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A Series Approximation for the Analytic Fourier–Feynman Transform on Wiener Space

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Abstract: In this paper, we first establish an evaluation formula to calculate Wiener integrals of functionals on Wiener space. We then apply our evaluation formula to carry out easy an calculation for the analytic Fourier–Feynman transform of the functionals. Some examples are furnished to illustrate the usefulness of the evaluation formula. Finally, using the evaluation formula, we establish the series approximation for the analytic Fourier–Feynman transform.

Keywords: evaluation formula; unbounded functionals; analytic Fourier–Feynman transform; series approximation

MSC: 42B10; 28C20; 34B16; 34C25

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Citation: Chung, H.S. A Series Approximation for the Analytic Fourier–Feynman Transform on Wiener Space. *Axioms* **2024**, *13*, 237. https://doi.org/10.3390/axioms13040237

Academic Editor: Gradimir V. Milovanović

Received: 8 March 2024
Revised: 2 April 2024
Accepted: 2 April 2024
Published: 3 April 2024

1. Introduction

For $T > 0$, let $(C_0[0,T], \mathcal{M}, m)$ denote the classical Wiener space, where $\mathcal{M}$ is the class of all Wiener measurable subsets of $C_0[0,T]$ and m is the Wiener measure. Then, $(C_0[0,T], \mathcal{M}, m)$ is a complete measure space. For an integrable functional F on $C_0[0,T]$, the Wiener integral of F is denoted by

$$\int_{C_0[0,T]} F(x)m(dx).$$

Some works and theories for the analytic Fourier–Feynman transform (FFT) on the Wiener space, initiated by Brue [1], have been developed in the various studies. Since it became known that Wiener integrals explain the movement of particles in quantum mechanics, many studies on Wiener integrals have been published. In particular, the Fourier–Feynman transform makes it possible to better explain the behavior of particles and thus make them more predictable. In addition, research is being conducted on a new form of Fourier–Feynman transformation. The analytic FFT and its properties are similar in many respects to the ordinary Fourier transform. For an elementary introduction to the analytic FFT [1,2] and the references cited therein, see [3–10]. Many mathematicians have been studied the analytic FFT of various functionals on Wiener space.

One of the many topics within the theory of the analytic FFT is concerned with the classes of all polynomial functionals [11,12]. These classes have been used to explain certain physical phenomena. However, there are some difficulties in evaluating analytic FFT for high-order polynomial functionals as follows: let $\langle v, x\rangle$ denote the Paley–Wiener–Zygmund (PWZ) stochastic integral. For each $n = 1, 2, \ldots$, let $G_n(x) = \langle v, x\rangle^n$ with $\|v\|_2 = 1$. To calculate the analytic FFT of G_n, we have to consider following Wiener integral:

$$\int_{C_0[0,T]} [\langle v, x\rangle + \langle v, y\rangle]^n m(dx). \tag{1}$$

One can see that it is not easy to calculate of the Wiener integral (1) because the Lebesgue integral

$$\left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} (u + \langle v, y \rangle)^n \exp\left\{-\frac{u^2}{2}\right\} du \tag{2}$$

appears in the calculation of the Wiener integral (1) whenever we apply the change of the variable theorem. In order to evaluate the Lebesgue integral (2), we have to use the integration by parts formulas repeatedly. However it is very difficult and complicated.

In this paper, we establish a new evaluation formula to figure out these difficulties and complications. Using the evaluation formula, we obtain various examples involving the analytic FFTs very easily. Finally, we give a series approximation for the analytic FFT.

2. Definitions and Preliminaries

We first list key some definitions and preliminaries that are needed to understand this paper.

A subset B of $C_0[0, T]$ is said to be scale-invariant measurable provided $\rho B \in \mathcal{M}$ for all $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to be hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G are equal s-a.e., we write $F \approx G$.

For $v \in L_2[0, T]$ and $x \in C_0[0, T]$, let $\langle v, x \rangle$ denote the PWZ stochastic integral. Then, we have the following assertions.

(i) For each $v \in L_2[0, T]$, $\langle v, x \rangle$ exists for a.e. $x \in C_0[0, T]$.
(ii) If $v \in L_2[0, T]$ is a function of bounded variation, $\langle v, x \rangle$ equals the Riemann–Stieltjes integral $\int_0^T v(t)dx(t)$ for s-a.e. $x \in C_0[0, T]$.
(iii) The $\langle v, x \rangle$ has the expected linearity property.
(iv) The $\langle v, x \rangle$ is a Gaussian random variable with mean 0 and variance $\|v\|_2^2$.

