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Szász–Durrmeyer Operators Involving Confluent Appell Polynomials

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Abstract: This article is concerned with the Durrmeyer-type generalization of Szász operators, including confluent Appell polynomials and their approximation properties. Also, the rate of convergence of the confluent Durrmeyer operators is found by using the modulus of continuity and Peetre's $\mathcal{K}$-functional. Then, we show that, under special choices of $A(t)$, the newly constructed operators reduce confluent Hermite polynomials and confluent Bernoulli polynomials, respectively. Finally, we present a comparison of newly constructed operators with the Durrmeyer-type Szász operators graphically.

Keywords: confluent Appell polynomials; confluent Bernoulli polynomials; confluent Hermite polynomials; Szász–Durrmeyer operators

MSC: 41A20; 41A25; 47A58

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1. Introduction

As a polynomial set, an Appell set [1] satisfies the following criteria: the determining function that enables us to have

$$A(t)e^{xt} = \sum_{k=0}^{\infty} P_k(x)\frac{t^k}{k!} \tag{1}$$

is an official power series as follows:

$$A(t) = \sum_{k=0}^{\infty} \theta_k \frac{t^k}{k!}, \quad (A(0) \neq 0).$$

For some $r > 0$, it is presumed that the series in (1) convergent in $|t| < r$. Another way to describe the Appell polynomials is the $P_k^{'}(x) = kP_{k-1}(x)$ recurrence formulas, where $k = 1, 2, \ldots$.

Theorem 1. *Consider the polynomial sequence* $\left\{P_k^{(a,b)}(x)\right\}_{k\geq 0}$, *with* $b \notin \{\ldots, -1, 0\}$. *Consequently, the ensuing statements are interchangeable.*

(i) $\left\{P_k^{(a,b)}(x)\right\}_{k\geq 0}$ *is a confluent sequence of Appell polynomials.*

(ii) $\left\{P_k^{(a,b)}(x)\right\}_{k\geq 0}$*'s generating function is granted by*

$$\sum_{k=0}^{\infty} P_k^{(a,b)}(x)\frac{t^k}{k!} = A(t)\,{}_1F_1(a;b;xt), \tag{2}$$

where $\{\theta_k\}$ is unrelated to n with $\theta_0 \neq 0$, an analytic function, $A(t)$ has an extension of power series

$$A(t) = \sum_{k=0}^{\infty} \theta_k \frac{t^k}{k!}, \tag{3}$$

and is an analytical function, and

$${}_1F_1(a;b;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!} \tag{4}$$

is a confluent hypergeometric function. For all finite z, this function converges, assuming $b \notin \{\ldots,-1,0\}$ [2]. Then,

$$\begin{cases} (a)_k = a(a+1)\ldots(a+k-1); & k \geq 1 \\ (a)_0 = 1; & 1 \end{cases}$$

gives the definition of the Pocchammer symbol [2].

Jakimovski and Leviatan [3] construct the operators as follows

$$J_\eta(f;x) = \frac{e^{-\eta x}}{A(1)} \sum_{k=0}^{\infty} P_k(\eta x) f\left(\frac{k}{\eta}\right).$$

Mazhar and Totik [4] define Durrmeyer-type Szász operators as follows:

$$Z_\eta(f;x) = \eta \sum_{k=0}^{\infty} e^{-\eta x} \frac{(\eta x)^k}{k!} \int_0^\infty e^{-\eta t} \frac{(\eta t)^k}{k!} f\left(\frac{k}{\eta}\right).$$

Recently, Özarslan and Çekim [5] define confluent Jakimovski–Leviatan operators as

$$L_\eta(f;x) = \frac{1}{A(1)_1F_1(a;b;\eta x)} \sum_{k=0}^{\infty} \frac{P_k^{(a,b)}(\eta x)}{k!} f\left(\frac{k}{\eta}\right), \quad b > a > 0$$

$\eta \in \mathbb{N}$ and $x \geq 0$.

Furthermore, it is expected that these operators fulfill the following requirements:

i. $0 \leq \frac{\theta_k}{A(1)}$, $k = 0, 1, \ldots$, and $A(1) \neq 0$.
ii. The series at (2) and (3) converge for $|t| < r (r > 1)$.

Recently, they have remarkable studies in operator theory [6–9], analytic function theory [10], and other fields [11,12].

Now, we define the Durrmeyer-type generalization of Szász operators involving confluent Appell polynomials

$$S_\eta(f;x) = \frac{\eta}{A(1)_1F_1(a;b;\eta x)} \sum_{k=0}^{\infty} \frac{P_k^{(a,b)}(\eta x)}{k!} \int_0^\infty e^{-\eta t} \frac{(\eta t)^k}{k!} f(t)dt, \quad b > a > 0$$

$P_k^{(a,b)}$ is given in (2), and $x \geq 0$, $\eta \in \mathbb{N}$.

