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Exact and Approximate Solutions for Some Classes of the Inhomogeneous Pantograph Equation

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Abstract: The standard pantograph delay equation (SPDDE) is one of the famous delay models. This standard model is basically homogeneous in nature and it has been extensively studied in the literature. However, the studies on the general inhomogeneous form of such a model seem rare. This paper considers the inhomogeneous pantograph delay equation (IPDDE) with a kind of arbitrary inhomogeneous term. This arbitrary inhomogeneous term is used in different forms to generate various classes of IPDDEs. The solutions of the present classes are obtained in closed series forms which satisfy the criteria of convergence. Also, the exact solutions are determined for these classes under a certain relation between the given initial condition of the model and the initial value of the inhomogeneous term. Several classes are generated and solved when the inhomogeneous term takes the form of trigonometric, exponential, and hyperbolic functions. Some existing results in the literature are recovered as special cases of the present ones. Moreover, the behaviors of the obtained solutions are demonstrated through graphs for various kinds of IPDDEs.

Keywords: pantograph; delay; inhomogeneous; exact solution; series solution

MSC: 34k06

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1. Introduction

Real-life phenomena are often modeled by means of ordinary differential equations (ODEs). However, the ODEs are not effective to model natural systems with memory effect as a consequence of locality of the ordinary derivative. Thus, physical phenomena with memory/hereditary properties are usually modeled by incorporating nonlocal components, such as delays, in the form of delay differential equations (DDEs). A famous DDE is known as the pantograph delay differential equation (PDDE). The standard pantograph delay differential equation (SPDDE) is in the form: $w'(r) = aw(r) + bw(cr), w(0) = \lambda$. This model has a particular application in the interaction between the pantograph device and the overhead wire when controlling the electric trains.

The pantograph problem gained the interest of many researchers and it has been solved utilizing different numerical techniques such as the Chebyshev polynomials [1], the collocation method [2,3], the Bernstein polynomials [4], and the spectral methods [5,6]. Furthermore, the SPDDE has been solved very recently using two analytical techniques based on the Adomian decomposition method (ADM) [7] and the homotopy perturbation method (HPM) [8]. In addition, a high-order version of the pantograph equation was investigated by the authors [9] in which a new rational approximation is proposed. Particular types of the SPDDEs have been addressed in [10]. Moreover, a novel closed solution is determined for the SPDDE by El-Zahar and Ebaid [11] using an ansatz method that was based on assuming the solution in a specific form. Also, the SPDDE was analyzed using the Laplace transform (LT) by Alrebdi and Al-Jeaid [12]. When $a = -1$, $b = c = 1/\gamma$ $(\gamma > 1)$, the SPDDE reduces to the Ambartusmian delay model, which has been studied by several authors [13–15].

In this paper, an inhomogeneous version of the pantograph delay differential equation (IPDDE) is considered in the form:

$$w'(r) = aw(r) + bw(cr) - af(r) - bf(cr) + f'(r), \quad c \neq 0, 1, \quad 0 \leq r \leq 1, \tag{1}$$

under the initial condition (IC):

$$w(0) = \lambda. \tag{2}$$

This class reduces to some models in the literature [10,16,17] at particular choices of λ, a, b, c, and $f(r)$ (should be continuous and differentiable on [0,1]).

The objective of our work is to find the general solution of the problem (1)–(2) in terms of $f(r)$ via a straightforward analysis. Our approach is mainly based on transforming the IPDDE (1)–(2) to the SPDDE by the aide of a suitable transformation. Accordingly, the solution of the present class will be constructed in a general closed series form in terms of $f(r)$. It will also be shown that the current closed series form satisfies the convergence criteria. Furthermore, it will be declared in a subsequent section that the results in the literature can be directly recovered as special cases of the current ones.

2. Analysis

Theorem 1. *The problem (1)–(2) reduces to:*

$$y'(r) = ay(r) + by(cr), \quad c \neq 0, 1, \tag{3}$$

with the IC:

$$y(0) = \lambda - f(0), \tag{4}$$

under the transformation:

$$w(r) = y(r) + f(r). \tag{5}$$

Proof. Let us rewrite Equation (1) as

$$w'(r) = a[w(r) - f(r)] + b[w(cr) - f(cr)] + f'(r). \tag{6}$$

Assume that

$$y(r) = w(r) - f(r), \tag{7}$$

then

$$w'(r) = y'(r) + f'(r), \quad y(cr) = w(cr) - f(cr), \tag{8}$$

and hence Equation (6) becomes

$$y'(r) = ay(r) + by(cr). \tag{9}$$

Substituting (2) into (7) gives

$$y(0) = \lambda - f(0), \tag{10}$$

which completes the proof. □

Theorem 2. *The power series solution (PSS) of the problem (3)–(4) is given by*

$$y(r) = (\lambda - f(0)) \sum_{i=0}^{\infty} \left(-\frac{b}{a} : c \right)_i \frac{(ar)^i}{i!}, \quad a \neq 0, \tag{11}$$

where $\left(-\frac{b}{a}:c\right)_i$ *is given by the product:*

$$\left(-\frac{b}{a}:c\right)_i=\prod_{k=1}^{i}\left(1+\frac{b}{a}c^{k-1}\right). \tag{12}$$

Proof. Following the authors [7,8], one can obtain the PSS of the problem (3)–(4) in the form:

$$y(r)=(\lambda-f(0))\left[1+\sum_{i=1}^{\infty}\left(\prod_{k=1}^{i}\left(a+bc^{k-1}\right)\right)\frac{r^i}{i!}\right], \tag{13}$$

which can be written as

$$y(r)=(\lambda-f(0))\sum_{i=0}^{\infty}\left(\prod_{k=1}^{i}\left(a+bc^{k-1}\right)\right)\frac{r^i}{i!}, \tag{14}$$

where the product $\prod_{k=1}^{i}\left(a+bc^{k-1}\right)=1$ for $i=0$. To express the solution $y(r)$ in terms of quantum calculus notation we can write

$$y(r)=(\lambda-f(0))\sum_{i=0}^{\infty}\left(\prod_{k=1}^{i}a\times\prod_{k=1}^{i}\left(1+\frac{b}{a}c^{k-1}\right)\right)\frac{r^i}{i!},\quad a\neq 0. \tag{15}$$

Using the definition:

$$(p:q)_i=\prod_{k=0}^{i-1}\left(1-pq^k\right)\quad \text{or}\quad (p:q)_i=\prod_{k=1}^{i}\left(1-pq^{k-1}\right), \tag{16}$$

for $p=-\frac{b}{a}$ and $q=c$, then

$$\prod_{k=1}^{i}\left(1+\frac{b}{a}c^{k-1}\right)=\left(-\frac{b}{a}:c\right)_i. \tag{17}$$

From (15) and (17) we obtain

$$y(r)=(\lambda-f(0))\sum_{i=0}^{\infty}\left(-\frac{b}{a}:c\right)_i\frac{(ar)^i}{i!},\quad a\neq 0, \tag{18}$$

where the equality $\prod_{k=1}^{i}a=a^i$ is used, this completes the proof. □

Remark 1. *The convergence of the series in the right hand side of Equations (11) or (18) can be easily justified by ratio test as follows. The series can be written as* $\sum_{i=0}^{\infty}d_i$, *where* $d_i=\left(-\frac{b}{a}:c\right)_i\frac{(ar)^i}{i!}$. *Following [16] and applying the ratio test we have*

$$\lim_{i\to\infty}\left|\frac{d_{i+1}}{d_i}\right|=\lim_{i\to\infty}\left|\frac{ar}{i+1}\frac{\prod_{k=1}^{i+1}\left(1+\frac{b}{a}c^{k-1}\right)}{\prod_{k=1}^{i}\left(1+\frac{b}{a}c^{k-1}\right)}\right|=\lim_{i\to\infty}\left|\frac{ar}{i+1}\left(1+\frac{b}{a}c^i\right)\right|=0\ \text{if}\ |c|<1,$$

and accordingly the series converges for all $0<r<1$.

Lemma 1. *The solution of the problem (1)–(2) is given by*

$$w(r) = f(r) + (\lambda - f(0)) \sum_{i=0}^{\infty} \left(-\frac{b}{a} : c\right)_i \frac{(ar)^i}{i!}, \quad a \neq 0. \tag{19}$$

Proof. The proof follows immediately by substituting the result of Theorem 2 given in Equation (11) into the transformation given by Equation (5). □

3. Solution for a Cass with $f(r)$ in Trigonometric Form

Lemma 2. *For all $a \in \mathbb{R} - \{0\}$, $b \in \mathbb{R}$ and $c \in \mathbb{R} - \{0, 1\}$, the solution of the IPDDE:*

$$w'(r) = aw(r) + bw(cr) - a\sin(r) - b\sin(cr) + \cos(r), \quad w(0) = \lambda, \tag{20}$$

is given by

$$w(r) = \sin(r) + \lambda \sum_{i=0}^{\infty} \left(-\frac{b}{a} : c\right)_i \frac{(ar)^i}{i!}. \tag{21}$$

