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On Equivalence Operators Derived from Overlap and Grouping Functions

Lei Du, Yingying Xu, Haifeng Song and Songsong Dai *

School of Electronics and Information Engineering, Taizhou University, Taizhou 318000, China; dulei2109@tzc.edu.cn (L.D.); yyxu@tzc.edu.cn (Y.X.); isshf@tzc.edu.cn (H.S.)
* Correspondence: ssdai@tzc.edu.cn

Abstract: This paper introduces the concept of equivalence operators based on overlap and grouping functions where the associativity property is not strongly required. Overlap functions and grouping functions are weaker than positive and continuous t-norms and t-conorms, respectively. Therefore, these equivalence operators do not necessarily satisfy certain properties, such as associativity and the neutrality principle. In this paper, two models of fuzzy equivalence operators are obtained by the composition of overlap functions, grouping functions and fuzzy negations. Their main properties are also studied.

Keywords: equivalence operators; overlap function; grouping function

MSC: 47S40; 03E72

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1. Introduction

In classical logic, the equivalence operator can be represented as

$$s \equiv t \stackrel{def}{=} (s \to t) \wedge (t \to s) = (s \wedge t) \vee (\neg s \wedge \neg t). \tag{1}$$

It is also interpreted as the bi-implication logical connective denoted by using a double-headed arrow $s \leftrightarrow t$.

Different versions of fuzzy equivalence operator have been investigated in the literature. Hájek [1] considered the equivalence connective as a derived connective which is defined by

$$s \equiv t \stackrel{def}{=} (s \to t)\&(t \to s) \tag{2}$$

in several fuzzy logics, where $\to$ is a fuzzy implication and & is a t-norm. Novák et al. [2,3] developed the EQ-logic and EQ-algebras in which fuzzy equivalence operator is a basic connective. Dombi et al. [4,5] studied several different types of equivalence operators in Pliant systems and nilpotent systems. Hu et al. [6] studied the asymmetric equivalences in fuzzy logics. Fuzzy equivalence operators are highly applied in fuzzy theories and fuzzy methods. Based on the Formula (2) wherein & represents the minimum t-norm, i.e., $\& = \wedge$, Pan et al. [7] explored the properties of robustness of the lattice-valued similarity, while Jin et al. [8] and Dai et al. [9] investigated the robustness of fuzzy reasoning, Georgescu [10] explored the similarity of fuzzy choice functions, Wang et al. [11] and Duan et al. [12] investigated fuzzy metric spaces.

Notably, Fodor and Keresztfalvi [13] and Bustince et al. [14,15] mentioned applications where the associativity properties of the t-norm and t-conorm are not rrquired. Consequently, Bustince et al. [14,15] introduced the overlap and grouping functions as two types of non-necessarily associative bivariate operators. By replacing t-norms and t-conorms with overlap and grouping functions respectively, researchers devoloped many new concepts, such as symmetric differences [16,17], R_O-implications [18,19], (G,N)-implications [20], binary relations $\leq_O$ [21], (IO, O)-fuzzy rough sets [22], and (O,G)-fuzzy rough sets [23] where G is a grouping function and O is an overlap function.

In this paper, we aim to define equivalence operators by considering (G,N)-implications, overlap functions and grouping functions in the previously mentioned formulas. This approach is particularly significant given that overlap and grouping functions are weaker than the positive and continuous t-norms and t-conorms. Therefore, the new equivalence operators may not necessarily satisfy certain properties, such as associativity and the neutrality principle.

The paper is organized as follows. Section 2 presents the the concept of overlap and grouping functions, fuzzy implication and (G, N)-implication. Section 3 focuses on the model $G(O_1(s,t), O_2(N_1(s), N_2(t)))$ of equivalence operator. Section 4 addresses the model $O(G_1(N_1(s),t), G_2(N_2(t),s))$ of the equivalence operator. Section 5 constructes fuzzy symmetric differences from fuzzy equivalences. Lastly, Section 6 provides a comparative study, and the conclusion is presented in Section 7.

2. Preliminaries

Definition 1 ([14])**.** *A binary function $O : [0,1]^2 \to [0,1]$ is said to be an overlap function if the following conditions hold: $\forall s,t \in [0,1]$,*

(O1) $O(s,t) = O(t,s)$;

(O2) $O(s,t) = 0 \iff st = 0$;

(O3) $O(s,t) = 1 \iff st = 1$;

(O4) *O is increasing;*

(O5) *O is continuous.*

Definition 2 ([15])**.** *A binary function $O : [0,1]^2 \to [0,1]$ is said to be a grouping function if the following conditions hold: $\forall s,t \in [0,1]$,*

(G1) $G(s,t) = G(t,s)$;

(G2) $G(s,t) = 0 \iff s = t = 0$;

(G3) $G(s,t) = 1 \iff s = 1$ or $t = 1$;

(G4) *G is increasing;*

(G5) *G is continuous.*

Definition 3 ([24])**.** *A binary function $I : [0,1]^2 \to [0,1]$ is said to be a fuzzy implication if the following conditions hold: $\forall s,t,u \in [0,1]$,*

(F1) $s \leq t \Longrightarrow I(t,u) \leq I(s,u)$;

(F2) $s \leq t \Longrightarrow I(u,s) \leq I(u,t)$;

(F3) $I(0,0) = 1$;

(F4) $I(1,1) = 1$;

(F5) $I(1,0) = 0$.

