation of interpolation and projection errors that arise due to the movement of grid points. It is shown that the cases when the time grid steps are doubled or halved have different stability properties. As an additional technique to improve the accuracy of discretizations on non-stationary space grids, it is recommended to use projection operators instead of interpolation operators. This technique is used to solve a test parabolic problem. The results of specially selected computational experiments are also presented, and they confirm the accuracy of all theoretical error estimates obtained in this article.

Keywords: finite-difference schemes; Galerkin schemes; non-uniform grids; adaptive grids; hyperbolic problems; parabolic problems; stability; interpolation errors; projection errors

MSC: 65M06; 65M12; 65M15

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1. Introduction

Numerical algorithms that form the basis for recent simulations of various complex processes in engineering, technologies, physics, and medicine are based on two of the most important theoretical topics. The first one is the approximation theory, and the second one is the stability analysis. The general convergence framework states that the stability and approximation properties guarantee the convergence of discrete solutions [1,2]. The deep, broad, and constructive theories of approximation and stability are developed, and they cover various important topics dealing with non-smooth data [3–5], weak solutions [6–8], energy and maximum principle stability estimates [1,2,9,10], ill-posed and inverse problems [11,12], and nonlocal mathematical models including fractional derivatives [13–16].

The topic of the stability of discrete methods for solving non-stationary linear PDEs is very important and actively investigated. In our paper, we restrict ourselves to a specific question dealing with the stability of finite-difference schemes with respect to interpolation and projection errors. In addition, here, we note the impact of A.V. Gulin on this field of research [17]. Again, in the presented review, we mainly restricted ourselves to Gulin's works connected to a research topic that is close to our paper, when second-order PDEs are solved with non-classical nonlocal boundary conditions [18,19].

Adaptive grids in both space and time are used to fit the grid points to the dynamics of the solution and to minimize the approximation error [1,20,21]. At the same time, in many cases, uniform grids have been used in recent big data projects due to two important properties: high-order approximations can be constructed directly on uniform grids, and the obtained structure of the grids is well suited for parallel computing techniques [22,23]. Thus, different modifications of the algorithms are proposed that try to preserve the uniformity of the grid as close as possible [24].

There are a huge number of different methods for solving linear second-order partial differential equations of parabolic and hyperbolic types. Still, we are not proposing new methods in this article. Our goal is to analyze the efficiency of the scheme that is constructed in [25]. It uses the interpolation technique to define initial conditions on the previous time level. Thus, our aim is to derive stability estimates with respect to this new source of discretization error. It is shown that such errors accumulate undesirably fast. As an alternative, we recall the Discontinuous Galerkin (DG) method, which uses the projection operator instead of the interpolation operator. The results of the computational experiments confirm the theoretical estimates. We note that the presented example of the time grid was selected only as a benchmark to compare the accumulation rates of interpolation and projection errors.

We conclude that, in this article, we present the stability and convergence analysis of a new three-level finite-difference scheme, which is used to solve a hyperbolic problem on a perturbed uniform time grid [25]. At some specific points, the length of the grid steps can be doubled or halved. The error analysis performed in this paper is based on the energy method and state that, in the worst case, changes of step lengths can lead to the estimates of the global error (see also [9]):

$$\|Z^n\|_E \leqslant (M_- + M_+)\big(\|Z^1\|_E + \sum_{k=1}^{n} \tau_k \|\Psi^n\|\big). \tag{1}$$

Here, Ψ^n is the truncation error of the discrete scheme and

$$M_- = 2^{m_-}, \quad M_+ = 2^{m_+},$$

where m_- is the number of times the time step is halved and m_+ is the number of times the time step is doubled. We note once more that our aim is to make a full stability analysis of the interpolation errors introduced by the proposed algorithm. It is proven that the cases of doubling and reducing twice the time steps lead to different error accumulation rates. Our main aim is not to develop the ideas proposed in [25], but to explain why this new finite-difference scheme is not working as good as can be expected from schemes constructed on a uniform grid. The estimates derived in our analysis agree well with the results of extensive computational experiments.

We also investigate the difference in the stability of the backward Euler (BE) finite-difference scheme and the DG finite-element scheme when both schemes are used to solve one-dimensional parabolic problems on dynamically shifted uniform space grids. A good review on the DG method is given in [20,21,26], and applications for parabolic and hyperbolic problems are described in [4,5,27] Our analysis also proves that the accumulation of interpolation and projection approximation errors is quite different. The stability of the DG scheme with respect to the projection error has much better properties. Numerical examples illustrate these theoretical results.

In Section 2, the semi-discrete hyperbolic problem is formulated, and a standard three-level finite difference scheme is constructed on the uniform time grid. The stability of this scheme is investigated by using the energy and spectral methods. Note that the spectral method will be the main tool in our theoretical analysis.

In Section 3, the three-level finite-difference scheme from [25] is considered. It is defined on modified uniform time grids when, at some points, the lengths of the grid steps are doubled or halved. The most valuable property of this scheme is that the approximation is performed on uniformly distributed grid points; thus, the basic advantages of such discrete schemes are preserved. In Section 3.1, the case when the sizes of the grid are doubled is considered. We prove that, in this case, the finite- difference scheme remains unconditionally stable and the second-order accuracy in time is valid. This estimate improves the result presented in [25]. The new global error estimate is connected to the fact that no additional approximation errors, such as an interpolation error, are introduced, and it is sufficient to analyze the stability of the scheme on uniform sub-grids only.