For a more detailed study of the PWZ stochastic integral, see [2,6,8,13–17].

We are ready to recall the definitions of analytic Feynman integral and analytic FFT on Wiener space [1–3].

Let $\mathbb{C}$, $\mathbb{C}_+$, and $\widetilde{\mathbb{C}}_+$ denote the set of complex numbers, complex numbers with a positive real part, and nonzero complex numbers with a nonnegative real part, respectively. For each $\lambda \in \mathbb{C}$, $\lambda^{1/2}$ denotes the principal square root of λ, i.e., $\lambda^{1/2}$ is always chosen to have positive real part, so that $\lambda^{-1/2} = (\lambda^{-1})^{1/2}$ is in $\mathbb{C}_+$ for all $\lambda \in \widetilde{\mathbb{C}}_+$. Let F be a $\mathbb{C}$-valued scale-invariant measurable functional on $C_0[0, T]$ such that

$$J(\lambda) \equiv \int_{C_0[0,T]} F(\lambda^{-1/2}x)m(dx)$$

exists as a finite number for all $\lambda > 0$. If a function $J^*(\lambda)$ analytic on $\mathbb{C}_+$ exists such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_0[0, T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$J^*(\lambda) = \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x)m(dx).$$

Let q be a nonzero real number, and let F be a functional such that $\int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x)m(dx)$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the analytic Feynman integral of F with parameter q and we write

$$\int_{C_0[0,T]}^{\mathrm{anf}_q} F(x)m(dx) = \lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}_+}} \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x)m(dx).$$

From the fact above with some notations in [4,7], we state the definition of the analytic FFT.

Definition 1. *For $\lambda \in \mathbb{C}_+$ and $y \in C_0[0,T]$, let*

$$T_\lambda(F)(y) = \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(y+x)m(dx).$$

We define the L_1 analytic Fourier–Feynman transform, $T_q^{(1)}(F)$ of F, by the formula

$$T_q^{(1)}(F)(y) = \lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}_+}} T_\lambda(F)(y)$$

for s-a.e. $y \in C_0[0,T]$ and a nonzero real number q.

We note that $T_q^{(1)}(F)$ is defined only s-a.e. We also note that if $T_q^{(1)}(F)$ exists and if $F \approx G$, then $T_q^{(1)}(G)$ exists and $T_q^{(1)}(G) \approx T_q^{(1)}(F)$.

The following Wiener integration formula is used several times in this paper. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be any complete orthonormal set of functions in $L_2[0,T]$, and let $h : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable. Then,

$$\begin{aligned} &\int_{C_0[0,T]} h(\langle\alpha_1, x\rangle, \cdots, \langle\alpha_n, x\rangle)m(dx) \\ &\qquad = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{\mathbb{R}^n} h(\vec{u}) \exp\left\{ -\sum_{j=1}^n \frac{u_j^2}{2} \right\} d\vec{u} \end{aligned} \tag{3}$$

in the sense that if either side of (3) exists, both sides exist and equality holds.

We finish this section by giving the functionals on Wiener space, which are used in this paper. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a complete orthonormal set in $L_2[0,T]$, and let F be a functional defined by the formula

$$F(x) = \langle\alpha_1, x\rangle^{2p_1} \times \cdots \times \langle\alpha_1, x\rangle^{2p_n} = \prod_{j=1}^n \langle\alpha_j, x\rangle^{2p_j} \tag{4}$$

where $p_1, p_2, \ldots, p_{n-1}$ and p_n are nonnegative integers. Then, one can see that the functionals defined in Equation (4) are unbounded functionals used in [11,12].

Remark 1. *Let $\mathcal{P}$ be the set of all functionals of the form*

$$H(x) = h(\langle\alpha_1, x\rangle, \ldots, \langle\alpha_n, x\rangle)$$

where h is a continuous function on $\mathbb{R}^n$. By the Bolzano–Weierstrass theorem, there is a sequence $\{f_n\}$ of polynomial functions such that $\|h - f_n\|_\infty = \sup_{\vec{u} \in \mathbb{R}^n} |h(\vec{u}) - f_n(\vec{u})| \to 0$ as $n \to \infty$. Thus, the polynomial functionals such as Equation (4) are meaningful objects to study the FFT. The usefulness of the functionals (4) will be explained in Section 5 below.

3. An Evaluation Formula

In this section, we give an evaluation formula for the Wiener integrals. To do this, we shall start by giving two lemmas. The first lemma is the formula for the Lebesgue integral.