Definition 4 ([24,25])**.** *A function $N : [0,1] \to [0,1]$ is said to be a fuzzy negation if it is decreasing and satisfies $N(0) = 1$ and $N(1) = 0$.*

Moreover, we say N is strong if $N(N(s)) = s$, $\forall s \in [0,1]$. The standard negation is $N(s) = 1 - s$, $\forall s \in [0,1]$.

The overlap function O given by

$$O(s,t) = N\Big(G\big(N(s), N(t)\big)\Big), \ \forall s,t \in [0,1]. \tag{3}$$

is the dual overlap function of G for N and, analogously, the grouping function G given by

$$G(s,t) = N\Big(O\big(N(s), N(t)\big)\Big),\ \forall s,t \in [0,1]. \tag{4}$$

is the dual grouping function of O for N.

Definition 5 ([20])**.** *Let G be a grouping function and N be a fuzzy negation, we say that the function $I_{G,N} : [0,1]^2 \to [0,1]$ defined by*

$$I_{G,N}(s,t) = G\big(N(s), t\big) \tag{5}$$

is a (G,N)-implication.

Example 1. *Some examples of overlap and grouping functions are given in [15,26]*

- $O_{nm}(s,t) = \min(s,t)\max(s^2,t^2)$, $G_{nm}(a,b) = 1 - \min(1-a, 1-b)\max((1-a)^2,(1-b)^2)$;
- $O_p(s,t) = s^p t^p$, $G_p(s,t) = 1-(1-s)^p(1-t)^p$, *where* $p > 0$;
- $O_{mp}(s,t) = \min(s^p,t^p)$, $G_{mp}(s,t) = 1 - \min\big((1-s)^p,(1-t)^p\big)$, *where* $p > 0$;
- $O_{Mp}(s,t) = 1 - \max\big((1-s)^p,(1-t)^p\big)$, $G_{Mp}(s,t) = \max(s^p,t^p)$, *where* $p > 0$.

3. The Model $G\Big(O_1(s,t), O_2\big(N_1(s), N_2(t)\big)\Big)$

In this section, we present our first model of the equivalence operator. We then illustrate its properties and conclude with two examples.

The analogue of the formula $a \equiv b \overset{def}{=} (a \wedge b) \vee (\neg a \wedge \neg b)$ in fuzzy set theory is

$$s \equiv_1 t \overset{def}{=} G(O_1(s,t), O_2(N_1(s), N_2(t))) \tag{6}$$

where, G is a grouping function, O_1 and O_2 are overlap functions, and N_1 and N_2 are fuzzy negations.

Theorem 1. *The function* $\equiv_1$: $[0,1]^2 \to [0,1]$ *given by Equation (6) satisfies:* $\forall s,t \in [0,1]$

(i) *If* $N_1 = N_2$, *then* $s \equiv_1 t = t \equiv_1 s$;

(ii) $1 \equiv_1 0 = 0 \equiv_1 1 = 0$;

(iii) $1 \equiv_1 1 = 0 \equiv_1 0 = 1$;

(iv) *If both* N_1 *and* N_2 *are continuous, then* $\equiv_1$ *is continuous;*

(v) *If* O_1 *has 1 as neutral element,* G *has 0 as neutral element, then* $1 \equiv_1 s = s \equiv_1 1 = s$;

(vi) *If* O_2 *has 1 as neutral element,* G *has 0 as neutral element, then* $0 \equiv_1 s = N_2(s)$ *and* $s \equiv_1 0 = N_1(s)$;

(vii) *If both* N_1 *and* N_2 *are defined as*

$$N_\top(s) = \begin{cases} 0, & \text{if } s = 1, \\ 1, & \text{if } s < 1. \end{cases} \tag{7}$$

then $s \equiv_1 s = 1$;

(viii) *If* $O_1 = O_2$ *and* N_1 *and* N_2 *are strong, then* $\equiv_1$ *is invariant with the pair of negations* (N_1, N_2), *i.e.,* $s \equiv_1 t = N_1(s) \equiv_1 N_2(t)$.

Proof. (i) If $N_1 = N_2$, then

$$\begin{aligned} & s \equiv_1 t \\ &= G\Big(O_1(s,t), O_2\big(N_1(s), N_1(t)\big)\Big), \ \textit{by assumption and Equation (6)} \\ &= G\Big(O_1(t,s), O_2\big(N_1(t), N_1(s)\big)\Big), \ \textit{by } (O1) \\ &= t \equiv_1 s, \ \textit{by Equation (6)}. \end{aligned}$$

(ii) Taking $s = 1$ and $t = 0$, then

$$\begin{aligned} & 1 \equiv_1 0 \\ &= G\Big(O_1(1,0), O_2\big(N_1(1), N_2(0)\big)\Big), \ \textit{by Equation (6)} \\ &= G\big(O_1(1,0), O_2(0,1)\big) \\ &= G(0,0), \ \textit{by } (O2) \\ &= 0, \ \textit{by } (G2). \end{aligned}$$

Similarly, we have $0 \equiv_1 1 = 0$.

(iii) Taking $s = 0$ and $t = 0$, then

$$\begin{aligned} & 0 \equiv_1 0 \\ &= G\Big(O_1(0,0), O_2\big(N_1(0), N_2(0)\big)\Big), \ \textit{by Equation (6)} \\ &= G\big(O_1(0,0), O_2(1,1)\big) \\ &= G(0,1), \ \textit{by } (O2) \textit{ and } (O3) \\ &= 1, \ \textit{by } (G3). \end{aligned}$$

Taking $s = t = 1$, then

$$\begin{aligned} & 1 \equiv_1 1 \\ &= G\Big(O_1(1,1), O_2\big(N_1(1), N_2(1)\big)\Big), \ \textit{by Equation (6)} \\ &= G\big(O_1(1,1), O_2(0,0)\big) \\ &= G(1,0), \ \textit{by } (O2) \textit{ and } (O3) \\ &= 1, \ \textit{by } (G3). \end{aligned}$$

(iv) It is easy to be obtained from the continuity of O_1, O_2, G, N_1 and N_2.