Proof. Comparing Equation (20) with Equation (1) implies that $f(r) = \sin(r)$ which yields $f(0) = 0$. On inserting these values into the result of Lemma 1 gives Equation (21). □

Lemma 3. *The exact solution of the IPDDE:*

$$w'(r) = aw(r) + bw(cr) - a\sin(r) - b\sin(cr) + \cos(r), \quad w(0) = 0, \tag{22}$$

is given by

$$w(r) = \sin(r). \tag{23}$$

Proof. Since $w(0) = 0$, then $\lambda = 0$ and hence the proof follows immediately by substituting this value into the result of Lemma 2 given by Equation (21). □

Remark 2. *The initial value problem (IVP) (22) has been analyzed in Ref. [10] using an efficient numerical method. However, the current analysis determined the exact solution of this IVP in a straightforward/direct manner.*

Examples in Generalized Trigonometric Forms

The above results can be generalized as follows. The class (20) with arbitrary argument for the trigonometric functions reads

$$w'(r) = aw(r) + bw(cr) - a\sin(\varpi r) - b\sin(c\varpi r) + \varpi\cos(\varpi r), \quad w(0) = \lambda, \tag{24}$$

where the solution takes the form:

$$w(r) = \sin(\varpi r) + \lambda \sum_{i=0}^{\infty} \left(-\frac{b}{a} : c\right)_i \frac{(ar)^i}{i!}, \quad \varpi \in \mathbb{R}. \tag{25}$$

It is clear that the IVP (24) reduces to the IVP (20) at $\varpi = 1$ and also the solution (25) agrees with the solution (21) in this case. One can also find that the solution of the class:

$$w'(r) = aw(r) + bw(cr) - a\cos(\Omega r) - b\cos(c\Omega r) - \Omega\sin(\Omega r), \quad w(0) = \lambda, \tag{26}$$

is given by

$$w(r) = \cos(\Omega r) + (\lambda - 1) \sum_{i=0}^{\infty} \left(-\frac{b}{a} : c\right)_i \frac{(ar)^i}{i!}, \quad \Omega \in \mathbb{R}. \tag{27}$$

Another advantage of the class (24) arises when $\varpi \to 0$. In this case, the class (24) becomes the SPDDE:

$$w'(r) = aw(r) + bw(cr), \quad w(0) = \lambda, \tag{28}$$

with the solution:

$$w(r) = \lambda \sum_{i=0}^{\infty} \left(-\frac{b}{a} : c\right)_i \frac{(ar)^i}{i!}. \tag{29}$$

4. Solution for a Class with $f(r)$ in Exponential Form

Theorem 3. *For $a, b \in \mathbb{R} - \{0\}$ and $c \in \mathbb{R} - \{0, 1\}$, the solution of the IPDDE:*

$$w'(r) = aw(r) + bw(cr) - he^{acr}, \quad w(0) = \lambda, \tag{30}$$

is given by

$$w(r) = \frac{h}{b}e^{ar} + \left(\lambda - \frac{h}{b}\right) \sum_{i=0}^{\infty} \left(-\frac{b}{a} : c\right)_i \frac{(ar)^i}{i!}. \tag{31}$$

Proof. Assume that $f(r)$ satisfies the equation

$$-af(r) + f'(r) = 0, \tag{32}$$

then the class in Equations (1) and (2) reduces to

$$w'(r) = aw(r) + bw(cr) - bf(cr), \quad w(0) = \lambda. \tag{33}$$

Solving Equation (32) for $f(r)$ gives

$$f(r) = \xi e^{ar}, \tag{34}$$

where ξ is arbitrary constant. Accordingly, Equation (33) becomes

$$w'(r) = aw(r) + bw(cr) - b\xi e^{acr}, \quad w(0) = \lambda. \tag{35}$$

The solution of the model (35) is determined by inserting (34) into the general solution (19), hence

$$w(r) = \xi e^{ar} + (\lambda - \xi) \sum_{i=0}^{\infty} \left(-\frac{b}{a} : c\right)_i \frac{(ar)^i}{i!}. \tag{36}$$