2. Approximation Properties

In this section, we give moments and central moments for our operator including confluent Appell polynomials.

Lemma 1. *For any $x \in [0,\infty)$, we obtain*

$$\begin{aligned}
S_\eta(1;x) &= 1, \\
S_\eta(t;x) &= \frac{(a)_1}{(b)_1}\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)}x + \frac{A'(1)}{\eta A(1)} + \frac{1}{\eta}, \\
S_\eta\left(t^2;x\right) &= \frac{(a)_2}{(b)_2}\frac{{}_1F_1(a+2;b+2;\eta x)}{{}_1F_1(a;b;\eta x)}x^2 + \frac{(a)_1}{(b)_1}\frac{1}{\eta}\left(\frac{2A'(1)}{A(1)}+3\right)\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)}x \\
&\quad + \frac{A''(1)+3A'(1)}{\eta^2 A(1)} + \frac{2}{\eta^2}.
\end{aligned}$$

Proof. One way to illustrate the proof is to use it as given in (2)

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{P_k^{(a,b)}(\eta x)}{k!} &= A(1){}_1F_1(a;b;\eta x), \\
\sum_{k=0}^{\infty}\frac{P_{k+1}^{(a,b)}(\eta x)}{k!} &= A'(1){}_1F_1(a;b;\eta x) + \frac{(a)_1}{(b)_1}\eta x A(1){}_1F_1(a+1;b+1;\eta x), \\
\sum_{k=0}^{\infty}\frac{P_{k+2}^{(a,b)}(\eta x)}{k!} &= A''(1){}_1F_1(a;b;\eta x) + A'(1){}_1F_1(a;b;\eta x) + 2\frac{(a)_1}{(b)_1}\eta x A'(1){}_1F_1(a+1;b+1;\eta x) \\
&\quad + \frac{(a)_2}{(b)_2}(\eta x)^2 A(1){}_1F_1(a+2;b+2;\eta x).
\end{aligned}$$

By using these equalities in the operator, we obtain the desired results. □

Theorem 2. *For $f \in C[0,\infty)$,*

$$\lim_{n\to\infty} S_\eta(f;x) = f(x)$$

uniformly converges in every compact subset of $[0,\infty)$.

Proof. From (4) and Lemma 1, we obtain

$$\lim_{\eta\to\infty} S_\eta(e_i;x) = e_i(x), \ \ i = 0,1,2.$$

So, from the well-known Korovkin theorem [13] the proof is completed.
□

Lemma 2. *The first and second central moments for S_η are given as follows:*

$$\begin{aligned}
S_\eta(t-x;x) &= \left(\frac{(a)_1}{(b)_1}\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)} - 1\right)x + \frac{A'(1)}{\eta A(1)} + \frac{1}{\eta}, \\
S_\eta\left((t-x)^2;x\right) &= \left(\frac{(a)_2}{(b)_2}\frac{{}_1F_1(a+2;b+2;\eta x)}{{}_1F_1(a;b;\eta x)} - 2\frac{(a)_1}{(b)_1}\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)} + 1\right)x^2 \\
&\quad + \left(\frac{(a)_1}{(b)_1}\frac{1}{\eta}\left(\frac{2A'(1)}{A(1)}+3\right)\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)} - \frac{2}{\eta}\left(\frac{A'(1)}{A(1)}+1\right)\right)x \\
&\quad + \frac{A''(1)+3A'(1)}{\eta^2 A(1)} + \frac{2}{\eta^2}.
\end{aligned}$$

Proof. From the linearity of the operator and Lemma 1,

$$\begin{aligned}
S_\eta(t-x;x) &= S_\eta(e_1;x) - xS_\eta(e_0;x), \\
S_\eta\left((t-x)^2;x\right) &= S_\eta(e_2;x) - 2xS_\eta(e_1;x) + x^2 S_\eta(e_0;x).
\end{aligned}$$