(v) Since O_1 has 1 as neutral element, G has 0 as neutral element, then

$$\begin{aligned} & 1 \equiv_1 s \\ &= G\Big(O_1(1,s), O_2\big(N_1(1), N_2(s)\big)\Big), \ \textit{by Equation (6)} \\ &= G\Big(O_1(1,s), O_2\big(0, N_2(s)\big)\Big) \\ &= G(s,0), \ \textit{by assumption and } (O2) \\ &= s, \ \textit{by assumption}. \end{aligned}$$

$$
\begin{aligned}
& s \equiv_1 1 \\
&= G\Big(O_1(s,1), O_2(N_1(s), N_2(1))\Big), \ \textit{by Equation } (6) \\
&= G(O_1(s,1), O_2(N_1(s), 0)) \\
&= G(s,0), \ \textit{by assumption and } (O2) \\
&= s, \ \textit{by assumption.}
\end{aligned}
$$

(vi) Since O_2 has 1 as neutral element, G has 0 as neutral element, then

$$
\begin{aligned}
& 0 \equiv_1 s \\
&= G\Big(O_1(0,s), O_2(N_1(0), N_2(s))\Big), \ \textit{by Equation } (6) \\
&= G(O_1(0,s), O_2(1, N_2(s))) \\
&= G(0, N_2(s)), \ \textit{by assumption and } (O2) \\
&= N_2(s), \ \textit{by assumption.}
\end{aligned}
$$

$$
\begin{aligned}
& s \equiv_1 0 \\
&= G\Big(O_1(s,0), O_2(N_1(s), N_2(0))\Big), \ \textit{by Equation } (6) \\
&= G(O_1(s,0), O_2(N_1(s), 1)) \\
&= G(0, N_1(s)), \ \textit{by assumption and } (O2) \\
&= N_1(s), \ \textit{by assumption.}
\end{aligned}
$$

(vii) If $N_1 = N_2 = N_{\mathbf{T}}$, case 1, if $s = 0$ or $s = 1$, then $s \equiv_1 s = 1$. Case 2, if $s \in (0,1)$, then

$$
\begin{aligned}
& s \equiv_1 s \\
&= G\Big(O_1(s,s), O_2(N_\top(s), N_\top(s))\Big), \ \textit{by Equation } (6) \\
&= G(O_1(s,s), O_2(1,1)) \\
&= G(O_1(s,s), 1), \ \textit{by } (03) \\
&= 1.
\end{aligned}
$$

(viii) If $O_1 = O_2$ and N_1 and N_2 are strong, then

$$
\begin{aligned}
& s \equiv_1 t \\
&= G\Big(O_1(s,t), O_1(N_1(s), N_2(t))\Big), \ \textit{by Equation } (6) \\
&= G(O_1(N_1(s), N_2(t)), O_1(s,t)), \ \textit{by } (G1) \\
&= G\Big(O_1(N_1(s), N_2(t)), O_1(N_1(N_1(s)), N_2(N_2(t)))\Big) \ (N_1 \textit{ and } N_2 \textit{ are strong}) \\
&= N_1(s) \equiv_1 N_2(t), \ \textit{by Equation } (6).
\end{aligned}
$$

□

Example 2. *Consider the* $O_1(s,t) = O_2(s,t) = O_{m0.5}(s,t) = \min\{\sqrt{s}, \sqrt{t}\}$*,* N_1 *and* N_2 *are standard negation, and* $G(s,t) = G_{M2}(s,t) = \max\{s^2, t^2\}$*. Then*

$$\begin{aligned}
& s \equiv_1 t \\
&= G\Big(O_1(s,t), O_2\big(N_1(s), N_2(t)\big)\Big) \\
&= \max\Big(\big(\min(\sqrt{s}, \sqrt{t})\big)^2, \big(\min(\sqrt{1-s}, \sqrt{1-t})\big)^2\Big) \\
&= \max\Big(\min(s,t), \min((1-s),(1-t))\Big).
\end{aligned} \tag{8}$$

The characteristics of this equivalence operator are shown in Figure 1. The lines are turning progressively yellower, indicating that the values are on the rise.

Example 3. *Consider the* $O_1(s,t) = O_{m0.5}(s,t) = \min\{\sqrt{s}, \sqrt{t}\}$*,* $O_2(s,t) = O_{p=3}(s,t) = s^3t^3$*,*$N_1(s) = N_2(s) = N(s) = 1 - s^2$*, and* $G(s,t) = G_{M2}(s,t) = \max\{s^2, t^2\}$*. Then*

$$\begin{aligned}
& s \equiv_1 t \\
&= G(O_1(s,t), O_2(N_1(s), N_2(t))) \\
&= \max\Big(\big(\min(\sqrt{s}, \sqrt{t})\big)^2, \big((1-s^2)^3(1-t^2)^3\big)^2\Big) \\
&= \max\Big(\min(s,t), (1-s^2)^6(1-t^2)^6\Big).
\end{aligned} \tag{9}$$

The characteristics of this equivalence operator are shown in Figure 2.