Note that $f(0) = \xi$ in this case. The model (35) and its solution (36) can be written in the following equivalent forms

$$w'(r) = aw(r) + bw(cr) - he^{acr}, \quad w(0) = \lambda, \tag{37}$$

and

$$w(r) = \frac{h}{b}e^{ar} + \left(\lambda - \frac{h}{b}\right) \sum_{i=0}^{\infty} \left(-\frac{b}{a} : c\right)_i \frac{(ar)^i}{i!}, \quad b \neq 0, \tag{38}$$

where $h = b\xi$, this completes the proof. □

Remark 3. *If the IC $w(0) = \lambda$ is chosen such that $\lambda = \frac{h}{b}$, then the class:*

$$w'(r) = aw(r) + bw(cr) - he^{acr}, \quad w(0) = \frac{h}{b}, \tag{39}$$

has the exact solution:

$$w(r) = \frac{h}{b}e^{ar}, \quad b \neq 0. \tag{40}$$

This remark is useful to derive the exact solution of the following examples.

Examples of Exponential Order

In Ref. [10], the authors considered the IPDDE:

$$w'(r) = -w(r) + \frac{1}{2}w(cr) - \frac{1}{2}e^{-cr}, \quad w(0) = 1, \tag{41}$$

where a numerical approach has been applied to solve the IVP (41). In addition, the authors [10] compared their numerical results with an available exact solution given by $w(r) = e^{-r}$. Here, we show that such an exact solution can be directly obtained through the current work. On comparing Equation (41) with Equation (39), we obtain $a = -1$, $b = \frac{1}{2}$ and $h = \frac{1}{2}$. Substituting these values into Equation (40), we directly obtain $w(r) = e^{-r}$ which is the desired exact solution.

Another example was discussed by the authors [17] in the form:

$$w'(r) = -w(r) + \frac{c}{2}w(cr) - \frac{c}{2}e^{-cr}, \quad w(0) = 1, \tag{42}$$

which has been solved using Taylor method. In [17], the authors mentioned that the Taylor method has far better results than the collocation method through comparisons with the exact solution $w(r) = e^{-r}$. However, this exact solution can also be evaluated in a similar manner. This case yields $a = -1$, $b = \frac{c}{2}$ and $h = \frac{c}{2}$. Implementing such values, one can find from Equation (40) that $w(r) = e^{-r}$, which is also the same exact solution.

In view of Remark 3, it is clear that the model given by the IVP:

$$w'(r) = aw(r) + bw(cr) - be^{acr}, \quad w(0) = 1, \tag{43}$$

has the exact solution:

$$w(r) = e^{ar}, \tag{44}$$

$\forall\, a, b \in \mathbb{R}$ and $c \in \mathbb{R} - \{0, 1\}$. Similarly, one can find the exact solution for the IVP:

$$w'(r) = aw(r) + bw(cr) + be^{acr}, \quad w(0) = -1, \tag{45}$$

in the form:

$$w(r) = -e^{ar}. \tag{46}$$

A final observation is about the case of trivial solution for the model (33) in the absence of the inhomogeneous term. If the coefficient h of the exponential function vanishes, it can be easily deduced that the IVP:

$$w'(r) = aw(r) + bw(cr), \quad w(0) = 0, \tag{47}$$

has the trivial solution $w(0) = 0\ \forall\, a, b \in \mathbb{R}$.

5. Solution for a Class with $f(r)$ in Hyperbolic Form

Following the analysis of the previous sections, one can determine the solution of the IPDDE:

$$w'(r) = aw(r) + bw(cr) - a\sinh(r) - b\sinh(cr) + \cosh(r), \quad w(0) = \lambda, \tag{48}$$

as

$$w(r) = \sinh(r) + \lambda \sum_{i=0}^{\infty} \left(-\frac{b}{a} : c\right)_i \frac{(ar)^i}{i!}. \tag{49}$$

This result is achieved via Lemma 1 and through comparing Equation (48) with Equation (1), this yields $f(r) = \sinh(r)$ and $f(0) = 0$.

Proceeding as above, the class (48) with arbitrary argument θ for the hyperbolic functions becomes

$$w'(r) = aw(r) + bw(cr) - a\sinh(\theta r) - b\sinh(c\theta r) + \theta\cosh(\theta r), \quad w(0) = \lambda, \tag{50}$$

and the solution reads

$$w(r) = \sinh(\theta r) + \lambda \sum_{i=0}^{\infty} \left(-\frac{b}{a} : c\right)_i \frac{(ar)^i}{i!}, \quad \theta \in \mathbb{R}. \tag{51}$$

