axioms

Article

Fermatean Fuzzy Fairly Aggregation Operators with Multi-Criteria Decision-Making

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Abstract: A Fermatean fuzzy set (FRFS) is the extension of a fuzzy set, an intuitionistic fuzzy set, and a Pythagorean fuzzy set, and is used in different fields. Unlike other fuzzy structures, the sum of cubes of membership grades in FRFSs approximates a unit interval, increasing uncertainty. In this study, we intend to provide unique operational rules and aggregation operators (AOs) inside a Fermatean fuzzy environment. To develop a fair remedy for the membership degree and non-membership degree features of "Fermatean fuzzy numbers (FRFNs)", our solution introduces new neutral or fair operating principles, which include the concept of proportional distribution. Based on the suggested operating principles, we provide the "Fermatean fuzzy fairly weighted average operator and the Fermatean fuzzy fairly ordered weighted averaging operator". Our suggested AOs provide more generalized, reliable, and exact data than previous techniques. Combining the recommended AOs with multiple decision-makers and partial weight information under FRFSs, we also devised a technique for "multi-criteria decision-making". To illustrate the application of our novel method, we provide an example of the algorithm's effectiveness in addressing decision-making challenges.

Keywords: aggregation operators; fairly operations; linear programming; decision-making

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1. Introduction

Decision-making is the process of selecting a feasible alternative from options that are available. It is a vital ability that serves a significant purpose in both our personal and professional lives. Analyzing the facts at hand, considering the merits of several courses of action, and settling on one that seems to provide the most potential for success are all steps involved in the decision-making process. Decision-making is crucial for personal growth and development. Making decisions allows individuals to take control of their lives and create a path toward success. It enables individuals to make informed choices that can help them achieve their goals and improve their lives. Without decision-making, an individual may end up living a life that is not aligned with their values and goals. For instance, deciding to pursue further education or training can lead to personal growth and development. This decision requires careful analysis of the available options and evaluating the potential benefits and drawbacks. Making an informed decision in this case can lead to a successful career, financial stability, and personal fulfillment [1].

Effective decision-making requires problem-solving skills. The process of decision-making involves analyzing information, evaluating options, and selecting the best course of action. This process requires critical thinking, analytical skills, and problem-solving abilities. Decision-making is critical for business success. Business leaders are often required to make decisions that can significantly impact their organizations. These decisions may involve investments, partnerships, product development, and hiring decisions. Making informed decisions based on careful analysis and evaluation can lead to business success and growth. Effective decision-making can also improve relationships. Decision-making requires individuals to consider the needs and opinions of others. By involving others in the decision-making process, individuals can build trust, foster collaboration, and improve communication. In personal relationships, effective decision-making can lead to better communication, increased understanding, and improved outcomes. In professional relationships, effective decision-making can lead to better teamwork, increased productivity, and improved performance [2].

Green supplier selection is the process of identifying and selecting suppliers who are committed to sustainable practices and environmental responsibility. It is a critical process that can significantly impact an organization's sustainability efforts. Effective green supplier selection requires careful analysis and evaluation of the available options, and this process involves decision-making. Effective decision-making can lead to improved sustainability performance, risk management, cost savings, improved stakeholder engagement, and a competitive advantage [2]. Organizations that prioritize sustainability and environmental responsibility should engage in effective decision-making when selecting suppliers. This process requires careful evaluation and analysis of potential suppliers' sustainability performance, which can help organizations select suppliers who meet their sustainability standards and values. Effective decision-making in green supplier selection can lead to a more sustainable and responsible business practices, and this can positively impact an organization's financial performance and reputation [3].

Real-world issues, such as agglomeration, segmentation, decision-making, and supplier evaluation, rely heavily on ambiguities. Without processing imprecise, confusing, or confusing data, DMs cannot acquire reliable outcomes. As per Zadeh [4], a fuzzy set (FS) is an extension of a classical set, and the membership function quantifies the membership levels of FS elements. Researchers have extensively examined Zadeh's fuzzy sets (FSs) from several multi-criteria decision-making (MCDM)-related perspectives. Atanassov [5] developed "intuitionistic fuzzy sets" (IFS) to broaden the concept of FS, where the sum of the membership degree (MSD) and non-membership degree (NMSD) is less than or equal to 1. This attribute is valued by many IFS experts who solve real-world situations. IFS is incapable of resolving complex decision-making challenges because MSD and NMSD must be equal. Yager's "Pythagorean fuzzy set" (PFS) is a significant generalization of the IFS that restricts the sum of squares of MSD and NMSD to 1 [6]. It has several applications, including selection. PFS is not always authorized. For instance, a panel of experts was divided into two groups in order to assess training schools. The first group of experts assigned an MSD rating of 0.92, whereas the second group assigned an NMSD rating of 0.84. The square sum of MSD and NMSD surpassed one. IFS and PFS were unable to illustrate this. To overcome this problem, Senapati and Yager [7] proposed the notion of FRFS as a logical extension of IFS and PFS. In an FRFS, the MSD and NMSD values of an object's MSD and NMSD are limited by 1 cube. The FRFS theory plays a significant role in several areas since it is a sound concept used for dealing with contradictory and incorrect data in an FRF framework.

Numerous research studies have focused exclusively on FRFSs. When faced with multiple potential resolutions to a given problem, the concept of AOs is essential in determining the most favorable choice. Mesiar and Pap [8], in their seminal work, introduced the notion of aggregating infinite sequences. Significant progress has been made in the field of FRFSs, as evidenced by the extensive body of research conducted by many scholars [9–13]. The notion of bipolar picture fuzzy was introduced by Riaz et al. [14]. Several researchers have introduced various aggregation operators (AOs) for different extensions of fuzzy sets (FSs). Jana et al. [15], Sitara et al. [16], Riaz and Hashmi [17], Iampan et al. [18], Riaz et al. [19], Ashraf and Abdullah [20], Saha et al. [21], and Farid and Riaz [22] are among those who have made significant contributions in this area.

Jana et al. [23] proposed the concept of trapezoidal neutrosophic AOs, in addition to bipolar fuzzy Dombi-prioritized AOs [24], neutrosophic Dombi-powered AOs [25], and trapezoidal neutrosophic normal AOs [26]. The work conducted by Yang et al. [27] introduced continuous ordered weighted averaging AOs as a means to handle interval-valued q-ROF information. The authors also demonstrated the application of these AOs in the assessment of aesthetic design quality for smartwatches. Chen et al. [28] proposed the utilization of enhanced ordered weighted averaging AOs in the context of MCDM. Chen et al. [29] introduced power-average aggregation operators for a proportional hesitant fuzzy linguistic word set. These operators were subsequently utilized in an online product recommender system designed to facilitate consumer decision-making. Jana and Pal presented a novel hybrid method that incorporates dynamical elements in the design of decision-making processes, drawing upon the GRA approach as a foundation [30]. Linear Diophantine fuzzy soft-max AOs [31] and a numerically validated approach to modeling water hammer phenomena are presented in [32].

We are aware that the present AOs, despite the fact that they address MCDM issues within the context of the FRF framework, do not always demonstrate impartiality when dealing with both MSD and NMSD scenarios. Existing AOs, for instance, are unable to discern between circumstances where a DM considers all MSD and NMSD possibilities, which results in biased decision-making. As a result, we require new operations that are capable of treating both MSD and NMSD choices in an equitable manner and ensuring that FRFN operations remain neutral. Thus, we came up with two brand new operations that assess positive, neutral, and negative memberships based on the proportional distribution rules of all functions. As a result, the primary purposes of this publication are to address these concerns and devise FRF-based operations that are fair and impartial.

The format of this paper is as follows: In Section 2, we delve into the fundamental concepts of FRFSs and provide a comprehensive explanation. In Section 3, we present a range of equitable FRFN procedures. In Section 4, using proposed AOs, a decision-making algorithm has developed. Now, let us proceed to Section 5, where a case study with numerical examples will be presented to showcase the benefits of the suggested approach. Section 6 of the study presents a thorough summary of the noteworthy discoveries derived from the research.

2. Preliminaries

Some fundamental topics linked with FRFSs are introduced in this section.

Definition 1. *An FRFS $\wp$ in the universal set $¥$ is given as*

$$\wp = \{\langle \check{\Upsilon}, \mathfrak{V}_{\wp}(\check{\Upsilon}), \mathscr{I}_{\wp}(\check{\Upsilon}) | \check{\Upsilon} \in ¥ \rangle\} \tag{1}$$

where $\mathfrak{V}_{\wp}(\check{\Upsilon}), \mathscr{I}_{\wp}(\check{\Upsilon}) \in [0,1]$, s.t. $0 \leq \mathfrak{V}^3_{\wp}(\check{\Upsilon}) + \mathscr{I}^3_{\wp}(\check{\Upsilon}) \leq 1 \; \forall \; \check{\Upsilon} \in ¥$. $\mathfrak{V}_{\wp}(\check{\Upsilon}), \mathscr{I}_{\wp}(\check{\Upsilon})$, we denote MPD, and NMPD, respectively, for some $\check{\Upsilon} \in ¥$.

we denote this pair as $\aleph^{\zeta} = (\mathfrak{V}_{\aleph^{\zeta}}, \mathscr{I}_{\aleph^{\zeta}})$, throughout this article, and call it FRFN.

Definition 2. *The "score function" (SF) corresponding to FRFN is presented, as shown in the following*

$$W(\aleph^{\zeta}) = \mathfrak{V}^3_{\aleph^{\zeta}} - \mathscr{I}^3_{\aleph^{\zeta}} \tag{2}$$

In certain circumstances, the aforementioned function may be unable to differentiate between FRFNs, making it difficult to tell which is greater. To address this, an "accuracy function" D of $\aleph^{\zeta}$ can be defined as follows:

$$D(\aleph^{\zeta}) = \mathfrak{V}^3_{\aleph^{\zeta}} + \mathscr{I}^3_{\aleph^{\zeta}} \tag{3}$$

Definition 3. *Let* $\aleph^{\zeta}{}_1 = \langle \mathfrak{V}_1, \mathscr{I}_1 \rangle$ *and* $\aleph^{\zeta}{}_2 = \langle \mathfrak{V}_2, \mathscr{I}_2 \rangle$ *be two FRFNs, then*

$$\aleph^{\zeta c}_1 = \left\langle \mathscr{I}_1, \mathfrak{V}_1 \right\rangle \tag{4}$$

$$\aleph^{\zeta}{}_1 \vee \aleph^{\zeta}{}_2 = \left\langle max\{\mathfrak{V}_1, \mathfrak{V}_2\}, min\{\mathscr{I}_1, \mathscr{I}_2\} \right\rangle \tag{5}$$

$$\aleph^{\zeta}{}_1 \wedge \aleph^{\zeta}{}_2 = \left\langle min\{\mathfrak{V}_1, \mathfrak{V}_2\}, max\{\mathscr{I}_1, \mathscr{I}_2\} \right\rangle \tag{6}$$

$$\aleph^{\zeta}{}_1 \oplus \aleph^{\zeta}{}_2 = \left\langle \sqrt[3]{\mathfrak{V}_1^3 + \mathfrak{V}_2^3 - \mathfrak{V}_1^3\mathfrak{V}_2^3}, \mathscr{I}_1\mathscr{I}_2 \right\rangle \tag{7}$$

$$\aleph^{\zeta}{}_1 \otimes \aleph^{\zeta}{}_2 = \left\langle \mathfrak{V}_1\mathfrak{V}_2, \sqrt[3]{\mathscr{I}_1^3 + \mathscr{I}_2^3 - \mathscr{I}_1^3\mathscr{I}_2^3} \right\rangle \tag{8}$$

$$\sigma\aleph^{\zeta}{}_1 = \left\langle \sqrt[3]{1 - (1 - \mathfrak{V}_1^3)^{\sigma}}, \mathscr{I}_1^{\sigma} \right\rangle \tag{9}$$

$$\aleph^{\zeta\sigma}_1 = \left\langle \mathfrak{V}_1^{\sigma}, \sqrt[3]{1 - (1 - \mathscr{I}_1^3)^{\sigma}} \right\rangle \tag{10}$$

To assist this sort of study, a new score function is added, whose value should always be between -1 and 1. The new scoring function is symbolized by $\nabla^{\gamma}(R) = \frac{1+\mathfrak{V}_R^3 - \mathscr{I}_R^3}{2}$ and $0 \le \nabla^{\gamma}(R) \le 1$.