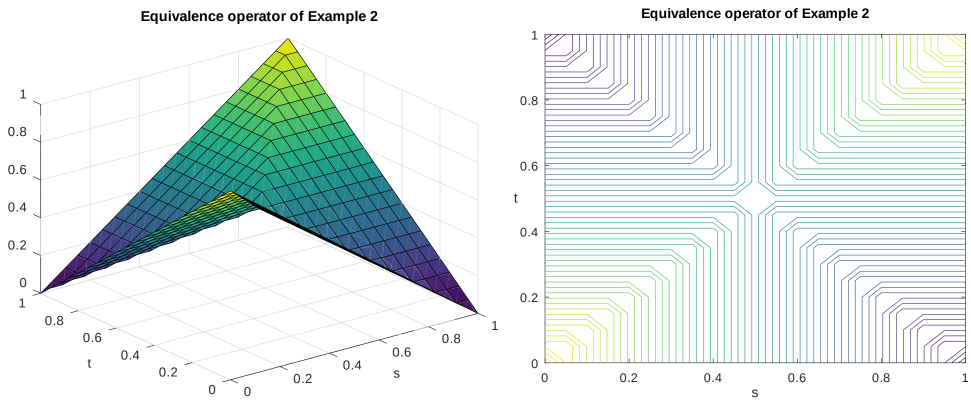

Figure 1. Characteristics of equivalence operator of Example 2 and its contour line.

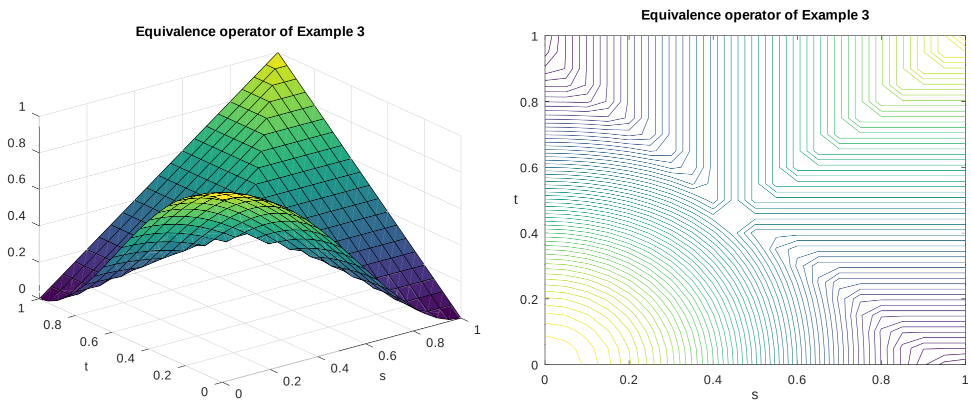

Figure 2. Characteristics of equivalence operator of Example 3 and its contour line.

4. The Model $O\big(G_1(N_1(s),t), G_2(N_2(t),s)\big)$

In this section, we present our second model of the equivalence operator. We then illustrate its properties and conclude with two examples.

The analogue of the formula $a \equiv b \stackrel{def}{=} (a \to b) \wedge (b \to a)$ in fuzzy set theory is

$$s \equiv_2 t \stackrel{def}{=} O\big((s \to_1 t), (t \to_2 s)\big) \tag{10}$$

where O is a overlap function and $\to_1$ and $\to_2$ are fuzzy implications. Here we consider the (G,N)-implication, i.e., $s \to t = G(N(s), t)$. Then we obtain the following model for fuzzy equivalence operator:

$$s \equiv_2 t \stackrel{def}{=} O\big(G_1(N_1(s),t), G_2(N_2(t),s)\big) \tag{11}$$

where, G_1 and G_2 are grouping functions, O is a overlap function, and N_1 and N_2 are fuzzy negations.

Theorem 2. *The function $\equiv_2$ given by formula (11) satisfies: $\forall s,t \in [0,1]$*

(i) *If $G_1 = G_2$ and $N_1 = N_2$, then $s \equiv_2 t = t \equiv_2 s$;*

(ii) $1 \equiv_2 0 = 0 \equiv_2 1 = 0$;

(iii) $1 \equiv_2 1 = 0 \equiv_2 0 = 1$;

(iv) *If N_1 and N_2 are continuous, then $\equiv_2$ is continuous;*

(v) *If O has 1 as neutral element, G_1 has 0 as neutral element, then $1 \equiv_2 s = s$ and $s \equiv_2 0 = N_1(s)$;*

(vi) *If O has 1 as neutral element, G_2 has 0 as neutral element, then $s \equiv_2 1 = s$ and $0 \equiv_2 s = N_2(s)$;*

(vii) *If $N_1 = N_2 = N_\top$, then $s \equiv_2 s = 1$;*

(viii) *If $G_1 = G_2$, $N_1 = N_2$, and N_1 is strong, then $\equiv_2$ is invariant with the negation N_1, i.e., $s \equiv_2 t = N_1(s) \equiv_2 N_1(t)$.*

Proof. (i) If $G_1 = G_2$ and $N_1 = N_2$, then

$$\begin{aligned}
& s \equiv_2 t \\
&= O\big(G_1(N_1(s),t), G_1(N_1(t),s)\big), \ \textit{by assumption and Equation } (6) \\
&= O\big(G_1(N_1(t),s), G_1(N_1(s),t)\big), \ \textit{by } (O1) \\
&= t \equiv_2 s, \ \textit{by assumption and Equation } (6).
\end{aligned}$$

(ii) Taking $s = 1$ and $t = 0$, then

$$\begin{aligned}
& 1 \equiv_2 0 \\
&= O\big(G_1(N_1(1),0), G_2(N_2(0),1)\big), \ \textit{by Equation } (11) \\
&= O\big(G_1(0,0), G_2(1,1)\big) \\
&= O(0,1), \ \textit{by } (G3) \textit{ and } (O2) \\
&= 0, \ \textit{by } (O2).
\end{aligned}$$

Similarly, we can obtain $0 \equiv_2 1 = 0$.