By using these equalities, the proof is completed. □

3. Rate of Convergence

In this section, we give the rate of convergence by the modulus of continuity, Peetre's-$\mathcal{K}$ functional, and the second modulus of continuity, respectively. The modulus of continuity is given by

$$\omega(f,\delta) := \sup_{|t-x|\leq\delta} \sup_{x\in[0,\infty)} |f(t)-f(x)|, \ \ \delta>0,$$

where $f \in C[0,\infty)$. It is due to the following feature of the modulus of continuity

$$|f(t)-f(x)| \leq \left(1+\frac{|t-x|}{\delta}\right)\omega(f,\delta).$$

Theorem 3. *For every $x \in [0,\infty)$ and $f \in C[0,\infty)$,*

$$\left|S_\eta(f;x)-f(x)\right| \leq 2\omega\left(f,\delta_\eta\right),$$

where

$$\begin{aligned}\delta_\eta(x) \ &= \ \Bigg\{\left(\frac{(a)_2}{(b)_2}\frac{{}_1F_1(a+2;b+2;\eta x)}{{}_1F_1(a;b;\eta x)} - 2\frac{(a)_1}{(b)_1}\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)}+1\right)x^2 \\ &+\left(\frac{(a)_1}{(b)_1}\frac{1}{\eta}\left(\frac{2A'(1)}{A(1)}+3\right)\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)} - \frac{2}{\eta}\left(\frac{A'(1)}{A(1)}+1\right)\right)x \\ &+\frac{A''(1)+3A'(1)}{\eta^2 A(1)}+\frac{2}{\eta^2}\Bigg\}^{\frac{1}{2}}. \end{aligned} \tag{5}$$

Proof. Using the operators S_η's linearity and from Lemma 2, we obtain

$$\begin{aligned}\left|S_\eta(f;x)-f(x)\right| \ &\leq \ \frac{\eta}{A(1)_1F_1(a;b;\eta x)}\sum_{k=0}^{\infty}\frac{P_k^{(a,b)}(\eta x)}{k!}\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}|f(t)-f(x)|dt \\ &\leq \frac{\eta}{A(1)_1F_1(a;b;\eta x)}\sum_{k=0}^{\infty}\frac{P_k^{(a,b)}(\eta x)}{k!}\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}\left(1+\frac{|t-x|}{\delta}\right)\omega(f,\delta)dt \\ &\leq \left\{1+\frac{\eta}{A(1)_1F_1(a;b;\eta x)}\frac{1}{\delta}\sum_{k=0}^{\infty}\frac{P_k^{(a,b)}(\eta x)}{k!}\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}|t-x|dt\right\}\omega(f,\delta). \end{aligned} \tag{6}$$

For integral by using the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}\left|S_\eta(f;x)-f(x)\right| \ &\leq \ \Bigg\{\left(\frac{\eta}{A(1)_1F_1(a;b;\eta x)}+1\right)\frac{1}{\delta}\sum_{k=0}^{\infty}\frac{P_k^{(a,b)}(\eta x)}{k!}\left(\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}dt\right)^{\frac{1}{2}} \\ &\left(\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}(t-x)^2dt\right)^{\frac{1}{2}}\Bigg\}\omega(f,\delta). \end{aligned}$$

Examining Cauchy–Schwarz disparity in summation, one can easily obtain

$$\begin{aligned}
|S_\eta(f;x)-f(x)| &\leq \left\{1+\frac{1}{\delta}\left(\frac{\eta}{A(1)_1F_1(a;b;\eta x)}\sum_{k=0}^{\infty}\frac{P_k^{(a,b)}(\eta x)}{k!}\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}dt\right)^{\frac{1}{2}}\right.\\
&\quad\left.\times\left(\frac{\eta}{A(1)_1F_1(a;b;\eta x)}\sum_{k=0}^{\infty}\frac{P_k^{(a,b)}(\eta x)}{k!}\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}(t-x)^2dt\right)^{\frac{1}{2}}\right\}\omega(f,\delta),\\
&= \left\{1+\frac{1}{\delta}(S_\eta(1;x))^{\frac{1}{2}}\left(S_\eta\left((t-x)^2;x\right)\right)^{\frac{1}{2}}\right\}\omega(f,\delta),\\
&= \left\{1+\frac{1}{\delta}(\delta_\eta(x))^{\frac{1}{2}}\right\}\omega(f,\delta)
\end{aligned}$$

where $\delta_\eta(x)$ is given by (5).

$$|S_\eta(f;x)-f(x)| \leq \left\{1+\frac{1}{\delta}\sqrt{\delta_\eta(x)}\right\}\omega(f,\delta)$$

can be obtained by considering this inequality in (6). If we choose $\delta = \sqrt{\delta_\eta(x)}$, we can obtain the desired result. □

Lemma 3. *For $x \in [0,\infty)$ and $f \in C_B[0,\infty)$, we have*

$$|S_\eta(f;x)| \leq \|f\|.$$

Proof. For S_η, we obtain

$$\begin{aligned}
|S_\eta(f;x)| &= \left|\frac{1}{A(1)_1F_1(a;b;\eta x)}\sum_{k=0}^{\infty}\frac{P_k^{(a,b)}(\eta x)}{k!}\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}f(t)dt\right|\\
&\leq \frac{1}{A(1)_1F_1(a;b;\eta x)}\sum_{k=0}^{\infty}\frac{P_k^{(a,b)}(\eta x)}{k!}\left|\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}f(t)dt\right|\\
&\leq \frac{1}{A(1)_1F_1(a;b;\eta x)}\sum_{k=0}^{\infty}\frac{P_k^{(a,b)}(\eta x)}{k!}\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}|f(t)|dt\\
&\leq \|f\|S_\eta(1;x)\\
&\leq \|f\|.
\end{aligned}$$