Definition 4. *Let* $\aleph^{\zeta}{}_1 = \langle \mathfrak{V}_1, \mathscr{I}_1 \rangle$ *and* $\aleph^{\zeta}{}_2 = \langle \mathfrak{V}_2, \mathscr{I}_2 \rangle$ *be two FRFNs and* $\chi^{\varrho}, \chi^{\varrho}{}_1, \chi^{\varrho}{}_2 > 0$ *be the real numbers, then we have,*

1. $\aleph^{\zeta}{}_1 \oplus \aleph^{\zeta}{}_2 = \aleph^{\zeta}{}_2 \oplus \aleph^{\zeta}{}_1$
2. $\aleph^{\zeta}{}_1 \otimes \aleph^{\zeta}{}_2 = \aleph^{\zeta}{}_2 \otimes \aleph^{\zeta}{}_1$
3. $\chi^{\varrho}(\aleph^{\zeta}{}_1 \oplus \aleph^{\zeta}{}_2) = (\chi^{\varrho}\aleph^{\zeta}{}_1) \oplus (\chi^{\varrho}\aleph^{\zeta}{}_2)$
4. $(\aleph^{\zeta}{}_1 \otimes \aleph^{\zeta}{}_2)^{\chi^{\varrho}} = \aleph^{\zeta\chi^{\varrho}}_1 \otimes \aleph^{\zeta\chi^{\varrho}}_2$
5. $(\chi^{\varrho}{}_1 + \chi^{\varrho}{}_2)\aleph^{\zeta}{}_1 = (\chi^{\varrho}{}_1\aleph^{\zeta}{}_1) \oplus (\chi^{\varrho}{}_2\aleph^{\zeta}{}_2)$
6. $\aleph^{\zeta\chi^{\varrho}{}_1+\chi^{\varrho}{}_2}_1 = \aleph^{\zeta\chi^{\varrho}{}_1}_1 \otimes \aleph^{\zeta\chi^{\varrho}{}_2}_2$

If $\mathfrak{V}_{\aleph^{\zeta}{}_1} = \mathscr{I}_{\aleph^{\zeta}{}_1}$ *and* $\mathfrak{V}_{\aleph^{\zeta}{}_2} = \mathscr{I}_{\aleph^{\zeta}{}_2}$*, then from Definition 3, we have* $\mathfrak{V}_{\aleph^{\zeta}{}_1\oplus\aleph^{\zeta}{}_2} \neq \mathscr{I}_{\aleph^{\zeta}{}_1\oplus\aleph^{\zeta}{}_2}$, $\mathfrak{V}_{\aleph^{\zeta}{}_1\otimes\aleph^{\zeta}{}_2} \neq \mathscr{I}_{\aleph^{\zeta}{}_1\otimes\aleph^{\zeta}{}_2}, \mathfrak{V}_{\chi^{\varrho}\aleph^{\zeta}{}_1} \neq \mathscr{I}_{\chi^{\varrho}\aleph^{\zeta}{}_1}, \mathfrak{V}_{\aleph^{\zeta\chi^{\varrho}}_1} \neq \mathscr{I}_{\aleph^{\zeta\chi^{\varrho}}_1}$*. Thus, none of the operations* $\aleph^{\zeta}{}_1 \oplus \aleph^{\zeta}{}_2$, $\aleph^{\zeta}{}_1 \otimes \aleph^{\zeta}{}_2, \chi^{\varrho}\aleph^{\zeta}{}_1, \aleph^{\zeta\chi^{\varrho}}_1$ *were found to be neutral or fair, indeed. Thus, our initial focus must be to develop some fair operations amongst FRFNs.*

3. Fairly AOs for FRFNs

In this section, the evolution and properties of fair AOs for FRFNs will be addressed.

3.1. FRFFWA Operator

Definition 5. *Consider that* $\aleph^{\zeta}{}_\Lambda = \langle \mathfrak{V}_\Lambda, \mathscr{I}_\Lambda \rangle$ *is the accumulation of FRFNs, and let FRFFWA,* $\Gamma^n \to \Gamma$*, be the mapping. if*

$$FRFFWA(\aleph^{\zeta}{}_1, \aleph^{\zeta}{}_2, \ldots \aleph^{\zeta}{}_e) = \left(\mathscr{A}^{\tau}{}_1 * \aleph^{\zeta}{}_1 \tilde{\oplus} \mathscr{A}^{\tau}{}_2 * \aleph^{\zeta}{}_2 \tilde{\oplus} \ldots, \tilde{\oplus} \mathscr{A}^{\tau}{}_e * \aleph^{\zeta}{}_e\right) \tag{11}$$

then the mapping FRFFWA is called the "Fermatean fuzzy fairly weighted averaging (FRFFWA) operator"; here, $\mathscr{L}^{\tau}{}_i$ *is the "weight vector" (WV) of* $\aleph^{\zeta}{}_\Lambda$ *with* $\mathscr{A}^{\tau}{}_\Lambda > 0$ *and* $\sum_{\Lambda=1}^{e} \mathscr{A}^{\tau}{}_\Lambda = 1$.

As demonstrated in the subsequent theorem, it is also possible to examine the FRFFWA operator from the perspective of operational principles.

Theorem 1. Consider $\aleph^{\zeta}{}_{\Lambda} = \langle \mathfrak{V}_{\Lambda}, \mathscr{I}_{\Lambda} \rangle$ is the accumulation of FRFNs; we can also find the FRFFWA operator by

$$FRFFWA(\aleph^{\zeta}{}_1, \aleph^{\zeta}{}_2, \ldots, \aleph^{\zeta}{}_e) = \begin{pmatrix} \sqrt[3]{\frac{\prod_{\Lambda=1}^{e} (\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{e} (\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}} + \prod_{\Lambda=1}^{e} (\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1 - \prod_{\Lambda=1}^{e} (1 - \mathfrak{V}^3{}_{\Lambda} - \mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)}, \\ \sqrt[3]{\frac{\prod_{\Lambda=1}^{e} (\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{e} (\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}} + \prod_{\Lambda=1}^{e} (\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1 - \prod_{\Lambda=1}^{e} (1 - \mathfrak{V}^3{}_{\Lambda} - \mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)}, \end{pmatrix}$$

Proof. By a mathematical induction.

For $e = 1$, we have $\aleph^{\zeta}{}_1 = \langle \mathfrak{V}_1, \mathscr{I}_1 \rangle$, and $\mathscr{A}^{\tau} = 1$.

$$FRFFWA(\aleph^{\zeta}{}_1) = \mathscr{A}^{\tau}{}_1 * \aleph^{\zeta}{}_1 = \begin{pmatrix} \sqrt[3]{\frac{(\mathfrak{V}^3{}_1)^{\mathscr{A}^{\tau}{}_1}}{(\mathfrak{V}^3{}_1)^{\mathscr{A}^{\tau}{}_1} + (\mathscr{I}^3{}_1)_1^{\mathscr{A}^{\tau}}} \times \left(1 - (1 - \mathfrak{V}^3{}_1 - \mathscr{I}^3{}_1)^{\mathscr{A}^{\tau}{}_1}\right)}, \\ \sqrt[3]{\frac{(\mathscr{I}^3{}_1)^{\mathscr{A}^{\tau}{}_1}}{(\mathfrak{V}^3{}_1)^{\mathscr{A}^{\tau}{}_1} + (\mathscr{I}^3{}_1)_1^{\mathscr{A}^{\tau}}} \times \left(1 - (1 - \mathfrak{V}^3{}_1 - \mathscr{I}^3{}_1)^{\mathscr{A}^{\tau}{}_1}\right)} \end{pmatrix}$$

The theory is valid for $e = 1$, and we make the assumption that it remains valid when $e = g$, i.e.,

$$FRFFWA(\aleph^{\zeta}{}_1, \aleph^{\zeta}{}_2, \ldots, \aleph^{\zeta}{}_g) = \begin{pmatrix} \sqrt[3]{\frac{\prod_{\Lambda=1}^{g} (\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{g} (\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}} + \prod_{\Lambda=1}^{g} (\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1 - \prod_{\Lambda=1}^{g} (1 - \mathfrak{V}^3{}_{\Lambda} - \mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)}, \\ \sqrt[3]{\frac{\prod_{\Lambda=1}^{g} (\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{g} (\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}} + \prod_{\Lambda=1}^{g} (\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1 - \prod_{\Lambda=1}^{g} (1 - \mathfrak{V}^3{}_{\Lambda} - \mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)} \end{pmatrix}$$

We will prove for $e = g + 1$.

$$FRFFWA(\aleph^{\zeta}{}_1, \aleph^{\zeta}{}_2, \ldots, \aleph^{\zeta}{}_{g+1}) = FRFFWA(\aleph^{\zeta}{}_1, \aleph^{\zeta}{}_2, \ldots, \aleph^{\zeta}{}_g) \tilde{\oplus} (\mathscr{A}^{\tau}{}_{g+1} * \aleph^{\zeta}{}_{g+1})$$

$$= \begin{pmatrix} \sqrt[3]{\frac{\prod_{\Lambda=1}^{g} (\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{g} (\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}} + \prod_{\Lambda=1}^{g} (\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1 - \prod_{\Lambda=1}^{g} (1 - \mathfrak{V}^3{}_{\Lambda} - \mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)}, \\ \sqrt[3]{\frac{\prod_{\Lambda=1}^{g} (\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{g} (\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}} + \prod_{\Lambda=1}^{g} (\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1 - \prod_{\Lambda=1}^{g} (1 - \mathfrak{V}^3{}_{\Lambda} - \mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)} \end{pmatrix} \tilde{\oplus}$$

$$\begin{pmatrix} \sqrt[3]{\left(\frac{\mathfrak{V}^{3\,\mathscr{A}^{\tau}{}_{g+1}}_{\aleph^{\zeta}{}_{g+1}}}{\mathfrak{V}^{3\,\mathscr{A}^{\tau}{}_{g+1}}_{\aleph^{\zeta}{}_{g+1}} + \mathscr{I}^{3\,\mathscr{A}^{\tau}{}_{g+1}}_{\aleph^{\zeta}{}_{g+1}}}\right) \times \left(1 - \left(1 - \mathfrak{V}^3{}_{\aleph^{\zeta}{}_{g+1}} - \mathscr{I}^3{}_{\aleph^{\zeta}{}_{g+1}}\right)^{\mathscr{A}^{\tau}{}_{g+1}}\right)}, \\ \sqrt[3]{\left(\frac{\mathscr{I}^{3\,\mathscr{A}^{\tau}{}_{g+1}}_{\aleph^{\zeta}{}_{g+1}}}{\mathfrak{V}^{3\,\mathscr{A}^{\tau}{}_{g+1}}_{\aleph^{\zeta}{}_{g+1}} + \mathscr{I}^{3\,\mathscr{A}^{\tau}{}_{g+1}}_{\aleph^{\zeta}{}_{g+1}}}\right) \times \left(1 - \left(1 - \mathfrak{V}^3{}_{\aleph^{\zeta}{}_{g+1}} - \mathscr{I}^3{}_{\aleph^{\zeta}{}_{g+1}}\right)^{\mathscr{A}^{\tau}{}_{g+1}}\right)} \end{pmatrix}$$