(iii) Taking $s = 0$ and $t = 0$, then

$$\begin{aligned}
& 0 \equiv_2 0 \\
&= O\big(G_1(N_1(0),0), G_2(N_2(0),0)\big), \ \textit{by Equation } (11) \\
&= O\big(G_1(1,0), G_2(1,0)\big) \\
&= O(1,1), \ \textit{by } (G3) \\
&= 1, \ \textit{by } (O3).
\end{aligned}$$

Taking $s = t = 1$, then

$$
\begin{aligned}
&1 \equiv_2 1 \\
&= O\big(G_1(N_1(1),1), G_2(N_2(1),1)\big), \ \textit{by Equation}\ (11) \\
&= O\big(G_1(0,1), G_2(0,1)\big) \\
&= O\big(1,1\big), \ \textit{by}\ (G3) \\
&= 1, \ \textit{by}\ (O3).
\end{aligned}
$$

(iv) It is easy to be obtained from the continuity of G_1, G_2, O, N_1 and N_2.
(v) Taking $t = 0$, then

$$
\begin{aligned}
&s \equiv_2 0 \\
&= O\big(G_1(N_1(s),0), G_2(N_2(0),s)\big), \ \textit{by Equation}\ (11) \\
&= O\big(G_1(N_1(s),0), G_2(1,s)\big) \\
&= O\big(N_1(s),1\big), \ \textit{by assumption and}\ (G3) \\
&= N_1(s), \ \textit{by assumption}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&1 \equiv_2 s \\
&= O\big(G_1(N_1(1),s), G_2(N_2(s),1)\big), \ \textit{by Equation}\ (11) \\
&= O\big(G_1(0,s), G_2(N_2(s),1)\big) \\
&= O\big(s,1\big), \ \textit{by assumption and}\ (G3) \\
&= s, \ \textit{by assumption}.
\end{aligned}
$$

(vi) Taking $t = 1$, then

$$
\begin{aligned}
&s \equiv_2 1 \\
&= O\big(G_1(N_1(s),1), G_2(N_2(1),s)\big), \ \textit{by Equation}\ (11) \\
&= O\big(G_1(N_1(s),1), G_2(0,s)\big) \\
&= O\big(1,s\big), \ \textit{by assumption and}\ (G3) \\
&= s, \ \textit{by assumption}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&0 \equiv_2 s \\
&= O\big(G_1(N_1(0),s), G_2(N_2(s),0)\big), \ \textit{by Equation}\ (11) \\
&= O\big(G_1(1,s), G_2(N_2(s),0)\big) \\
&= O\big(1, N_2(s)\big), \ \textit{by assumption and}\ (G3) \\
&= N_2(s), \ \textit{by assumption}.
\end{aligned}
$$

(vii) If $N_1 = N_2 = N_\top$, there are two cases. In case 1, if $s = 0$ or $s = 1$, then $s \equiv_2 s = 1$. In case 2, if $s \in (0,1)$, then

$$
\begin{aligned}
&s \equiv_2 s \\
&= O\big(G_1(N_1(s),s), G_2(N_2(s),s)\big), \ \textit{by Equation}\ (11) \\
&= O\big(G_1(1,s), G_2(1,s)\big) \\
&= O(1,1), \ \textit{by}\ (G3) \\
&= 1, \ \textit{by}\ (O3).
\end{aligned}
$$

(viii) If $G_1 = G_2$, $N_1 = N_2$, and N_1 is strong, then

$$\begin{aligned}
& s \equiv_2 t \\
&= O\big(G_1(N_1(s), t), G_1(N_1(t), s)\big), \ \textit{by Equation (11)} \\
&= O\Big(G_1(s, N_1(t)), G_1\big(t, N_1(s)\big)\Big), \ \textit{by (G1) and (O1)} \\
&= O\Big(G_1\big(N_1(N_1(s)), N_1(t)\big), G_1\big(N_1(N_1(t)), N_1(s)\big)\Big) \ (N_1 \textit{ and } N_2 \textit{ are strong}) \\
&= N_1(s) \equiv_1 N_1(t), \ \textit{by Equation (6)}.
\end{aligned}$$

This completes the proof. □

Example 4. *Consider the* $G_1(s,t) = G_2(s,t) = \max\{s^2, t^2\}$, $O(s,t) = O_{m2}(s,t) = \min\{s^2, t^2\}$, N_1 *and* N_2 *are the standard negation. Then*

$$\begin{aligned}
& s \equiv_2 t \\
&= O\big(G_1(N_1(s), t), G_2(N_2(t), s)\big) \\
&= \min\Big(\big(\max((1-s)^2, t^2)\big)^2, \big(\max((1-t)^2, s^2)\big)^2\Big) \\
&= \min\Big(\max\big((1-s)^4, t^4\big), \max\big((1-t)^4, s^4\big)\Big).
\end{aligned} \tag{12}$$

The characteristics of this equivalence operator are shown in Figure 3.