Lemma 1. *Let s be a nonnegative integer. Then, we have*

$$\int_{\mathbb{R}} u^s \exp\left\{-\frac{u^2}{2}\right\} du = 2^{\frac{s-1}{2}}(1 + (-1)^s)\Gamma\left(\frac{1+s}{2}\right) \tag{5}$$

where Γ *denotes the gamma function defined by the formula*

$$\Gamma(r) = \int_0^\infty t^{r-1}e^{-r}dt$$

for a complex number r *with* $Re(r) > 0$, *see [15,16].*

We now state some properties of the Gamma function Γ. For any positive integer n, let $n! = n \times (n-1) \times (n-2) \times \cdots \times 1$, and let $(2n-1)!! = (2n-1) \times (2n-3) \times (2n-5) \times \cdots \times 3 \times 1$ and set $(-1)!! = 1$. Then,

(i) $\Gamma(n) = (n-1)!$ for all positive integers n.
(ii) $\Gamma(s+1) = s\Gamma(1)$ for all positive real numbers s.
(iii) $\Gamma(n+\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$ for all positive integers n.

In our next lemma, we establish an Wiener integration formula.

Lemma 2. *Let* p *be a nonnegative integer, and let* α *be an element of* $L_2[0,T]$ *with* $\|\alpha\|_2 = 1$. *Then, for all nonzero real numbers* γ *and* β, *we have*

$$\begin{aligned} &\int_{C_0[0,T]} [\gamma\langle\alpha,x\rangle + \beta\langle\alpha,y\rangle]^{2p} m(dx) \\ &= \sum_{s=0}^{p} {}_{2p}C_{2s}(2s-1)!!\gamma^{2s}\beta^{2p-2s}\langle\alpha,y\rangle^{2p-2s} \end{aligned} \tag{6}$$

for $y \in C_0[0,T]$, *where* ${}_nC_k = \frac{n!}{k!(n-k)!}$ *for nonnegative integers* n *and* k *with* $n \geq k$.

Proof. For $y \in C_0[0,T]$, let $v = \langle\alpha,y\rangle$. Then, using Equation (3) for all nonzero real numbers γ and β and $y \in C_0[0,T]$, we have

$$\int_{C_0[0,T]} [\gamma\langle\alpha,x\rangle + \beta\langle\alpha,y\rangle]^{2p} m(dx) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} [\gamma u + \beta v]^{2p} \exp\left\{-\frac{u^2}{2}\right\}du.$$

Using the binomial formula

$$(au+bv)^n = \sum_{k=0}^{n} {}_nC_k(au)^k(bv)^{n-k} = \sum_{k=0}^{n} {}_nC_k a^k b^{n-k} u^k v^{n-k},$$

Equation (5), and some properties of the Gamma function, we have

$$\begin{aligned} &\int_{C_0[0,T]} [\gamma\langle\alpha,x\rangle + \beta\langle\alpha,y\rangle]^{2p} m(dx) \\ &= \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \sum_{k=0}^{2p} {}_{2p}C_k\gamma^k\beta^{2p-k}v^{2p-k} \int_{\mathbb{R}} u^k \exp\left\{-\frac{u^2}{2}\right\}du \\ &= \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \sum_{k=0}^{2p} {}_{2p}C_k\gamma^k\beta^{2p-k}v^{2p-k}2^{\frac{k-1}{2}}(1+(-1)^k)\Gamma\left(\frac{1+k}{2}\right) \\ &= \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \sum_{s=0}^{p} {}_{2p}C_{2s}\gamma^{2s}\beta^{2p-2s}v^{2p-2s}2^{\frac{2s-1}{2}}(1+(-1)^{2s})\Gamma\left(\frac{1+2s}{2}\right) \\ &= \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \sum_{s=0}^{p} {}_{2p}C_{2s}\gamma^{2s}\beta^{2p-2s}v^{2p-2s}2^{s+\frac{1}{2}}\Gamma\left(s+\frac{1}{2}\right) \\ &= \left(\frac{1}{\pi}\right)^{\frac{1}{2}} \sum_{s=0}^{p} {}_{2p}C_{2s}\gamma^{2s}\beta^{2p-2s}v^{2p-2s}2^{s}\frac{(2s-1)!!}{2^s}\sqrt{\pi} \\ &= \sum_{s=0}^{p} {}_{2p}C_{2s}(2s-1)!!\gamma^{2s}\beta^{2p-2s}v^{2p-2s}, \end{aligned} \tag{7}$$

which completes the proof of the lemma as desired. □

Using Equation (6) in Lemma 2, we can establish the evaluation formula for the Wiener integral.

