axioms

Article

Improvements of Integral Majorization Inequality with Applications to Divergences

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Abstract: Within the recent wave of research advancements, mathematical inequalities and their practical applications play a notably significant role across various domains. In this regard, inequalities offer a captivating arena for scholarly endeavors and investigational pursuits. This research work aims to present new improvements for the integral majorization inequalities using an interesting aproach. Certain previous improvements have been achieved for the Jensen inequality as direct outcomes of the main results. Additionally, estimates for the Csiszár divergence and its cases are provided as applications of the main results. The circumstances under which the principal outcomes offer enhanced estimations for majorization differences are also underscored and emphasized.

Keywords: convex function; majorization inequality; Jensen's inequality; information theory

MSC: 26A51, 26D15, 68P30

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1. Introduction

The importance of mathematical inequalities and their practical applications lies in their ability to describe and tackle real-world problems, particularly in scenarios where the relationships are usually ambiguous or uneven [1–3]. The beauty of mathematical inequalities is found not just in their capacity to measure relationships between numbers but also in their power to reveal the intricacies of complex systems [4–6]. Because of their widespread applicability across various mathematical domains and real-world situations, mathematical inequalities serve as a focal point for launching new investigations [7–9]. A critical factor contributing to the advancement and widespread acceptance of inequalities lies in the robust and influential concept known as convexity [10–12]. It has been observed that convex functions and inequalities share a closely intertwined relationship, as many achievements would be challenging or even unattainable without the guiding principles provided by convexity [13–15]. The Hermite-Hadamard [16], Jensen-Mercer [17], Slater [18], and Jensen-Steffensen [19] inequalities are striking examples where the concept of convexity plays a pivotal role, providing a solid foundation for their formulation and proof. One of the most captivating inequalities that emerge with the aid of convexity is the renowned Jensen inequality [20]. The beauty of the Jensen inequality lies not only in its elegance and simplicity but also in its versatility and widespread applicability [21–23]. It serves as a cornerstone for numerous mathematical proofs and has paved the way for deeper exploration and understanding of convexity and related concepts [24–26].

The continuous form of the Jensen inequality is defined as follows [27]:

Suppose that $p, f : [\alpha_1, \alpha_2] \to [\beta_1, \beta_2]$ are integrable functions with $p \geq 0$ on $[\alpha_1, \alpha_2]$ and $p^* := \int_{\alpha_1}^{\alpha_2} p(\varrho)d\varrho > 0$. Further, let $\psi : [\beta_1, \beta_2] \to \mathbb{R}$ be a convex function and $\psi \circ f$ be integrable. Then

$$\psi\left(\frac{1}{p^*}\int_{\alpha_1}^{\alpha_2} p(\varrho)f(\varrho)d\varrho\right) \leq \frac{1}{p^*}\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho. \tag{1}$$

The concave function ψ reverses the validity of the inequality (1).

Jensen's inequality offers a straightforward and uncomplicated approach to understanding and proving various concepts. Due to these facts and figures, it's evident that Jensen's inequality is a highly studied topic in the realm of research activities. Kian [28] introduced an operator-based representation of the Jensen inequality specifically designed for super-quadratic functions, along with a discussion on the applications stemming from the obtained results. Matković et al. [29] presented a variant of Jensen's inequality for operators by applying the notion of convexity, which serves as a generalization of Mercer's result. In 2008, Zhu and Yang [30] utilized Jensen's inequality to analyze and discuss the stability of discrete-time delay systems. Ullah et al. [31] introduced some improvements of the discrete as well as integral Jensen's inequality by implementing $4-$convexity. You et al. [32] utilized the integral version of Jensen's inequality and established improvements of the Slater inequality in both discrete and integral forms.

In the remaining part of the current section, we want to divert our attention to the concept of majorization.

Majorization is a mathematical concept that quantifies the comparison of two vectors or sequences in a specific order based on certain criteria [33,34]. In essence, majorization compares the "spread" or arrangement of elements in one vector with that of another, determining if one vector is more spread out than the other according to a particular criterion [35]. At this moment, we give the definition of majorization [36]: Let $\mathbf{x} = (\varrho_1, \varrho_2, \cdots, \varrho_n)$ and $\mathbf{y} = (\sigma_1, \sigma_2, \cdots, \sigma_n)$ be any two $n-$tuples, and

$$\varrho_{[1]} \geq \varrho_{[2]} \geq \cdots \geq \varrho_{[n]}, \quad \sigma_{[1]} \geq \sigma_{[2]} \geq \cdots \geq \sigma_{[n]}$$

be the decreasing order of the tuples $\mathbf{x}$ and $\mathbf{y}$ respectively. Then $\mathbf{y}$ is said to be majorized by $\mathbf{x}$, if

$$\sum_{i=1}^{k} \varrho_{[i]} \geq \sum_{i=1}^{k} \sigma_{[i]}, \qquad k = 1, 2, \cdots, n-1,$$
$$\sum_{i=1}^{n} \varrho_i = \sum_{i=1}^{n} \sigma_i.$$

In symbols, majorization can be represented as $\mathbf{x} \succ \mathbf{y}$.

In 1932, Karamata [37] formulated a relation for majorized tuples utilizing principles of convexity, and this relation is well-recognized as the majorization inequality in literature. The Karamata relation for majorized tuples can be expressed as:

Suppose that $\mathbf{x} = (\varrho_1, \varrho_2, \cdots, \varrho_n)$, $\mathbf{y} = (\sigma_1, \sigma_2, \cdots, \sigma_n) \in [\alpha_1, \alpha_2]^n$ are $n-$tuples such that $\mathbf{x} \succ \mathbf{y}$, and further presume that $\psi : [\alpha_1, \alpha_2] \to \mathbb{R}$ is a convex function. Then

$$\sum_{i=1}^{n} \psi(\varrho_i) \geq \sum_{i=1}^{n} \psi(\sigma_i). \tag{2}$$

The inequality (2) flips when considering the concave function ψ.

