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Finite Chaoticity and Pairwise Sensitivity of a Strong-Mixing Measure-Preserving Semi-Flow

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Abstract: Chaos is a common phenomenon in nature and social sciences. As is well known, chaos has multiple definitions, and there are both differences and connections between them. The unique properties of chaotic systems can be leveraged to address challenges in communication, security, data processing, system analysis, and control across different domains. For semi-flows, this paper introduces two important concepts corresponding to discrete dynamical systems, finitely chaotic and pairwise sensitivity. Since Tent map and its induced suspended semi-flows both have these two properties, then these two concepts on the semi-flows have extensive and important applications and meanings in information security, finance, artificial intelligence and other fields. This paper extends the vast majority of corresponding results in discrete dynamical systems to semi-flows.

Keywords: finitely chaotic set; strong-mixing; measure-preserving semi-flow

MSC: 37B55; 37B45; 54H20

Citation: Li, R.; Pi, J.; Li, Y.; Lu, T.; Wang, J.; Ding, X. Finite Chaoticity and Pairwise Sensitivity of a Strong-Mixing Measure-Preserving Semi-Flow. *Axioms* **2023**, *12*, 860. https://doi.org/10.3390/axioms12090860

Academic Editors: Gheorghita Zbaganu and Emanuel Guariglia

Received: 22 June 2023
Revised: 26 August 2023
Accepted: 29 August 2023
Published: 7 September 2023

1. Introduction

To understand chaos in a topologically transitive semi-flows, we first recall some key concepts involved in this paper. A semi-flow on a (nonempty) set M is a map $\varphi : I \times M \to M$, $I \subseteq \mathbb{R}$, satisfying $\varphi_0(x) = x$ and $\varphi_s(\varphi_t(x)) = \varphi_{s+t}(x)$. It is a mathematical abstraction used to describe a continuous-time dynamical system. It consists of a set (phase space) and a family of continuous functions that represent the evolution of the system over time. Each function describes the behavior of the system for positive time values. In the context of dynamical systems, finite chaoticity refers to the property of a system where chaotic behavior is observed only for a finite subset of the phase space while outside this region, the behavior may be regular or non-chaotic. Pairwise sensitivity of a strong-mixing measure refers to a property of a dynamical system, specifically a measure-preserving system, which captures the sensitivity of the system to small changes in initial conditions in a pairwise manner. A measure-preserving system means that this measure captures the probability distribution or density of points in the phase space and is preserved by the dynamics of the system. Strong mixing is a property of measure-preserving systems that indicates the rapid and complete mixing of subsets of the phase space as time progresses. Strong mixing implies that the system's dynamics thoroughly mix the points in the phase space, resulting in a high level of randomness and unpredictability. Pairwise sensitivity refers to the sensitivity of a dynamical system to small changes in initial conditions in a pairwise manner. So, even though the overall dynamics of the system may be mixing and chaotic, the pairwise sensitivity property specifically focuses on the divergence of nearby trajectories.

Smítal [1] proved that for the tent map $H(x) = 1- \mid 1-2x \mid$ defined on $I = [0,1]$, there exists a set C with Lebesgue outer measure $\mu(C) = 1$ such that for any $u, v \in C : u \neq v$ and a periodic point p of H with $p \neq u$,

$$\limsup_{m\to\infty} |H^m(u) - H^m(v)| = 1,$$

$$\limsup_{m\to\infty} |H^m(u) - H^m(p)| \geq \frac{1}{2},$$

and

$$\liminf_{m\to\infty} |H^m(u) - H^m(v)| = 0.$$

where $m \in \mathbb{N} = \{1, 2, \cdots\}$. Periodic point is $p \in I \subseteq \mathbb{R}^+ = [0, +\infty)$ such that $H^{t+s}(p) = H^t(p)$ for every $t \in I$. s is called a period of H if it is the minimum positive integer which satisfying $H^{t+s}(p) = H^t(p)$ $(\forall t \in I)$. And $\{p, H(p), H^2(p), \cdots, H^{s-1}(p)\}$ is a period orbit of H. A stronger result than Smítal's has been proven by Chen [2] for the tent map on space I (resp. the shift of symbolic space). Later, in [3], Xiong and Chen discusses chaotic behavior caused by strong mixing preserving measure mappings and proves that the above result in [1] is a special case. The main result is then applied to one-dimensional mappings. They also proved a statement for a topological space M satisfying the second countability axiom and an outer measure μ on M satisfying the following conditions (i)–(iii). If $f : M \to M$ is a strong-mixing measure-preserving map on the probability space $(M, \mathcal{B}(M), \mu)$, and $\{s_j\}_{j\in\mathbb{N}} \in \mathbb{R}^+$ is a strictly increasing sequence of positive real numbers, then there is a set $C \subset M : \mu(C) = 1$ such that C is finitely chaotic with respect to the sequence $\{s_j\}_{j\in\mathbb{N}}$. Where,

(i) any nonempty open subset $V \subset M$ is μ-measurable with $\mu(V) > 0$;
(ii) the restriction of μ to the sigma-algebra $\mathcal{B}(M)$ of Borel subsets of M is a probability space;
(iii) for each subset V of M, there exists a Borel set $A \in \mathcal{B}(M)$ satisfying $A \supset V$ and if a Borel set $B \supset V$, then $A \subseteq B$. And then, $\mu(V) = \mu(A)$.