Theorem 1. *Let F be as in Equation (4) above. Then, for all nonzero real numbers γ and β, we have*

$$\int_{C_0[0,T]} F(\gamma x + \beta y) m(dx) = \prod_{j=1}^{n} \left[\sum_{s=0}^{p_j} {}_{2p_j}C_{2s}(2s-1)!!\gamma^{2s}\beta^{2p_j-2s}\langle \alpha_j, y\rangle^{2p_j-2s} \right] \tag{8}$$

for $y \in C_0[0,T]$.

Proof. We first note that for each $j = 1, 2, \ldots, n$, let $X_j(y) = \langle \alpha_j, y\rangle$. Then, X_j's are independent Gaussian random variables. Thus, for any Lebesgue measurable function h on $\mathbb{R}$, $h(X_j)$'s are also independent Gaussian random variables. Then, for all nonzero real numbers γ and β, and $y \in C_0[0,T]$,

$$\begin{aligned} &\int_{C_0[0,T]} F(\gamma x + \beta y) m(dx) \\ &= \int_{C_0[0,T]} \prod_{j=1}^{n} [\gamma\langle \alpha_j, x\rangle + \beta\langle \alpha_j, y\rangle]^{2p_j} m(dx) \\ &= \prod_{j=1}^{n} \left[\int_{C_0[0,T]} [\gamma\langle \alpha_j, x\rangle + \beta\langle \alpha_j, y\rangle]^{2p_j} m(dx) \right]. \end{aligned}$$

Finally, using Equation (6) n-times repeatedly, we can establish Equation (8) as desired. □

4. Some Formulas for the Analytic FFT via the Evaluation Formula

In this section, we give an application of our evaluation formula. Theorem 2 is one of the main results in this paper.

Theorem 2. *Let F be as in Theorem 1 above, and let q be a nonzero real number. Then, the analytic FFT $T_q^{(1)}(F)$ of F exists and is given by the formula*

$$T_q^{(1)}(F)(y) = \prod_{j=1}^{n} \left[\sum_{s=0}^{p_j} {}_{2p_j}C_{2s}(2s-1)!! \left(\frac{i}{q}\right)^{s} \langle \alpha_j, y\rangle^{2p_j-2s} \right] \tag{9}$$

for s-a.e. $y \in C_0[0,T]$.

Proof. In Equation (8), set $\gamma = \lambda^{-\frac{1}{2}}$ and $\beta = 1$ for $\lambda > 0$. Then, it follows that for all $\lambda > 0$ and s-a.e. $y \in C_0[0,T]$, we have

$$T_\lambda(F)(y) = \prod_{j=1}^{n} \left[\sum_{s=0}^{p_j} {}_{2p_j}C_{2s}(2s-1)!!\lambda^{-s}\langle \alpha_j, y\rangle^{2p_j-2s} \right]. \tag{10}$$

From this, we observe that $T_\lambda(F)(y)$ of F exists for all $\lambda > 0$. We will show that the analytic FFT $T_q^{(1)}(F)(y)$ of F exists. To do this, for $\lambda \in \mathbb{C}_+$, let

$$J^*(\lambda) = \prod_{j=1}^{n} \left[\sum_{s=0}^{p_j} {}_{2p_j}C_{2s}(2s-1)!!\lambda^{-s}\langle \alpha_j, y\rangle^{2p_j-2s} \right].$$

Then, $J(\lambda) = J^*(\lambda)$ for all λ. Let Λ be any simple closed contour in $\mathbb{C}_+$. Then, using the Cauchy theorem, we have

$$\int_\Lambda J^*(\lambda)d\lambda = \int_\Lambda \prod_{j=1}^{n}\bigg[\sum_{s=0}^{p_j} {}_{2p_j}C_{2s}(2s-1)!!\lambda^{-s}\langle\alpha_j, y\rangle^{2p_j-2s}\bigg]d\lambda = 0$$

because the function $\sum_{s=0}^{p_j} {}_{2p_j}C_{2s}(2s-1)!!\lambda^{-s}\langle\alpha_j, y\rangle^{2p_j-2s}$ is an analytic function of λ in $\mathbb{C}_+$. Hence, using Morera's theorem, we conclude that $J^*(\lambda)$ is analytic on $\mathbb{C}_+$. It remains to show that

$$\lim_{\substack{\lambda\to -iq \\ \lambda\in\mathbb{C}_+}} J^*(\lambda) = \prod_{j=1}^{n}\bigg[\sum_{s=0}^{p_j} {}_{2p_j}C_{2s}(2s-1)!!\left(\frac{i}{q}\right)^s\langle\alpha_j, y\rangle^{2p_j-2s}\bigg].$$

However, it is an immediate consequence of the fact that the functions λ^s, $s = 1, 2, \ldots$ are continuous and analytic on $\mathbb{C}_+$. Thus, we complete the proof of Theorem 2 as desired. $\square$

We now give some formulas for the analytic FFT via the evaluation formula obtained by Equation (9). We first give several formulas for the 1-dimensional functionals in Table 1.