In Section 3.2, the case when the sizes of the grid are halved is considered. The obtained stability estimates give a possibility to define the convergence rate for different asymptotics of the number of times the time step is halved. In particular, it is proven that, due to the accumulation of interpolation errors, the convergence order of the global error is reduced to the first-order if the grid sizes are halved only at a finite number of points and the discrete solution is not converging at all if this number is proportional to $O(1/\tau)$. The results of the computational experiments agree well with these theoretical conclusions.

In Section 4, the accumulation of interpolation errors is demonstrated also for parabolic problems. It is proven that, if the space grid depends on time (in a discontinuous fashion), then the stability of the implicit backward Euler (BE) finite-difference scheme with respect to the interpolation error leads only to conditional convergence rates. As a possibility to avoid this negative effect, it is recommended to use DG schemes, when the interpolation operator is changed to the projection operator. The DG scheme is stable with respect to the projection error. The results of the computational experiments agree well with these conclusions.

Some final conclusions are given in Section 5.

2. Problem Formulation

Let $\Omega = (0,1)^d$ be an open and bounded domain $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$. Define a self-adjoint linear elliptic diffusion operator:

$$\mathcal{A}u = -\mathrm{div}(\mathcal{K}\nabla u) \quad \text{in } \Omega \tag{2}$$

with $\mathcal{K}(x) \in \mathbb{R}^{d\times d}$ symmetric and the uniformly positive definite $d \times d$ matrix. Operator $\mathcal{A}$ is supplemented with homogeneous Dirichlet boundary conditions on $\partial\Omega$.

Next, by using the finite-volume or finite-element methods, we approximate operator $\mathcal{A}$ by discrete operator A_h, which is defined in a real finite-dimensional Hilbert space H:

$$A_h = A_h^* \geqslant \alpha_A\, I, \quad \alpha_A > 0, \tag{3}$$

where I is the identity operator in H. In order to simplify the notation for discrete operators, we restrict ourselves to $d = 1$.

Consider a semi-discrete hyperbolic problem for the function $u(t) \in H$:

$$\frac{d^2u}{dt^2} + \beta\frac{du}{dt} + A_h u = f(t), \quad t > 0 \tag{4}$$

$$u(0) = u_0, \quad \frac{du}{dt}(0) = v_0, \quad u_0, v_0 \in H, \tag{5}$$

where $\beta > 0$. Then, the following a priori estimate of the solution of (4) can be proven directly by using the energy method (see also [1,9,25]):

$$\left\|\frac{du}{dt}(t)\right\|^2 + \|u(t)\|_{A_h} \leqslant \|v_0\|^2 + \|u_0\|^2_{A_h} + \frac{1}{2\beta}\int_0^t \|f(s)\|^2 ds, \tag{6}$$

where, for any self-adjoint positive definite operator B, a Hilbert space H_B is defined with the inner product and the norm:

$$(u,v)_B = (Bu, v), \quad \|u\|_B = (u,u)_B^{1/2}.$$

First, let us define a uniform time grid

$$\omega_t = \{t^n : \; t^n = t^{n-1} + \tau, \;\; n = 1,\ldots,N, \;\; t^0 = 0, \; t^N = T\}.$$

The discrete function $U^n = U(t^n)$ gives an approximation of the exact solution $u(t^n)$. The differential problem (4) is approximated by the following standard implicit symmetrical three-level scheme:

$$\begin{aligned} &\frac{U^{n+1} - 2U^n + U^{n-1}}{\tau^2} + \beta \frac{U^{n+1} - U^{n-1}}{2\tau} + A_h \frac{U^{n+1} + U^{n-1}}{2} = F^n, \\ &U^0 = u_0, \quad U^1 = u_0 + \tau v_0. \end{aligned} \tag{7}$$

The unconditional stability of this scheme can be proven by using the energy and spectral methods. They give similar general information on the stability of the discrete solution, but still can give estimates of the accumulation of truncation and interpolation errors in different norms. This possibility enables us to follow the dynamics of the interpolation errors in more detail.

Let us start from the application of the standard energy method [9]. If $\beta > 0$, then it is easy to obtain the following stability estimate:

$$\begin{aligned} &\left\| \frac{U^{n+1} - U^n}{\tau} \right\|^2 + \frac{1}{2}\|U^{n+1}\|^2_{A_h} + \frac{1}{2}\|U^n\|^2_{A_h} \\ &\quad \leqslant \left\| \frac{U^n - U^{n-1}}{\tau} \right\|^2 + \frac{1}{2}\|U^n\|^2_{A_h} + \frac{1}{2}\|U^{n-1}\|^2_{A_h} + \frac{\tau}{2\beta}\|F^n\|^2. \end{aligned} \tag{8}$$

First, the uniform space grid is used:

$$\omega_x = \{x_j : \ x_0 = 0, \ x_J = 1, \ \ x_j = jh\}.$$

Then, discrete functions $U_j = U(x_j)$, $x_j \in \omega_x$ can be defined. Let us assume that functions U satisfy the homogeneous boundary conditions:

$$U_0 = 0, \quad U_J = 0.$$

The inner product in the Hilbert space H is defined in a standard way:

$$(U, V) = \sum_{j=1}^{J-1} U_j V_j h.$$

Then, the second-order derivative $-\frac{\partial^2 u}{\partial x^2}$ is approximated by the discrete operator:

$$A_h U = -\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2}.$$

The eigenvectors $\{\varphi_k(x_j) = \sqrt{2}\sin(\pi k x_j)\}$ of A_h make a full basis set of orthonormal vectors [1,9]:

$$A_h \sin(\pi k x_j) = \lambda_k \sin(\pi k x_j), \quad \lambda_j = \frac{4\sin^2(\pi k h/2)}{h^2}, \ \ k = 1, \dots, J-1.$$

It follows that A_h is a self-adjoint and positive definite operator in H.