Furthermore, for every finite subset $B \subset C$ and each map $G : B \to M$, there exists a subsequence $\{t_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ of $\{s_j\}_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} f^{t_j}(b) = G(b)$$

for all $b \in B$.

One of the necessary conditions for various definitions of chaos [4–8] is sensitivity, which characterizes the unpredictability of chaos in dynamical systems, as is widely acknowledged. In recent years, there has been a widespread interest in the concept of sensitive dependence on initial conditions [5,9–13], which has been formalized in various ways by several authors [14–18]. In order to illustrate the exponential rate of divergence of neighboring point trajectories, Ref. [19] first used the phrase "sensitive dependence on initial conditions". In a more general sense, the sensitivity property is a characterization of the dynamical behavior in which even a small change in the initial condition can result in a significant change in the resulting trajectory.

Based on the earlier work of Guckenheimer [20], Devaney [6] gave a mathematical definition of sensitivity as follows. Let H be a continuous self-map on a metric space (M, ρ). H is sensitive if there is an $\eta > 0$ such that for any $u \in M$ and $\varepsilon > 0$, there always exists a $v \in M$ with $\rho(u, v) < \varepsilon$ satisfying the condition that

$$\rho(H^m(u), H^m(v)) \geq \eta$$

for some integer m. In the past few years, some authors have proposed sufficient conditions on H and (M, ρ) to ensure sensitivity (see [5,10–14,20–22]). In [14], the authors demonstrate that on a Borel probability space $(M, \rho, \mathcal{B}(M), \mu)$ of supp $\mu = M$, if the measure preserving map H is topologically mixing (or weakly mixing), then for all non-empty open set $U \subset M$, there exists a sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers with positive upper density satisfying that

$$U \bigcap (\bigcap_{k\geq 0} H^{-n_k} U) \neq \emptyset,$$

then it is sensitive. By proving weak-mixing to be sufficient, in [10] He reduced the conditions required for sensitive dependence on a metric probability space $(M, \rho, \mathcal{B}(M), \mu)$ with supp $\mu = M$ for both measure-preserving maps and measure-preserving semi-flows. Recently, Gu [23] showed that H is sensitive on a nontrivial compact metric space M if the pair (H, μ) satisfies the large deviation theorem (where H is continuous and topological strongly ergodic). Moreover, according to [24] (Proposition 7.2.14), topological mixing of the mapping H in [23] provides sensitivity. In addition, Ref. [16] offers some sufficient conditions for sensitivity that are more general and relaxed than the conditions presented in [10,14,23–25]. However, in chaos theory, sensitivity to initial conditions was first proposed with the measure of divergence of neighboring point trajectories, similar to the butterfly effect described by Lorenz (see [6]). To avoid the asymmetry in the definition of sensitivity mentioned above, the authors [18] gave the definition of pairwise sensitivity by using the tools from ergodic theory. Where, a map $H : M \to M$ is topologically ergodic (briefly, ergodic) if $N_H(A, B) = \{m \in \mathbb{N} : H^m(A) \cap B \neq \emptyset\}$ has positive upper density for any non-empty open sets $A, B \subset M$. That is, $\limsup_{k\to\infty} \frac{1}{k} \mid N_H(A, B) \mid > 0$.

The first purpose of the current research is to define the notion of finite chaos with respect to a sequence for a semi-flow (which may not be continuous), and to extend the results of Xiong and Chen in [3] to semi-flows. In particular, the following conditions are satisfied by the establishment of a topological space M with the second axiom of countability and an outer measure μ on it.

(1) any nonempty open subset $V \subset M$ is μ-measurable with $\mu(V) > 0$;
(2) the restriction of μ to the sigma-algebra $\mathcal{B}(M)$ of Borel subsets of M is a probability space;
(3) if $(M, \mathcal{B}(M), \mu)$ is a probability space with a strong-mixing measure-preserving semi-flow $\varphi : \mathbb{R}^+ \times M \to M$, then for any subset V of M, there is a Borel set $A \in \mathcal{B}(M)$: $A \supset V, \mu(A) = \mu(V)$. And then, there exists a set $C \subset M$ with $\mu(C) = 1$ that is finitely chaotic with respect to a strictly increasing sequence $\{s_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$. That is, for any finite subset $B \subset C$ and every map $G : B \to M$ there exists a subsequence $\{t_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ of $\{s_j\}_{j\in\mathbb{N}}$ with

$$\lim_{j\to\infty} \varphi_{t_j}(v) = G(v)$$

for any $v \in B$.

Additionally, the notion of pairwise sensitivity is presented for semi-flows on Lebesgue metric spaces in this paper. The link between pairwise sensitivity, weak mixing, and pairwise sensitivity for semi-flows is examined, along with the positiveness of metric entropy. Moreover, we also calculated the maximum sensitivity constants for specific semi-flows.