Table 1. Formulas for the 1-dimensional functionals.

$n = 1, p_j = j$	analytic FFT of $F_j, j = 1, 2, 3, 4$
$F_1(x) = \langle\alpha_1, x\rangle^2$	$\langle\alpha_1, y\rangle^2 + \frac{i}{q}$
$F_2(x) = \langle\alpha_1, x\rangle^4$	$\langle\alpha_1, y\rangle^4 + \frac{6i}{q}\langle\alpha_1, y\rangle^2 - \frac{3}{q^2}$
$F_3(x) = \langle\alpha_1, x\rangle^6$	$\langle\alpha_1, y\rangle^6 + \frac{15i}{q}\langle\alpha_1, y\rangle^4 - \frac{45}{q^2}\langle\alpha_1, y\rangle^2 - \frac{15i}{q^3}$
$F_4(x) = \langle\alpha_1, x\rangle^8$	$\langle\alpha_1, y\rangle^8 + \frac{28i}{q}\langle\alpha_1, y\rangle^6 - \frac{210}{q^2}\langle\alpha_1, y\rangle^4 - \frac{320i}{q^3}\langle\alpha_1, y\rangle^2 + \frac{105}{q^4}$

From now on, we next give a formula for the analytic FFT with the multi-dimensional functionals.

Example 1. *Let* $F_5(x) = \langle\alpha_1, x\rangle^2\langle\alpha_2, x\rangle^4$ (*set* $n = 2; p_1 = 1, p_2 = 2$ *in Equation* (4)). *Then, for s-a.e.* $y \in C_0[0, T]$, *we have*

$$\begin{aligned} T_q^{(1)}(F_5)(y) &= \prod_{j=1}^{2}\bigg[\sum_{s=0}^{p_j} {}_{2p_j}C_{2s}\left(\frac{i}{q}\right)^s\langle\alpha_j, y\rangle^{2p_j-2s}(2s-1)!!\bigg] \\ &= \bigg[\sum_{s=0}^{1} {}_2C_{2s}(2s-1)!!\left(\frac{i}{q}\right)^s\langle\alpha_1, y\rangle^{2-2s}\bigg] \\ &\quad\times\bigg[\sum_{s=0}^{2} {}_4C_{2s}(2s-1)!!\left(\frac{i}{q}\right)^s\langle\alpha_2, y\rangle^{4-2s}\bigg] \\ &= \bigg[\langle\alpha_1, y\rangle^2 + \frac{i}{q}\bigg]\bigg[\langle\alpha_2, y\rangle^4 + \frac{6i}{q}\langle\alpha_2, y\rangle^2 - \frac{3}{q^2}\bigg]. \end{aligned}$$

Remark 2. *From the definition of analytic FFT, one can observe that for $\lambda > 0$,*

$$
\begin{aligned}
T_\lambda(F_5)(y) &= \int_{C_0[0,T]} [\lambda^{-\frac{1}{2}}\langle \alpha_1, x\rangle + \langle \alpha_1, y\rangle]^2 [\lambda^{-\frac{1}{2}}\langle \alpha_2, x\rangle + \langle \alpha_2, y\rangle]^4 m(dx) \\
&= \int_{C_0[0,T]} [\lambda^{-1}\langle \alpha_1, x\rangle^2 + 2\lambda^{-\frac{1}{2}}\langle \alpha_1, x\rangle\langle \alpha_1, y\rangle + \langle \alpha_1, y\rangle^2] \\
&\qquad \times [\lambda^{-2}\langle \alpha_2, x\rangle^4 + 4\lambda^{-\frac{3}{2}}\langle \alpha_2, x\rangle^3\langle \alpha_2, y\rangle + 6\lambda^{-1}\langle \alpha_2, x\rangle^2\langle \alpha_2, y\rangle^2 \\
&\qquad\quad + 4\lambda^{-\frac{1}{2}}\langle \alpha_2, x\rangle\langle \alpha_2, y\rangle^3 + \langle \alpha_2, y\rangle^4] m(dx) \\
&= \int_{C_0[0,T]} [\lambda^{-3}\langle \alpha_1, x\rangle^2\langle \alpha_2, x\rangle^4 + 6\lambda^{-2}\langle \alpha_1, x\rangle^2\langle \alpha_2, x\rangle^2\langle \alpha_2, y\rangle^2 \\
&\qquad\qquad + \lambda^{-1}\langle \alpha_1, x\rangle^2\langle \alpha_2, y\rangle^4 + \lambda^{-2}\langle \alpha_2, x\rangle^4\langle \alpha_1, y\rangle^2 \\
&\qquad\qquad\quad + 6\lambda^{-1}\langle \alpha_2, x\rangle^2\langle \alpha_1, y\rangle^2\langle \alpha_2, y\rangle^2 + \langle \alpha_1, y\rangle^2\langle \alpha_2, y\rangle^4] m(dx) \\
&= 3\lambda^{-3} + 6\lambda^{-2}\langle \alpha_2, y\rangle^2 + \lambda^{-1}\langle \alpha_2, y\rangle^4 + 3\lambda^{-2}\langle \alpha_1, y\rangle^2 \\
&\qquad\qquad\quad + 6\lambda^{-1}\langle \alpha_1, y\rangle^2\langle \alpha_2, y\rangle^2 + \langle \alpha_1, y\rangle^2\langle \alpha_2, y\rangle^4.
\end{aligned}
$$