When $\theta = 1$, the IVP (50) is equivalent to the IVP (48) while the solution (51) takes the form (49). Moreover, the solution of the class:

$$w'(r) = aw(r) + bw(cr) - a\cosh(\rho r) - b\cosh(c\rho r) - \rho\sinh(\rho r), \quad w(0) = \lambda, \tag{52}$$

is given by

$$w(r) = \cosh(\rho r) + (\lambda - 1) \sum_{i=0}^{\infty} \left(-\frac{b}{a} : c\right)_i \frac{(ar)^i}{i!}, \quad \rho \in \mathbb{R}. \tag{53}$$

Remark 4. *Although the solutions in this section are obtained for classes containing two specific hyperbolic functions,* sinh *and* cosh, *the solutions for additional classes containing other types of hyperbolic functions can also be generated by following the same procedure explained above.*

6. Main Results

In the previous sections, we showed that the exact solutions of several classes can be obtained under certain constraints of the parameters contained in each of these classes. If such constraints are not satisfied, then the exact solutions will not be available. In these cases, the approximate solutions can be employed instead of the exact ones which may not be available. For the numerical purpose, the general series solution given by Equation (19) for the problem (1)–(2) with the arbitrary function $f(r)$ should contains a finite number of terms instead of infinity. By this, the infinity in Equation (19) is replaced by a finite number n of the series. Hence, the n-term approximate solution $\Phi_n(r)$ for the problem (1)–(2) is expressed as

$$\Phi_n(r) = f(r) + (\lambda - f(0)) \sum_{i=0}^{n-1} \left(-\frac{b}{a} : c\right)_i \frac{(ar)^i}{i!}, \quad a \neq 0. \tag{54}$$

Here, it should be noted that $\Phi_n(r)$ $(n > 1)$ represents the approximations for the class:

$$w'(r) = aw(r) + bw(cr) - af(r) - bf(cr) + f'(r), \quad c \neq 0, 1, \quad w(0) = \lambda, \quad 0 \leq r \leq 1, \tag{55}$$

where $\Phi_n(r)$ transforms to exact solution if the function $f(r)$ is chosen to satisfy the condition $f(0) = \lambda$. So, this section aims to extract samples of numerical solutions for the classes discussed in the previous sections such that $f(0) \neq \lambda$. It may be also important to point out that the values of the parameters a, b, and c are specified to extract numerical solutions such that they ensure the convergence of the series (54), see remark 1 for such conditions.

Let us begin with the case of the trigonometric function $f(r) = \sin(r)$. In this case the approximations $\Phi_n(r)$ for the class:

$$w'(r) = aw(r) + bw(cr) - a\sin(r) - b\sin(cr) + \cos(r), \quad w(0) = \lambda, \tag{56}$$

is

$$\Phi_n(r) = \sin(r) + \lambda \sum_{i=0}^{n-1} \left(-\frac{b}{a} : c\right)_i \frac{(ar)^i}{i!}, \quad a \neq 0, \quad n > 1. \tag{57}$$

The curves of the approximations $\Phi_n(r)$ at $n = 4, 5, 6, 7$ are displayed in Figure 1 which shows the satisfaction of convergence of these curves to a certain one. Similarly, the approximations $\Phi_n(r)$ at $n = 5, 6, 7, 8$ are depicted in Figure 2 for the problem (55) with $f(r) = \sin(\varpi r)$, where

$$\Phi_n(r) = \sin(\varpi r) + \lambda \sum_{i=0}^{n-1} \left(-\frac{b}{a} : c\right)_i \frac{(ar)^i}{i!}, \quad a \neq 0, \quad n > 1, \tag{58}$$

for the IVP:

$$w'(r) = aw(r) + bw(cr) - a\sin(\varpi r) - b\sin(c\varpi r) + \varpi\cos(\varpi r), \quad w(0) = \lambda. \tag{59}$$

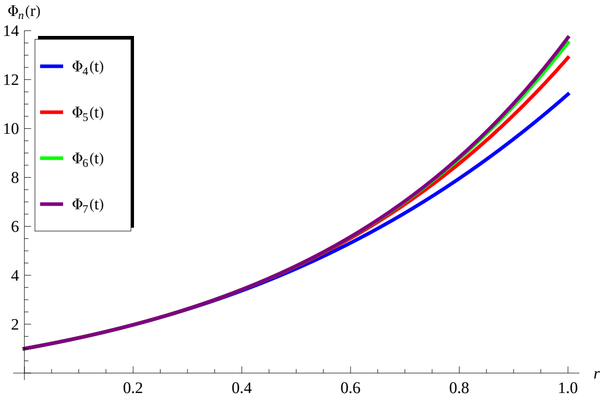

Figure 1. Plots of the approximations $\Phi_n(r)$ $(n = 4, 5, 6, 7)$ in Equation (54) when $a = 2$, $b = 1$, $c = \frac{1}{2}$, $\lambda = 1$, and $f(r) = \sin(r)$.