The discrete majorization inequality plays a vital role in comparing, ordering, and analyzing discrete sequences or vectors, making it a valuable tool in various mathematical and applied domains [38,39].

Now, we give the definition of majorization for functions [40]:

Assume that f and g are any functions on $[\alpha_1, \alpha_2]$, then function f is said to majorizes g (abbreviated as $f \succ g$), if both f and g are decreasing and satisfy:

$$\int_{\alpha_1}^{\sigma} f(\varrho)d\varrho \geq \int_{\alpha_1}^{\sigma} g(\varrho)d\varrho, \qquad \sigma \in [\alpha_1, \alpha_2]$$

and

$$\int_{\alpha_1}^{\alpha_2} f(\varrho)d\varrho = \int_{\alpha_1}^{\alpha_2} g(\varrho)d\varrho.$$

Inequality (2) can be used for comparing individual elements. However, in recent years, a question has been posed regarding how to effectively compare function values over a set using the concept of majorization. This question is answered by developing the integral version of majorization inequality. The integral majorization inequality can be verbalized as follows [40]:

Let $f, g : [\alpha_1, \alpha_2] \to [\beta_1, \beta_2]$ be decreasing functions such that their integral exists, and $f \succ g$. If ψ is convex on $[\beta_1, \beta_2]$, then

$$\int_{\alpha_1}^{\alpha_2} \psi(f(\varrho))d\varrho \geq \int_{\alpha_1}^{\alpha_2} \psi(g(\varrho))d\varrho. \tag{3}$$

In the case of a concave function ψ, inequality (3) is satisfied in the reverse direction.

In 1995, Maligranda et al. [41] gave the weighted version of (3), which states that: Assume that $\psi : [\beta_1, \beta_2] \to \mathbb{R}$ is convex and $p, f, g : [\alpha_1, \alpha_2] \to [\beta_1, \beta_2]$ are continuous functions such that $p(\varrho) \geq 0$, $\varrho \in [\alpha_1, \alpha_2]$ with

$$\int_{\alpha_1}^{\sigma} p(\varrho)f(\varrho)d\varrho \geq \int_{\alpha_1}^{\sigma} p(\varrho)g(\varrho)d\varrho, \qquad \sigma \in [\alpha_1, \alpha_2] \tag{4}$$

and

$$\int_{\alpha_1}^{\alpha_2} p(\varrho)f(\varrho)d\varrho = \int_{\alpha_1}^{\alpha_2} p(\varrho)g(\varrho)d\varrho.$$

If g is decreasing, then

$$\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho \geq \int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(g(\varrho))d\varrho. \tag{5}$$

If f is increasing, then (5) true in reverse direction.

Several mathematicians have focused on majorization inequality and used different approaches for the derivation of results for majorization inequality. In [36,40], Dragomir used a new approach of applying Chebyshev's inequality with the help of synchronous functions as well as sequences without using condition (4) and obtained the weighted version of majorization inequality. Niezgoda was inspired by the idea of Dragomir and utilized the generalized Chebyshev's inequality for separable sequences [42]. These generalized results of majorization have been discussed for certain orthogonal bases. Also, Niezgoda [43] used the concept of majorization and presented enhanced results for the Jensen-Mercer inequality, while these results have been further applied for the derivation of interesting conticrete Hermite-Hadarmard type inequalities [44] which attracted the attention of other mathematicians. The book [45] contains numerous results on majorization and its applications. Particularly, these results are associated with different interpolation polynomials and Green functions. Several identities for majorization difference have been obtained that contain the n^{th} derivative of the function. By virtue of n–convexity and some features of interpolation polynomials, several generalized majorization inequalities have been derived. Further, these results have been discussed for certain tuples and functions which have been utilized for obtaining the classical majorization inequalities. Generalized mean value theorems, log convexity, and exponential convexity have also been proved for the generalized results of majorization. Moreover, the generalized results of majorization have been devoted to applications in information theory. In 2021, Bradanović [46] obtained some improvements of majorization inequalities with the help of superquadratic functions, which is in fact the generalized concept of convexity. The results are also associated with different entropies. The work of Chin and Huh is very remarkable with respect to applications of majorization for linear optical networks [47]. Very recently, the integral majorization inequalities have been studied and derived the Hermite-Hadamard-Fejér-Type inequalities and several applications have presented in information theory [48].

2. Main Results

This section focuses on the key results, which will offer estimations for majorization and Jensen differences. The primary aim of this section is to obtain estimates for the majorization difference through the use of functions that are twice differentiable. The estimates we aim to obtain can be achieved by utilizing the concepts of convexity, Hölder's inequality, the well-recognized Jensen's inequality, and the renowned power mean inequality. We will examine the direct consequences of each outcome with respect to the Jensen differences. We commence this section with the subsequent lemma, which introduces an identity linked to the majorization difference.