We also consider a general nonuniform space grid:

$$\omega_x = \{x_j : \ \ x_0 = 0, \ \ x_{J_n} = 1, \ \ x_j = x_{j-1} + h_{j-0.5}, \ j = 1, \dots, J-1\}.$$

By using the finite-volume method [1,9], the following operator A_h can be defined on this grid:

$$A_h U = -\frac{1}{h_j}\left(\frac{U_{j+1} - U_j}{h_{j+0.5}} - \frac{U_j - U_{j-1}}{h_{j-0.5}} \right),$$

where $h_j = (h_{j+0.5} + h_{j-0.5})/2$. The inner product in the Hilbert space H is defined by

$$(U, V) = \sum_{j=1}^{J-1} U_j V_j h_j.$$

Again, it easy to prove that A_h is a self-adjoint and positive definite operator in H, i.e., the estimates (3) are valid.

As a basic technique for the stability analysis of discrete schemes proposed in this paper, we use the spectral method. It was efficiently used for non-stationary problems with nonlocal fractional-order elliptic operators [16,28,29].

Functions $U^n \in H$ can be written as

$$U_j^n = \sum_{k=1}^{J-1} c_k^n \varphi_k(x_j), \quad j = 1, \ldots, J-1,$$

where $c_k^n = (U^n, \varphi_k)$.

By using the Fourier method, we write discrete problems for each coefficient c_k^n:

$$\begin{aligned} &\frac{c_k^{n+1} - 2c_k^n + c_k^{n-1}}{\tau^2} + \beta \frac{c_k^{n+1} - c_k^{n-1}}{2\tau} + \lambda_k \frac{c_k^{n+1} + c_k^{n-1}}{2} = f_k^n, \\ &c_k^0 = \widetilde{u}_k, \quad c_k^1 = \widetilde{u}_k + \tau \widetilde{v}_k, \end{aligned} \tag{9}$$

where

$$\begin{aligned} F_j^n &= \sum_{k=1}^{J-1} f_k^n \varphi_k(x_j), \quad j = 1, \ldots, J-1, \\ u_j^0 &= \sum_{k=1}^{J-1} \widetilde{u}_k \varphi_k(x_j), \quad v_j^0 = \sum_{k=1}^{J-1} \widetilde{v}_k \varphi_k(x_j). \end{aligned}$$

Lemma 1. *Let us assume that $\beta \geqslant 0$, then the discrete scheme (9) is unconditionally stable.*

Proof. The solution of the homogeneous version of Equation (9) can be written as

$$c_k^n = \gamma_{k1} q_1^n + \gamma_{k2} q_2^n,$$

where q_1 and q_2 are solutions of the characteristic equations:

$$\left(1 + \frac{\tau}{2}\beta + \frac{\tau^2}{2}\lambda_k\right) q^2 - 2q + \left(1 - \frac{\tau}{2}\beta + \frac{\tau^2}{2}\lambda_k\right) = 0.$$

Next, we write this equation in a standard form:

$$q^2 - \frac{2}{1 + \frac{\tau}{2}\beta + \frac{\tau^2}{2}\lambda_k} q + \frac{1 - \frac{\tau}{2}\beta + \frac{\tau^2}{2}\lambda_k}{1 + \frac{\tau}{2}\beta + \frac{\tau^2}{2}\lambda_k} = 0. \tag{10}$$

It follows from the Hurwitz criterion that $|q_{1,2}| \leqslant 1$ if and only if

$$\frac{1 - \frac{\tau}{2}\beta + \frac{\tau^2}{2}\lambda_k}{1 + \frac{\tau}{2}\beta + \frac{\tau^2}{2}\lambda_k} \leqslant 1, \quad \frac{2}{1 + \frac{\tau}{2}\beta + \frac{\tau^2}{2}\lambda_k} \leqslant \frac{2 + \tau^2 \lambda_k}{1 + \frac{\tau}{2}\beta + \frac{\tau^2}{2}\lambda_k}.$$

Both inequalities are unconditionally satisfied. The proof is finished. □

3. Nonuniform Grids

Let us consider a general nonuniform grid:

$$\omega_t = \{t^n : \ t^n = t^{n-1} + \tau_{n-1/2}, \ \ n = 1, \ldots, N, \ \ t^0 = 0, \ t^N = T\} \tag{11}$$

and denote $t^{n-1/2} = t^{n-1} + 0.5\tau_{n-1/2}$.

3.1. The Time Steps Are Doubled at Some Grid Points.

In [25], special weakly nonuniform grids are considered, when the step size of the grid can be doubled or halved at a finite number of points. Let us consider the case that, when starting at time $t = t^n$, the grid step is doubled $\tau_{n+1} = 2\tau_n$. Here, τ_n denotes the length of the discrete step till the grid point t^n, and τ_{n+1} is a modified step for the following sequence of uniformly distributed grid points.