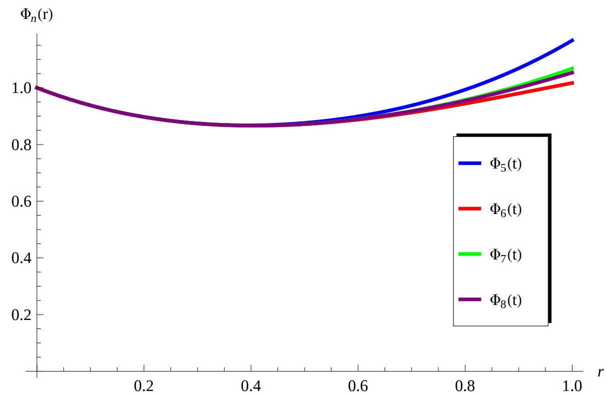

Figure 2. Plots of the approximations $\Phi_n(r)$ $(n = 5, 6, 7, 8)$ in Equation (54) when $\varpi = \frac{\pi}{2}$, $a = -2$, $b = 1$, $c = -\frac{1}{2}$, $\lambda = 1$, and $f(r) = \sin(\varpi r)$.

The convergence of the approximations $\Phi_5(r)$, $\Phi_6(r)$, $\Phi_7(r)$, and $\Phi_8(r)$ is also clear in Figure 2.

The other Figures 3–7 confirm the conclusion that the approximation (54) gives accurate approximate solution for the problem (55) with other cases for the function $f(r)$, where $f(r) = \cos(\Omega r)$ (Figures 3 and 4), $f(r) = \frac{h}{b}e^{ar}$ (Figure 5), $f(r) = \sinh(\theta r)$ (Figure 6), and $f(r) = \cosh(\rho r)$ (Figure 7).

The last part of this discussion focuses on estimating the error through the residual errors $RE_n(r)$ defined by

$$RE_n(r) = \left|a\Phi_n(r) + b\Phi_n(cr) - af(r) - bf(cr) + f'(r)\right|, \quad 0 \leq r \leq 1. \tag{60}$$

Figures 8–10 display the numerical calculations of the residuals $RE_n(r)$ at $n = 20, 21, 22$ for three different cases of the function $f(r)$ with selected values of the other parameters. The results show acceptable error and the advantage of the proposed approach is obvious.

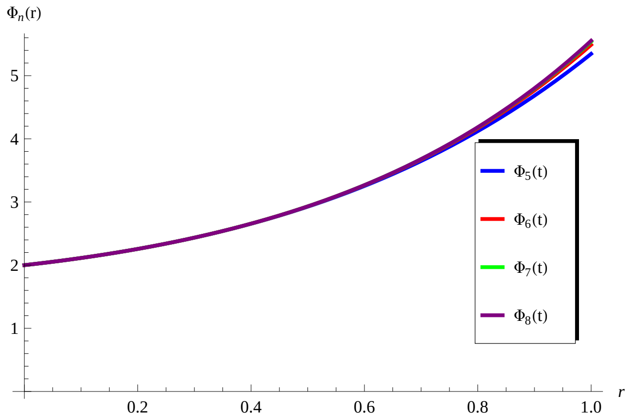

Figure 3. Plots of the approximations $\Phi_n(r)$ $(n = 5, 6, 7, 8)$ in Equation (54) when $\Omega = \frac{\pi}{8}$, $a = 2$, $b = -1$, $c = -\frac{2}{3}$, $\lambda = 2$, and $f(r) = \cos(\Omega r)$.

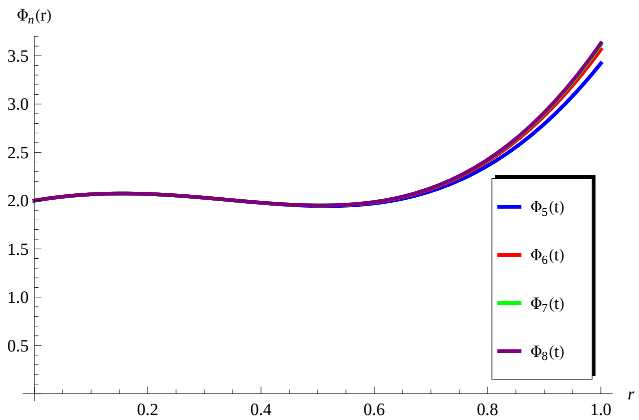

Figure 4. Plots of the approximations $\Phi_n(r)$ $(n = 5, 6, 7, 8)$ in Equation (54) when $\Omega = -\pi$, $a = 2$, $b = -1$, $c = -\frac{2}{3}$, $\lambda = 2$, and $f(r) = \cos(\Omega r)$.