Lemma 1. *Let $\psi : (\beta_1, \beta_2) \to \mathbb{R}$ be a twice differentiable function such that ψ'' is integrable and $f, g : [\alpha_1, \alpha_2] \to (\beta_1, \beta_2)$, $p : [\alpha_1, \alpha_2] \to [0, \infty)$ be integrable functions. Then*

$$\begin{aligned}\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho - \int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(g(\varrho))d\varrho \\ = \int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho) - f(\varrho)\Big)^2 \int_0^1 t\psi''\Big(tg(\varrho) + (1-t)f(\varrho)\Big)dtd\varrho \\ - \int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho) - f(\varrho)\Big)\psi'(g(\varrho))d\varrho.\end{aligned} \tag{6}$$

Proof. By avoiding unnecessary generalizations, let us suppose that $g(\varrho)$ and $f(\varrho)$ are distinct for every ϱ in $[\alpha_1, \alpha_2]$. Employing the technique of integration by parts, we can deduce the following identity:

$$\begin{aligned}&\int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho) - f(\varrho)\Big)^2 \int_0^1 t\psi''\Big(tg(\varrho) + (1-t)f(\varrho)\Big)dtd\varrho \\ &= \int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho) - f(\varrho)\Big)^2 \Big(\frac{t}{g(\varrho) - f(\varrho)}\psi'\Big(tg(\varrho) + (1-t)f(\varrho)\Big)\Big|_0^1 \\ &\quad - \frac{1}{g(\varrho) - f(\varrho)}\int_0^1 \psi'\Big(tg(\varrho) + (1-t)f(\varrho)\Big)dt\Big)d\varrho \\ &= \int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho) - f(\varrho)\Big)^2 \Big(\frac{\psi'(g(\varrho))}{g(\varrho) - f(\varrho)} - \frac{1}{\Big(g(\varrho) - f(\varrho)\Big)^2}\psi\Big(tg(\varrho) + (1-t)f(\varrho)\Big)\Big|_0^1\Big)d\varrho \\ &= \int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho) - f(\varrho)\Big)^2 \Big(\frac{\psi'(g(\varrho))}{g(\varrho) - f(\varrho)} - \frac{1}{\Big(g(\varrho) - f(\varrho)\Big)^2}\Big(\psi(g(\varrho)) - \psi(f(\varrho))\Big)\Big)d\varrho \\ &= \int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho) - f(\varrho)\Big)\psi'(g(\varrho))d\varrho - \int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(\psi(g(\varrho)) - \psi(f(\varrho))\Big)d\varrho.\end{aligned} \tag{7}$$

We arrive to the identity (8) by rearranging the terms of (7):

$$\begin{aligned}\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho - \int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(g(\varrho))d\varrho \\ = \int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho) - f(\varrho)\Big)^2 \int_0^1 t\psi''\Big(tg(\varrho) + (1-t)f(\varrho)\Big)dtd\varrho \\ - \int_{\alpha_1}^{\alpha_2} p(\varrho)(g(\varrho) - f(\varrho))\psi'(g(\varrho))d\varrho.\end{aligned} \tag{8}$$

Without a doubt, (6) and (7) are one and the same. □

We leverage the Hölder inequality and the notion of a convex function to establish an improvement concern to the majorization inequality, which is verbalized in the upcoming theorem.

Theorem 1. *Let $\psi : (\beta_1, \beta_2) \to \mathbb{R}$ be a twice differentiable function such that ψ'' is integrable and $|\psi''|^q$ is convex for $q > 1$. Also, assume that $f, g : [\alpha_1, \alpha_2] \to (\beta_1, \beta_2)$, $p : [\alpha_1, \alpha_2] \to [0, \infty)$ are integrable functions. Then*

$$\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho - \int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(g(\varrho))d\varrho \leq -\int_{\alpha_1}^{\alpha_2} p(\varrho)(g(\varrho) - f(\varrho))\psi'(g(\varrho))d\varrho + \int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho) - f(\varrho)\Big)^2 \left(\frac{(q+1)\left|\psi''(g(\varrho))\right|^q + \left|\psi''(f(\varrho))\right|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}} d\varrho. \tag{9}$$

Proof. Since, $\psi(\varrho) \leq |\psi(\varrho)|$ holds true for all ϱ in $[\alpha_1, \alpha_2]$. Therefore, from the identity (6), we can infer the following expression:

$$\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho - \int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(g(\varrho))d\varrho \leq -\int_{\alpha_1}^{\alpha_2} p(\varrho)(g(\varrho) - f(\varrho))\psi'(g(\varrho))d\varrho + \int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho) - f(\varrho)\Big)^2 \int_0^1 \left|t\psi''\Big(tg(\varrho) + (1-t)f(\varrho)\Big)\right| dt d\varrho. \tag{10}$$

Upon applying the Hölder inequality to the second term on the right-hand side of (10), we arrive at the following inequality:

$$\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho - \int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(g(\varrho))d\varrho \leq -\int_{\alpha_1}^{\alpha_2} p(\varrho)(g(\varrho) - f(\varrho))\psi'(g(\varrho))d\varrho + \int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho) - f(\varrho)\Big)^2 \left(\int_0^1 t^q \left|\psi''\Big(tg(\varrho) + (1-t)f(\varrho)\Big)\right|^q dt\right)^{\frac{1}{q}} d\varrho. \tag{11}$$

Now, by making use of convexity of the function $|\psi''|^q$ on the right-hand side of (11), we obtain:

$$\begin{aligned} &\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho - \int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(g(\varrho))d\varrho \\ &\quad \leq -\int_{\alpha_1}^{\alpha_2} p(\varrho)(g(\varrho) - f(\varrho))\psi'(g(\varrho))d\varrho + \int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho) - f(\varrho)\Big)^2 \\ &\quad \times \left(|\psi''(g(\varrho))|^q \int_0^1 t^{q+1} dt + |\psi''(f(\varrho))|^q \int_0^1 t^q (1-t) dt\right)^{\frac{1}{q}} d\varrho. \end{aligned} \tag{12}$$

Through the evaluation of integrals given in inequality (12), we are led to the improvement presented in (9). □

The following corollary stems from Theorem 1, presenting an improvement to the Jensen inequality.