The original algorithm is defined by

$$\frac{U^{n+1/2} - 2U^n + U^{n-1}}{\tau_n^2} + \beta \frac{U^{n+1/2} - U^{n-1}}{2\tau_n} + A_h \frac{U^{n+1/2} + U^{n-1}}{2} = F^n,$$
$$\frac{U^{n+1} - 2U^{n+1/2} + U^n}{\tau_n^2} + \beta \frac{U^{n+1} - U^n}{2\tau_n} + A_h \frac{U^{n+1} + U^n}{2} = F^{n+1/2};$$

next, U^{n+2} is computed using the standard three-level scheme on the uniform grid with the step τ_{n+1}:

$$\frac{U^{n+2} - 2U^{n+1} + U^n}{\tau_{n+1}^2} + \beta \frac{U^{n+2} - U^n}{2\tau_{n+1}} + A_h \frac{U^{n+2} + U^n}{2} = F^{n+1}. \tag{12}$$

We present a slightly modified version of the original discrete scheme when temporary grid points are not used. First, the solution U^{n+1} is computed

$$\frac{U^{n+1} - 2U^n + U^{n-2}}{\tau_{n+1}^2} + \beta \frac{U^{n+1} - U^{n-2}}{2\tau_{n+1}} + A_h \frac{U^{n+1} + U^{n-2}}{2} = F^n \tag{13}$$

Next, the uniform grid version of the discrete scheme (12) is used to compute U^{n+2}.

The stability and convergence analysis is based on the results of Lemma 1.

Theorem 1. *The solution of the finite-difference scheme* (12) *and* (13) *converges to order* $O(\tau^2)$.

Proof. We restrict ourselves to the analysis of one time moment where the time step is doubled. First, starting at time point t^n, the discrete scheme is again defined as a three-level scheme on a uniform grid with a doubled time step $\tau_{n+1} = 2\tau_n$. Thus, the scheme remains unconditionally stable.

Second, the initial conditions, i.e., discrete solutions on layers t^n and t^{n-2}, are calculated by using solutions derived by a more accurate scheme with the time step τ_n. As a conclusion, we obtain that the discrete solution converges to order $O(\tau^2)$. In the case of more time moments, when the step size of the grid ω_t is doubled, the same arguments are iteratively applied. □

In the computational experiments, we compared the accuracy of the constructed combined discrete scheme (12) and (13) with a popular benchmark scheme. This three-level finite-difference scheme is constructed on a general non-uniform time grid (11):

$$\frac{1}{\tau_n}\Big(\frac{U^{n+1} - U^n}{\tau_{n+1/2}} - \frac{U^n - U^{n-1}}{\tau_{n-1/2}}\Big) + \beta \frac{U^{n+1} - U^{n-1}}{2\tau_n}$$
$$+ A_h\Big[U^n + \frac{1}{2}\tau_n\Big(\frac{U^{n+1} - U^n}{\tau_{n+1/2}} - \frac{U^n - U^{n-1}}{\tau_{n-1/2}}\Big)\Big] = F^n. \tag{14}$$

Here, $\tau_n = \frac{1}{2}(\tau_{n+1/2} + \tau_{n-1/2})$.

We present the results of the computational experiments. The differential problem (4) is solved for $\beta = 1$ till the final time moment $T = 1$. The initial and boundary data and $f(x,t)$ are chosen so that the solution $u(x,t)$ is the function:

$$u(x,t) = e^t \sin(\pi x).$$

The space grid ω_x is uniform, and the number of points is equal to $J = 20$. The time grid is generated by dividing the time interval $[0, 1]$ into five subintervals:

$$[(k-1)/5,\ k/5], \quad k = 1, \dots, 5.$$

In each subinterval, uniform grids are generated with time step sizes of

$$\tau(k) = 2^{k-5}/N, \quad k = 1, \dots, 5.$$

Let us denote $Z_j^n = u(x_j, t^n) - U_j^n$ as the global error of the discrete solution. The maximum norm of a discrete function Z^n is defined as

$$\|Z(t^n)\|_\infty = \max_{0<j<J} |Z_j^n|.$$

The experimental convergence rate $\rho(\tau)$ is defined as

$$\rho(\tau) = \log_2 \Big(\frac{\|Z(2\tau)\|_\infty}{\|Z(\tau)\|_\infty} \Big).$$

In order to show that the constructed three-level discrete scheme on this special non-uniform time grid ω_t is stable and additional grid points really reduce the global error of the discrete solution, we give also the errors Z_3 of the classical three-level discrete scheme (14) when the time grid is uniform and it has N discrete points. The results of the computational experiments are presented in Table 1, where Z_1 is the error for the discrete solution of the scheme (12) and (13), Z_2 is the error for the discrete solution of the classical finite-difference scheme (14) on non-uniform time grids, and Z_3 is the error of the solution of the symmetrical scheme (7) when the time grid is uniform in $[0, 1]$ and it has N points.

Table 1. Errors $\|Z_1\|_\infty$ and experimental convergence rates $\rho(\tau)$ at $T = 1$ for the discrete solution of the scheme (12) and (13) and errors $\|Z_2\|_\infty$ and experimental convergence rates $\rho(\tau)$ for the discrete solution of the finite-difference scheme (14) for a sequence of time steps τ. $\|Z_3\|_\infty$ is the error of the discrete solution of the the symmetrical scheme (7) when the time grid is uniform and it has N points.