2. Preliminaries

Let M a topological space and $f : M \to M$ be a map (which may not be continuous).

Definition 1 ([3]). *A set $B \subset M$ is said to be finitely chaotic with respect to the strictly increasing sequence $\{s_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$, if for all finite subsets $A \subset B$ and for every map $G : A \to M$ there exists a subsequence $\{t_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ of $\{s_j\}_{j\in\mathbb{N}}$ satisfying that*

$$\lim_{j\to\infty} f^{t_j}(u) = G(u)$$

for all $u \in A$.

The corresponding concept for a semi-flow is given as follow.

Definition 2. *Let $\varphi : \mathbb{R}^+ \times M \to M$ be a semi-flow. A set $A \subset M$ is said to be finitely chaotic with respect to the strictly increasing sequence $\{s_i\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$, if for all finite subset $B \subset A$ and every map $G : B \to M$ there exists a subsequence $\{t_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ of $\{s_j\}_{j\in\mathbb{N}}$ satisfying that*

$$\lim_{j\to\infty} \varphi_{t_j}(v) = G(v)$$

for all $v \in B$.

Remark 1. *According to the definition, a set $A \subset M$ is finitely chaotic with respect to the sequence $\{s_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ if and only if any finite subset $B \subset A$ is finitely chaotic with respect to the sequence $\{s_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$.*

By the first statement of Proposition 1 in [3], a set with finite chaos with respect to a sequence of positive integers is chaotic in the sense of Li-Yorke [26–28]. From the second statement of Proposition 1 in [3], for certain distinct sequences of positive integers, sets with finite chaos satisfy the conditions described by Smítal in [1]. By utilizing Proposition 1 and Remark 3 in [3], it is established that a set possessing finite chaos with respect to a sequence of positive integers is categorized as strongly chaotic (as outlined in [29]), but not necessarily chaotic (for further information on the concept of chaos, please refer to [30]). In regards to semi-flows, the following result is offered, which is similar to Proposition 1 in [3].

Proposition 1. *Let $\varphi : \mathbb{R}^+ \times M \to M$ be a semi-flow, ρ is a metric on M. And let $A \subset M$ be a set, finitely chaotic with respect to a strictly increasing sequence $\{s_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$. Then, for any two different points $a, b \in A$ and any periodic point p of φ,*

$$\limsup_{j\to\infty} \rho(\varphi_{s_j}(a), \varphi_{s_j}(b)) = |M|,$$

$$\liminf_{j\to\infty} \rho(\varphi_{s_j}(a), \varphi_{s_j}(b)) = 0$$

and

$$\limsup_{j\to\infty} \rho(\varphi_{s_j}(a), \varphi_{s_j}(p)) > 0$$

are held, where $|M|$ is the diameter of M.

Proof. Let $a, b \in A : a \neq b$. Choose $u_j, v_j \in M$ $(j \in \{1, 2, \cdots\})$ satisfying

$$\lim_{j\to\infty} \rho(u_j, v_j) = |M|.$$

For any $j \in \{1, 2, \cdots\}$, a map $G_j : \{a, b\} \to M$ is defined by $G_j(a) = u_j$ and $G_j(b) = v_j$. For any $i \in \{1, 2, \cdots\}$, by Definition 2, there is a subsequence $\{s_{j,i}\} \subset \mathbb{R}^+$ of $\{s_j\}_{j\in\mathbb{N}}$ satisfying that

$$\lim_{j\to\infty} \varphi_{s_{j,i}}(a) = u_i$$

and

$$\lim_{j\to\infty} \varphi_{s_{j,i}}(b) = v_i.$$

And because

$$\limsup_{j\to\infty} \rho(\varphi_{s_j}(a), \varphi_{s_j}(b)) \geq \rho(u_i, v_i),$$

then

$$\limsup_{j\to\infty} \rho(\varphi_{s_j}(a), \varphi_{s_j}(b)) = |M|.$$

Define a map $G_0 : \{a, b\} \to M$ by $G_0(a) = b$ and $G_0(b) = b$. Then, there is a subsequence $\{s_{j,0}\} \subset \mathbb{R}^+$ of the sequence $\{s_j\}_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} \varphi_{s_{j,0}}(a) = b$$

and

$$\lim_{j\to\infty} \varphi_{s_{j,0}}(b) = b.$$

Hence, one has

$$\liminf_{j\to\infty} \rho(\varphi_{s_j}(a), \varphi_{s_j}(b)) = 0.$$

Let p be a periodic point of φ with $p \neq a$. It is clear that if M is a periodic orbit of φ then

$$\limsup_{j\to\infty} \rho(\varphi_{s_j}(a), \varphi_{s_j}(p)) > 0.$$

Assuming that M does not represent a periodic orbit of φ, it is feasible to locate a point $z \in M$ that is not included in the periodic orbit of φ where p is included. Then, there exists a subsequence $\{t_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ of $\{s_j\}_{j\in\mathbb{N}}$ satisfying that