Corollary 1. *Let $\psi : (\beta_1, \beta_2) \to \mathbb{R}$ be a twice differentiable function such that ψ'' is integrable and $|\psi''|^q$ is convex for $q > 1$. Also, assume that $f : [\alpha_1, \alpha_2] \to (\beta_1, \beta_2)$, $p : [\alpha_1, \alpha_2] \to [0, \infty)$ are integrable functions and $p^* := \int_{\alpha_1}^{\alpha_2} p(\varrho)d\varrho > 0$, $\bar{f} := \frac{1}{p^*}\int_{\alpha_1}^{\alpha_2} p(\varrho)f(\varrho)d\varrho$. Then*

$$\begin{aligned} &\frac{1}{p^*}\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho - \psi(\bar{f}) \\ &\quad \leq \int_{\alpha_1}^{\alpha_2} p(\varrho)(\bar{f} - f(\varrho))^2 \left(\frac{(q+1)|\psi''(\bar{f})|^q + |\psi''(f(\varrho))|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}} d\varrho. \end{aligned} \tag{13}$$

Proof. Replacing $g(\varrho)$ with $\overline{f}$ in inequality (9) leads us directly to the anticipated outcome (13). □

Remark 1. *In inequality (4), Khan et al. [49] demonstrated an enhancement similar to the improvement outlined in Corollary 1, which provides an estimate for the absolute Jensen difference.*

The upcoming theorem presents an inequality for majorization, potentially attainable through the utilization of the Hölder inequality and the convex function formulation.

Theorem 2. *Let us assume that all the conditions of Theorem 1 hold true, and additionally, consider the scenario that* $\frac{1}{p} + \frac{1}{q} = 1, p, q > 1$. *Then*

$$\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho - \int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(g(\varrho))d\varrho \leq -\int_{\alpha_1}^{\alpha_2} p(\varrho)(g(\varrho) - f(\varrho))\psi'(g(\varrho))d\varrho$$
$$+\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho) - f(\varrho)\Big)^2 \left(\frac{|\psi''(g(\varrho))|^q + |\psi''(f(\varrho))|^q}{2}\right)^{\frac{1}{q}} d\varrho. \tag{14}$$

Proof. We can easily reach to (15), by employing the Hölder inequality on the right side of inequality (10):

$$\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho - \int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(g(\varrho))d\varrho \leq -\int_{\alpha_1}^{\alpha_2} p(\varrho)(g(\varrho) - f(\varrho))\psi'(g(\varrho))d\varrho$$
$$+\int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho) - f(\varrho)\Big)^2 \left(\int_0^1 t^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|\psi''\Big(tg(\varrho) + (1-t)f(\varrho)\Big)\right|^q dt\right)^{\frac{1}{q}} d\varrho$$
$$= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho) - f(\varrho)\Big)^2 \left(\int_0^1 \left|\psi''\Big(tg(\varrho) + (1-t)f(\varrho)\Big)\right|^q dt\right)^{\frac{1}{q}} d\varrho$$
$$-\int_{\alpha_1}^{\alpha_2} p(\varrho)(g(\varrho) - f(\varrho))\psi'(g(\varrho))d\varrho.$$

Simply applying the convex function definition to the right side of (15), we drive

$$\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho - \int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(g(\varrho))d\varrho \leq -\int_{\alpha_1}^{\alpha_2} p(\varrho)(g(\varrho) - f(\varrho))\psi'(g(\varrho))d\varrho$$
$$+\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho) - f(\varrho)\Big)^2 \left(|(g(\varrho))|^q \int_0^1 t dt + |\psi''(f(\varrho))|^q \int_0^1 (1-t)dt\right)^{\frac{1}{q}} d\varrho. \tag{16}$$

Now, upon evaluating the integrals in (16), we obtain (14). □

The following corollary establishes an improvement for the Jensen inequality, which can be followed straightly from Theorem 2.

Corollary 2. *Presume that the hypotheses of Corollary 1 are fulfilled, and additionally, let* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\frac{1}{p^*}\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho - \psi(\overline{f}) \quad \leq \quad \frac{1}{p^*}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \int_{\alpha_1}^{\alpha_2} p(\varrho)(\overline{f} - f(\varrho))^2$$
$$\times\left(\frac{|\psi''(\overline{f})|^q + |\psi''(f(\varrho))|^q}{2}\right)^{\frac{1}{q}} d\varrho. \tag{17}$$

Proof. By choosing $g(\varrho)$ as substitute of $\overline{f}$ in (14), we arrive at the inequality (17). □

Remark 2. *A comparable improvement to the one presented on the right side of (17) is also achieved in inequality (9) by Khan et al. in [49].*

The ensuing theorem elegantly provides an improvement for the integral majorization inequality by harnessing the power of the Hölder's and Jensen's inequalities.

Theorem 3. *Let $\psi : (\beta_1, \beta_2) \to \mathbb{R}$ be a twice differentiable function such that ψ'' is integrable and $|\psi''|$ is concave. Also, assume that $f, g : [\alpha_1, \alpha_2] \to (\beta_1, \beta_2)$, $p : [\alpha_1, \alpha_2] \to [0, \infty)$ are integrable functions. Then*

$$\begin{aligned}\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho - \int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(g(\varrho))d\varrho \\ \leq \frac{1}{2}\int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho) - f(\varrho)\Big)^2\left|\psi''\left(\frac{2g(\varrho) + f(\varrho)}{3}\right)\right|d\varrho \\ - \int_{\alpha_1}^{\alpha_2} p(\varrho)(g(\varrho) - f(\varrho))\psi'(g(\varrho))d\varrho. \end{aligned} \tag{18}$$