N	$\|Z_1\|_\infty$	$\rho_1(\tau)$	$\|Z_2\|_\infty$	$\rho_2(\tau)$	$\|Z_3\|_\infty$
20	1.018×10^{-3}	—	1.279×10^{-3}	—	3.611×10^{-3}
40	2.429×10^{-4}	2.067	3.255×10^{-4}	1.974	9.042×10^{-4}
80	5.914×10^{-5}	2.038	8.204×10^{-5}	1.988	2.261×10^{-4}
160	1.458×10^{-5}	2.020	2.052×10^{-5}	1.999	5.654×10^{-5}
320	3.620×10^{-6}	2.010	5.131×10^{-6}	2.000	1.414×10^{-5}

As expected, the new three-level discrete scheme is stable and preserves the second order of convergence. We also note that additional grid points decrease the error; thus, the application of such a modified time grid is justified.

3.2. The Time Steps Are Halved at Some Grid Points

For the case when the time step size is halved $\tau_{n+1} = \frac{1}{2}\tau_n$, the following algorithm is proposed in [25]. The auxiliary solution $\widetilde{U}^{n+2}$ is computed using the standard three-level scheme:

$$\frac{\widetilde{U}^{n+2} - 2U^n + U^{n-1}}{\tau_n^2} + \beta \frac{\widetilde{U}^{n+2} - U^{n-1}}{2\tau_n} + A_h \frac{\widetilde{U}^{n+2} + U^{n-1}}{2} = F^n. \tag{15}$$

Then, solution U^{n+1} is computed by using the linear interpolation algorithm:

$$U^{n+1} = \frac{\widetilde{U}^{n+2} + U^n}{2}. \tag{16}$$

The final value of the solution U^{n+2} is obtained on the uniform grid with time step τ_{n+1}:

$$\frac{U^{n+2} - 2U^{n+1} + U^n}{\tau_{n+1}^2} + \beta \frac{U^{n+2} - U^n}{2\tau_{n+1}} + A_h \frac{U^{n+2} + U^n}{2} = F^{n+1}. \quad (17)$$

Again, we propose a modification of this algorithm when an auxiliary solution is not required. It is sufficient to use $U^{n-1/2}$, which is obtained by the linear interpolation:

$$\frac{U^{n+1} - 2U^n + U^{n-1/2}}{\tau_{n+1}^2} + \beta \frac{U^{n+1} - U^{n-1/2}}{2\tau_{n+1}} + A_h \frac{U^{n+1} + U^{n-1/2}}{2} = F^n. \quad (18)$$

In order to simplify the stability analysis, we take $\beta = 0$. Note, that it follows from the results given above that the real part of the solutions of characteristic Equation (10) is decreased for $\beta > 0$.

Theorem 2. *Let us assume that the interpolation error of* (16) *can be bounded by*

$$|\Psi_I| \leqslant C\tau^2.$$

If M is the number of times the time step of grid ω_t is halved, then the following estimate of the global error of scheme (18) *is valid:*

$$\|Z^n\| \leqslant M\tau. \quad (19)$$

In particular, if M is finite, then the error of the solution of discrete scheme (18) *is estimated by*

$$\|Z^n\| \leqslant C\tau. \quad (20)$$

If $M = C/\sqrt{\tau}$, then we have the estimate:

$$\|Z^n\| \leqslant C\sqrt{\tau}. \quad (21)$$

If $M = C/\tau$, then the discrete solution of (18) *is not converging at all:*

$$\|Z^n\| \leqslant O(1). \quad (22)$$

Proof. It is sufficient to consider the following problem for the Fourier coefficients of the error vector:

$$z_k^n = \gamma_{k1} q_{k1}^n + \gamma_{k2} q_{k2}^n, \quad k = 1, \dots, J-1,$$
$$z_k^0 = 0, \quad z_k^1 = d, \quad |d| = C\tau^2,$$

where q_{k1} and q_{k2} are solutions of the characteristic equation:

$$\left(1 + \frac{\tau^2}{2}\lambda_k\right)(q_k^{n+1})^2 - 2q_k^n + \left(1 + \frac{\tau^2}{2}\lambda_k\right) = 0.$$

Let us denote

$$b = \frac{1}{1 + \frac{\tau^2}{2}\lambda_k}.$$

Simple computations give

$$q_{k,1,2} = b \pm i\sqrt{1 - b^2},$$
$$z_k^n = \frac{d}{\sqrt{1 - b^2}} \sin(\varphi n),$$

where

$$b \pm i\sqrt{1 - b^2} = \cos(\varphi) \pm i\sin(\varphi).$$

Then, it follows that

$$\sqrt{1-b^2} = \sqrt{1 - 1/\left(1 + \frac{\tau^2}{2}\lambda_k\right)^2}$$
$$= \frac{\tau\sqrt{\lambda_k}\sqrt{1+\tau^2\lambda_k}}{1+\tau^2\lambda_k/2}. = C\tau.$$

Thus, taking into account the estimate of the interpolation error and the bound for $\sqrt{1-b^2}$, we obtain that the global error estimate (19) is valid. The remaining error estimates (20)–(22) follow directly. □

Next, we present the results of computational experiments. The time grid is generated by dividing the time interval $[0,1]$ into five subintervals:

$$[(k-1)/5,\ k/5], \quad k = 1, \dots, 5.$$

In each subinterval, uniform grids are generated with step sizes

$$\tau(k) = 2^{1-k}/N, \quad k = 1, \dots, 5,$$

where N is the selected number of time points in the first subinterval. The results of the computational experiments are given in Table 2, where Z_1 is the error for the discrete solution of the scheme (18), Z_2 is the error for the discrete solution of the classical finite-difference scheme (14) on non-uniform time grids, and $\rho_{1,2}(\tau)$ are experimental convergence rates.