$$\lim_{j\to\infty} \varphi_{t_j}(a) = z.$$

For each $j \in \{1, 2, \cdots\}$, since $\varphi_{t_j}(p)$ is in the periodic orbit of φ which p belongs to, then

$$\limsup_{j\to\infty} \rho(\varphi_{s_j}(a), \varphi_{s_j}(p)) \geq \alpha,$$

where α is the distance between p and the periodic orbit of φ containing the points z. One has that

$$\limsup_{j\to\infty} \rho(\varphi_{s_j}(a), \varphi_{s_j}(p)) > 0.$$

Thus, the proof is ended. □

In the following there is always assume that (M, ρ) represents a nontrivial metric space. A probability measure on $(M, \mathcal{B}(M))$ is denoted by μ with $\mathcal{B}(M)$ serving as the sigma-algebra of Borel subsets of M. Thus, the space M is identified as a metric probability space and can be represented as $(M, \rho, \mathcal{B}(M), \mu)$ or $(M, \mathcal{B}(M), \mu)$.

A measurable mapping F is said to be measure-preserving on $(M, \mathcal{B}(M), \mu)$ if $\mu(A) = \mu(F^{-1}(A))$ for all $A \in \mathcal{B}(M)$ [31]. A measurable semi-flow φ is said to be measure-preserving on $(M, \mathcal{B}(M), \mu)$ if

$$\mu(A) = \mu(\varphi_k^{-1}(A))$$

for all $A \in \mathcal{B}(M)$ and all $k \in \mathbb{R}^+$.

A measure-preserving mapping F on $(M, \mathcal{B}(M), \mu)$ is said to be strong-mixing if for every U, V of $\mathcal{B}(M)$,

$$\lim_{j\to\infty} \mu(U \cap F^{-j}(V)) = \mu(U)\mu(V)$$

is held.

For semi-flows, one has the following analogous definition.

Definition 3. *Provided that $(M, \mathcal{B}(M), \mu)$ is a probability Lebesgue space with a metric ρ, a measure-preserving semi-flow φ can be considered as strong-mixing when*

$$\lim_{t\to\infty} \mu(U \cap \varphi_t^{-1}(V)) = \mu(U)\mu(V)$$

holds for all U, V of $\mathcal{B}(M)$.

By [31], a measure-preserving mapping F on $(M, \mathcal{B}(M), \mu)$ is said to be weakly mixing if for every U and V of $\mathcal{B}(M)$,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} \mid \mu(U \bigcap F^{-i}(V)) - \mu(U)\mu(V) \mid = 0.$$

Similarly, a measure-preserving semi-flow φ on $(M, \mathcal{B}(M), \mu)$ is said to be weakly mixing if for every U and V of $\mathcal{B}(M)$, one has

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mid \mu(U \bigcap \varphi_s^{-1}(V)) - \mu(U)\mu(V) \mid ds = 0.$$

where $|\cdot|$ denotes absolute value.

A mapping $F : M \to M$ (resp. a semi-flow φ) is sensitive if there exists an $\eta > 0$ satisfying that for every $a \in M$ and every neighborhood $V_a \ni a$, there exists $m \in \mathbb{N}$ (resp. $k \in \mathbb{R}^+$) such that $sup\{\rho(F^m(a), F^m(b)) : b \in V_a\} > \eta$ (resp. $sup\{\rho(\varphi_k(a), \varphi_k(b)) : b \in V_a\} > \eta$).

The following definition is analogous to the definition of paired sensitivities for measure-preserving maps as defined by B. Cadre and P. Jacob in [31].

Definition 4. *A semi-flow φ on a probability Lebesgue space $(M, \mathcal{B}(M), \mu)$ with a metric ρ is called to be paired sensitivities (relative to the initial condition) if there is an $\eta > 0$ such that for almost everywhere measure $\mu^{\otimes 2}$ in $M \times M$ and $(u, v) \in M^2$, there exist some proper $r \geq 0$ satisfying*

$$\rho(\varphi_r(u), \varphi_r(v)) \geq \eta.$$

A semi-flow φ is said to be pairwise sensitive if there is an $\eta > 0$ satisfying that

$$\mu^{\otimes 2}\left(\bigcap_{k\geq 0} (\overline{\varphi}_k)^{-1}(A_\eta)\right) = 0,$$

where $\overline{\varphi} = \varphi \times \varphi$ is the semi-flow on M^2 defined as

$$\overline{\varphi}(k, (u, v)) = (\varphi(k, u), \varphi(k, v))$$

for any $(u, v) \in M^2$ and any $k \geq 0$, and

$$A_r = \{(u, v) \in M^2 : \rho(u, v) \leq r\}$$

for any $r > 0$.

3. Main Results

In order to derive the principal outcomes of this paper, we require the following three lemmas, which are semi-flow versions of Lemmas 1–3 initially presented in [3].