Proof. By employing the property of the absolute function, we can rewrite identity (6) as:

$$\begin{aligned}\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho - \int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(g(\varrho))d\varrho \\ \leq \int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho) - f(\varrho)\Big)^2 \int_0^1 \left|t\psi''\Big(tg(\varrho) + (1-t)f(\varrho)\Big)\right|dtd\varrho \\ - \int_{\alpha_1}^{\alpha_2} p(\varrho)(g(\varrho) - f(\varrho))\psi'(g(\varrho))d\varrho. \end{aligned} \tag{19}$$

At the moment, leveraging Jensen's inequality on the right side of (19), we achieve:

$$\begin{aligned}\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho - \int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(g(\varrho))d\varrho \leq -\int_{\alpha_1}^{\alpha_2} p(\varrho)(g(\varrho) - f(\varrho))\psi'(g(\varrho))d\varrho \\ +\frac{1}{2}\int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho) - f(\varrho)\Big)^2\left|\psi''\left(\frac{\int_0^1 t\Big(tg(\varrho) + (1-t)f(\varrho)\Big)dt}{\int_0^1 tdt}\right)\right|d\varrho. \end{aligned} \tag{20}$$

Upon simplification of the inequality (20), we attain the improvement articulated in (18). □

A further improvement of the Jensen inequality is outlined in the forthcoming corollary, which can directly be followed from Theorem 3.

Corollary 3. *Presume that the hypotheses of Corollary 1 are fulfilled but instead of $|\psi''|^q$ convexity, assume that $|\psi''|$ is a concave function. Then*

$$\frac{1}{p^*}\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho - \psi(\bar{f}) \leq \frac{1}{2p^*}\int_{\alpha_1}^{\alpha_2} p(\varrho)(\bar{f} - f(\varrho))^2\left|\psi''\left(\frac{2\bar{f} + f(\varrho)}{3}\right)\right|d\varrho. \tag{21}$$

Proof. By substituting $\bar{f}$ for $g(\varrho)$ in the inequality (18), we arrive at (21). □

Remark 3. *The improvement of the Jensen inequality derived in inequality (12) in the article [49] will resemble the improvement presented in (21), when we consider $q = 1$ in (12).*

The upcoming theorem offers a further refinement of the majorization inequality, which can be derived by employing both the Hölder inequality and Jensen's inequality.

Theorem 4. *Let us suppose that all statements in Theorem 2 hold true but instead of $|\psi''|^q$ convexity assume that $|\psi''|^q$ is concave function for $q > 1$. Then*

$$\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho - \int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(g(\varrho))d\varrho \leq -\int_{\alpha_1}^{\alpha_2} p(\varrho)(g(\varrho) - f(\varrho))\psi'(g(\varrho))d\varrho$$

$$+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\int_{\alpha_1}^{\alpha_2}p(\varrho)\Big(g(\varrho)-f(\varrho)\Big)^2\left|\psi''\left(\frac{g(\varrho)+f(\varrho)}{2}\right)\right|d\varrho. \tag{22}$$

Proof. The inequality (23) can be deduced by employing the Hölder inequality to the expression on the right-hand side of (19):

$$\begin{aligned}
&\int_{\alpha_1}^{\alpha_2}p(\varrho)\psi(f(\varrho))d\varrho-\int_{\alpha_1}^{\alpha_2}p(\varrho)\psi(g(\varrho))d\varrho\leq-\int_{\alpha_1}^{\alpha_2}p(\varrho)(g(\varrho)-f(\varrho))\psi'(g(\varrho))d\varrho\\
&+\int_{\alpha_1}^{\alpha_2}p(\varrho)\Big(g(\varrho)-f(\varrho)\Big)^2\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\psi''\Big(tg(\varrho)+(1-t)f(\varrho)\Big)\right|^q dt\right)^{\frac{1}{q}}d\varrho\\
&=\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\int_{\alpha_1}^{\alpha_2}p(\varrho)\Big(g(\varrho)-f(\varrho)\Big)^2\left(\int_0^1\left|\psi''\Big(tg(\varrho)+(1-t)f(\varrho)\Big)\right|^q dt\right)^{\frac{1}{q}}d\varrho\\
&\quad-\int_{\alpha_1}^{\alpha_2}p(\varrho)(g(\varrho)-f(\varrho))\psi'(g(\varrho))d\varrho.
\end{aligned}\tag{23}$$

To arrive at (24), simply apply Jensen's inequality to the expression on the right-hand side of (23):

$$\begin{aligned}
&\int_{\alpha_1}^{\alpha_2}p(\varrho)\psi(f(\varrho))d\varrho-\int_{\alpha_1}^{\alpha_2}p(\varrho)\psi(g(\varrho))d\varrho\leq-\int_{\alpha_1}^{\alpha_2}p(\varrho)(g(\varrho)-f(\varrho))\psi'(g(\varrho))d\varrho\\
&+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\int_{\alpha_1}^{\alpha_2}p(\varrho)\Big(g(\varrho)-f(\varrho)\Big)^2\left|\psi''\left(\int_0^1\Big(tg(\varrho)+(1-t)f(\varrho)\Big)dt\right)\right|d\varrho.
\end{aligned}\tag{24}$$

By evaluating the integral on the right-hand side of (24), we reach to (22). □

The subsequent corollary articulates another improvement of the Jensen inequality stemming from Theorem 4.

Corollary 4. *Presume that all the hypotheses of Corollary 2 are fulfilled but instead of* $|\psi''|^q$ *convexity, assume that* $|\psi''|^q$ *is concave function. Then*

$$\begin{aligned}
&\frac{1}{p^*}\int_{\alpha_1}^{\alpha_2}p(\varrho)\psi(f(\varrho))d\varrho-\psi(\overline{f})\\
&\qquad\leq\frac{1}{p^*}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\int_{\alpha_1}^{\alpha_2}p(\varrho)(\overline{f}-f(\varrho))^2\left|\psi''\left(\frac{\overline{f}+f(\varrho)}{2}\right)\right|d\varrho.
\end{aligned}\tag{25}$$

Proof. By simply changing $g(\varrho)$ with $\overline{f}$ in inequality (22), we acquire (25). □

Remark 4. *Khan et al. [49] presented an improvement of the Jensen inequality in (16) in the absolute sense, which is alike to our improvement of the Jensen inequality stated in (25).*

Theorem 5 elegantly leverages the definition of convex functions along with the power mean inequality, furnishing an improvement for the discernible majorization inequality.