$$= \left(\begin{array}{c} \sqrt[3]{\frac{\prod_{\Lambda=1}^{g}(\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}} \times (\mathfrak{V}^3{}_{g+1})^{\mathscr{A}^{\tau}{}_{g+1}}}{\prod_{\Lambda=1}^{g}(\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}} \times (\mathfrak{V}^3{}_{g+1})^{\mathscr{A}^{\tau}{}_{g+1}} + \prod_{\Lambda=1}^{g}(\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}} \times (\mathscr{I}^3{}_{g+1})^{\mathscr{A}^{\tau}{}_{g+1}}} \times \left(1 - \prod_{\Lambda=1}^{g}\left(1 - \mathfrak{V}^3{}_{\Lambda} - \mathscr{I}^3{}_{\Lambda}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}} \times \left(1 - \mathfrak{V}^3{}_{g+1} - \mathscr{I}^3{}_{g+1}\right)^{\mathscr{A}^{\tau}{}_{g+1}}\right)}, \\ \sqrt[3]{\frac{\prod_{\Lambda=1}^{g}(\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}} \times (\mathscr{I}^3{}_{g+1})^{\mathscr{A}^{\tau}{}_{g+1}}}{\prod_{\Lambda=1}^{g}(\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}} \times (\mathfrak{V}^3{}_{g+1})^{\mathscr{A}^{\tau}{}_{g+1}} + \prod_{\Lambda=1}^{g}(\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}} \times (\mathscr{I}^3{}_{g+1})^{\mathscr{A}^{\tau}{}_{g+1}}} \times \left(1 - \prod_{\Lambda=1}^{g}\left(1 - \mathfrak{V}^3{}_{\Lambda} - \mathscr{I}^3{}_{\Lambda}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}} \times \left(1 - \mathfrak{V}^3{}_{g+1} - \mathscr{I}^3{}_{g+1}\right)^{\mathscr{A}^{\tau}{}_{g+1}}\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[3]{\frac{\prod_{\Lambda=1}^{g+1}(\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{g+1}(\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}} + \prod_{\Lambda=1}^{g+1}(\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1 - \prod_{\Lambda=1}^{g+1}\left(1 - \mathfrak{V}^3{}_{\Lambda} - \mathscr{I}^3{}_{\Lambda}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)}, \\ \sqrt[3]{\frac{\prod_{\Lambda=1}^{g+1}(\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{g+1}(\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}} + \prod_{\Lambda=1}^{g+1}(\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1 - \prod_{\Lambda=1}^{g+1}\left(1 - \mathfrak{V}^3{}_{\Lambda} - \mathscr{I}^3{}_{\Lambda}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)} \end{array} \right)$$

Hence, the aforementioned statement is applicable to the equation e = g + 1. Consequently, in accordance with the principle of induction on the variable 'e,' the aforementioned conclusion is valid for any conceivable value of 'e.' □

Example 1. *Consider that* $\aleph^{\zeta}{}_1 = \langle 0.28, 0.28 \rangle$, $\aleph^{\zeta}{}_2 = \langle 0.45, 0.45 \rangle$, $\aleph^{\zeta}{}_3 = \langle 0.29, 0.29 \rangle$, *and* $\aleph^{\zeta}{}_3 = \langle 0.56, 0.56 \rangle$ *are four FRFNs with WV* $\aleph^{\zeta} = (0.10, 0.53, 0.10, 0.27)$, *then*

$$\sqrt[3]{\frac{\prod_{\Lambda=1}^{4}(\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{4}(\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}} + \prod_{\Lambda=1}^{4}(\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1 - \prod_{\Lambda=1}^{4}(1 - \mathfrak{V}^3{}_{\Lambda} - \mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)} = 0.470121,$$

$$\sqrt[3]{\frac{\prod_{\Lambda=1}^{4}(\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{4}(\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}} + \prod_{\Lambda=1}^{4}(\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1 - \prod_{\Lambda=1}^{4}(1 - \mathfrak{V}^3{}_{\Lambda} - \mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)} = 0.470121$$

$$\begin{aligned} & FRFFWA(\aleph^{\zeta}{}_1, \aleph^{\zeta}{}_2, \aleph^{\zeta}{}_3) \\ &= \left(\begin{array}{c} \sqrt[3]{\frac{\prod_{\Lambda=1}^{4}(\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{4}(\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}} + \prod_{\Lambda=1}^{4}(\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1 - \prod_{\Lambda=1}^{4}(1 - \mathfrak{V}^3{}_{\Lambda} - \mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)}, \\ \sqrt[3]{\frac{\prod_{\Lambda=1}^{4}(\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{4}(\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}} + \prod_{\Lambda=1}^{4}(\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1 - \prod_{\Lambda=1}^{4}(1 - \mathfrak{V}^3{}_{\Lambda} - \mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)} \end{array} \right) \\ &= (0.470121, 0.470121) \end{aligned}$$

The fundamental advantage of our proposed AOs is that the aggregation answer has equal MSD and NMSD; this is visible in the aggregation answer. If we use any other AOs for the aggregation of FRFNs, we would be unable to obtain the same MSD and NMSD when all input FRFNs have the same MSD and NMSD.

The suggested AO satisfies certain specific conditions, which will now be expounded upon in the subsequent section through the formulation of theorems.

Theorem 2. *Consider that $\aleph^{\zeta}{}_{\Lambda} = \langle \mathfrak{V}_{\Lambda}, \mathscr{I}_{\Lambda} \rangle$ and $\aleph^{\zeta}{}_{i^*} = \langle \mathfrak{V}_{i^*}, \mathscr{I}_{i^*} \rangle$ are the families of FRFNs, and also consider*

$$FRFFWA(\aleph^{\zeta}{}_{1}, \aleph^{\zeta}{}_{2}, \dots \aleph^{\zeta}{}_{e}) = \aleph^{\zeta} = \langle \mathfrak{V}, \mathscr{I}, \tau \rangle$$

and

$$FRFFWA(\aleph^{\zeta}{}_{1^*}, \aleph^{\zeta}{}_{2^*}, \dots \aleph^{\zeta}{}_{e^*}) = \aleph^{\zeta}{}_{*} = \langle \mathfrak{V}_{*}, \mathscr{I}_{*} \rangle.$$

Then,

$$\mathfrak{V}^3 + \mathscr{I}^3 \leq \mathfrak{V}^3{}_{*} + \mathscr{I}^3{}_{*}, \quad \textit{if} \quad \mathfrak{V}^3{}_{\Lambda} + \mathscr{I}^3{}_{\Lambda} \leq \mathfrak{V}^3{}_{i^*} + \mathscr{I}^3{}_{i^*}$$

Proof. By applying Theorem 1 on both collections of FRFNs, namely, $\aleph^{\zeta}{}_{\Lambda} = \langle \mathfrak{V}_{\Lambda}, \mathscr{I}_{\Lambda} \rangle$ and $\aleph^{\zeta}{}_{i^*} = \langle \mathfrak{V}_{i^*}, \mathscr{I}_{i^*} \rangle$, we have

$$\begin{aligned}
\mathfrak{V}^3 &= \frac{\prod_{\Lambda=1}^{e} \left(\mathfrak{V}^3{}_{\Lambda}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{e} \left(\mathfrak{V}^3{}_{\Lambda}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}} + \prod_{\Lambda=1}^{e} \left(\mathscr{I}^3{}_{\Lambda}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1 - \prod_{\Lambda=1}^{e} \left(1 - \mathfrak{V}^3{}_{\Lambda} - \mathscr{I}^3{}_{\Lambda}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}\right) \\
\mathscr{I}^3 &= \frac{\prod_{\Lambda=1}^{e} \left(\mathscr{I}^3{}_{\Lambda}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{e} \left(\mathfrak{V}^3{}_{\Lambda}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}} + \prod_{\Lambda=1}^{e} \left(\mathscr{I}^3{}_{\Lambda}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1 - \prod_{\Lambda=1}^{e} \left(1 - \mathfrak{V}^3{}_{\Lambda} - \mathscr{I}^3{}_{\Lambda}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{V}^3{}_{*} &= \frac{\prod_{\Lambda=1}^{e} \left(\mathfrak{V}^3{}_{i^*}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{e} \left(\mathfrak{V}^3{}_{i^*}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}} + \prod_{\Lambda=1}^{e} \left(\mathscr{I}^3{}_{i^*}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1 - \prod_{\Lambda=1}^{e} \left(1 - \mathfrak{V}^3{}_{i^*} - \mathscr{I}^3{}_{i^*}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}\right) \\
\mathscr{I}^3{}_{*} &= \frac{\prod_{\Lambda=1}^{e} \left(\mathscr{I}^3{}_{i^*}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{e} \left(\mathfrak{V}^3{}_{i^*}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}} + \prod_{\Lambda=1}^{e} \left(\mathscr{I}^3{}_{i^*}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1 - \prod_{\Lambda=1}^{e} \left(1 - \mathfrak{V}^3{}_{i^*} - \mathscr{I}^3{}_{i^*}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)
\end{aligned}$$

By this, if $\mathfrak{V}^3{}_{\Lambda} + \mathscr{I}^3{}_{\Lambda} \leq \mathfrak{V}^3{}_{i^*} + \mathscr{I}^3{}_{i^*}$ then we have,

$$\mathfrak{V}^3 + \mathscr{I}^3 = 1 - \prod_{\Lambda=1}^{e} \left(1 - \left\{\mathfrak{V}^3{}_{\Lambda} + \mathscr{I}^3{}_{\Lambda}\right\}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}} \leq 1 - \prod_{\Lambda=1}^{e} \left(1 - \left\{\mathfrak{V}^3{}_{i^*} + \mathscr{I}^3{}_{i^*}\right\}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}} \leq \mathfrak{V}^3{}_{*} + \mathscr{I}^3{}_{*}$$

□

Theorem 3. *Let $\aleph^{\zeta}{}_{\Lambda} = \langle \mathfrak{V}_{\Lambda}, \mathscr{I}_{\Lambda} \rangle$ be the accumulation of FRFNs.*
Then for $FRFFWA(\aleph^{\zeta}{}_{1}, \aleph^{\zeta}{}_{2}, \dots, \aleph^{\zeta}{}_{e}) = \langle \mathfrak{V}_{x}, \mathscr{I}_{x} \rangle$, we have

$$\min_{\Lambda}\left\{\mathfrak{V}^3{}_{\Lambda} + \mathscr{I}^3{}_{\Lambda}\right\} \leq \mathfrak{V}^3{}_{x} + \mathscr{I}^3{}_{x} \leq \max_{\Lambda}\left\{\mathfrak{V}^3{}_{\Lambda} + \mathscr{I}^3{}_{\Lambda}\right\}$$