Table 2. Errors $\|Z_1\|_\infty$ and experimental convergence rates $\rho(\tau)$ at $T = 1$ for the discrete solution of the scheme (18) and errors $\|Z_2\|_\infty$ and experimental convergence rates $\rho(\tau)$ for the discrete solution of the finite-difference scheme (14) for a sequence of time steps τ.

N	$\|Z_1\|_\infty$	$\rho_1(\tau)$	$\|Z_2\|_\infty$	$\rho_2(\tau)$
20	6.600×10^{-3}	—	2.205×10^{-4}	—
40	3.200×10^{-3}	1.044	5.787×10^{-5}	1.930
80	1.573×10^{-3}	1.025	1.481×10^{-5}	1.966
160	7.798×10^{-4}	1.012	3.743×10^{-6}	1.984
320	3.881×10^{-4}	1.007	0.941×10^{-7}	1.991

The presented results agree well with the theoretical convergence rate $O(\tau)$ given in Theorem 2.

In the final computational experiment, the length of the time grid steps was allowed to be doubled or halved. The length of sub-blocks is equal to 4τ, $\tau = 1/N$, and the grid points are distributed as

$$n = 12m, \quad m = 0, 1, \dots$$
$$t^{n+k} = t^{n+k-1} + \tau, \quad k = 1, 2, 3, 4,$$
$$t^{n+4+k} = t^{n+4+k-1} + \tau/2, \quad k = 1, \dots, 8.$$

The results of the computational experiments are given in Table 3, and they agree well with the theoretical estimates.

It follows from the presented results that, as is stated in Theorem 2, the discrete solution is not converging at all for such a modified time grid. Still the solution of the finite-difference scheme (14) is converging to quadratic order.

Table 3. Errors $\|Z_1\|_\infty$ and experimental convergence rates $\rho(\tau)$ at $T = 1$ for the discrete solution of the scheme (18) and errors $\|Z_2\|_\infty$ and experimental convergence rates $\rho(\tau)$ for the discrete solution of the finite-difference scheme (14) for a sequence of time steps τ.

N	$\|Z_1\|_\infty$	$\rho_1(\tau)$	$\|Z_2\|_\infty$	$\rho_2(\tau)$
20	1.036×10^{-2}	—	1.819×10^{-3}	—
40	9.391×10^{-3}	0.142	4.245×10^{-4}	2.099
80	9.033×10^{-3}	0.056	1.084×10^{-4}	1.969
160	8.964×10^{-3}	0.011	2.737×10^{-5}	1.986
320	8.958×10^{-3}	0.010	6.873×10^{-6}	1.994

4. Parabolic Interpolation

Let us consider one-dimensional parabolic problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + f(x,t), \ \ (x,t) \in Q, \\ 0 < x < 1, \ 0 < t \leqslant T, \\ u(0,t) = \mu_0(t), \ u(1,t) = \mu_1(t), \\ u(x,0) = u_0(x), \ 0 \leqslant x \leqslant 1, \end{cases} . \tag{23}$$

where $Q = \{(x,t) : \ 0 < x < 1, \ 0 < t \leqslant T\}$ and functions $f(x,t)$, $u_0(x)$, $\mu_0(t)$, and $\mu_1(t)$ are sufficiently smooth.

In addition to the uniform time grid ω_t:

$$\omega_t = \{t^n : \ t^n = n\tau, \ \ n = 1, 2, \ldots, N\},$$

we define a nonuniform space grid, which can depend on time:

$$\omega_x(t^n) = \{x_j^n : \ x_0^n = 0, \ x_{J_n}^n = 1, \ \ x_j^n = x_{j-1}^n + h_{j-0.5}^n\}.$$

The grid $\omega_h(t^k)$ is not constant in time; thus, the number of grid points J_n and the position of each point may depend on t^n.

4.1. Finite-Difference Scheme

The main aim of this subsection is to show that the accumulation of the classical truncation error and of the additional interpolation error can be very different [1,9]. Generally, we are interested in investigating the stability of the BE finite difference scheme with respect to different types of local approximation errors.

In order to simplify our analysis, we assume that, at each time level that the space grid $\omega_x(t^n)$ is uniform, only the number of grid points J_n can vary from one step to another. The differential problem (23) is approximated by the implicit backward Euler (BE) scheme:

$$\begin{cases} \dfrac{U_j^n - I_{n-1}^n U_j^{n-1}}{\tau} = U_{\bar{x}x}^n + f(x_j^n, t^n), \quad 0 < j < J_n, \\ U_0^n = \mu_0(t^n), \quad U_{J_n}^n = \mu_1(t^n), \\ U^0(x_j) = U_0(x_j), \quad x_j^0 \in \omega_k(t^0). \end{cases} \tag{24}$$

Here, we denote the discrete solution $U_j^k = U(x_j^k, t^k)$, and the discrete operator:

$$U_{\bar{x}x} = \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2}, \quad j = 1, \ldots, J-1$$

is used to approximate the second-order derivative in space. Then, the linear interpolation operator I_{n-1}^n:

$$I_{n-1}^n U_j^{n-1} = \frac{x_j^n - x_l^{n-1}}{x_{l+1}^{n-1} - x_l^{n-1}} U_{l+1}^{n-1} + \frac{x_{l+1}^{n-1} - x_j^n}{x_{l+1}^{n-1} - x_l^{n-1}} U_l^{n-1}$$

is applied to compute the values of a discrete solution U^{n-1} at grid points $x_j^n \in \omega_x(t^n)$, where $x_l^{n-1} \leqslant x_j^n \leqslant x_{l+1}^{n-1}$.