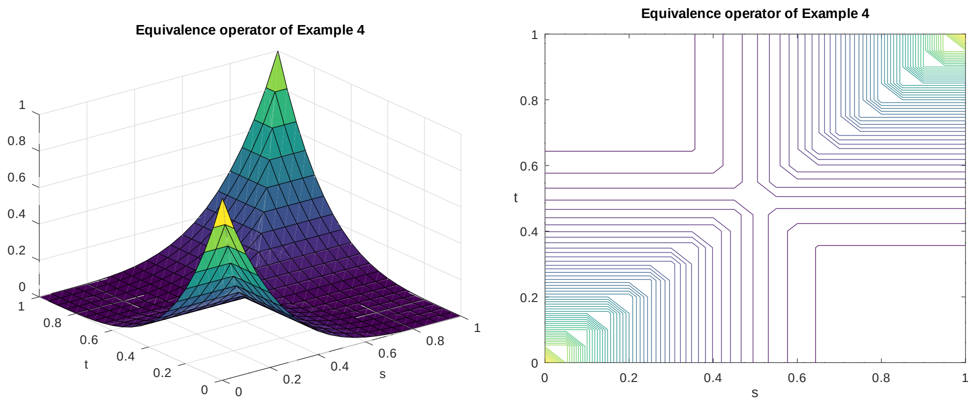

Figure 3. Characteristics of equivalence operator of Example 4 and its contour line.

Example 5. *Consider the* $G_1(s,t) = \max\{s^2, t^2\}$, $G_2(s,t) = \max\{s^3, t^3\}$, $O(s,t) = O_{p=2}(s,t) = s^2t^2$, N_1 *and* N_2 *are the standard negation. Then*

$$\begin{aligned}
& s \equiv_2 t \\
&= O\big(G_1(N_1(s), t), G_2(N_2(t), s)\big) \\
&= \big(\max((1-s)^2, t^2)\big)^2 \cdot \big(\max((1-t)^3, s^3)\big)^2 \\
&= \max\big((1-s)^4, t^4\big) \cdot \max\big((1-t)^6, s^6\big).
\end{aligned} \tag{13}$$

The characteristics of this equivalence operator are shown in Figure 4.

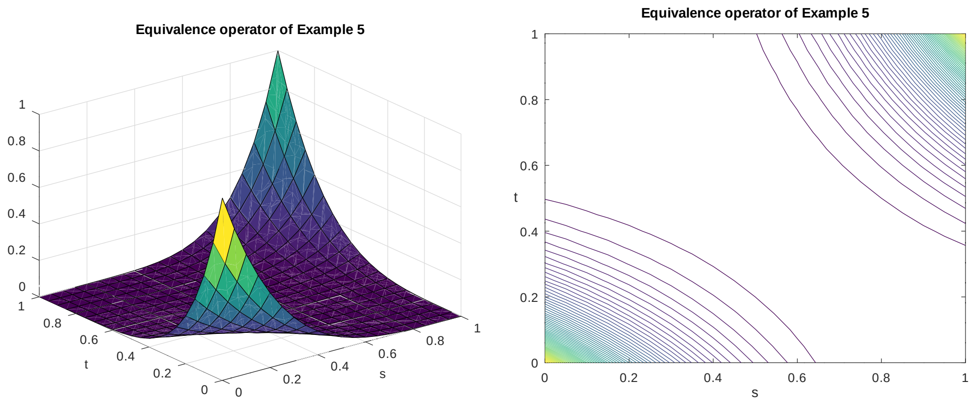

Figure 4. Characteristics of equivalence operator of Example 5 and its contour line.

5. Comparative Study

In this section, we show a short comparison of the proposed equivalence operators with some existing equivalence operators.

In [27], Li et al. defined the fuzzy equivalence operator as a binary function $EN{:}[0,1]^2 \to [0,1]$ satisfying

$(EN1)$ $E(s,t) = E(t,s)$;

$(EN2)$ $E(s,1) = s$;

$(EN3)$ $E(0,0) = 0$.

Li et al. [27] gave the following two models,

$$EN_1(s,t) = S\Big(T(s,t), T\big(N(s), N(t)\big)\Big); \tag{14}$$

$$EN_2(s,t) = T\big(S(N(s),t), S(N(t),s)\big). \tag{15}$$

In [4,5], Dombi, Csiszár listed some important properties for equivalence operators: $\forall s,t,u \in [0,1]$

(E1) Symmetry, $s \equiv t = t \equiv s$;
(E2) Compatibility, $0 \equiv 1 = 1 \equiv 0 = 0$ and $0 \equiv 0 = 1 \equiv 1 = 1$;
(E3) Reflexivity, $s \equiv s = 1$;
(E4) Associativity, $s \equiv (t \equiv u) = (s \equiv t) \equiv u$;
(E5) Neutrality principle, $1 \equiv s = s$.

The comparison is demonstrated in the following:

Remark 1. *$\equiv_1$ and $\equiv_2$ are two developed models of EN_1 and EN_2 by replacing t-norms and t-conorms with overlap and grouping functions respectively.*

Remark 2. *$\equiv_1$, $\equiv_2$ and fuzzy equivalences in [6] drop symmetry (E1). Fuzzy equivalences in [4,5,27] satisfy symmetry.*

Remark 3. *$\equiv_1$, $\equiv_2$ and fuzzy equivalences in [4–6,27] satisfy the compatibility (E2), so they are generalizations of classical equivalence.*

Remark 4. *Both $\equiv_1$ and $\equiv_2$ drop reflexivity (E3). Fuzzy equivalences in [4–6,27] satisfy reflexivity.*

Remark 5. *$\equiv_1$, $\equiv_2$ and fuzzy equivalences in [5] drop associativity (E4). Fuzzy equivalences in [4,27] satisfy associativity.*

Remark 6. *Both $\equiv_1$ and $\equiv_2$ drop the neutrality principle (E5). Fuzzy equivalences in [4–6,27] satisfy the eutrality principle.*

Table 1 provides a comprehensive comparison of various fuzzy equivalences. As can be seen from the table, fuzzy equivalences $\equiv_1$ and $\equiv_2$ introduced in this paper have few restrictions, thus providing greater flexibility and functionality.