Theorem 5. *Presume that the stated assumptions in Theorem 1 hold true, then*

$$\begin{aligned}
&\int_{\alpha_1}^{\alpha_2}p(\varrho)\psi(f(\varrho))d\varrho-\int_{\alpha_1}^{\alpha_2}p(\varrho)\psi(g(\varrho))d\varrho\leq-\int_{\alpha_1}^{\alpha_2}p(\varrho)(g(\varrho)-f(\varrho))\psi'(g(\varrho))d\varrho\\
&\quad+\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\int_{\alpha_1}^{\alpha_2}p(\varrho)\Big(g(\varrho)-f(\varrho)\Big)^2\left(\frac{2|\psi''(g(\varrho))|^q+|\psi''(f(\varrho))|^q}{6}\right)^{\frac{1}{q}}d\varrho.
\end{aligned}\tag{26}$$

Proof. When the power mean inequality is applied to the expression on the right side of (10), then we reach to the below inequality:

$$\begin{aligned}
&\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho - \int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(g(\varrho))d\varrho \leq -\int_{\alpha_1}^{\alpha_2} p(\varrho)(g(\varrho)-f(\varrho))\psi'(g(\varrho))d\varrho \\
&\quad + \int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho)-f(\varrho)\Big)^2 \left(\int_0^1 t dt\right)^{1-\frac{1}{q}} \left(\int_0^1 t\left|\psi''\Big(tg(\varrho)+(1-t)f(\varrho)\Big)\right|^q dt\right)^{\frac{1}{q}} d\varrho \\
&\quad = \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho)-f(\varrho)\Big)^2 \left(\int_0^1 t\left|\psi''\Big(tg(\varrho)+(1-t)f(\varrho)\Big)\right|^q dt\right)^{\frac{1}{q}} d\varrho \\
&\quad\quad - \int_{\alpha_1}^{\alpha_2} p(\varrho)(g(\varrho)-f(\varrho))\psi'(g(\varrho))d\varrho. \qquad (27)
\end{aligned}$$

By leveraging the convex nature of the function $|\psi''|^q$ on the right side of (27), we arrive at:

$$\begin{aligned}
&\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho - \int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(g(\varrho))d\varrho \\
&\qquad \leq -\int_{\alpha_1}^{\alpha_2} p(\varrho)(g(\varrho)-f(\varrho))\psi'(g(\varrho))d\varrho + \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \int_{\alpha_1}^{\alpha_2} p(\varrho)\Big(g(\varrho)-f(\varrho)\Big)^2 \\
&\qquad\quad \times \left(|\psi''(\sigma)|^q \int_0^1 t^2 dt + |\psi''(\varrho)|^q \int_0^1 t(1-t)dt\right)^{\frac{1}{q}} d\varrho. \qquad (28)
\end{aligned}$$

Now, arriving at the improvement elucidated in (26) is a straightforward process, simply by evaluating the integrals provided in (28). □

The subsequent corollary provides an improvement of the Jensen inequality as an immediate consequence of Theorem 5.

Corollary 5. *Assuming the stipulated assumptions of Corollary 1 hold, then*

$$\begin{aligned}
\frac{1}{p^*}\int_{\alpha_1}^{\alpha_2} p(\varrho)\psi(f(\varrho))d\varrho - \psi(\bar{f}) \quad &\leq \quad \frac{1}{p^*}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \int_{\alpha_1}^{\alpha_2} p(\varrho)(\bar{f}-f(\varrho))^2 \\
&\quad \times \left(\frac{2|\psi''(\bar{f})|^q + |\psi''(f(\varrho))|^q}{6}\right)^{\frac{1}{q}} d\varrho. \qquad (29)
\end{aligned}$$

Proof. Upon changing $g(\varrho)$ with $\bar{f}$ in inequality (26), we gain (29). □

Remark 5. *In inequality (19), Khan and colleagues [49] achieved an improvement of the Jensen inequality in an absolute sense, akin to the improvement outlined in (29).*

Remark 6. *In the article [50], Basir et al. discovered the discrete versions of the above aforementioned results.*

3. Analysis of the Superiority of Key Findings

The objective of this section is to highlight the scenarios under which the main results, given in Theorem 1 to Theorem 5 concerning majorization inequalities will become more accurate and excellent. The discussion regarding the superiority and bitterness of the main results primarily hinges on the integral value: $\int_{\alpha_1}^{\alpha_2} p(\varrho)\big(g(\varrho)-f(\varrho)\big)\psi'\big(g(\varrho)\big)d\varrho$. The critical question here is under which context the results will be fine. So, the answer is that the improvements will be attractive if the integral: $\int_{\alpha_1}^{\alpha_2} p(\varrho)\big(g(\varrho)-f(\varrho)\big)\psi'\big(g(\varrho)\big)d\varrho$ is non-negative. In this segment, we will investigate the conditions that render the integral: $\int_{\alpha_1}^{\alpha_2} p(\varrho)\big(g(\varrho)-f(\varrho)\big)\psi'\big(g(\varrho)\big)d\varrho$ non-negative.