Lemma 1. *Let $\varphi : \mathbb{R}^+ \times M \to M$ be a semi-flow (which may not be continuous), where M is a topological space, and $\{s_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ is a strictly increasing sequence. Let $Y \subset M$ be dense in M and $A \subset M$ be a given set. If for all finite subsets $B \subset A$ and all mappings $G : B \to Y$ there exists a subsequence $\{t_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ of $\{s_j\}_{j\in\mathbb{N}}$ satisfying that*

$$\lim_{j\to\infty} \varphi_{t_j}(b) = G(b)$$

for any $b \in B$, then A is a finite chaotic set with respect to the sequence $\{s_j\}_{j\in\mathbb{N}}$.

Proof. Let $G : B \to M$ be a mapping, where $B \subset A$ is a finite set. For every $b \in B$, let $\{V_j(b)\}_{j=1}^{\infty}$ be a locally countable base of $G(B)$ such that $V_j(b)$ is open, and

$$V_j(b) \supset V_{j+1}(b)$$

for any $j \in \{1, 2, \cdots\}$. For each $j \in \{1, 2, \cdots\}$, take

$$u_j(b) \in V_j(b) \cap Y.$$

Define $G_j : B \to Y$ by

$$G_j(b) = u_j(b)$$

for any $b \in B$. By the conditions of this Lemma, there is a subsequence $\{t_j(i)\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ of $\{s_j\}_{j\in\mathbb{N}}$ satisfying that

$$\lim_{j\to\infty} \varphi_{t_j(i)}(b) = G_i(b)$$

for every $b \in B$ and any $i \in \{1, 2, \cdots\}$.

Define a subsequence $\{t_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ of $\{s_j\}_{j\in\mathbb{N}}$ as follows. By assumption of a base, one can choose some $s_k \in \{s_j\}_{j\in\mathbb{N}}$ such that $\varphi_{s_k} \in V_1(b)$ and set $t_1 = s_k$. Then, for a given real number $t_{j-1} \in \{s_j\}_{j\in\mathbb{N}}$ and for $j > 1$, choose $t_j \in \{s_j\}_{j\in\mathbb{N}}$ such that $t_j > t_{j-1}$ and $\varphi_{t_j}(b) \in V_j(b)$ for every $b \in B$. One has that

$$\lim_{j\to\infty} \varphi_{t_j}(b) = G(b)$$

for any $b \in B$. So, A is a finite chaotic set with respect to the sequence $\{s_j\}_{j\in\mathbb{N}}$. □

Lemma 2. *Let $\varphi : \mathbb{R}^+ \times M \to M$ be a strong-mixing semi-flow, where$(M, \mathcal{B}(M), \mu)$ be a probability space. Then, for each $A \in \mathcal{B}(M)$ with $\mu(A) > 0$ and any strictly increasing sequence $\{s_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$,*

$$\mu\left(\bigcup_{j=1}^{\infty} \varphi_{s_j}^{-1}(A)\right) = 1.$$

Proof. Let $\mu(A) > 0$, where $A \in \mathcal{B}(M)$. Then

$$\bigcup_{j=1}^{\infty} \varphi_{s_j}^{-1}(A) \in \mathcal{B}(M).$$

Since $\varphi : \mathbb{R}^+ \times M \to M$ is a strong-mixing semi-flow, then

$$\begin{aligned} \mu(A) = & \lim_{j\to\infty} \mu\left(\varphi_{s_j}^{-1}(A)\right) \\ = & \lim_{j\to\infty} \mu\left(\varphi_{s_j}^{-1}(A) \bigcap \left(\bigcup_{j=1}^{\infty} \varphi_{s_j}^{-1}(A)\right)\right) \\ = & \mu(A)\mu(\bigcup_{j=1}^{\infty} \varphi_{s_j}^{-1}(A)), \end{aligned}$$

which implies that

$$\mu\left(\bigcup_{j=1}^{\infty} \varphi_{s_j}^{-1}(A)\right) = 1.$$

□

Lemma 3. *Let $(M, \mathcal{B}(M), \mu)$ be a Borel probability space satisfying the condition that $\mu(A)$ is positive for each nonempty open subset $A \subset M$ which satisfies the second axiom of countability.*

Let $\{s_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ be a strictly increasing sequence, and $\varphi : \mathbb{R}^+ \times M \to M$ be a strong-mixing semi-flow. If C is finitely chaotic with respect to the sequence $\{s_j\}_{j\in\mathbb{N}}$, and if C is countable, then there is a set $E(C) \in \mathcal{B}(M)$ satisfying that

(1) $\mu(E(C)) = 1$;
(2) *for any $c \in E(C)$, $C \cup \{c\}$ is finitely chaotic with respect to the sequence $\{s_j\}_{j\in\mathbb{N}}$.*

Proof. The discussion is divided into two cases.

(i) Assume that C is a finite set and finitely chaotic with respect to the sequence $\{s_j\}_{j\in\mathbb{N}}$. Let $\mathcal{A}$ is a family of sets which includes all non-empty open sets of a countable base of M. Let Y be a countable dense set of M. Consider $\mathcal{F}$ to be the collection of all maps from C into Y. Clearly, $\mathcal{F}$ is countable.