Proof. We start with

$$\begin{aligned}
\min_{\Lambda}\left\{\mathfrak{V}^3{}_\Lambda+\mathscr{I}^3{}_\Lambda\right\} &= 1-\left(1-\min_{\Lambda}\left\{\mathfrak{V}^3{}_\Lambda+\mathscr{I}^3{}_\Lambda\right\}\right)\\
&= 1-\left(1-\min_{\Lambda}\left\{\mathfrak{V}^3{}_\Lambda+\mathscr{I}^3{}_\Lambda\right\}\right)^{\sum_{\Lambda=1}^{e}\mathscr{A}^\tau{}_\Lambda}\\
&= 1-\prod_{\Lambda=1}^{e}\left(1-\min_{\Lambda}\left\{\mathfrak{V}^3{}_\Lambda+\mathscr{I}^3{}_\Lambda\right\}\right)^{\mathscr{A}^\tau{}_\Lambda}\\
&\leq 1-\prod_{\Lambda=1}^{e}\left(1-\left\{\mathfrak{V}^3{}_\Lambda+\mathscr{I}^3{}_\Lambda\right\}\right)^{\mathscr{A}^\tau{}_\Lambda}\\
&\leq 1-\prod_{\Lambda=1}^{e}\left(1-\max_{\Lambda}\left\{\mathfrak{V}^3{}_\Lambda+\mathscr{I}^3{}_\Lambda\right\}\right)^{\mathscr{A}^\tau{}_\Lambda}\\
&= 1-\left(1-\max_{\Lambda}\left\{\mathfrak{V}^3{}_\Lambda+\mathscr{I}^3{}_\Lambda\right\}\right)^{\sum_{\Lambda=1}^{e}\mathscr{A}^\tau{}_\Lambda}\\
&= \max_{\Lambda}\left\{\mathfrak{V}^3{}_\Lambda+\mathscr{I}^3{}_\Lambda\right\}
\end{aligned}$$

By Theorem 1, we have

$$\begin{aligned}
\mathfrak{V}_x &= \sqrt[3]{\frac{\prod_{\Lambda=1}^{e}(\mathfrak{V}^3{}_\Lambda)^{\mathscr{A}^\tau{}_\Lambda}}{\prod_{\Lambda=1}^{e}(\mathfrak{V}^3{}_\Lambda)^{\mathscr{A}^\tau{}_\Lambda}+\prod_{\Lambda=1}^{e}(\mathscr{I}^3{}_\Lambda)^{\mathscr{A}^\tau{}_\Lambda}}\times\left(1-\prod_{\Lambda=1}^{e}(1-\mathfrak{V}^3{}_\Lambda-\mathscr{I}^3{}_\Lambda)^{\mathscr{A}^\tau{}_\Lambda}\right)}\\
\mathscr{I}_x &= \sqrt[3]{\frac{\prod_{\Lambda=1}^{e}(\mathscr{I}^3{}_\Lambda)^{\mathscr{A}^\tau{}_\Lambda}}{\prod_{\Lambda=1}^{e}(\mathfrak{V}^3{}_\Lambda)^{\mathscr{A}^\tau{}_\Lambda}+\prod_{\Lambda=1}^{e}(\mathscr{I}^3{}_\Lambda)^{\mathscr{A}^\tau{}_\Lambda}}\times\left(1-\prod_{\Lambda=1}^{e}(1-\mathfrak{V}^3{}_\Lambda-\mathscr{I}^3{}_\Lambda)^{\mathscr{A}^\tau{}_\Lambda}\right)}
\end{aligned}$$

From this, we have

$$\mathfrak{V}^3{}_x+\mathscr{I}^3{}_x=\left(1-\prod_{\Lambda=1}^{e}\left(1-\mathfrak{V}^3{}_\Lambda-\mathscr{I}^3{}_\Lambda\right)^{\mathscr{A}^\tau{}_\Lambda}\right)$$

Consequently,

$$\min_\Lambda\{\mathfrak{V}^3{}_\Lambda+\mathscr{I}^3{}_\Lambda\}\leq\mathfrak{V}^3{}_x+\mathscr{I}^3{}_x\leq\max_\Lambda\{\mathfrak{V}^3{}_\Lambda+\mathscr{I}^3{}_\Lambda\}$$

□

Theorem 4. *Let* $\aleph^\zeta{}_\Lambda=\langle\mathfrak{V}_\Lambda,\mathscr{I}_\Lambda\rangle$ *be the accumulation of FRFNs and* $\aleph^\zeta{}_\diamond=\langle\mathfrak{V}_\diamond,\mathscr{I}_\diamond\rangle$ *be the FRFNs, such that,* $\aleph^\zeta{}_\Lambda=\aleph^\zeta{}_\diamond\forall\Lambda$*. Then*

$$FRFFWA(\aleph^\zeta{}_1,\aleph^\zeta{}_2,\ldots,\aleph^\zeta{}_e)=\aleph^\zeta{}_\diamond \tag{12}$$

Proof. Given that $\aleph^\zeta{}_\Lambda=\aleph^\zeta{}_\diamond\forall\Lambda$, by this, $\mathfrak{V}_\Lambda=\mathfrak{V}_\diamond,\ \ \mathscr{I}_\Lambda=\mathscr{I}_\diamond$ and $\tau_\Lambda=\tau_\diamond\forall\Lambda$

$$FRFFWA(\aleph^{\zeta}{}_1, \aleph^{\zeta}{}_2, \ldots, \aleph^{\zeta}{}_e)$$

$$= \left(\begin{array}{c} \sqrt[3]{\frac{\prod_{\Lambda=1}^{e}(\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{e}(\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}+\prod_{\Lambda=1}^{e}(\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1-\prod_{\Lambda=1}^{e}(1-\mathfrak{V}^3{}_{\Lambda}-\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)}, \\ \sqrt[3]{\frac{\prod_{\Lambda=1}^{e}(\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{e}(\mathfrak{V}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}+\prod_{\Lambda=1}^{e}(\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1-\prod_{\Lambda=1}^{e}(1-\mathfrak{V}^3{}_{\Lambda}-\mathscr{I}^3{}_{\Lambda})^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[3]{\frac{\prod_{\Lambda=1}^{e}(\mathfrak{V}^3{}_{\diamond})^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{e}(\mathfrak{V}^3{}_{\diamond})^{\mathscr{A}^{\tau}{}_{\Lambda}}+\prod_{\Lambda=1}^{e}(\mathscr{I}^3{}_{\diamond})^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1-\prod_{\Lambda=1}^{e}(1-\mathfrak{V}^3{}_{\diamond}-\mathscr{I}^3{}_{\diamond})^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)}, \\ \sqrt[3]{\frac{\prod_{\Lambda=1}^{e}(\mathscr{I}^3{}_{\diamond})^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{e}(\mathfrak{V}^3{}_{\diamond})^{\mathscr{A}^{\tau}{}_{\Lambda}}+\prod_{\Lambda=1}^{e}(\mathscr{I}^3{}_{\diamond})^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1-\prod_{\Lambda=1}^{e}(1-\mathfrak{V}^3{}_{\diamond}-\mathscr{I}^3{}_{\diamond})^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[3]{\frac{(\mathfrak{V}^3{}_{\diamond})^{\sum_{\Lambda=1}^{e}\mathscr{A}^{\tau}{}_{\Lambda}}}{(\mathfrak{V}^3{}_{\diamond})^{\sum_{\Lambda=1}^{e}\mathscr{A}^{\tau}{}_{\Lambda}}+(\mathscr{I}^3{}_{\diamond})^{\sum_{\Lambda=1}^{e}\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1-(1-\mathfrak{V}^3{}_{\diamond}-\mathscr{I}^3{}_{\diamond})^{\sum_{\Lambda=1}^{e}\mathscr{A}^{\tau}{}_{\Lambda}}\right)}, \\ \sqrt[3]{\frac{(\mathscr{I}^3{}_{\diamond})^{\sum_{\Lambda=1}^{e}\mathscr{A}^{\tau}{}_{\Lambda}}}{(\mathfrak{V}^3{}_{\diamond})^{\sum_{\Lambda=1}^{e}\mathscr{A}^{\tau}{}_{\Lambda}}+(\mathscr{I}^3{}_{\diamond})^{\sum_{\Lambda=1}^{e}\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1-(1-\mathfrak{V}^3{}_{\diamond}-\mathscr{I}^3{}_{\diamond})^{\sum_{\Lambda=1}^{e}\mathscr{A}^{\tau}{}_{\Lambda}}\right)} \end{array} \right)$$

$$= \left(\sqrt[3]{\frac{(\mathfrak{V}^3{}_{\diamond})}{(\mathfrak{V}^3{}_{\diamond})+(\mathscr{I}^3{}_{\diamond})} \times (1-(1-\mathfrak{V}^3{}_{\diamond}-\mathscr{I}^3{}_{\diamond}))}, \sqrt[3]{\frac{(\mathscr{I}^3{}_{\diamond})}{(\mathfrak{V}^3{}_{\diamond})+(\mathscr{I}^3{}_{\diamond})} \times (1-(1-\mathfrak{V}^3{}_{\diamond}-\mathscr{I}^3{}_{\diamond}))} \right)$$

$$= \langle \mathfrak{V}_{\diamond}, \mathscr{I}_{\diamond} \rangle = \aleph^{\zeta}{}_{\diamond}$$

□

3.2. FRFFOWA Operator

Definition 6. *Let $\aleph^{\zeta}{}_{\Lambda} = \langle \mathfrak{V}_{\Lambda}, \mathscr{I}_{\Lambda} \rangle$ represent the accumulation of FRFNs, and FRFFOWA $\mathfrak{V}^n \to \mathfrak{V}$ denote the n-dimensionial mapping. If*

$$FRFFOWA(\aleph^{\zeta}{}_1, \aleph^{\zeta}{}_2, \ldots \aleph^{\zeta}{}_e) = \left(\mathscr{A}^{\tau}{}_1 * \aleph^{\zeta}{}_{\xi_{(1)}} \tilde{\bigoplus} \mathscr{A}^{\tau}{}_2 * \aleph^{\zeta}{}_{\xi_{(2)}} \tilde{\bigoplus} \ldots, \tilde{\bigoplus} \mathscr{A}^{\tau}{}_e * \aleph^{\zeta}{}_{\xi_{(e)}}\right) \tag{13}$$

then the mapping FRFFOWA is called the "Fermatean fuzzy fairly ordered weighted averaging (FRFFOWA) operator", $\xi : 1, 2, 3, \ldots, n \to 1, 2, 3, \ldots, n$ is a permutation map s.t. $\aleph^{\zeta}{}_{\xi_{(i-1)}} \geq \aleph^{\zeta}{}_{\xi_{(i)}}$

Theorem 5. *Let $\aleph^{\zeta}{}_{\Lambda} = \langle \mathfrak{V}_{\Lambda}, \mathscr{I}_{\Lambda} \rangle$ be the accumulation of FRFNs, we can also find FRFFOWA by*

$$FRFFOWA(\aleph^{\zeta}{}_1, \aleph^{\zeta}{}_2, \ldots, \aleph^{\zeta}{}_e)$$

$$= \left(\begin{array}{c} \sqrt[3]{\frac{\prod_{\Lambda=1}^{e}\left(\mathfrak{V}^3{}_{\xi_{(i)}}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{e}\left(\mathfrak{V}^3{}_{\xi_{(i)}}\right)^{\mathscr{A}^{\tau}{}_{\xi_{(i)}}}+\prod_{\Lambda=1}^{e}\left(\mathscr{I}^3{}_{\xi_{(i)}}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1-\prod_{\Lambda=1}^{e}\left(1-\mathfrak{V}^3{}_{\xi_{(i)}}-\mathscr{I}^3{}_{\xi_{(i)}}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)}, \\ \sqrt[3]{\frac{\prod_{\Lambda=1}^{e}\left(\mathscr{I}^3{}_{\xi_{(i)}}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}}{\prod_{\Lambda=1}^{e}\left(\mathfrak{V}^3{}_{\xi_{(i)}}\right)^{\mathscr{A}^{\tau}{}_{\xi_{(i)}}}+\prod_{\Lambda=1}^{e}\left(\mathscr{I}^3{}_{\xi_{(i)}}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}} \times \left(1-\prod_{\Lambda=1}^{e}\left(1-\mathfrak{V}^3{}_{\xi_{(i)}}-\mathscr{I}^3{}_{\xi_{(i)}}\right)^{\mathscr{A}^{\tau}{}_{\Lambda}}\right)} \end{array} \right)$$

where $\mathscr{A}^{\tau}{}_{\Lambda}$ *is the WV of* $\aleph^{\zeta}{}_{\Lambda}$ *with* $\mathscr{A}^{\tau}{}_{\Lambda} > 0$ *and* $\sum_{\Lambda=1}^{e} \mathscr{A}^{\tau}{}_{\Lambda} = 1$.