It can be analytically continued on $\mathbb{C}_+$, and so letting $\lambda \to -iq$, we have

$$
T_q^{(1)}(F_5)(y) = \left[\langle \alpha_1, y\rangle^2 + \frac{i}{q}\right]\left[\langle \alpha_2, y\rangle^4 + \frac{6i}{q}\langle \alpha_2, y\rangle^2 - \frac{3}{q^2}\right].
$$

It is evident from the preceding discussion that the calculating process is a challenging task. Therefore, the development of our evaluation formula holds significant value in addressing this difficulty and providing a practical solution.

We give more explicit formulas for the analytic FFT with the multi-dimensional functionals.

Example 2. *Let $F_6(x) = \langle \alpha_1, x\rangle^4\langle \alpha_2, x\rangle^2\langle \alpha_3, x\rangle^6$ (set $n = 3; p_1 = 2, p_2 = 1, p_3 = 3$ in Equation (4)). Then, for s-a.e. $y \in C_0[0,T]$ we have*

$$
\begin{aligned}
T_q^{(1)}(F_6)(y) &= \prod_{j=1}^{3}\left[\sum_{s=0}^{p_j} {}_{2p_j}C_{2s}(2s-1)!!\left(\frac{i}{q}\right)^s \langle \alpha_j, y\rangle^{2p_j-2s}\right] \\
&= \left[\sum_{s=0}^{2} {}_4C_{2s}(2s-1)!!\left(\frac{i}{q}\right)^s \langle \alpha_1, y\rangle^{4-2s}\right] \\
&\quad \times \left[\sum_{s=0}^{1} {}_2C_{2s}(2s-1)!!\left(\frac{i}{q}\right)^s \langle \alpha_2, y\rangle^{2-2s}\right] \\
&\qquad \times \left[\sum_{s=0}^{3} {}_6C_{2s}(2s-1)!!\left(\frac{i}{q}\right)^s \langle \alpha_3, y\rangle^{6-2s}\right] \\
&= \left[\langle \alpha_1, y\rangle^4 + \frac{6i}{q}\langle \alpha_1, y\rangle^2 - \frac{3}{q^2}\right]\left[\langle \alpha_2, y\rangle^2 + \frac{i}{q}\right] \\
&\quad \times \left[\langle \alpha_3, y\rangle^6 + \frac{15i}{q}\langle \alpha_3, y\rangle^4 - \frac{45}{q^2}\langle \alpha_3, y\rangle^2 - \frac{15i}{q^3}\right].
\end{aligned}
$$

Example 3. *Let* $F_7(x) = \langle \alpha_1, x\rangle^4 \langle \alpha_3, x\rangle^2 \langle \alpha_4, x\rangle^4$ *(set* $n = 4; p_1 = 2, p_2 = 0, p_3 = 1, p_4 = 3$ *in Equation (4)). Then, for s-a.e.* $y \in C_0[0, T]$*, we have*