Since the set C is finitely chaotic with respect to the sequence $\{s_j\}_{j\in\mathbb{N}}$, then for any map $G \in \mathcal{F}$ there exists a subsequence $\{t_j(G)\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ of the sequence $\{s_j\}_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} \varphi_{t_j(G)}(u) = G(u)$$

for any $u \in C$. Write

$$E_G = \bigcap_{A\in\mathcal{A}} B_A,$$

where

$$B_A = \bigcap_{n=1}^{\infty} B_{n,A} \text{ and } B_{n,A} = \bigcup_{j=1}^{n} \varphi^{-1}_{t_j(G)}(A).$$

Then $B_{n,A} \in \mathcal{B}(M)$ for every $n \geq 1$. So, $B_A \in \mathcal{B}(M)$, which implies $E_G \in \mathcal{B}(M)$. By Lemma 2, $\mu(B_{n,A}) = 1$ for any $n \geq 1$. This means $\mu(E_G) = 1$ for any map $G \in \mathcal{F}$. By the definition of the set E_G, it follows that for any $s \in E_G$ and any $y \in M$ there exists a subsequence $\{q_j(G)\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ of $\{t_j(G)\}_{j\in\mathbb{N}} \subset \{s_j\}_{j\in\mathbb{N}}$ such that $\varphi_{q_j(G)}(s) = y$. So, $\varphi_{q_j(G)}(u) = G(u)$ for any $u \in C$.

For each $c \in E_G$, let

$$G_1 : C \cup \{u\} \to Y$$

be a map such that $G_1|_C = G$. Then, by the above argument, there is a subsequence $\{q_j(G_1)\}$ of $\{s_j\}_{j\in\mathbb{N}}$ satisfying that

$$\lim_{j\to\infty} \varphi_{q_j(G_1)}(v) = G_1(v)$$

for any $v \in C \cup \{u\}$. By Lemma 1 and Definition 2, $C \bigcup \{u\}$ is finitely chaotic with respect to the sequence $\{s_j\}_{j\in\mathbb{N}}$ for any $u \in E(C)$, where

$$E(C) = \bigcap_{G\in\mathcal{F}} E_G.$$

Then,

$$E(C) \in \mathcal{B}(M) \text{ and } \mu(E(C)) = 1.$$

(ii) Assume that C is a countable set and finitely chaotic with respect to the sequence $\{s_j\}_{j\in\mathbb{N}}$. Let $\mathcal{P}$ denotes the collection of all finite subsets of C which is evidently countable. Write

$$E(C) = \bigcap_{C_1\in\mathcal{P}} E(C_1).$$

Then, by the above argument and the definition of $E(C)$, one has that

$$E(C) \in \mathcal{B}(M) \text{ and } \mu(E(C)) = 1.$$

For any $u \in E(C)$, let

$$B \subset E(C) \cup \{u\}$$

be a finite set and $G : B \to M$ be a map. If $B \subset C$, then B is finitely chaotic with respect to some sequence $\{s_j\}_{j\in\mathbb{N}}$. If B is not a subset of C and

$$B = C_1 \cup \{c\}$$

for some $c \in B$ and some $C_1 \subset C$, then, by the above argument, B is finitely chaotic with respect to the sequence $\{s_j\}_{j\in\mathbb{N}}$. So, there is a subsequence $\{q_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ of the sequence $\{s_j\}_{j\in\mathbb{N}}$ satisfying that

$$\lim_{j\to\infty} \varphi_{q_j(G_1)}(v) = G_1(v)$$

for any $v \in B$. Thus, $E(C)$ is required. □

Now Theorem 1 in [3] can be extended to semi-flows as follow.

Theorem 1. *Let $(M, \mathcal{B}(M), \mu)$ be a probability space satisfies that, $\mu(A) > 0$ for all nonempty open subset $A \subset M$ fit for the second axiom of countability. Let $\{s_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ be a strictly increasing sequence and $\varphi : \mathbb{R}^+ \times M \to M$ be a strong-mixing semi-flow. Then there is a set $C \in \mathcal{B}(M)$ such that*

(1) $\mu(C) = 1$;
(2) for any $b \in M$ and any $a \in C$, there exists a subsequence $\{t_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ of $\{s_j\}_{j\in\mathbb{N}}$ satisfying

$$\lim_{j\to\infty} \varphi_{t_j}(a) = b.$$

Proof. (1) Consider the collection $\mathcal{A}$ to be comprised of all non-empty open sets that form a countable base for M. Write

$$C = \bigcap_{A\in\mathcal{A}} B_A,$$

where

$$B_A = \bigcap_{n=1}^{\infty} B_{n,A} \text{ and } B_{n,A} = \bigcup_{j=1}^{n} \varphi^{-1}_{t_j(F)}(A).$$

Then $B_{n,A} \in \mathcal{B}(M)$ for any integer $n \geq 1$. So, $B_A \in \mathcal{B}(M)$, which implies $C \in \mathcal{B}(M)$. By Lemma 2,

$$\mu(B_{n,A}) = 1$$

for any integer $n \geq 1$. This means $\mu(C) = 1$. Thus, the set C is required.