Theorem 6. *Let* $\aleph^{\zeta}{}_{\Lambda} = \langle \mathfrak{V}_{\Lambda}, \mathscr{I}_{\Lambda} \rangle$ *be the accumulation of FRFNs and* $\aleph^{\zeta}{}_{\diamond} = \langle \mathfrak{V}_{\diamond}, \mathscr{I}_{\diamond} \rangle$ *be the FRFNs, such that* $\aleph^{\zeta}{}_{\Lambda} = \aleph^{\zeta}{}_{\diamond} \forall \Lambda$. *Then*

$$FRFFOWA(\aleph^{\zeta}{}_{1}, \aleph^{\zeta}{}_{2}, \ldots, \aleph^{\zeta}{}_{e}) = \aleph^{\zeta}{}_{\diamond} \quad (14)$$

Theorem 7. *Let* $\aleph^{\zeta}{}_{\Lambda} = \langle \mathfrak{V}_{\Lambda}, \mathscr{I}_{\Lambda} \rangle$ *be the accumulation of FRFNs. Then for* $FRFFOWA(\aleph^{\zeta}{}_{1}, \aleph^{\zeta}{}_{2}, \ldots, \aleph^{\zeta}{}_{e}) = \langle \mathfrak{V}_{x}, \mathscr{I}_{x} \rangle$, *we have*

$$\min_{\xi_{(i)}} \left\{ \mathfrak{V}^{3}{}_{\xi_{(i)}} + \mathscr{I}^{3}{}_{\xi_{(i)}} \right\} \leq \mathfrak{V}^{3}{}_{x} + \mathscr{I}^{3}{}_{x} \leq \max_{\xi_{(i)}} \left\{ \mathfrak{V}^{3}{}_{\xi_{(i)}} + \mathscr{I}^{3}{}_{\xi_{(i)}} \right\}$$

Theorem 8. *Consider that* $\aleph^{\zeta}{}_{\Lambda} = \langle \mathfrak{V}_{\Lambda}, \mathscr{I}_{\Lambda} \rangle$ *and* $\aleph^{\zeta}{}_{i^*} = \langle \mathfrak{V}_{i^*}, \mathscr{I}_{i^*} \rangle$ *are the families of FRFNs, and also consider*

$$FRFFOWA(\aleph^{\zeta}{}_{1}, \aleph^{\zeta}{}_{2}, \ldots \aleph^{\zeta}{}_{e}) = \aleph^{\zeta} = \langle \mathfrak{V}, \mathscr{I} \rangle$$

and

$$FRFFOWA(\aleph^{\zeta}{}_{1^*}, \aleph^{\zeta}{}_{2^*}, \ldots \aleph^{\zeta}{}_{e^*}) = \aleph^{\zeta}{}_{*} = \langle \mathfrak{V}_{*}, \mathscr{I}_{*} \rangle.$$

Then

$$\mathfrak{V}^{3} + \mathscr{I}^{3} \leq \mathfrak{V}^{3}{}_{*} + \mathscr{I}^{3}{}_{*}, \quad \textit{if} \quad \mathfrak{V}^{3}{}_{\xi_{(i)}} + \mathscr{I}^{3}{}_{\xi_{(i)}} \leq \mathfrak{V}^{3}{}_{\xi^*_{(i)}} + \mathscr{I}^{3}{}_{\xi^*_{(i)}}$$

4. Decision-Making Algorithm

The methodology outlined in this part enables the retrieval of different alternatives $\mathscr{M}^{\xi}{}_{j}$ from a specialist D_k, utilizing the parameter $\mathscr{P}^{\zeta}{}_{i}$ in the form of an FRF context. The evaluation outcomes are denoted as FRFNs, symbolized as $\aleph^{\zeta}{}^{p}_{ji} = \left\langle \mathfrak{V}^{p}_{ji}, \mathscr{I}^{p}_{ji} \right\rangle$.

Moreover, let ϖ_t represent the weight assigned to the property $\mathscr{P}^{\zeta}{}_{\Lambda}$ given the specified requirements, where ϖ_t is greater than or equal to zero and the total of all ϖ_t from $t = 1$ to m is equal to one. After the identification of the most desirable alternative, a proposed operator is utilized to construct an MCDM process that integrates the FRF information. This procedure encompasses the subsequent phases:

Step 1:

In order to assess the significance of DMs in terms of their relevance, as indicated in FRFNs, it is possible to employ linguistic terms (LTs) outlined in Table 1. Consider the FRFN, denoted as Υ_k, which consists of the pair $\left\langle \mathfrak{V}_{k}, \mathscr{I}_{k} \right\rangle$. This FRFN represents the significance of the k-th DM. In order to obtain the weight ζ_k of the k-th DM, the equation that follows can be employed:

$$\zeta_k = \frac{\Upsilon_k}{\sum_{k=1}^{p} \Upsilon_k}, k = 1, 2, 3, \ldots, p \quad (15)$$

where $\Upsilon_k = \mathfrak{V}_k + \sqrt[3]{(1 - \mathfrak{V}^{3}{}_{k} - \mathscr{I}^{3}{}_{k})} \left(\frac{\mathscr{I}_k}{\mathfrak{V}_k + \mathscr{I}_k} \right)$ and Clearly $\sum_{k=1}^{p} \zeta_k = 1$.

Table 1. Linguistic terms for DMs.

LTs	FRFNs
Very admissible	0.85, 0.05
Admissible	0.75, 0.10
Medium admissible	0.60, 0.20
Inadmissible	0.30, 0.35
Very inadmissible	0.20, 0.45

Step 2:

Obtain a decision matrix, denoted as $D_{(p)}$, which consists of FRFNs from the DMs. The matrix $D_{(p)}$ is represented as $(\mathfrak{Y}_{ji}^{(p)})_{n\times m}$.

$$\begin{array}{ccc} & & \begin{array}{cccc} \mathscr{P}^{\zeta}{}_1 & \mathscr{P}^{\zeta}{}_2 & & \mathscr{P}^{\zeta}{}_m \end{array} \\ \begin{array}{c} D_1 \\ \\ \\ \\ D_2 \\ \\ \\ \\ D_p \\ \\ \\ \\ \end{array} & \begin{array}{c} \mathscr{M}^{\xi}{}_1 \\ \mathscr{M}^{\xi}{}_2 \\ \\ \mathscr{M}^{\xi}{}_n \\ \mathscr{M}^{\xi}{}_1 \\ \mathscr{M}^{\xi}{}_2 \\ \\ \mathscr{M}^{\xi}{}_n \\ \mathscr{M}^{\xi}{}_1 \\ \mathscr{M}^{\xi}{}_2 \\ \\ \mathscr{M}^{\xi}{}_n \end{array} & \begin{bmatrix} (\mathfrak{V}^1_{11},\mathscr{I}^1_{11}) & (\mathfrak{V}^1_{12},\mathscr{I}^1_{12}) & \cdots\cdots & (\langle\mathfrak{V}^1_{1m},\mathscr{I}^1_{1m}) \\ (\mathfrak{V}^1_{21},\mathscr{I}^1_{21}) & (\mathfrak{V}^1_{22},\mathscr{I}^1_{22}) & \cdots\cdots & (\mathfrak{V}^1_{2m},\mathscr{I}^1_{2m}) \\ \vdots & \vdots & \ddots\ddots & \vdots \\ (\mathfrak{V}^1_{n1},\mathscr{I}^1_{n1}) & (\mathfrak{V}^1_{n2},\mathscr{I}^1_{n2}) & \cdots\cdots & (\mathfrak{V}^1_{nm},\mathscr{I}^1_{nm}) \\ (\mathfrak{V}^2_{11},\mathscr{I}^2_{11}) & (\mathfrak{V}^2_{12},\mathscr{I}^2_{12}) & \cdots\cdots & (\mathfrak{V}^2_{1m},\mathscr{I}^2_{1m}) \\ (\mathfrak{V}^2_{21},\mathscr{I}^2_{21}) & (\mathfrak{V}^2_{22},\mathscr{I}^2_{22}) & \cdots\cdots & (\mathfrak{V}^2_{2m},\mathscr{I}^2_{2m}) \\ \vdots & \vdots & \ddots\ddots & \vdots \\ (\mathfrak{V}^2_{n1},\mathscr{I}^2_{n1}) & (\mathfrak{V}^2_{n2},\mathscr{I}^2_{n2}) & \cdots\cdots & (\mathfrak{V}^2_{nm},\mathscr{I}^2_{nm}) \\ (\mathfrak{V}^p_{11},\mathscr{I}^p_{11}) & (\mathfrak{V}^p_{12},\mathscr{I}^p_{12}) & \cdots\cdots & (\mathfrak{V}^p_{1m},\mathscr{I}^p_{1m}) \\ (\mathfrak{V}^p_{21},\mathscr{I}^p_{21}) & (\mathfrak{V}^p_{22},\mathscr{I}^p_{22}) & \cdots\cdots & (\mathfrak{V}^p_{2m},\mathscr{I}^p_{2m}) \\ \vdots & \vdots & \ddots\ddots & \vdots \\ (\mathfrak{V}^p_{n1},\mathscr{I}^p_{n1}) & (\mathfrak{V}^p_{n2},\mathscr{I}^p_{n2}) & \cdots\cdots & (\mathfrak{V}^p_{nm},\mathscr{I}^p_{nm}) \end{bmatrix} \end{array}$$

Step 3:

In order to build an aggregated FRF judgment matrix, it is necessary to amalgamate the individual FRF judgment matrices contributed by each decision maker into a unified matrix that accurately represents the collective preferences of the entire group. In order to accomplish this task, the utilization of the suggested AOs and their respective impacts on the ultimate determination can be employed. The AOs have the following contributions:

Let $H = \left(H_{ji}\right)_{n\times m}$ be the aggregated FRF decision matrix, where

$$H_{ji} = FRFFWA\ \left(\mathfrak{Y}_{ji}^{(1)}, \mathfrak{Y}_{ji}^{(2)}, \ldots, \mathfrak{Y}_{ji}^{(p)}\right)$$

or

$$H_{ji} = FRFFOWA\ \left(\mathfrak{Y}_{ji}^{(1)}, \mathfrak{Y}_{ji}^{(2)}, \ldots, \mathfrak{Y}_{ji}^{(p)}\right)$$

For convenience, we take H_{ji} as $H_{ji} = \left\langle \mathfrak{V}_{ji}, \mathscr{I}_{ji} \right\rangle$.