$$\begin{aligned}
T_q^{(1)}(F_7)(y) &= \prod_{j=1}^{4} \left[\sum_{s=0}^{p_j} {}_{2p_j}C_{2s}(2s-1)!! \left(\frac{i}{q}\right)^s \langle \alpha_j, y\rangle^{2p_j - 2s} \right] \\
&= \left[\sum_{s=0}^{2} {}_4C_{2s}(2s-1)!! \left(\frac{i}{q}\right)^s \langle \alpha_1, y\rangle^{4-2s} \right] \\
&\quad \times \left[\sum_{s=0}^{0} {}_0C_{2s}(2s-1)!! \left(\frac{i}{q}\right)^s \langle \alpha_2, y\rangle^{-2s} \right] \\
&\qquad \times \left[\sum_{s=0}^{1} {}_2C_{2s}(2s-1)!! \left(\frac{i}{q}\right)^s \langle \alpha_3, y\rangle^{2-2s} \right] \\
&\qquad\quad \times \left[\sum_{s=0}^{2} {}_4C_{2s}(2s-1)!! \left(\frac{i}{q}\right)^s \langle \alpha_4, y\rangle^{6-2s} \right] \\
&= \left[\langle \alpha_1, y\rangle^4 + \frac{6i}{q}\langle \alpha_1, y\rangle^2 - \frac{3}{q^2} \right] \\
&\quad \times \left[\langle \alpha_3, y\rangle^2 + \frac{i}{q} \right] \left[\langle \alpha_4, y\rangle^4 + \frac{6i}{q}\langle \alpha_1, y\rangle^2 - \frac{3}{q^2} \right].
\end{aligned}$$

Remark 3. *We considered only three functionals. But, we can obtain various functionals with high dimensionals.*

5. Series Approximation for the Analytic FFT

In this section, using Equation (9) we shall establish a series approximation for the analytic FFT through several steps.

Step 1: Let $h \in C^\infty(\mathbb{R}^n)$ with

$$\int_{\mathbb{R}^n} |h(\vec{u})| \exp\left\{ -a \sum_{j=1}^{n} u_j^2 \right\} d\vec{u} < \infty \tag{11}$$

for all $a > 0$. Then, the Maclaurin series expansion of h is given by the formula

$$h(\vec{u}) = h(\vec{0}) + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\sum_{i_1, \cdots, i_k = 1}^{n} h_{u_{i_1} \cdots u_{i_k}}(\vec{0}) u_{i_1} \cdots u_{i_k} \right) \tag{12}$$

where $h_{u_{i_1} \cdots u_{i_k}}$ is k-th the derivative of h. Assume that $h_{u_{i_1} \cdots u_{i_k}}(\vec{0}) = 1$ for all derivatives of h (in fact, all of the processes of this development can be applied in the case that any k-th order partial derivatives $h_{u_{i_1} \cdots u_{i_k}}$'s have the same value when $\vec{0}$ is constant). Then, Equation (12) can be written by

$$\begin{aligned}
h(\vec{u}) &= h(\vec{0}) + \sum_{k=1}^{\infty} \frac{1}{k!} (u_1 + \cdots + u_n)^k \\
&= \lim_{r \to \infty} h_r(\vec{u})
\end{aligned} \tag{13}$$

where $h_r(\vec{u}) = h(\vec{0}) + \sum_{k=1}^{r} \frac{1}{k!}(u_1 + \cdots + u_n)^k, r = 1, 2, \ldots$. Hence, we have $|h(\vec{u}) - h_r(\vec{u})| \to 0$ as $r \to \infty$.

Step 2: For each $r = 1, 2, \ldots$, let $H_r(x) = h_r(\langle \alpha_1, x\rangle, \ldots, \langle \alpha_n, x\rangle)$ and let $H(x) = h(\langle \alpha_1, x\rangle, \ldots, \langle \alpha_n, x\rangle)$. Then, one can check that for all $\rho > 0$,

$$\int_{C_0[0,T]} |H(\rho x) - H_r(\rho x)| m(dx) \to 0$$

as $r \to \infty$ because for all $\rho > 0$, we see that

$$
\begin{aligned}
L_r &\equiv \int_{C_0[0,T]} |H(\rho x) - H_r(\rho x)| m(dx) \\
&= \int_{C_0[0,T]} |h(\rho\langle \alpha_1, x\rangle, \ldots, \rho\langle \alpha_n, x\rangle) - h_r(\rho\langle \alpha_1, x\rangle, \ldots, \rho\langle \alpha_n, x\rangle)| m(dx) \\
&= \left(\frac{1}{\sqrt{2\pi\rho^2}}\right)^n \int_{\mathbb{R}^n} |h(\vec{u}) - h_r(\vec{u})| \exp\left\{-\sum_{j=1}^n \frac{u_j^2}{2\rho^2}\right\} d\vec{u} \\
&\leq \left(\frac{1}{\sqrt{2\pi\rho^2}}\right)^n \int_{\mathbb{R}} |h(\vec{u})| \exp\left\{-\sum_{j=1}^n \frac{u_j^2}{2\rho^2}\right\} d\vec{u} \\
&\qquad + \left(\frac{1}{\sqrt{2\pi\rho^2}}\right)^n \int_{\mathbb{R}} |h_r(\vec{u})| \exp\left\{-\sum_{j=1}^n \frac{u_j^2}{2\rho^2}\right\} d\vec{u} < \infty
\end{aligned}
$$

for all $r = 1, 2, \cdots$. Hence, we can conclude that L_r tends to zero as $r \to 0$ from the dominated convergence theorem.