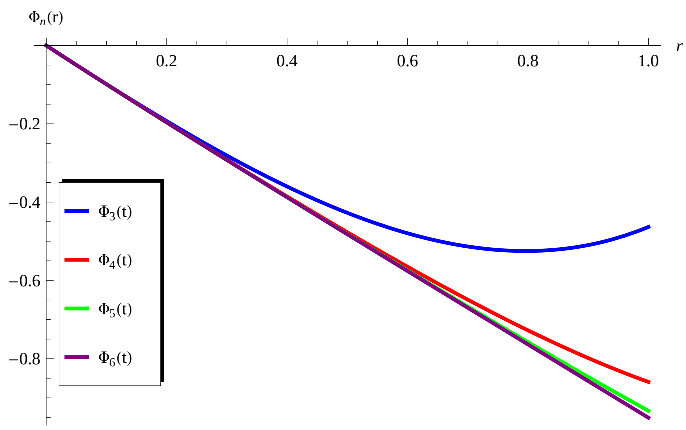

Figure 5. Plots of the approximations $\Phi_n(r)$ $(n = 3, 4, 5, 6)$ in Equation (54) when $h = 1$, $a = 1$, $b = \frac{1}{2}$, $c = -\frac{4}{5}$, $\lambda = 0$, and $f(r) = \frac{h}{b}e^{ar}$.

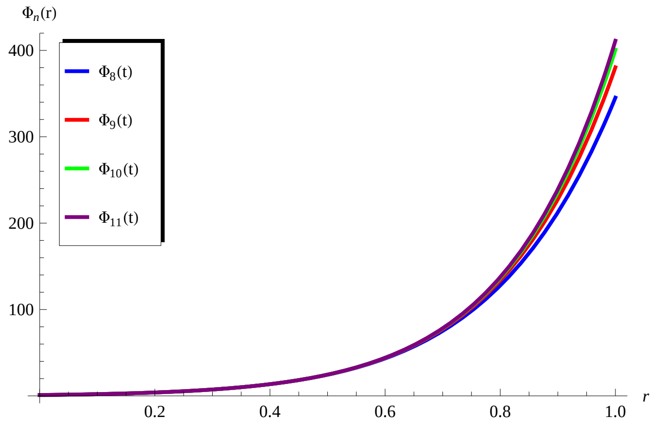

Figure 6. Plots of the approximations $\Phi_n(r)$ $(n = 8, 9, 10, 11)$ in Equation (54) when $\theta = \frac{\pi}{5}$, $a = 5$, $b = 2$, $c = \frac{3}{4}$, $\lambda = 1$, and $f(r) = \sinh(\theta r)$.

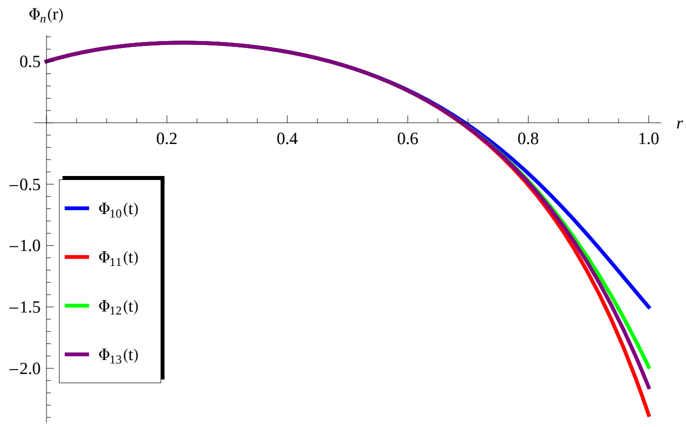

Figure 7. Plots of the approximations $\Phi_n(r)$ $(n = 10, 11, 12, 13)$ in Equation (54) when $\rho = \frac{\pi}{5}$, $a = -5$, $b = 2$, $c = -\frac{3}{4}$, $\lambda = \frac{1}{2}$, and $f(r) = \cosh(\rho r)$.