Step 4:

If necessary, the FRFNs can be normalized by transforming all cost-type parameters (τ_c) into benefit-type characteristics (τ_b), using the following formula:

$$(\aleph_{ji}^{N})_{n\times m} = \begin{cases} (H_{ji})^c; & i \in \tau_c \\ H_{ji}; & i \in \tau_b. \end{cases} \tag{16}$$

where $(H_{ji})^c$ shows the compliment of (H_{ji}). The normalized decision matrix will be $\Gamma_N = \left(\aleph_{ji}^{N}\right)_{n\times m} = \left(\breve{\mathfrak{V}}_{ji}, \breve{\mathscr{I}}_{ji}\right)_{n\times m}$.

Step 5:

Construct the score matrix, by utilizing the SF of FRFNs as $\Psi = \left(\nabla^{\gamma}\left(\aleph^{N}_{ji}\right)\right)_{n\times m}$

$$\begin{array}{cc} & \begin{array}{cccccc} \mathscr{P}^{\zeta}{}_{1} & \mathscr{P}^{\zeta}{}_{2} & \mathscr{P}^{\zeta}{}_{3} & \cdots & \mathscr{P}^{\zeta}{}_{m} \end{array} \\ \begin{array}{c} \mathscr{M}^{\xi}{}_{1} \\ \mathscr{M}^{\xi}{}_{2} \\ \mathscr{M}^{\xi}{}_{3} \\ \vdots \\ \mathscr{M}^{\xi}{}_{m} \end{array} & \begin{bmatrix} \nabla^{\gamma}(\aleph^{N}_{11}) & \nabla^{\gamma}(\aleph^{N}_{12}) & \nabla^{\gamma}(\aleph^{N}_{13}) & \cdots & \nabla^{\gamma}(\aleph^{N}_{1m}) \\ \nabla^{\gamma}(\aleph^{N}_{21}) & \nabla^{\gamma}(\aleph^{N}_{22}) & \nabla^{\gamma}(\aleph^{N}_{23}) & \cdots & \nabla^{\gamma}(\aleph^{N}_{2m}) \\ \nabla^{\gamma}(\aleph^{N}_{31}) & \nabla^{\gamma}(\aleph^{N}_{32}) & \nabla^{\gamma}(\aleph^{N}_{33}) & \cdots & \nabla^{\gamma}(\aleph^{N}_{3m}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\aleph^{N}_{n1}) & \nabla^{\gamma}(\aleph^{N}_{n2}) & \nabla^{\gamma}(\aleph^{N}_{n3}) & \cdots & \nabla^{\gamma}(\aleph^{N}_{nm}) \end{bmatrix} \end{array}$$

Step 6:

Based on the underlying assumption of the score matrix Ψ, a calculated aggregate of the scores for each alternative $\mathscr{M}^{\xi}{}_{j}$ is determined by a process of assigning weights

$$\yen(\mathscr{M}^{\xi}{}_{j}) = \sum_{\Lambda=1}^{m} \varpi^{\gamma}_{\Lambda}\, \nabla^{\gamma}\left(\aleph^{N}_{ji}\right), \quad (j = 1, 2, \ldots, n).$$

where $\varpi^{\gamma}_{1}, \varpi^{\gamma}_{2}, \ldots \varpi^{\gamma}_{m}$ is the WV of the given criterion.

Given the absence of specified weights, it is desired to represent a fraction of these weights using the $\widehat{\coprod}$ coefficients. To calculate the indeterminate weights, we utilize the mathematical framework presented in this context.

$$Max\ \ g = \sum_{\Lambda=1}^{m} \yen(\mathscr{M}^{\xi}{}_{j})$$

With conditions $\sum_{\Lambda=1}^{m} \varpi^{\gamma}_{\Lambda} = 1$, the normalized word vector (WV) can be obtained using this methodology. In order to assess the relative significance of each criterion within the given constraints, a "linear programming model" is employed.

Step 7:

The determination of the "aggregated weighted FRF decision matrix" involves the utilization of the Γ_N matrix and the WV ϖ^{γ}. We incorporated the recommended action items (AOs), as indicated below.

$$FRFFWA(\aleph^{N}_{j1}, \aleph^{N}_{j2}, \ldots, \aleph^{N}_{jm}) = \left(\begin{array}{c} \dfrac{\prod_{j=1}^{m}\left(\breve{\mathfrak{V}}^{3}{}_{j\xi_{(i)}}\right)^{\varpi^{\gamma}{}_{\Lambda}}}{\prod_{j=1}^{m}\left(\breve{\mathfrak{V}}^{3}{}_{j\xi_{(i)}}\right)^{\varpi^{\gamma}{}_{\Lambda}} + \prod_{j=1}^{m}\left(\breve{\mathscr{I}}^{3}{}_{j\xi_{(i)}}\right)^{\varpi^{\gamma}{}_{\Lambda}}} \times \left(1 - \prod_{j=1}^{m}\left(1 - \breve{\mathfrak{V}}^{3}{}_{j\xi_{(i)}} - \breve{\mathscr{I}}^{3}{}_{j\xi_{(i)}}\right)^{\varpi^{\gamma}_{\Lambda}}\right), \\ \\ \dfrac{\prod_{j=1}^{m}\left(\breve{\mathscr{I}}^{3}{}_{j\xi_{(i)}}\right)^{\varpi^{\gamma}{}_{\Lambda}}}{\prod_{j=1}^{m}\left(\breve{\mathfrak{V}}^{3}{}_{j\xi_{(i)}}\right)^{\varpi^{\gamma}{}_{\Lambda}} + \prod_{j=1}^{m}\left(\breve{\mathscr{I}}^{3}{}_{j\xi_{(i)}}\right)^{\varpi^{\gamma}{}_{\Lambda}}} \times \left(1 - \prod_{j=1}^{m}\left(1 - \breve{\mathfrak{V}}^{3}{}_{j\xi_{(i)}} - \breve{\mathscr{I}}^{3}{}_{j\xi_{(i)}}\right)^{\varpi^{\gamma}_{\Lambda}}\right) \end{array} \right)$$

or

$$FRFFOWA(\aleph_{j1}^{N}, \aleph_{j2}^{N}, \ldots, \aleph_{jm}^{N}) = \left(\begin{array}{c} \frac{\prod_{j=1}^{m} \left(\breve{\mathfrak{V}}^{3}{}_{j\xi_{(i)}}\right)^{\varpi^{\Upsilon}\Lambda}}{\prod_{j=1}^{m} \left(\breve{\mathfrak{V}}^{3}{}_{j\xi_{(i)}}\right)^{\varpi^{\Upsilon}\Lambda} + \prod_{j=1}^{m} \left(\breve{\mathscr{I}}^{3}{}_{j\xi_{(i)}}\right)^{\varpi^{\Upsilon}\Lambda}} \times \left(1 - \prod_{j=1}^{m} \left(1 - \breve{\mathfrak{V}}^{3}{}_{j\xi_{(i)}} - \breve{\mathscr{I}}^{3}{}_{j\xi_{(i)}}\right)^{\varpi_{\Lambda}^{\Upsilon}}\right), \\ \frac{\prod_{j=1}^{m} \left(\breve{\mathscr{I}}^{3}{}_{j\xi_{(i)}}\right)^{\varpi^{\Upsilon}\Lambda}}{\prod_{j=1}^{m} \left(\breve{\mathfrak{V}}^{3}{}_{j\xi_{(i)}}\right)^{\varpi^{\Upsilon}\Lambda} + \prod_{j=1}^{m} \left(\breve{\mathscr{I}}^{3}{}_{j\xi_{(i)}}\right)^{\varpi^{\Upsilon}\Lambda}} \times \left(1 - \prod_{j=1}^{m} \left(1 - \breve{\mathfrak{V}}^{3}{}_{j\xi_{(i)}} - \breve{\mathscr{I}}^{3}{}_{j\xi_{(i)}}\right)^{\varpi_{\Lambda}^{\Upsilon}}\right) \end{array} \right)$$

Step 8:

The SF formula can be employed to ascertain the score value associated with the cumulative total value. Subsequently, based on the stated selection framework, we evaluate and assign ratings to each available alternative, and subsequently decide the option(s) with the highest preference levels.

5. Case Study

The management of supply chains has emerged as a crucial component in contemporary corporate operations. Organizations have recognized the need for optimizing their supply chains in order to maintain competitiveness, minimize expenses, improve operational effectiveness, and augment customer satisfaction. In light of the increasingly intricate nature of global supply chains, it is imperative for organizations to embrace contemporary technologies and adhere to established best practices in order to maintain smooth and efficient operations. This essay explores the significance of supply chain management in the contemporary world and underscores crucial elements that organizations must consider to enhance the efficiency of their supply chains [33].

Supply chain management encompasses the synchronization and orchestration of various activities pertaining to the creation, movement, and provision of goods and services. The scope of this undertaking encompasses the oversight of various aspects, including the procurement and handling of primary resources, coordination with suppliers, supervision of manufacturing procedures, effective management of inventories, efficient logistical operations, and the provision of satisfactory customer service. The primary objective of supply chain management is to guarantee the efficient and economical delivery of items or services to clients within specified time frames. Supply chain management encompasses multiple stakeholders, such as suppliers, manufacturers, distributors, retailers, and customers.

The significance of supply chain management in the contemporary period is of utmost importance and should not be underestimated. Several compelling justifications exist for the indispensability of supply chain management of contemporary enterprises that operate inside a highly competitive milieu.

- Increased efficiency and productivity: One of the main benefits of supply chain management is increased efficiency and productivity. By optimizing their supply chains, companies can streamline their operations, reduce waste, and increase output. For example, by implementing lean manufacturing practices, companies can reduce production time, minimize inventory levels, and eliminate non-value-added activities. This, in turn, leads to increased productivity and efficiency [34].
- Reduced costs: Another critical benefit of supply chain management is reduced costs. By optimizing their supply chains, companies can identify areas where costs can be reduced, such as inventory holding costs, transportation costs, and labor costs.
- Improved customer service: The role of supply chain management is crucial in enhancing customer service. Through the process of supply chain optimization, organizations can effectively and efficiently provide products or services to clients, ensuring both

timeliness and cost-effectiveness. Consequently, this results in heightened levels of consumer satisfaction and loyalty. By incorporating a resilient logistics infrastructure, organizations may guarantee the timely and pristine delivery of goods to their clientele. Consequently, this results in enhanced levels of consumer satisfaction and loyalty.
- Enhanced supply chain visibility: Supply chain management also enables companies to enhance supply chain visibility. With the growing complexity of global supply chains, it is essential for companies to have real-time visibility into their operations. By leveraging modern technologies, such as the Internet of Things (IoT), companies can track their products and monitor their operations in real time. This, in turn, leads to increased transparency, improved decision-making, and reduced risks.
- Improved collaboration: Effective supply chain management requires collaboration among various stakeholders, including suppliers, manufacturers, distributors, and retailers. By fostering collaboration and communication among these stakeholders, companies can improve supply chain performance, reduce costs, and enhance customer service. For example, by implementing a supplier relationship management (SRM) program, companies can build strong relationships with their suppliers, negotiate better terms, and improve the quality of raw materials.

Supply chain optimization is essential for companies that want to remain competitive in the global marketplace. This paper will discuss the factors that companies need to consider when optimizing their supply chains.