At this juncture, we aim to elucidate and shed light on the specific scenario in which the integral:

$$\int_{\alpha_1}^{\alpha_2} p(\varrho)(g(\varrho)-f(\varrho))\psi'(g(\varrho))d\varrho$$

leading to a state where its value is non-negative:

- **A** By employing the proof concept outlined in [41], one can substantiate that the expression "$\int_{\alpha_1}^{\alpha_2} p(\varrho)\big(g(\varrho)-f(\varrho)\big)\psi'\big(g(\varrho)\big)d\varrho$" remains non-negative under specified conditions that the function ψ is convex, with further restrictions:

  (i) $g(\varrho)$ is a monotonically decreasing function that fulfills

$$\int_{\alpha_1}^{k} p(\varrho)f(\varrho)d\varrho \leq \int_{\alpha_1}^{k} p(\varrho)g(\varrho)d\varrho, \qquad k\in[\alpha_1,\alpha_2] \tag{30}$$

  and

$$\int_{\alpha_1}^{\alpha_2} p(\varrho)f(\varrho)d\varrho = \int_{\alpha_1}^{\alpha_2} p(\varrho)g(\varrho)d\varrho. \tag{31}$$

  OR

  (ii) $g(\varrho)$ is a monotonically increasing function with

$$\int_{\alpha_1}^{k} p(\varrho)f(\varrho)d\varrho \geq \int_{\alpha_1}^{k} p(\varrho)g(\varrho)d\varrho, \qquad k\in[\alpha_1,\alpha_2] \tag{32}$$

  and

$$\int_{\alpha_1}^{\alpha_2} p(\varrho)f(\varrho)d\varrho = \int_{\alpha_1}^{\alpha_2} p(\varrho)g(\varrho)d\varrho. \tag{33}$$

- **B** If ψ is a convex function and both g and $f-g$ exhibit monotonicity in the same direction and satisfy the condition expressed as

$$\int_{\alpha_1}^{\alpha_2} p(\varrho)f(\varrho)d\varrho = \int_{\alpha_1}^{\alpha_2} p(\varrho)g(\varrho)d\varrho, \tag{34}$$

  then by following the proof methodology outlined in the Theorem 6 stated in [40], it becomes evident that the expression "$\int_{\alpha_1}^{\alpha_2} p(\varrho)\big(g(\varrho)-f(\varrho)\big)\psi'\big(g(\varrho)\big)d\varrho$" is non-negative.
- **C** Applying the proof methodology of Theorem 7 given in [40], considering the stipulations that the function ψ is both increasing and convex and further assuming that g and $f-g$ are monotonicity functions in a similar direction, while also satisfying the inequality given by:

$$\int_{\alpha_1}^{\alpha_2} p(\varrho)f(\varrho)d\varrho \geq \int_{\alpha_1}^{\alpha_2} p(\varrho)g(\varrho)d\varrho, \tag{35}$$

  one can establish that $\int_{\alpha_1}^{\alpha_2} p(\varrho)\big(g(\varrho)-f(\varrho)\big)\psi'\big(g(\varrho)\big)d\varrho \geq 0$.

4. Applications in Information Theory

Information theory is a mathematical and conceptual framework that quantifies the storage, transmission, and processing of information. It originated with the work of Claude Shannon in the mid-20th century and has since become a fundamental discipline in various fields, including computer science, electrical engineering, telecommunications, linguistics, and neuroscience.

The present section is dedicated to elucidating the practical implications of key findings within the realm of information theory. The envisaged applications will encompass the provisioning of rigorous bounds for the esteemed Csiszár and Kullback-Leibler divergences, Shannon entropy, and Bhattacharyya coefficient. To present the desired estimates for the aforementioned concepts, we first define them.

Definition 1. *Let $\psi:[\beta_1,\beta_2]\to\mathbb{R}$, $f:[\alpha_1,\alpha_2]\to\mathbb{R}$, $p:[\alpha_1,\alpha_2]\to(0,\infty)$ be integrable functions such that $\psi\circ\frac{f}{p}$ is integrable and $\frac{f(\varrho)}{p(\varrho)}\in[\beta_1,\beta_2]$ for $\varrho\in[\alpha_1,\alpha_2]$. Then, the Csiszár divergence is defined as follows:*

$$C_\psi(p,f) = \int_{\alpha_1}^{\alpha_2} p(\varrho)\psi\left(\frac{f(\varrho)}{p(\varrho)}\right)d\varrho.$$

Definition 2. *Let $p, f : [\alpha_1, \alpha_2] \to \mathbb{R}$ be any positive probability density functions. Then*

- *The* **Shannon entropy** *is defined by:*

$$\boldsymbol{S}_p = -\int_{\alpha_1}^{\alpha_2} p(\varrho)\log p(\varrho)d\varrho.$$

- *The* **Kullback–Liebler divergence** *is defined as follows:*

$$\boldsymbol{K}_{p;f} = \int_{\alpha_1}^{\alpha_2} p(\varrho)\log\left(\frac{p(\varrho)}{f(\varrho)}\right)d\varrho.$$

- *The* **Bhattacharyya coefficient** *is defined as:*

$$\boldsymbol{B}_{p;f} = \int_{\alpha_1}^{\alpha_2} \sqrt{p(\varrho)f(\varrho)}d\varrho.$$