Let us denote $Z_j^n = u(x_j^n, t^k) - U_j^n$ as the error of the discrete solution; it satisfies the following discrete problem:

$$\begin{cases} \dfrac{Z_j^n - Z_j^{n-1}}{\tau} = Z_{\bar{x}x}^n + \Psi_A^n + \Psi_I^n, \\ Z_0^n = 0, \quad Z_{J_n}^n = 0, \end{cases} \tag{25}$$

where Ψ_A^n is the truncation error of the discrete scheme and Ψ_I^n is the interpolation error.

Lemma 2. *Let us assume that $u(x,t) \in C_4^2(\overline{Q})$. The truncation error Ψ_A^n of the discrete scheme* (24) *and the interpolation error Ψ_I^n can be estimated by*

$$|\Psi_A| \leqslant C(h^2 + \tau), \quad |\Psi_I| \leqslant \frac{Ch^2}{\tau}. \tag{26}$$

The proof of these estimates is based on the Taylor expansion technique and is given in many classical textbooks; see, e.g., [1].

Let us define the maximum norm of a discrete function Z, which satisfies the homogeneous boundary conditions:

$$\|Z\|_\infty = \max_{0<j<J} |Z_j|.$$

Theorem 3. *The solution of the discrete scheme* (24) *converges to the solution of the differential problem* (23)*, and the following error estimate is valid:*

$$\|Z^n\|_\infty \leqslant C\Big(\tau + h^2 + \frac{h^2}{\tau}\Big), \quad n = 1, \ldots, N.$$

Proof. By applying the maximum principle to the solution of the problem (25), we obtain the stability estimate:

$$\|Z^n\|_\infty \leqslant \|Z^{n-1}\|_\infty + \tau\Big(\|\Psi_A^n\|_\infty + \|\Psi_I^n\|_\infty.\Big)$$

By applying this stability inequality iteratively, we show that

$$\|Z^n\|_\infty \leqslant \|Z^0\|_\infty + \sum_{k=1}^{n} \tau\Big(\|\Psi_A^k\|_\infty + \|\Psi_I^k\|_\infty.\Big)$$

The required estimates of the global error are obtained by using the estimates of Lemma 2. □

As one interesting conclusion from Theorem 3, we provide the accuracy estimates when the discrete time step τ is asymptotically decreased with respect to the space grid step h:

$$\|Z^n\|_\infty \leqslant C \begin{cases} h, \ if \ \ \tau = O(h), \\ \sqrt{h}, \ if \ \ \tau = O(h^{1.5}) \\ O(1), \ if \ \ \tau = O(h^2). \end{cases}$$

In order to show the accuracy of the obtained theoretical estimates, we present the results of the computational experiments. The data $u_0(x)$, μ_0, μ_1, and $f(x,t)$ were chosen so that the solution $u(x,t)$ is the function:

$$u(x,t) = e^t \sin(\pi x).$$

The given test problem is solved till $T = 1$.

The space grids are defined as a sequence of two uniform/almost uniform grids in the following way:

$$\omega_x(t^n) = \begin{cases} \omega_{x1} = \{x_j: \ x_0 = 0, \ x_J = 1, \ \ x_j = jh, \ j = 1,\dots,J-1\}, & \text{if } \ n = 2m, \\ \omega_{x2} = \{x_j: \ x_0{=}0, \ x_{J+1}{=}1, \ x_j{=}(j+\frac{1}{2})h, j = 1,\dots,J\}, & \text{if } \ n{=}2m{+}1. \end{cases}$$

We see that the lengths of the steps of both grids are equal, but the grid points are shifted by $\frac{1}{2}h$ relative to each other. In the case of odd time layers, the approximation of the second-order derivatives near the boundaries is performed by using the standard discrete operators as was described in the previous section. The second order of the truncation error is preserved also for this modified discrete scheme.

The results of the computational experiments are presented in Table 4, where Z is the error for the discrete solution of the BE scheme (24) and $\rho(h)$ denotes the experimental convergence rate.

Table 4. Errors $\|Z\|_\infty$ and experimental convergence rates $\rho(h)$ at $T = 1$ for the discrete solution of the BE scheme (24) for a sequence of time and space steps τ, h.

	J	τ	$\|Z\|_\infty$	$\rho(h)$
$\tau = 4h$	80	0.05	5.55×10^{-3}	—
	160	0.025	2.71×10^{-3}	1.034
	320	0.0125	1.34×10^{-3}	1.016
	640	0.00625	6.65×10^{-4}	1.011
$\tau = 2h^{1,5}$	160	3.12×10^{-3}	3.37×10^{-3}	—
	320	1.10×10^{-3}	2.56×10^{-3}	0.397
	640	3.91×10^{-4}	1.87×10^{-3}	0.453
	1280	1.38×10^{-4}	1.34×10^{-3}	0.481
$\tau = 40h^2$	80	6.25×10^{-3}	6.65×10^{-3}	—
	160	1.56×10^{-3}	7.41×10^{-3}	−0.156
	320	3.91×10^{-4}	7.62×10^{-3}	−0.040
	640	9.76×10^{-5}	7.67×10^{-3}	−0.009

The presented results of the computational experiments agree well with the theoretical estimates obtained above.