- Demand forecasting: Demand forecasting plays a crucial role in the optimization of supply chain operations. In order to effectively manage inventory and minimize expenses associated with excess inventory, companies must engage in precise demand forecasting to ensure sufficient stock availability while avoiding overstocking. Accurate demand forecasting can additionally assist organizations in discerning trends and patterns in consumer behavior, thereby providing valuable insight into future company strategies.
- Inventory management: The optimization of the supply chain necessitates careful consideration of inventory management as a crucial element. In order to effectively meet consumer demand while mitigating the risk of excess inventory, it is imperative for companies to maintain appropriate amounts of inventory. The act of maintaining excessive inventory can result in additional costs, whilst maintaining insufficient inventory might lead to stockouts, causing a loss in sales and dissatisfaction among customers.
- Transportation management: Transportation management refers to the systematic approach of strategizing, implementing, and overseeing the logistics involved in the transfer of commodities from their source to their final destination. In order to achieve cost reduction in transportation and maintain punctual delivery of goods, it is imperative for companies to optimize their transportation management. This process encompasses the selection of an appropriate transportation method, the optimization of shipping routes, and the minimization of transit duration.
- Supplier management: Supplier management is the process of managing relationships with suppliers to ensure that they meet quality, delivery, and cost requirements. Companies need to select the right suppliers, negotiate favorable contracts, and monitor supplier performance to ensure that they meet their obligations. Effective supplier management can help companies reduce costs, improve quality, and mitigate supply chain risks.
- Information technology: Information technology (IT) is an essential component of supply chain optimization. Companies need to invest in IT systems that can capture, store, and analyze supply chain data to identify opportunities for improvement.
- Risk management: Companies need to be aware of the risks that can affect their supply chains, including natural disasters, geopolitical events, and supplier bankruptcies.

Effective risk management can help companies mitigate these risks and ensure the continuity of their supply chains.

- Sustainability: Sustainability is an increasingly important factor to consider when optimizing the supply chain. Companies need to ensure that their supply chains are environmentally and socially responsible. This can involve reducing carbon emissions, using sustainable materials, and ensuring that suppliers comply with ethical and social responsibility standards.
- Lean manufacturing: Lean manufacturing is a philosophy that emphasizes the elimination of waste and the continuous improvement of processes. Companies need to adopt lean manufacturing principles to reduce lead times, improve quality, and increase efficiency. This can involve implementing just-in-time (JIT) inventory systems, reducing setup times, and improving process flow.
- Collaboration: Collaboration is an essential factor in optimizing the supply chain. Companies need to collaborate with suppliers, customers, and other stakeholders to improve supply chain efficiency and effectiveness. This can involve sharing data, coordinating production schedules, and aligning business objectives.
- Continuous improvement: Continuous improvement is the process of making incremental improvements to processes and systems over time. Companies need to adopt a culture of continuous improvement to ensure that their supply chains remain competitive. This can involve measuring performance, identifying areas for improvement, and implementing changes to processes and systems.

Green supplier selection is an important consideration for companies that want to improve their sustainability and environmental impacts. This case study will examine a hypothetical company, the XYZ Corporation, and the steps it took to select green suppliers for its manufacturing operations. The XYZ Corporation is a global manufacturer of consumer electronics products, including smartphones, tablets, and laptops. The company has a strong commitment to sustainability and has a set goal of reducing its carbon footprint by 50% by 2030. As part of this commitment, the XYZ Corporation decided to evaluate its suppliers and select those that have a strong environmental record.

The XYZ Corporation began its green supplier selection process by developing a set of criteria that suppliers needed to meet to be considered for selection. The criteria included the following:

- Environmental management systems (${\mathscr{P}^{\zeta}}_1$): Suppliers needed to have an established environmental management system that was certified to ISO 14001 or equivalent.
- Environmental performance (${\mathscr{P}^{\zeta}}_2$): Suppliers needed to demonstrate a commitment to environmental performance, including reducing carbon emissions, minimizing waste, and conserving natural resources.
- Green products (${\mathscr{P}^{\zeta}}_3$): Suppliers needed to offer a range of green products, including those made from sustainable materials and those that had a low environmental impact.
- Social responsibility (${\mathscr{P}^{\zeta}}_4$): Suppliers needed to demonstrate a commitment to social responsibility, including fair labor practices and community engagement.

The procurement manager is the first decision-maker who is central to the green supplier selection process. The sustainability agenda is inextricably linked to the procurement manager's vital responsibility of acquiring the materials and services necessary for the organization's operations. This position requires an in-depth comprehension of both conventional procurement metrics and sustainable procuring procedures. In order to strike a balance between economic and ecological objectives, the procurement manager must reconcile cost considerations with environmental and social criteria. In our case study, this person's decisions will impact the selection of suppliers based on their eco-friendliness, carbon footprint, ethical labor practices, and compliance with environmental standards. The sustainability officer is the second crucial decision-maker in the domain of green supplier selection. As the leader of the organization's sustainability strategy, this position entails formulating and implementing environmental stewardship-aligned policies. The sustainability officer advocates for responsible procurement and ensures that the organization's supplier network adheres to predetermined sustainability standards. Their decisions are influenced by the evaluation of the green credentials of prospective suppliers, such as their use of renewable resources, waste reduction initiatives, and adherence to circular economy principles. In our case study, the sustainability of an officer's decisions will have a significant impact on the value chain integration of environmentally conscious suppliers. The supreme decision-maker is the executive leadership, which consists of senior management and the board of directors. These stakeholders hold the final authority to endorse and allocate resources to the green supplier selection process. The executive leadership bases their decisions on a thorough understanding of the organization's strategic objectives, financial viability, and reputation management. In our case study, the decisions of the executive leadership will determine the extent to which the organization embraces a supplier ecosystem with a concentration on sustainability and the degree to which this commitment aligns with the organization's broader corporate goals.

The XYZ Corporation then developed a supplier scorecard that would be used to evaluate potential suppliers. The scorecard included metrics for each of the criteria listed above and was used to rank suppliers based on their environmental records.

After developing the scorecard, the XYZ Corporation began the process of evaluating its suppliers. The company requested information from all of its suppliers regarding their environmental management systems, environmental performance, green products, social responsibility, and innovation. Suppliers were also asked to provide data on their carbon emissions, waste generation, and energy use.
We take five companies, namely $\mathscr{M}^{\xi}{}_1$, $\mathscr{M}^{\xi}{}_2$, $\mathscr{M}^{\xi}{}_3$, $\mathscr{M}^{\xi}{}_4$, and $\mathscr{M}^{\xi}{}_5$. To assess the companies based on the aforementioned criteria, it is necessary to designate three DMs.

5.1. Decision-Making Process

Step 1:

The LTs corresponding to each DM are provided in Table 2. The weights of the DMs may be determined using the LTs and Equation (15). The resulting weights for the DMs are as follows: $\zeta_1 = 0.30$, $\zeta_2 = 0.35$, and $\zeta_3 = 0.35$.

Table 2. LTs for DMs.

DM	LTs
D_1	Medium admissible
D_2	Very admissible
D_3	Admissible

Step 2:

To obtain the decision matrix $D_{(p)}$ in the form of FRFNs from DMs, it is necessary to follow a certain procedure. The judgment values provided by three decision-makers are presented in Tables 3–5.

Table 3. Assessment matrix acquired from D_1.

	$\mathscr{P}^{\zeta}{}_1$	$\mathscr{P}^{\zeta}{}_2$	$\mathscr{P}^{\zeta}{}_3$	$\mathscr{P}^{\zeta}{}_4$
$\mathscr{M}^{\xi}{}_1$	(0.524, 0.365)	(0.353, 0.356)	(0.359, 0.115)	(0.333, 0.243)
$\mathscr{M}^{\xi}{}_2$	(0.455, 0.264)	(0.333, 0.412)	(0.291, 0.412)	(0.434, 0.222)
$\mathscr{M}^{\xi}{}_3$	(0.302, 0.288)	(0.155, 0.387)	(0.258, 0.335)	(0.235, 0.325)
$\mathscr{M}^{\xi}{}_4$	(0.444, 0.230)	(0.484, 0.368)	(0.473, 0.281)	(0.335, 0.265)
$\mathscr{M}^{\xi}{}_5$	(0.264, 0.480)	(0.481, 0.360)	(0.281, 0.415)	(0.125, 0.375)

Table 4. Assessment matrix acquired from D_2.

	$\mathscr{P}^{\zeta}{}_{1}$	$\mathscr{P}^{\zeta}{}_{2}$	$\mathscr{P}^{\zeta}{}_{3}$	$\mathscr{P}^{\zeta}{}_{4}$
$\mathscr{M}^{\xi}{}_{1}$	(0.420, 0.353)	(0.260, 0.235)	(0.452, 0.150)	(0.635, 0.520)
$\mathscr{M}^{\xi}{}_{2}$	(0.625, 0.235)	(0.265, 0.645)	(0.447, 0.435)	(0.542, 0.621)
$\mathscr{M}^{\xi}{}_{3}$	(0.333, 0.151)	(0.245, 0.240)	(0.255, 0.325)	(0.323, 0.245)
$\mathscr{M}^{\xi}{}_{4}$	(0.231, 0.260)	(0.340, 0.240)	(0.345, 0.225)	(0.323, 0.145)
$\mathscr{M}^{\xi}{}_{5}$	(0.461, 0.271)	(0.321, 0.240)	(0.580, 0.115)	(0.224, 0.663)

Table 5. Assessment matrix acquired from D_3.

	$\mathscr{P}^{\zeta}{}_{1}$	$\mathscr{P}^{\zeta}{}_{2}$	$\mathscr{P}^{\zeta}{}_{3}$	$\mathscr{P}^{\zeta}{}_{4}$
$\mathscr{M}^{\xi}{}_{1}$	(0.275, 0.355)	(0.355, 0.262)	(0.565, 0.313)	(0.525, 0.135)
$\mathscr{M}^{\xi}{}_{2}$	(0.235, 0.155)	(0.255, 0.335)	(0.240, 0.163)	(0.335, 0.453)
$\mathscr{M}^{\xi}{}_{3}$	(0.570, 0.475)	(0.145, 0.120)	(0.145, 0.162)	(0.315, 0.231)
$\mathscr{M}^{\xi}{}_{4}$	(0.320, 0.175)	(0.460, 0.225)	(0.415, 0.162)	(0.315, 0.451)
$\mathscr{M}^{\xi}{}_{5}$	(0.243, 0.464)	(0.180, 0.568)	(0.140, 0.425)	(0.311, 0.122)

Step 3:

Let $H = \left(H_{ji}\right)_{5\times 4}$ be the aggregated FRF decision matrix, where

$$H_{ji} = FRFFWA\ \left(\mathfrak{Y}_{ji}^{(1)}, \mathfrak{Y}_{ji}^{(2)}, \mathfrak{Y}_{ji}^{(3)}\right) = \left(\zeta_1 * \mathfrak{Y}_{ji}^{(1)} \tilde{\oplus} \zeta_2 * \mathfrak{Y}_{ji}^{(2)} \tilde{\oplus} \zeta_3 * \mathfrak{Y}_{ji}^{(3)}\right).$$

The aggregated SF decision matrix is given in Table 6.

Table 6. Aggregated FRF decision matrix.

	$\mathscr{P}^{\zeta}{}_{1}$	$\mathscr{P}^{\zeta}{}_{2}$
$\mathscr{M}^{\xi}{}_{1}$	(0.513819, 0.150840)	(0.531197, 0.119700)
$\mathscr{M}^{\xi}{}_{2}$	(0.517475, 0.114851)	(0.319155, 0.311010)
$\mathscr{M}^{\xi}{}_{3}$	(0.384447, 0.198998)	(0.175971, 0.335171)
$\mathscr{M}^{\xi}{}_{4}$	(0.359910, 0.395513)	(0.333581, 0.340343)
$\mathscr{M}^{\xi}{}_{5}$	(0.350903, 0.141431)	(0.340919, 0.334980)
	$\mathscr{P}^{\zeta}{}_{3}$	$\mathscr{P}^{\zeta}{}_{4}$
$\mathscr{M}^{\xi}{}_{1}$	(0.533589, 0.131038)	(0.533317, 0.148339)
$\mathscr{M}^{\xi}{}_{2}$	(0.317873, 0.310173)	(0.509047, 0.141500)
$\mathscr{M}^{\xi}{}_{3}$	(0.344751, 0.143313)	(0.385810, 0.137947)
$\mathscr{M}^{\xi}{}_{4}$	(0.318483, 0.109315)	(0.393379, 0.341111)
$\mathscr{M}^{\xi}{}_{5}$	(0.353798, 0.198190)	(0.171313, 0.191884)

Step 4:

There is no cost type attribute; thus, the normalized decision matrix will be $\Gamma_N = \left(\aleph_{ji}^{N}\right)_{n\times m} = \left(\breve{\mathfrak{V}}_{ji}, \breve{\mathscr{I}}_{ji}, \breve{\mathfrak{r}}_{ji}\right)_{5\times 4}$, given in Table 7.