Table 1. Comparison of fuzzy equivalences.

Property	$\equiv_1$	$\equiv_2$	Fuzzy Equivalences in [6]	Fuzzy Equivalences in [5]	Fuzzy Equivalences in [4,27]
E_1	$\times$	$\times$	$\times$	$\surd$	$\surd$
E_2	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
E_3	$\times$	$\times$	$\surd$	$\surd$	$\surd$
E_4	$\times$	$\times$	$\surd$	$\times$	$\surd$
E_5	$\times$	$\times$	$\surd$	$\surd$	$\surd$

6. Fuzzy Symmetric Differences from Fuzzy Equivalences

Fuzzy symmetric difference is a dual concept of fuzzy equivalence. We can define two kinds od fuzzy symmetric differences as

$$s\Delta_1 t \overset{def}{=} N(s \equiv_1 t) = N\Big(G\Big(O_1(s,t), O_2\big(N_1(s), N_2(t)\big)\Big)\Big); \tag{16}$$

$$s\Delta_2 t \overset{def}{=} N(s \equiv_2 t) = N\Big(O\big(G_1(N_1(s),t), G_2(N_2(t),s)\big)\Big). \tag{17}$$

Fuzzy symmetric differences Δ_1 and Δ_2 are generalizations of classical symmetric difference since $1\Delta_1 1 = 0\Delta_1 0 = 1\Delta_2 1 = 0\Delta_2 0 = 0$ and $0\Delta_1 1 = 1\Delta_1 0 = 0\Delta_2 1 = 1\Delta_2 0 = 1$.

Hu et al. [16] introduced the following two models of fuzzy symmetric differences

$$s\blacktriangle_1 t \overset{def}{=} G\Big(O_1\big(s, N_1(t)\big), O_2\big(N_2(s), t\big)\Big); \tag{18}$$

$$s\blacktriangle_2 t \overset{def}{=} O\Big(G(s,t), N\big(O(s,t)\big)\Big). \tag{19}$$

We show that they are connected as follows.

Theorem 3. *If $N = N_1 = N_2$ is a strong negation, let $O = O_1 = O_2$ and G is the dual grouping functions of O for N. Then $s\Delta_1 t = s\blacktriangle_2 t$.*

Proof. From the assumptions, it follows that

$$\begin{aligned} s \equiv_1 t &= N\Big(G\Big(O(s,t), O\big(N(s), N(t)\big)\Big)\Big) \\ &= O\Big(N\big(O(s,t)\big), N\Big(O\big(N(s), N(t)\big)\Big)\Big) \\ &= O\Big(N\big(O(s,t)\big), G\big(N(N(s)), N(N(t))\big)\Big) \\ &= O\Big(N\big(O(s,t)\big), G(s,t)\Big) \\ &= O\Big(G(s,t), N\big(O(s,t)\big)\Big) \\ &= s\blacktriangle_2 t. \end{aligned} \tag{20}$$

This completes the proof. $\square$

Theorem 4. *If $N = N_1 = N_2$ is a strong negation, let G is the dual grouping function of O for N and O_1 and O_2 are the dual overlap functions of G_1 and G_2 for N respectively. Then* $s\Delta_2 t = G\Big(O_1\big(s, N(t)\big), O_2\big(t, N(s)\big)\Big)$.

Proof. From the assumptions, it follows that

$$\begin{aligned} & s \equiv_2 t \\ = & N\Big(O\big(G_1(N(s),t), G_2(N(t),s)\big)\Big) \\ = & G\Big(N\big(G_1(N(s),t)\big), N\big(G_2(N(t),s)\big)\Big) \\ = & G\Big(N\big(G_1(N(s),t)\big), N\big(G_2(N(t),s)\big)\Big) \\ = & G\Big(O_1\big(N(N(s)), N(t)\big), O_2\big(N(N(t)), N(s)\big)\Big) \\ = & G\Big(O_1\big(s, N(t)\big), O_2\big(t, N(s)\big)\Big). \end{aligned} \tag{21}$$

This completes the proof. □

Corollary 1. *If $N = N_1 = N_2$ is a strong negation, let $O = O_1 = O_2$, $G = G_1 = G_2$ and G is the dual grouping function of O for N. Then $s\Delta_2 t = s\blacktriangle_1 t$.*

7. Conclusions

This paper introduces fuzzy equivalences based on overlap functions and grouping functions instead of t-norms and t-conorms, aiming to provide more flexible fuzzy equivalences that do not necessarily conform to certain properties such as associativity and the neutrality principle. The fuzzy equivalences proposed in this paper are more flexible due to their limited restrictions, compared to those based on t-norms and t-conorms (refer to Table 1). The study of two models of equivalence operators has revealed the potential for further investigation.

For further work, some possible topics are given.

(1) To expand on this research, it would be valuable to explore different types of overlap and grouping functions, including interval-valued, complex-valued, and lattice-valued functions. This exploration could lead to the development of new interval-valued, complex-valued, and lattice-valued operators based on these functions.
(2) Equivalence operators play a critical role in constructing similarity measures that are essential for various aspects of fuzzy theory and applications. They contribute to the robustness of fuzzy reasoning and the stabilization of fuzzy control systems. The equivalence operators presented in this paper offer promising prospects for enhancing fuzzy theory and applications.
(3) The additive and multiplicative generators of overlap and grouping functions are given in [28,29], suggesting that further exploration into the use of these generators to obtain equivalence operators would be a worthwhile pursuit.