□

$C_B^2[0,\infty)$ is the space of the functions f, for which f, f', and f'' are continuous on $[0,\infty)$. The norm on the space $C_B^2[0,\infty)$ is given by [14]

$$\|g\|_{C_B^2[0,\infty)} := \|g\|_{C_B[0,\infty)} + \|g'\|_{C_B[0,\infty)} + \|g''\|_{C_B[0,\infty)}.$$

Now, we define classical Peetre's-$\mathcal{K}$ functional as follows:

$$\mathcal{K}(f,\lambda) := \inf_{g\in C_B^2[0,\infty)}\{\|f-g\|+\lambda\|g''\|\}$$

where $\lambda > 0$.

Theorem 4. *Let $x \in [0,\infty)$ and $f \in C_B[0,\infty)$. Then, we have for all $\eta \in \mathbb{N}$,*

$$|S_\eta(f;x)-f(x)| \leq 2\mathcal{K}(f;\lambda_\eta(x)),$$

where

$$\lambda_\eta(x) = \left(\frac{A''(1) + A'(1)(3+2\eta)}{2\eta^2 A(1)} + \frac{3\eta+2}{2\eta^2} \right).$$

Proof. For a given function $g \in C_B^2[0,\infty)$, we have the following Taylor expansion

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-s)g''(s)ds. \tag{7}$$

Applying S_η operator to Equation (7), we obtain

$$\begin{aligned} \left|S_\eta(g;x) - g(x)\right| &= \left|S_\eta((t-x)g'(x);x)\right| + \left|S_\eta\left(\int_x^t (t-s)g''(s)ds;x\right)\right| \\ &\le ||g'||_{C_B[0,\infty)}\left|S_\eta(t-x;x)\right| + ||g''||_{C_B[0,\infty)}\left|S_\eta\left(\int_x^t (t-s)ds;x\right)\right| \\ &\le ||g'||_{C_B[0,\infty)}\left|S_\eta(t-x;x)\right| + \frac{1}{2}||g''||_{C_B[0,\infty)} S_\eta\left((t-x)^2;x\right). \end{aligned}$$

So,

$$\left|S_\eta(g;x) - g(x)\right| \le \lambda_\eta ||g||_{C_B^2[0,\infty)}.$$

Using the above inequality and Lemma 3, we obtain

$$\begin{aligned} \left|S_\eta(f;x) - f(x)\right| &= \left|S_\eta(f;x) - f(x) + S_\eta(g;x) - S_\eta(g;x) + g(x) - g(x)\right| \\ &\le ||f-g||_{C_B[0,\infty)}\left|S_\eta(1;x)\right| + ||f-g||_{C_B[0,\infty)} + \left|S_\eta(g;x) - g(x)\right| \\ &\le 2\left(||f-g||_{C_B[0,\infty)} + \lambda_\eta ||g||_{C_B^2[0,\infty)}\right) \\ &\le 2\mathcal{K}(f;\lambda_\eta). \end{aligned}$$

As a result,

$$|S_\eta(f;x) - f(x)| \le 2\mathcal{K}(f;\lambda_\eta). \tag{8}$$

Thus, the proof is completed. □

For $f \in C_B[0,\infty)$, the second modulus of continuity is explained by

$$\omega_2(f,\delta) = \sup_{0<t\le\delta} \|f(.+2t) - 2f(.+t) + f(.)\|_{C_B[0,\infty)}.$$

The relationship between Peetre's-$\mathcal{K}$ functional and the second modulus of continuity is given as follows:

$$\mathcal{K}(f;\delta) \le A\left(\omega_2\left(f,\sqrt{\delta}\right) + \min(1,\delta)\|f\|_{C_B[0,\infty)}\right), \tag{9}$$

where the constant A is unaffected by the values of f and δ from [15]. From (8) and (9),

$$|S_\eta(f;x) - f(x)| \le 2A\left(\omega_2\left(f,\sqrt{\lambda_\eta}\right) + \min(1,\lambda_\eta)\|f\|_{C_B[0,\infty)}\right).$$

4. Special Cases

In this section, we define Durrmeyer–Szász operators including confluent Bernoulli polynomials $S_\eta^{\mathcal{B}}$ and Durrmeyer–Szász operators including confluent Hermite polynomials $S_\eta^{\mathcal{H}}$ by selecting $A(t) = \frac{t}{e^t-1}$ and $A(t) = e^{-\frac{t^2}{2}}$ in (2), respectively.