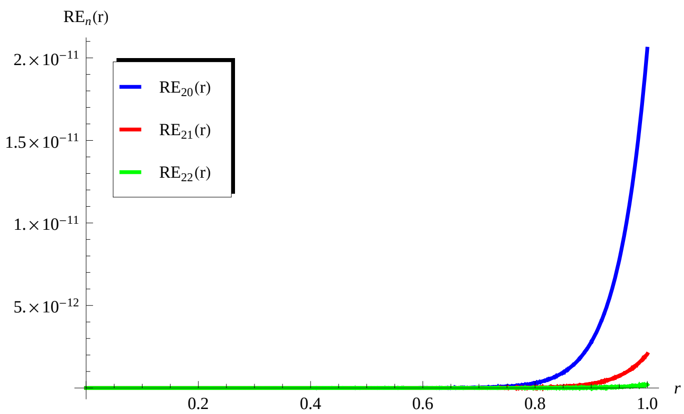

Figure 8. The residual errors $RE_n(r)$ $(n = 20, 21, 22)$ in Equation (60) when $a = 2$, $b = 1$, $c = \frac{1}{2}$, $\lambda = 1$, and $f(r) = \sin(r)$.

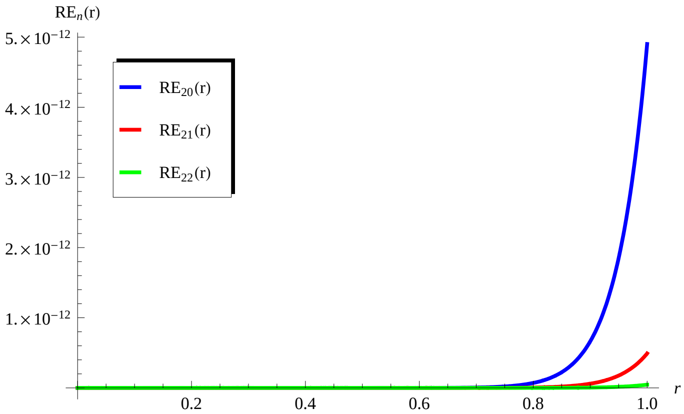

Figure 9. The residual errors $RE_n(r)$ $(n = 20, 21, 22)$ in Equation (60) when $\varpi = \frac{\pi}{2}$, $a = -2$, $b = 1$, $c = -\frac{1}{2}$, $\lambda = 1$, and $f(r) = \sin(\varpi r)$.

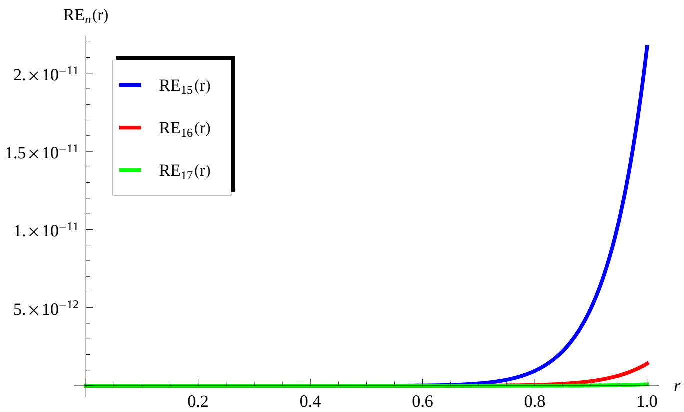

Figure 10. The residual errors $RE_n(r)$ $(n = 15, 16, 17)$ in Equation (60) when $h = 1$, $a = 1$, $b = \frac{1}{2}$, $c = -\frac{4}{5}$, $\lambda = 0$, and $f(r) = \frac{h}{b}e^{ar}$.

7. Conclusions

In this paper, a class of inhomogeneous pantograph delay equations (IPDDEs) with an arbitrary inhomogeneous term was analyzed. Different forms of such arbitrary inhomogeneous term were implemented to generate various classes of IPDDEs. The solutions of several classes were successfully accomplished in closed-series forms that satisfy the convergence criteria. It was also shown that the exact solutions for the considered classes are available if a certain relation between the given initial condition of the model and the initial value of the inhomogeneous term is satisfied. The current approach was capable to generate several classes for different forms of the inhomogeneous term such as trigonometric, exponential, and hyperbolic functions. One of the advantages of our analysis is that the existing results in the literature were efficiently recovered as special cases of the present ones. Finally, various plots were introduced for the behaviors of the obtained solutions for various kinds of IPDDEs.

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