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References

1. Hájek, P. *Metamathematics of Fuzzy Logic*; Kluwer: Dordrecht, The Netherlands, 1998.
2. Dyba, M.; Novák, V. EQ-logic: Non-commutative fuzzy logic based on fuzzy equality. *Fuzzy Sets Syst.* **2011**, *172*, 13–32. [CrossRef]
3. Novák, V.; Baets, B.D. EQ-algebras. *Fuzzy Sets Syst.* **2009**, *160*, 2956–2978. [CrossRef]
4. Dombi, J. Equivalence operators that are associative. *Inf. Sci.* **2014**, *281*, 281–294. [CrossRef]
5. Dombi, J.; Csiszár, O. Equivalence operators in nilpotent systems. *Fuzzy Sets Syst.* **2016**, *299*, 113–129. [CrossRef]
6. Hu, B.; Bi, L.; Li, S.; Dai, S. Asymmetric Equivalences in Fuzzy Logic. *Symmetry* **2017**, *9*, 224. [CrossRef]
7. Pan, H.; Cao, Y.; Zhang, M.; Chen, Y. Simulation for lattice-valued doubly labeled transition systems. *Int. J. Approx. Reason.* **2014**, *55*, 797–811 [CrossRef]
8. Jin, J.; Li, Y.; Li, C. Robustness of fuzzy reasoning via logically equivalence measure. *Inf. Sci.* **2007**, *177*, 5103–5117. [CrossRef]
9. Dai, S.; Pei, D.; Guo, D. Robustness analysis of full implication inference method. *Int. J. Approx. Reason.* **2013**, *54*, 653–666. [CrossRef]
10. Georgescu, I. Similarity of fuzzy choice functions. *Fuzzy Sets Syst.* **2007**, *158*, 1314–1326. [CrossRef]
11. Wang, G.; Duan, J. Two types of fuzzy metric spaces suitable for fuzzy reasoning. *Sci. China Inf. Sci.* **2014**, *44*, 623–632.
12. Duan, J.; Li, Y. Robustness analysis of logic metrics on F(X). *Int. J. Approx. Reason.* **2015**, *61*, 33–42. [CrossRef]
13. Fodor, J.C.; Keresztfalvi, T. Nonstandard conjunctions and implications in fuzzy logic. *Int. J. Approx. Reason.* **1995**, *12*, 69–84. [CrossRef]
14. Bustince, H.; Fernandez, J.; Mesiar, R.; Montero, J.; Orduna, R. Overlap functions. *Nonlinear Anal. Theory Methods Appl.* **2010**, *72*, 1488–1499. [CrossRef]
15. Bustince, H.; Pagola, M.; Mesiar, R.; Hüllermeier, E.; Herrera, F. Grouping, overlaps, and generalized bientropic functions for fuzzy modeling of pairwise comparisons. *IEEE Trans. Fuzzy Syst.* **2012**, *20*, 405–415. [CrossRef]
16. Hu, B.; He, D.; Dai, S. Symmetric Difference Operators Derived from Overlap and Grouping Functions. *Symmetry* **2023**, *15*, 1569. [CrossRef]
17. Dai, S.; Song, H.; Xu, Y.; Du, L. Fuzzy difference operators derived from overlap functions. *J. Intell. Fuzzy Syst.* **2024**, *46*, 247–255. [CrossRef]
18. Dimuro, G.P.; Bedregal, B. On residual implications derived from overlap functions. *Inf. Sci.* **2015**, *312*, 78–88. [CrossRef]
19. Qiao, J. R_O-implications on finite scales. *Int. J. Approx. Reason.* **2023**, *159*, 108921. [CrossRef]
20. Dimuro, G.P.; Bedregal, B.; Santiago, R.H.N. On (G,N)-implications derived from grouping functions. *Inf. Sci.* **2014**, *279*, 1–17. [CrossRef]
21. Qiao, J. On binary relations induced from overlap and grouping functions. *Int. J. Approx. Reason.* **2019**, *106*, 155–171. [CrossRef]
22. Qiao, J. On (IO, O)-fuzzy rough sets based on overlap functions. *Int. J. Approx. Reason.* **2021**, *132*, 26–48. [CrossRef]
23. Jiang, H.; Hu, B. On (O,G)-fuzzy rough sets based on overlap and grouping functions over complete lattices. *Int. J. Approx. Reason.* **2022**, *144*, 18–50. [CrossRef]
24. Baczyński, M.; Jayaram, B. *Fuzzy Implications*; Springer: Berlin/Heidelberg, Germany, 2008.
25. Klement, E.P.; Mesiar, R.; Pap, E. *Triangular Norms*; Kluwer: Dordrecht, The Netherlands, 2000.
26. Bedregal, B.; Dimuro, G.P.; Bustince, H.; Barrenechea, E. New results on overlap and grouping functions. *Inf. Sci.* **2013**, *249*, 148–170. [CrossRef]
27. Li, Y.; Qin, K.; He, X. Fuzzy XNOR connectives in fuzzy logic. *Soft Comput.* **2011**, *15*, 2457–2465. [CrossRef]
28. Dimuro, G.P.; Bedregal, B.; Bustince, H.; Asiáin, M.J.; Mesiar, R. On additive generators of overlap functions. *Fuzzy Sets Syst.* **2016**, *287*, 76–96. [CrossRef]
29. Qiao, J.; Hu, B.Q. On multiplicative generators of overlap and grouping functions. *Fuzzy Sets Syst.* **2018**, *332*, 1–24. [CrossRef]