4.1. Approximation Properties for $S_\eta^{\mathcal{B}}$

Choosing $A(t) = \frac{t}{e^t-1}$ in (2) we obtain $\mathcal{B}_k^{(a,b)}$. The confluent Bernoulli polynomials have

$$\frac{t}{e^t-1}\,{}_1F_1(a;b;xt) = \sum_{k=0}^{\infty} \mathcal{B}_k^{(a,b)}(x)\frac{t^k}{k!}, \quad |t| < \pi$$

as their generating function, where $b \notin \{\ldots,-1,0\}$.

The Szász–Durrmeyer operators including confluent Bernoulli polynomials are presented as

$$S_\eta^{\mathcal{B}}(f;x) = \frac{\eta(e-1)}{{}_1F_1(a;b;\eta x)} \sum_{k=0}^{\infty} \frac{\mathcal{B}_k^{(a,b)}(\eta x)}{k!} \int_0^{\infty} f(t)\frac{(\eta t)^k}{k!}e^{-\eta t}dt.$$

Now, we give moments, central moments, and modulus of continuity for our operator including confluent Bernoulli polynomials.

Lemma 4. *For $x \in [0,\infty)$, we have the moments for $S_\eta^{\mathcal{B}}$ as follows:*

$$\begin{aligned}
S_\eta^{\mathcal{B}}(1;x) &= 1,\\
S_\eta^{\mathcal{B}}(t;x) &= \frac{(a)_1}{(b)_1}\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)}x + \frac{e-2}{\eta(e-1)},\\
S_\eta^{\mathcal{B}}\left(t^2;x\right) &= \frac{(a)_2}{(b)_2}\frac{{}_1F_1(a+2;b+2;\eta x)}{{}_1F_1(a;b;\eta x)}x^2 + \frac{(a)_1}{(b)_1}\left(\frac{3e-5}{\eta(e-1)}\right)\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)}x\\
&\quad + \frac{e^2-4e+5}{\eta^2(e-1)^2}.
\end{aligned}$$

Lemma 5. *For every $x \in [0,\infty)$ and by Lemma 4, the following identities verify*

$$\begin{aligned}
S_\eta^{\mathcal{B}}(t-x;x) &= \left(\frac{(a)_1}{(b)_1}\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)} - 1\right)x + \frac{e-2}{\eta(e-1)},\\
S_\eta^{\mathcal{B}}\left((t-x)^2;x\right) &= \left(\frac{(a)_2}{(b)_2}\frac{{}_1F_1(a+2;b+2;\eta x)}{{}_1F_1(a;b;\eta x)} - 2\frac{(a)_1}{(b)_1}\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)} + 1\right)x^2\\
&\quad + \left(\frac{(a)_1}{(b)_1}\left(\frac{3e-5}{\eta(e-1)}\right)\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)} + \frac{4-2e}{\eta(e-1)}\right)x + \frac{e^2-4e+5}{\eta^2(e-1)^2}.
\end{aligned}$$

Theorem 5. *For every $x \in [0,\infty)$ and $f \in C[0,\infty)$,*

$$\left|S_\eta^{\mathcal{B}}(f;x) - f(x)\right| \le 2\omega(f,\lambda_\eta).$$

Here,

$$\begin{aligned}
\lambda_\eta(x) &= \Bigg\{\left(\frac{(a)_2}{(b)_2}\frac{{}_1F_1(a+2;b+2;\eta x)}{{}_1F_1(a;b;\eta x)} - 2\frac{(a)_1}{(b)_1}\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)} + 1\right)x^2\\
&\quad + \left(\frac{(a)_1}{(b)_1}\left(\frac{3e-5}{\eta(e-1)}\right)\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)} + \frac{4-2e}{\eta(e-1)}\right)x + \frac{e^2-4e+5}{\eta^2(e-1)^2}\Bigg\}^{\frac{1}{2}}.
\end{aligned} \tag{10}$$

Proof. Using linearity of the operators $S_\eta^{\mathcal{B}}$, we obtain

$$\begin{aligned}\left|S_\eta^{\mathcal{B}}(f;x)-f(x)\right| &\leq \frac{\eta(e-1)}{{}_1F_1(a;b;\eta x)}\sum_{k=0}^{\infty}\frac{\mathcal{B}_k^{(a,b)}(\eta x)}{k!}\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}|f(t)-f(x)|dt \\ &\leq \frac{\eta(e-1)}{{}_1F_1(a;b;\eta x)}\sum_{k=0}^{\infty}\frac{\mathcal{B}_k^{(a,b)}(\eta x)}{k!}\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}\left(1+\frac{|t-x|}{\delta}\right)\omega(f,\delta)dt \\ &\leq \left\{1+\frac{\eta(e-1)}{{}_1F_1(a;b;\eta x)}\frac{1}{\delta}\sum_{k=0}^{\infty}\frac{\mathcal{B}_k^{(a,b)}(\eta x)}{k!}\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}|t-x|dt\right\}\omega(f,\delta). \qquad (11)\end{aligned}$$