Step 3**:** One can see that

$$
\begin{aligned}
&\int_{C_0[0,T]} H(x) m(dx) \\
&= h(\vec{0}) + \sum_{l=1}^{\infty} \frac{1}{(2l)!} \int_{C_0[0,T]} (\langle \alpha_1, x\rangle + \cdots + \langle \alpha_n, x\rangle)^{2l} m(dx) \\
&= h(\vec{0}) + \sum_{l=1}^{\infty} I_{2l}
\end{aligned}
$$

for $l = 1, 2, \ldots$, where

$$
\begin{aligned}
I_{2l} &= \sum_{2p_1 + \cdots + 2p_n = 2l} \frac{1}{(2p_1)! \cdots (2p_n)!} \int_{C_0[0,T]} \langle \alpha_1, x\rangle^{2p_1} \cdots \langle \alpha_n, x\rangle^{2p_n} m(dx) \\
&= \sum_{2p_1 + \cdots + 2p_n = 2l} \frac{1}{(2p_1)! \cdots (2p_n)!} \int_{C_0[0,T]} F(x) m(dx),
\end{aligned}
$$

where F is given by Equation (4) above. This means that we can give the formula for analytic FFT as the series approximation by using Equation (9) in Theorem 2.

Step 4**:** We can conclude that

$$
T_q^{(1)}(H_r) \to T_q^{(1)}(H) \tag{14}
$$

in the sense $L_1(C_0[0,T])$ as $r \to \infty$. In fact, for each $\lambda > 0$, we have

$$
\begin{aligned}
&\int_{C_0[0,T]} |T_\lambda(H)(y) - T_\lambda(H_r)(y)| m(dy) \\
&\leq \int_{C_0[0,T]} \int_{C_0[0,T]} |H(\lambda^{-\frac{1}{2}} x + y) - H_r(\lambda^{-\frac{1}{2}} x + y)| m(dx) m(dy) \\
&= \int_{C_0[0,T]} |H(\sqrt{\lambda^{-1} + 1} z) - H_r(\sqrt{\lambda^{-1} + 1} z)| m(dz) \\
&\to 0
\end{aligned}
$$

as $r \to \infty$. The equality is obtained from the condition (11) and the Fubini theorem for the Wiener integrals. Also, by using the uniqueness of the analytic extension and the limit, we obtain Equation (14) as desired. Hence, the series approximation of the analytic FFT of functional H is given by the formula

$$T_q^{(1)}(H)(y) = \lim_{r\to\infty} T_q^{(1)}(H_r)(y)$$

in the sense $L_1(C_0[0,T])$, where

$$\begin{aligned} &T_q^{(1)}(H_r)(y) \\ &= h(\vec{0}) + \sum_{l=1}^{r}\bigg(\sum_{2p_1+\cdots+2p_n=2l} \frac{1}{(2p_1)!\cdots(2p_n)!} \\ &\qquad\qquad \times \prod_{j=1}^{l}\bigg[\sum_{s=0}^{p_j} {}_{2p_j}C_{2s}\Big(\frac{i}{q}\Big)^{s}\langle\alpha_j, y\rangle^{2p_j-2s}(2s-1)!!\bigg]\bigg) \end{aligned}$$

for s-a.e. $y \in C_0[0,T]$.

6. Conclusions

We finish this paper by giving Section 6 with a remark.

Remark 4. *In order to establish the series approximation with respect to the analytic FFT, we gave the condition as Equation* (11) *above. There are many functions that satisfy condition* (11)*. For example, all of the polynomial functions on* $\mathbb{R}^n$*, the exponential functions* $\exp\{a\sum_{j=1}^{n} u_j\}$ *with* $a > 0$*, and the trigonometric function* $\sin(\sum_{j=1}^{n} u_j), \cos(\sum_{j=1}^{n} u_j^2)$*. One can see that those functions satisfy condition* (11)*. Hence, we can establish many formulas for the analytic FFT as the series approximation. Furthermore, these functionals can be used in various fields such as the finance, engineering, or date anaysis. One can easily find the FFT of the conversion for all functions that are highly applicable.*

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflicts of interest.

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