(2) By the definition of the set C, the conclusion follows immediately. □

The following theorem is the semi-flow version of Theorem 2 from [3].

Theorem 2. *Let $(M, \mathcal{B}(M), \mu)$ be a probability space. $\mu(A)$ is positive for all nonempty open subset $A \subset M$ satisfying the second axiom of countability. Let $\{s_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ be a strictly increasing sequence and $\varphi : \mathbb{R}^+ \times M \to M$ be a strong-mixing semi-flow. A set $C \subset M$ that is finitely chaotic with respect to the sequence $\{s_j\}_{j\in\mathbb{N}}$ can be found such that $D \in \mathcal{B}(M)$ and $\mu(D) > 0$ imply $D \cap C \neq \emptyset$.*

Proof. The proof is similar to that of Theorem 2 in [3]. For the completeness, we provide its proof here.

Since the space M satisfies the second axiom of countability, the cardinal number $|M|$ is smaller than that of a continuum. By the reference [8] in [3], the cardinal number $|B(M)|$ is smaller than that of a continuum. Therefore, the cardinal number $|\mathcal{F}| = |\{W \subset M : W \in B(M), \mu(W) > 0\}|$ is also smaller than that of a continuum. So, $\mathcal{F}$ can be denoted by $\mathcal{F} = \{W_\nu\}_{1\leq\nu<\Delta}$, where Δ represents a finite number m, or ∞, or the first uncountable ordinal number Ω.

Define points $y_\nu \in W_\nu$ for every ν with $1 \leq \nu < \Delta$ by induction (in the case $\Delta = \Omega$, by transfinite induction) which satisfies that $\{y_\nu\}_{1\leq\nu<\theta}$ is finitely chaotic with respect to the sequence $\{S_j\}_{j\in\mathbb{N}}$ for each θ with $1 \leq \theta < \Delta$.

Assume that E is a set which satisfies conditions (1) and (2) of Lemma 3. By condition (1) of Theorem 1, $W_1 \cap E \neq \emptyset$. Pick $y_1 \in W_1 \cap E$. By condition (1) of Theorem 1, $\{y_1\}$ is finitely chaotic with respect to the sequence $\{S_j\}_{j\in\mathbb{N}}$. Since $1 \leq \theta < \Delta$, then one can pick a point y_ν for every $\nu(1 \leq \nu < \theta)$ such that $\{y_\nu\}_{1\leq\nu<\lambda}$ is finitely chaotic with respect to the sequence $\{S_j\}_{j\in\mathbb{N}}$ for each λ with $\nu(1 \leq \lambda < \theta)$. Obviously, $C = \{y_\nu\}_{1\leq\nu<\theta}$ is countable. It is easy to see that the set C is finitely chaotic with respect to the sequence $\{S_j\}_{j\in\mathbb{N}}$. By Lemma 3, there exists a set $E(C)$ which satisfies conditions (1) and (2). By condition (1) of Lemma 3, $W_\theta \cap E(C) \neq \emptyset$. Take $y_\theta \in W_\theta \cap E(C)$. By condition (1) of Lemma 3, the set $C = \{y_\nu\}_{1\leq\nu<\Delta}$ is finitely chaotic with respect to the sequence $\{S_j\}_{j\in\mathbb{N}}$. Clearly, for each $D \in \mathcal{F}, C \cap D \neq \emptyset$. □

The following result is the semi-flow version of Theorem 3 in [3].

Theorem 3. *Assume that M is a topological space that satisfies the second axiom of countability, μ is an outer measure on M satisfying $\mu(M) = 1$ and $\mu(A) > 0$ for any nonempty open subset $A \subset M$, and $\varphi : \mathbb{R}^+ \times M \to M$ is a strong-mixing semi-flow. If*

(1) each open subset of M is μ-measurable, that is, each $A \in \mathcal{B}(M)$ is μ-measurable;
(2) for any subset $Y \subset M$ there exists a set $A \in \mathcal{B}(M)$ with $Y \subset A$ and

$$\mu(A) = \mu(Y),$$

then, for any strictly increasing sequence $\{s_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$, there exists a $C \subset M$ with $\mu(C) = 1$ which is finitely chaotic with respect to the sequence $\{s_j\}_{j\in\mathbb{N}}$.