By applying the Cauchy–Schwarz inequality to the last integral, we obtain

$$\begin{aligned}\left|S_\eta^{\mathcal{B}}(f;x)-f(x)\right| \leq & \left\{\left(\frac{\eta(e-1)}{{}_1F_1(a;b;\eta x)}+1\right)\frac{1}{\delta}\sum_{k=0}^{\infty}\frac{\mathcal{B}_k^{(a,b)}(\eta x)}{k!}\left(\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}dt\right)^{\frac{1}{2}}\right. \\ & \left.\left(\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}(t-x)^2dt\right)^{\frac{1}{2}}\right\}\omega(f,\delta).\end{aligned}$$

Considering Cauchy–Schwarz inequality for summation and from Lemma 5, one can easily obtain

$$\begin{aligned}\left|S_\eta^{\mathcal{B}}(f;x)-f(x)\right| \leq & \left\{1+\frac{1}{\delta}\left(\frac{\eta(e-1)}{{}_1F_1(a;b;\eta x)}\sum_{k=0}^{\infty}\frac{\mathcal{B}_k^{(a,b)}(\eta x)}{k!}\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}dt\right)^{\frac{1}{2}}\right. \\ & \left.\times\left(\frac{\eta(e-1)}{{}_1F_1(a;b;\eta x)}\sum_{k=0}^{\infty}\frac{\mathcal{B}_k^{(a,b)}(\eta x)}{k!}\int_0^\infty (t-x)^2e^{-\eta t}\frac{(\eta t)^k}{k!}dt\right)^{\frac{1}{2}}\right\}\omega(f,\delta), \\ = & \left\{1+\frac{1}{\delta}\left(S_\eta^{\mathcal{B}}(1;x)\right)^{\frac{1}{2}}\left(S_\eta^{\mathcal{B}}\left((t-x)^2;x\right)\right)^{\frac{1}{2}}\right\}\omega(f,\delta), \\ = & \left\{1+\frac{1}{\delta}(\lambda_\eta(x))^{\frac{1}{2}}\right\}\omega(f,\delta)\end{aligned}$$

where $\lambda_\eta(x)$ is given by (10).

$$\left|S_\eta^{\mathcal{B}}(f;x)-f(x)\right| \leq \left\{1+\frac{1}{\delta}\sqrt{\lambda_\eta(x)}\right\}\omega(f,\delta)$$

can be obtained by considering this inequality in (11). If we choose $\delta=\sqrt{\lambda_\eta(x)}$, we achieve the desired result. □

4.2. Approximation Properties for $S_\eta^{\mathcal{H}}$

Choosing $A(t)=e^{-\frac{t^2}{2}}$ in (2), then we obtain $\mathcal{H}_k^{(a,b)}$. The confluent Hermite polynomials have

$$e^{-\frac{t^2}{2}}{}_1F_1(a;b;xt)=\sum_{k=0}^{\infty}\mathcal{H}_k^{(a,b)}(x)\frac{t^k}{k!},$$

as their generating function, where $b\notin\{\ldots,-1,0\}$.

The Szász–Durrmeyer operators including confluent Hermite polynomials are shown as

$$S_\eta^{\mathcal{H}}(f;x)=\frac{\eta e^{\frac{1}{2}}}{{}_1F_1(a;b;\eta x)}\sum_{k=0}^{\infty}\frac{\mathcal{H}_k^{(a,b)}(\eta x)}{k!}\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}f(t)dt.$$

Now, we give moments, central moments, and modulus of continuity for our operator including confluent Hermite polynomials.