4.2. Discontinuous Galerkin Method

In this subsection, we solve the same parabolic problem by applying the discontinuous Galerkin (DG) method [20,27]. Let us consider the time intervals:

$$I_n = \{t: \quad t^{n-1} \leqslant t \leqslant \ t^n\}.$$

A space of discrete solutions is defined as

$$W^{(0)} = \{U(x,t): \quad U|_{I_n} \in S^1_{h,n}\},$$

where $S^1_{h,n}$ is a space of piecewise linear in x functions:

$$S^1_{h,n} = \{v(x,t): \ v(x,t) = \sum_{j=0}^{J_n} c^n_j \varphi^n_j(x)\}$$

and the basis functions φ_j are defined as

$$\varphi_j = \begin{cases} \frac{x - x_{j-1}}{h_{j-1/2}}, & x_{j-1} \leqslant x \leqslant x_j, \\ \frac{x_{j+1} - x}{h_{j+1/2}}, & x_j \leqslant x \leqslant x_{j+1}, \end{cases} \quad , \quad 0 \leqslant j \leqslant J.$$

By using the discontinuous Galerkin method, we define a discrete function $U \in W^{(0)}$, which is constant in time t on each time interval I_n and satisfies the equation:

$$\tau\Big(\frac{dU^n}{dx}, \frac{dv}{dx}\Big) + ([U^{n-1}], v^+_{n-1}) = \int_{t^{n-1}}^{t^n} (f, v) dt, \qquad \forall v \in S^1_{h,n}, \tag{27}$$

where

$$\begin{aligned} &[U^{n-1}] = U^n - U^{n-1}, \\ &v^{\pm}_n = v(t^n \pm 0), \quad v^n = v^-_n = v^+_{n-1}, \\ &U^-_0 = u_0. \end{aligned}$$

From (27), we obtain the discrete scheme:

$$\frac{U^n - P_h U^{n-1}}{\tau} = U^n_{\bar{x}x} + \frac{1}{\tau} \int_{t^{n-1}}^{t^n} (P_h f) dt, \tag{28}$$

where $P_h f$ defines the L_2 projection:

$$(P_h f, v) = (f, v), \quad \forall v \in S^1_{h,n}.$$

By comparing the DG scheme (27) with the BE finite-difference scheme (24), we see that the main difference is in the way in which the solution values on the previous time level are computed. In the DG scheme, instead of the interpolation operator, the projection operator is used.

By applying the convergence analysis techniques described, e.g., in [20], the following result is proven directly.

Theorem 4. *The solution of the DG scheme* (28) *converges to the solution of the differential problem* (23)*, and the error estimate is valid:*

$$\|u(t^k) - U^k\| \leqslant C\Big(2 + \log\Big(\frac{t^k}{\tau}\Big)\Big) \max_{1 \leqslant k \leqslant K} \Big(\|h^2_k f\|_{I_k} + \|\tau f\|_{I_k} + \|[U_{k-1}]\|$$
$$+ \Big\|\frac{h^2_k}{\tau}[U_{k-1}]\Big\|^*\Big) \leqslant C(\tau + h^2).$$

The term $\|\cdot\|^*$ arises only if $S^1_{h,n-1} \nsubseteq S^1_{h,n}$.

Next, in Table 5, we present the results of the computational experiments. The same test problem is solved as for the BE scheme (24). Here, Z is the error for the discrete solution of the DG scheme (27) and $\rho(h)$ denotes the experimental convergence rate.

Table 5. Errors $\|Z\|_\infty$ and experimental convergence rates $\rho(h)$ at $T = 1$ for the discrete solution of the DG scheme (27) for a sequence of time and space steps τ, h.

	J	τ	$\|Z\|_\infty$	$\rho(h)$
$\tau = 4h$	80	0.05	1.12×10^{-2}	—
	160	0.025	6.19×10^{-3}	0.855
	320	0.0125	3.11×10^{-3}	0.993
	640	0.00625	1.55×10^{-3}	0.998
$\tau = 2h^{1.5}$	160	3.12×10^{-3}	3.98×10^{-4}	—
	320	1.10×10^{-3}	1.40×10^{-4}	1.507
	640	3.91×10^{-4}	4.93×10^{-5}	1.506
	1280	1.38×10^{-4}	1.74×10^{-5}	1.503
$\tau = 40h^2$	80	6.25×10^{-3}	8.12×10^{-4}	—
	160	1.56×10^{-3}	2.03×10^{-4}	2.00
	320	3.91×10^{-4}	5.08×10^{-5}	2.00
	640	9.76×10^{-5}	1.27×10^{-5}	2.00

It follows from the presented results that the the solution of the DG scheme (27) is unconditionally converging to order $O(\tau + h^2)$.

5. Conclusions

In this paper, we investigated the stability of two finite-difference and finite-element schemes constructed for the solution of hyperbolic and parabolic problems. The main result shows that the accumulation of the classical truncation errors and the accumulation of the interpolation errors are quite different. The accumulation of the interpolation errors gives only conditional estimates, and the application of the discrete scheme with smaller time steps can lead to not smaller, but larger global errors.

A more accurate stability analysis was performed for the three-level discrete scheme, which was presented in a recent paper [25]. It was proven that, for almost uniform time grids with a possibility to double the step sizes of the grid at some time moments, the second-order convergence rates are preserved. In the case when the time grid step sizes are halved at some time moments, additional interpolation errors are introduced. A detailed spectral stability analysis was used to estimate the asymptotics of the global error in this case.

The results of extensive computational experiments were presented, they confirmed the accuracy of the obtained theoretical convergence estimates.

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