Theorem 6. *Let $\psi : (\beta_1, \beta_2) \to \mathbb{R}$ be a twice differentiable function such that ψ'' is integrable and $|\psi''|^q$ $(q > 1)$ is convex. Also, assume that $f, g : [\alpha_1, \alpha_2] \to \mathbb{R}$, $p : [\alpha_1, \alpha_2] \to (0, \infty)$ are integrable functions and $\frac{f(\varrho)}{p(\varrho)}, \frac{g(\varrho)}{p(\varrho)} \in (\beta_1, \beta_2)$ for $\varrho \in [\alpha_1, \alpha_2]$. Then*

$$\begin{aligned} &C_\psi(p,g) - C_\psi(p,f) \\ &\quad\le \int_{\alpha_1}^{\alpha_2} \frac{(g(\varrho)-f(\varrho))^2}{p(\varrho)}\left(\frac{(q+1)\left|\psi''\left(\frac{g(\varrho)}{p(\varrho)}\right)\right|^q + \left|\psi''\left(\frac{f(\varrho)}{p(\varrho)}\right)\right|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}} d\varrho \\ &\qquad - \int_{\alpha_1}^{\alpha_2} (g(\varrho)-f(\varrho))\psi'\left(\frac{g(\varrho)}{p(\varrho)}\right)d\varrho. \end{aligned} \tag{36}$$

Proof. To arrive at the inequality (36), substitute $\frac{f(\varrho)}{p(\varrho)}$ for $f(\varrho)$ and $\frac{g(\varrho)}{p(\varrho)}$ for $g(\varrho)$ in (9). □

Corollary 6. *Presume that $p, g : [\alpha_1, \alpha_2] \to (0, \infty)$ are integrable functions with $\int_{\alpha_1}^{\alpha_2} p(\varrho)d\varrho = 1$ and $q > 1$, then*

$$\begin{aligned} \boldsymbol{S}_p - \int_{\alpha_1}^{\alpha_2} p(\varrho)\log\left(\frac{g(\varrho)}{p(\varrho)}\right)d\varrho \;\le\; & \int_{\alpha_1}^{\alpha_2} \frac{(g(\varrho)-1)^2}{p(\varrho)}\left(\frac{(q+1)(\frac{p(\varrho)}{g(\varrho)})^{2q} + x_j^{2q}}{(q+1)(q+2)}\right)^{\frac{1}{q}} d\varrho \\ & + \int_{\alpha_1}^{\alpha_2} (g(\varrho)-1)\left(\frac{p(\varrho)}{g(\varrho)}\right)d\varrho. \end{aligned} \tag{37}$$

Proof. By taking $\psi(\varrho) = -\log\varrho$, $\varrho > 0$ and $f(\varrho) = 1$ in (36), we receive (37). □

Corollary 7. *Let $p, f, g : [\alpha_1, \alpha_2] \to (0, \infty)$ be integrable functions with $\int_{\alpha_1}^{\alpha_2} p(\varrho)d\varrho = \int_{\alpha_1}^{\alpha_2} g(\varrho)$ $d\varrho = \int_{\alpha_1}^{\alpha_2} f(\varrho)d\varrho = 1$. If $q > 1$, then*

$$\begin{aligned} \boldsymbol{K}_{p,g} - \boldsymbol{K}_{p,f} \;\le\; & \int_{\alpha_1}^{\alpha_2} \frac{(g(\varrho)-f(\varrho))^2}{p(\varrho)}\left(\frac{(q+1)(\frac{p(\varrho)}{g(\varrho)})^{2q} + (\frac{p(\varrho)}{f(\varrho)})^{2q}}{(q+1)(q+2)}\right)^{\frac{1}{q}} d\varrho \\ & + \int_{\alpha_1}^{\alpha_2} (g(\varrho)-f(\varrho))\left(\frac{p(\varrho)}{g(\varrho)}\right)d\varrho. \end{aligned} \tag{38}$$

Proof. By applying inequality (36) for $\psi(\varrho) = -\log\varrho$, $\varrho > 0$, we derive (38). □

Corollary 8. *Under the assumptions of Corollary 7, the below inequality holds:*

$$\begin{aligned} &\boldsymbol{B}_{p,f} - \boldsymbol{B}_{p,g} \\ &\leq \frac{1}{4}\int_{\alpha_1}^{\alpha_2} \frac{\big(g(\varrho)-f(\varrho)\big)^2}{p(\varrho)} \left(\frac{(q+1)\left(\frac{p(\varrho)}{g(\varrho)}\right)^{\frac{3q}{2}} + \left(\frac{p(\varrho)}{f(\varrho)}\right)^{\frac{3q}{2}}}{(q+1)(q+2)}\right)^{\frac{1}{q}} d\varrho \\ &+\frac{1}{2}\int_{\alpha_1}^{\alpha_2} \big(g(\varrho)-f(\varrho)\big)\sqrt{\frac{p(\varrho)}{g(\varrho)}}d\varrho. \end{aligned} \tag{39}$$

Proof. Assume $\psi(\varrho) = -\sqrt{\varrho}$, $\varrho > 0$ in (36), we achieve (39). □

Remark 7. *Likewise, we can present applications of Theorem 2, Theorem 3, Theorem 4, and Theorem 5 for the Csiszár divergences, Kullback–Leibler divergence, Bhattacharyya coefficient, and Shannon entropy.*

5. Conclusions

Mathematical inequalities keep a fundamental and vital role in mathematical analysis, optimization, statistics, economics, and an array of other disciplines. They serve as potent instruments for demonstrating and addressing a diverse spectrum of both mathematical and real-world challenges spanning different domains. The elegant and foundational concept of convexity serves as a key catalyst for the advancement and formulation of inequalities. In this article, we introduced a series of improvements for majorization inequality in the continuous sense through twice differentiable functions. The principal roles that have been played in the development of desired improvements are the notion of convexity, Jensen inequality, Hölder inequality, and power mean inequality. It is worth noting that certain previous enhancements of the Jensen inequality are deduced as direct consequences of our principal findings. We have also provided criteria that define the conditions under which these improvements yield superior and more accurate estimates for the majorization difference. Furthermore, we have explored the applications of our principal discovery within the realm of information theory. These applications present bounds for the Csiszár and Kullback–Leibler divergences, Bhattacharyya coefficient and Shannon entropy.

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