Table 7. Normalized FRF decision matrix.

	$\mathscr{P}^{\zeta}{}_{1}$	$\mathscr{P}^{\zeta}{}_{2}$
$\mathscr{M}^{\xi}{}_{1}$	(0.513819, 0.150840)	(0.531197, 0.119700)
$\mathscr{M}^{\xi}{}_{2}$	(0.517475, 0.114851)	(0.319155, 0.311010)
$\mathscr{M}^{\xi}{}_{3}$	(0.384447, 0.198998)	(0.175971, 0.335171)
$\mathscr{M}^{\xi}{}_{4}$	(0.359910, 0.395513)	(0.333581, 0.340343)
$\mathscr{M}^{\xi}{}_{5}$	(0.350903, 0.141431)	(0.340919, 0.334980)
	$\mathscr{P}^{\zeta}{}_{3}$	$\mathscr{P}^{\zeta}{}_{4}$
$\mathscr{M}^{\xi}{}_{1}$	(0.533589, 0.131038)	(0.533317, 0.148339)
$\mathscr{M}^{\xi}{}_{2}$	(0.317873, 0.310173)	(0.509047, 0.141500)
$\mathscr{M}^{\xi}{}_{3}$	(0.344751, 0.143313)	(0.385810, 0.137947)
$\mathscr{M}^{\xi}{}_{4}$	(0.318483, 0.109315)	(0.393379, 0.341111)
$\mathscr{M}^{\xi}{}_{5}$	(0.353798, 0.198190)	(0.171313, 0.191884)

Step 5:

Construct the score matrix by utilizing the SF of FRFNs as follows: $\Psi = \left(\nabla^{\gamma}\left(\aleph_{ji}^{N}\right)\right)_{5\times 4}$.

$$\begin{array}{c} \\ \mathscr{M}^{\xi}{}_{1} \\ \mathscr{M}^{\xi}{}_{2} \\ \mathscr{M}^{\xi}{}_{3} \\ \mathscr{M}^{\xi}{}_{4} \\ \mathscr{M}^{\xi}{}_{5} \end{array} \begin{array}{c} \begin{array}{cccc} \mathscr{P}^{\zeta}{}_{1} & \mathscr{P}^{\zeta}{}_{2} & \mathscr{P}^{\zeta}{}_{3} & \mathscr{P}^{\zeta}{}_{4} \end{array} \\ \begin{pmatrix} 0.5733 & 0.2367 & 0.1266 & 0.3624 \\ 0.8223 & 0.1435 & 0.3103 & 0.5121 \\ 0.4916 & 0.2509 & 0.2186 & 0.3685 \\ 0.6516 & 0.2895 & 0.1255 & 0.2570 \\ 0.7331 & 0.4450 & 0.3098 & 0.8106 \end{pmatrix} \end{array}$$

Step 6:

Consider that the DMs provide the following partial weight details about the attribute weights:

$$\Psi = 0.10 \le \varpi_1^{\gamma} \le 0.50, 0.25 \le \varpi_2^{\gamma} \le 0.60, 0.20 \le \varpi_3^{\gamma} \le 0.60, 0.20 \le \varpi_4^{\gamma} \le 0.80$$

Relying on these data, the following optimization framework can be developed: $Max\ g$

$$\begin{aligned} = \quad & 0.5733\varpi_1^{\gamma} + 0.8223\varpi_1^{\gamma} + 0.4916\varpi_1^{\gamma} + 0.6516\varpi_1^{\gamma} + 0.7331\varpi_1^{\gamma} \\ & 0.2367\varpi_2^{\gamma} + 0.1435\varpi_2^{\gamma} + 0.2509\varpi_2^{\gamma} + 0.2895\varpi_2^{\gamma} + 0.4450\varpi_2^{\gamma} \\ & 0.1266\varpi_3^{\gamma} + 0.3103\varpi_3^{\gamma} + 0.2186\varpi_3^{\gamma} + 0.1255\varpi_3^{\gamma} + 0.3098\varpi_3^{\gamma} \\ & 0.3624\varpi_4^{\gamma} + 0.5121\varpi_4^{\gamma} + 0.3685\varpi_4^{\gamma} + 0.2570\varpi_4^{\gamma} + 0.8106\varpi_4^{\gamma} \end{aligned}$$

such that,

$$\begin{gathered} 0.10 \le \varpi_1^{\gamma} \le 0.50, \qquad 0.25 \le \varpi_2^{\gamma} \le 0.60, \\ 0.20 \le \varpi_3^{\gamma} \le 0.60, \qquad 0.20 \le \varpi_4^{\gamma} \le 0.80, \\ \varpi_1^{\gamma} + \varpi_2^{\gamma} + \varpi_3^{\gamma} + \varpi_4^{\gamma} = 1, \qquad \varpi_1^{\gamma}, \varpi_2^{\gamma}, \varpi_3^{\gamma}, \varpi_4^{\gamma} \ge 0. \end{gathered}$$

By solving this model, we have $\varpi_1^{\gamma} = 0.15, \varpi_2^{\gamma} = 0.30, \varpi_3^{\gamma} = 0.25, \varpi_4^{\gamma} = 0.30$

Step 7:

Evaluate the aggregated weighted SF decision matrix by using the proposed AOs given by Table 8.

Table 8. Aggregated weighted SF decision matrix.

$\mathscr{M}^{\xi}{}_1$	(0.9382, 0.3783)
$\mathscr{M}^{\xi}{}_2$	(0.2247, 0.4099)
$\mathscr{M}^{\xi}{}_3$	(0.3396, 0.5861)
$\mathscr{M}^{\xi}{}_4$	(0.7442, 0.2465)
$\mathscr{M}^{\xi}{}_5$	(0.2796, 0.2902)

Step 8:

Compute the score values of all alternatives.

$$\begin{aligned}
\nabla^{\gamma}\left(\mathscr{M}^{\xi}{}_1\right) &= 0.641242 \\
\nabla^{\gamma}\left(\mathscr{M}^{\xi}{}_2\right) &= 0.532322 \\
\nabla^{\gamma}\left(\mathscr{M}^{\xi}{}_3\right) &= 0.244450 \\
\nabla^{\gamma}\left(\mathscr{M}^{\xi}{}_4\right) &= 0.181823 \\
\nabla^{\gamma}\left(\mathscr{M}^{\xi}{}_5\right) &= 0.356035
\end{aligned}$$

At the end, the final ranking will be

$$\mathscr{M}^{\xi}{}_1 \succ \mathscr{M}^{\xi}{}_2 \succ \mathscr{M}^{\xi}{}_5 \succ \mathscr{M}^{\xi}{}_3 \succ \mathscr{M}^{\xi}{}_4.$$

As a result, the $\mathscr{M}^{\xi}{}_1$'s status is the most suitable supplier.

5.2. Comparative Analysis

This section contrasts several planned AOs with existing AOs. Using preexisting AOs to solve the data yields an optimal solution equivalent to our findings. This demonstrates the durability and efficacy of the AO. Our method is more applicable and superior compared to a number of previously reported AOs. Several current operators are used to validate our optimal solution. Our optimal choices are identical, demonstrating the validity of our proposed AOs given in Table 9.

Table 9. Comparison analysis.

Authors	AOs	Ranking of Alternatives
Garg et al. [12]	FRFYWA	$\mathscr{M}^{\xi}{}_1 \succ \mathscr{M}^{\xi}{}_5 \succ \mathscr{M}^{\xi}{}_2 \succ \mathscr{M}^{\xi}{}_3 \succ \mathscr{M}^{\xi}{}_4$
	FRFYWG	$\mathscr{M}^{\xi}{}_1 \succ \mathscr{M}^{\xi}{}_2 \succ \mathscr{M}^{\xi}{}_5 \succ \mathscr{M}^{\xi}{}_3 \succ \mathscr{M}^{\xi}{}_4$
Hadi et al. [35]	FRFHWA	$\mathscr{M}^{\xi}{}_1 \succ \mathscr{M}^{\xi}{}_2 \succ \mathscr{M}^{\xi}{}_5 \succ \mathscr{M}^{\xi}{}_3 \succ \mathscr{M}^{\xi}{}_4$
	FRFHWG	$\mathscr{M}^{\xi}{}_1 \succ \mathscr{M}^{\xi}{}_2 \succ \mathscr{M}^{\xi}{}_5 \succ \mathscr{M}^{\xi}{}_3 \succ \mathscr{M}^{\xi}{}_4$
Senapati and Yager [9]	FRFWA	$\mathscr{M}^{\xi}{}_1 \succ \mathscr{M}^{\xi}{}_3 \succ \mathscr{M}^{\xi}{}_5 \succ \mathscr{M}^{\xi}{}_2 \succ \mathscr{M}^{\xi}{}_4$
	FRFWG	$\mathscr{M}^{\xi}{}_1 \succ \mathscr{M}^{\xi}{}_4 \succ \mathscr{M}^{\xi}{}_5 \succ \mathscr{M}^{\xi}{}_3 \succ \mathscr{M}^{\xi}{}_2$
Proposed	FRFFWA	$\mathscr{M}^{\xi}{}_1 \succ \mathscr{M}^{\xi}{}_2 \succ \mathscr{M}^{\xi}{}_5 \succ \mathscr{M}^{\xi}{}_3 \succ \mathscr{M}^{\xi}{}_4$
	FRFFOWA	$\mathscr{M}^{\xi}{}_1 \succ \mathscr{M}^{\xi}{}_2 \succ \mathscr{M}^{\xi}{}_5 \succ \mathscr{M}^{\xi}{}_3 \succ \mathscr{M}^{\xi}{}_4$

6. Conclusions

According to current research, when a DM provides an equal number of evaluations for membership and non-membership objects, the aggregate scores are not evenly distributed. Using FRFSs and proportional distribution rules, we implemented innovative neutrality and fairness procedures in this scenario. Even when influenced by the DM's disposition, we focused on ensuring correctness and applicability in decision-making. The "Fermatean fuzzy fairly weighted averaging (FRFFWA) operator" and "Fermatean fuzzy fairly ordered weighted averaging (FRFFOWA) operator" were introduced to the FRFN information based on the principles of fuzzy operations. We extensively analyzed the proposed AO characteristics. Not only do the proposed operators facilitate interaction between dissimilar FRFNs, but they also aid in the study of DM attitude characteristics, allowing for a categorical approach to the degrees of FRFSs. The MCDM problem's proposed solution has been validated.

We anticipate that our discoveries will prove beneficial to scholars engaged in the study of supply chain analysis, information aggregation, decision analysis, supply management, and environmental science. The potential for future research lies in the methodological advancements pertaining to supply chain analyses. It is our contention that the convergence of these essential climate-centric organizational study domains presents significant potential for growth and the acquisition of knowledge about our world.

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