Lemma 6. *For $x \in [0,\infty)$, we obtain the moments for $S_\eta^{\mathcal{H}}$ as follows:*

$$\begin{aligned}
S_\eta^{\mathcal{H}}(1;x) &= 1,\\
S_\eta^{\mathcal{H}}(t;x) &= \frac{(a)_1}{(b)_1}\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)}x,\\
S_\eta^{\mathcal{H}}\left(t^2;x\right) &= \frac{(a)_2}{(b)_2}\frac{{}_1F_1(a+2;b+2;\eta x)}{{}_1F_1(a;b;\eta x)}x^2+\frac{(a)_1}{(b)_1}\frac{1}{\eta}\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)}x-\frac{1}{\eta^2}.
\end{aligned}$$

Lemma 7. *For every $x \in [0,\infty)$ and by Lemma 6, the following identities verify*

$$\begin{aligned}
S_\eta^{\mathcal{H}}(t-x;x) &= \left(\frac{(a)_1}{(b)_1}\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)}-1\right)x,\\
S_\eta^{\mathcal{H}}\left((t-x)^2;x\right) &= \left(\frac{(a)_2}{(b)_2}\frac{{}_1F_1(a+2;b+2;\eta x)}{{}_1F_1(a;b;\eta x)}-2\frac{(a)_1}{(b)_1}\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)}+1\right)x^2\\
&\quad+\left(\frac{(a)_1}{(b)_1}\frac{1}{\eta}\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)}\right)x-\frac{1}{\eta^2}.
\end{aligned}$$

Theorem 6. *For every $x \in [0,\infty)$ and $f \in C[0,\infty)$,*

$$\left|S_\eta^{\mathcal{H}}(f;x)-f(x)\right| \le 2\omega(f,\gamma_\eta),$$

where

$$\begin{aligned}
\gamma_\eta(x) &= \left(\frac{(a)_2}{(b)_2}\frac{{}_1F_1(a+2;b+2;\eta x)}{{}_1F_1(a;b;\eta x)}-2\frac{(a)_1}{(b)_1}\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)}+1\right)x^2\\
&\quad+\left(\frac{(a)_1}{(b)_1}\frac{1}{\eta}\frac{{}_1F_1(a+1;b+1;\eta x)}{{}_1F_1(a;b;\eta x)}\right)x-\frac{1}{\eta^2}.
\end{aligned}\tag{12}$$

Proof. From the linearity of the operators $S_\eta^{\mathcal{H}}$, we obtain

$$\begin{aligned}
\left|S_\eta^{\mathcal{H}}(f;x)-f(x)\right| &\le \frac{\eta e^{\frac{1}{2}}}{{}_1F_1(a;b;\eta x)}\sum_{k=0}^{\infty}\frac{\mathcal{H}_k^{(a,b)}(\eta x)}{k!}\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}|f(t)-f(x)|dt\\
&\le \frac{\eta e^{\frac{1}{2}}}{{}_1F_1(a;b;\eta x)}\sum_{k=0}^{\infty}\frac{\mathcal{H}_k^{(a,b)}(\eta x)}{k!}\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}\left(1+\frac{|t-x|}{\delta}\right)\omega(f,\delta)dt\\
&\le \left\{1+\frac{\eta e^{\frac{1}{2}}}{{}_1F_1(a;b;\eta x)}\frac{1}{\delta}\sum_{k=0}^{\infty}\frac{\mathcal{H}_k^{(a,b)}(\eta x)}{k!}\int_0^\infty |t-x|e^{-\eta t}\frac{(\eta t)^k}{k!}dt\right\}\omega(f,\delta).
\end{aligned}\tag{13}$$

For integration, we apply the Cauchy–Schwarz inequality and obtain

$$\begin{aligned}
\left|S_\eta^{\mathcal{H}}(f;x)-f(x)\right| \le{}& \left\{1+\frac{\eta e^{\frac{1}{2}}}{{}_1F_1(a;b;\eta x)}\frac{1}{\delta}\sum_{k=0}^{\infty}\frac{\mathcal{H}_k^{(a,b)}(\eta x)}{k!}\left(\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}dt\right)^{\frac{1}{2}}\right.\\
&\left.\left(\int_0^\infty e^{-\eta t}\frac{(\eta t)^k}{k!}(t-x)^2dt\right)^{\frac{1}{2}}\right\}\omega(f,\delta).
\end{aligned}$$

Examining Cauchy–Schwarz disparity in summation and from Lemma 7, one can easily obtain

$$
\begin{aligned}
\left|S_\eta^{\mathcal{H}}(f;x)-f(x)\right| \leq & \left\{1+\frac{1}{\delta}\left(\frac{\eta e^{\frac{1}{2}}}{{}_1F_1(a;b;\eta x)}\sum_{k=0}^{\infty}\frac{\mathcal{H}_k^{(a,b)}(\eta x)}{k!}\int_0^{\infty}e^{-\eta t}\frac{(\eta t)^k}{k!}dt\right)^{\frac{1}{2}}\right. \\
& \left.\times\left(\frac{\eta e^{\frac{1}{2}}}{{}_1F_1(a;b;\eta x)}\sum_{k=0}^{\infty}\frac{\mathcal{H}_k^{(a,b)}(\eta x)}{k!}\int_0^{\infty}e^{-\eta t}\frac{(\eta t)^k}{k!}(t-x)^2dt\right)^{\frac{1}{2}}\right\}\omega(f,\delta), \\
= & \left\{1+\frac{1}{\delta}\left(S_\eta^{\mathcal{H}}(1;x)\right)^{\frac{1}{2}}\left(S_\eta^{\mathcal{H}}\left((t-x)^2;x\right)\right)^{\frac{1}{2}}\right\}\omega(f,\delta), \\
= & \left\{1+\frac{1}{\delta}(\gamma_\eta(x))^{\frac{1}{2}}\right\}\omega(f,\delta)
\end{aligned}
$$

where $\gamma_\eta(x)$ is given by (12).

$$
\left|S_\eta^{\mathcal{H}}(f;x)-f(x)\right| \leq \left\{1+\frac{1}{\delta}\sqrt{\gamma_\eta(x)}\right\}\omega(f,\delta)
$$

can be obtained by considering this inequality in (13). If we choose $\delta=\sqrt{\gamma_\eta(x)}$, we obtain the desired result. □

5. Graphical Analysis

In this section, we will examine the approximations of both Durrmeyer-type Szász operators and the newly defined confluent Szász–Durrmeyer operators to a function f.

Let the function f be

$$
f(x)=(0.5)exp\left(\frac{x}{2}\right).
$$

Then, we plot the convergence of the newly constructed S_η confluent Szász–Durrmeyer operators and Z_η Durrmeyer-type Szász operators [4] to the function f in Figure 1 for $A(t) = 1$. In Figure 1, we give three different illustrations for selected values $\{\eta=5, a=0.5, b=1.5\}$, $\{\eta=10, a=0.5, b=1.5\}$, and $\{\eta=12, a=1, b=2\}$, respectively.

By choosing $f(x) = \frac{\log(x+1)}{1000}$, we show the error estimation of confluent Szász–Durrmeyer operators S_η via the way of the modulus of continuity in Table 1.

Table 1. Error approximation for S_η by using the modulus of continuity.

η	$S_\eta(f;x)$
10	0.00204334
10^2	0.00138627
10^3	0.00089236
10^4	0.00054954
10^5	0.00032734
10^6	0.00019062
10^7	0.00010942

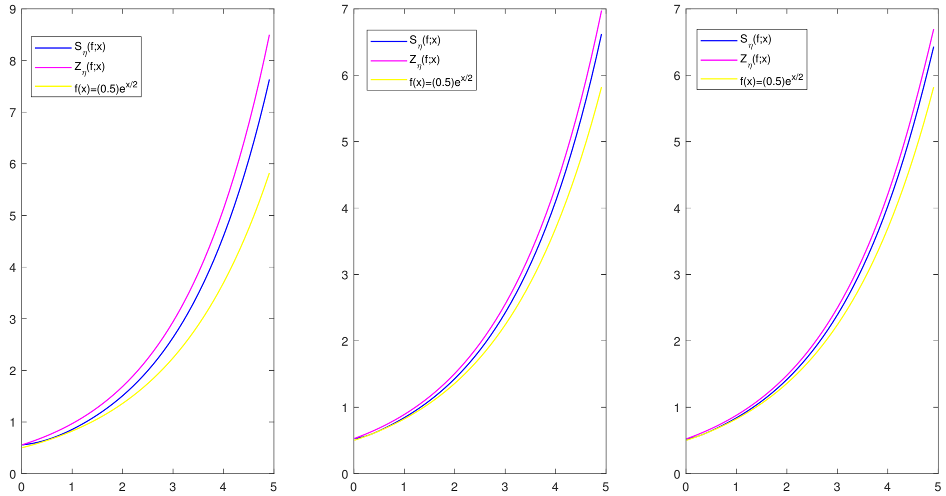

Figure 1. Illustration of approximation to the function $f(x) = (0.5)exp(\frac{x}{2})$ for selected values $\{\eta = 5, a = 0.5, b = 1.5\}$, $\{\eta = 10, a = 0.5, b = 1.5\}$, and $\{\eta = 12, a = 1, b = 2\}$, respectively.

6. Conclusions

In this study, Durrmeyer-type generalization of confluent Szász operators is constructed. The central moments of the newly constructed operators S_η are obtained. Furthermore, the rate of convergence is investigated by using the modulus of continuity and Peetre's K-functional. The relationship between the newly constructed operators with $\mathcal{B}_k^{(\mathsf{a},\mathsf{b})}$ and $\mathcal{H}_k^{(\mathsf{a},\mathsf{b})}$ are given, respectively. Finally, the convergence of the confluent Szász–Durrmeyer operators S_η and the classical Szász–Durrmeyer operators Z_η to the selected functions are illustrated. The comparison of convergence is given by numerical examples.

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