Proof. By Theorem 2, there exists a set C such that $C \cap D \neq \emptyset$ for any $D \in B(M)$ with $\mu(D) > 0$. So, there exists a $\widetilde{C} \in B(M)$ which satisfies $\mu(\widetilde{C}) = \mu(C)$ and $\widetilde{C} \supset C$. If $\mu(M - \widetilde{C}) > 0$, then $(M - \widetilde{C}) \cap C \neq \emptyset$. It is a contradiction. This means that $\mu(M - \widetilde{C}) = 0$. Hence, $\mu(C) = \mu(\widetilde{C}) = 1$. In addition, conditions (1) and (2) in Theorem 3 are easy to satisfy by an outer measure (This can be obtained from the last part of the proof of Theorem 3 in [3]). Thus, the conclusion is held. □

According to the definition of sensitive, if $\eta > 0$ is a sensitivity constant for a semi-flow φ, then $\eta' \leq \eta$ also is a sensitivity constant. Write

$$\triangle(\varphi) = \sup\{\eta : \eta \text{ is a sensitivity constant for } \varphi\}$$

and

$$\beta(\mu) = \sup\{\eta : \mu^{\otimes 2}(A_\eta) < 1\}.$$

Since the support of μ is not a single point, there is a $\eta > 0$ such that

$$\mu^{\otimes 2}(A_\eta) < 1.$$

Consequently, $\beta(\mu) > 0$.

In order to demonstrate Theorem 4, which presents the semi-flow version of Theorem 2.1 in [18], the following lemma, which corresponds to the semi-flow version of Lemma 2.1 in [18], is required.

Lemma 4. Assume that φ is a pairwise sensitive semi-flow on $(M, \rho, \mathcal{B}(M), \mu)$, where M is a nontrivial metric space. Then $\beta(\mu) = \text{diam}(\text{supp } \mu)$.

Proof. Since

$$\mu^{\otimes 2}(A_{D+\delta}) = 1$$

for any $\delta > 0$, where D =diam(supp μ), $\beta(\mu) \leq D$. Write $H_\delta = \{(x,y) \in \text{supp } \mu^{\otimes 2} : \rho(x,y) \leq D - \frac{1}{2}\delta\}$ for any $\delta > 0$. Due to H_δ is a closed subset and $H_\delta \neq \text{supp } \mu^{\otimes 2} = \text{supp } \mu \times \text{supp } \mu$, one has

$$\mu^{\otimes 2}(H_\delta) < 1.$$

And because

$$A_{D-\delta} \subset H_\delta, \ \ \mu^{\otimes 2}(A_{D-\delta}) < 1,$$

then, $\beta(\mu) \geq D$. Consequently, $\beta(\mu) = \text{diam}(\text{supp } \mu)$. $\square$

Theorem 4. *Assume that φ is a pairwise sensitive semi-flow on $(M, \rho, \mathcal{B}(M), \mu)$, where M is a nontrivial metric space. Then the following hold.*

(1) there is an $\eta > 0$, for almost everywhere measure $\mu^{\otimes 2}$ and $(a,b) \in M^2$, there is a sequence $(r_k)_{k \geq 0}$ satisfying that

$$\rho(\varphi_{r_k}(a), \varphi_{r_k}(b)) \geq \eta$$

for any integer $k \geq 0$;

(2) for almost everywhere measure $\mu^{\otimes 2}$ and $(a,b) \in M^2$,

$$\sup_{k \geq 0} \rho(\varphi_{r_k}(a), \varphi_{r_k}(b)) \geq \triangle(\varphi);$$

(3) $\triangle(\varphi) \leq$ diam(supp μ).

Proof. (1) For $r \geq 0$, let

$$M_r^2 = \bigcup_{r \geq 0} \overline{\varphi}_r^{-1}(a,b).$$

Write

$$F = M^2 \cap (\bigcup_{r \geq 0} \overline{\varphi}_r^{-1}(a,b)).$$

If $(a,b) \in F$, then there exists a real number sequence $0 < r_1 < r_2 < \cdots$ satisfying

$$\overline{\varphi}_{r_i}((a,b)) \in M^2$$

for any integer $i \geq 0$. Since

$$\overline{\varphi}_{r_j - r_i}(\overline{\varphi}_{r_i}((x,y))) \in M^2$$

for any integer $j \geq 0$, then for each $i \geq 0$, one has

$$\overline{\varphi}_{r_i}((a,b)) \in F.$$

It remains to show

$$\mu^{\otimes 2}(F) = \mu^{\otimes 2}(M^2).$$

In fact, since $\overline{\varphi}$ is a measure-preserving semi-flow,

$$\mu^{\otimes 2}(M_r^2) = M_0^2$$

for all $r \geq 0$. Clearly, $M_0^2 \supset M^2$. Therefore, one can get that

$$\mu^{\otimes 2}(F) = \mu^{\otimes 2}(M^2).$$

(2) If φ is pairwise sensitive, it follows that for some sufficiently small $\varepsilon > 0$,

$$\mu^{\otimes 2}\left(\bigcap_{t \geq 0} \overline{\varphi}_t^{-1}(A_{\triangle(\varphi) - \varepsilon})\right) = 0.$$

Therefore, based on a monotonicity argument, it can be concluded that

$$\mu^{\otimes 2}\left(\bigcup_{\varepsilon>0}\bigcap_{t\geq 0}\overline{\varphi}_t^{-1}(A_{\triangle(\varphi)-\varepsilon})\right) = \lim_{\varepsilon\to 0}\mu^{\otimes 2}\left(\bigcap_{t\geq 0}\overline{\varphi}_t^{-1}(A_{\triangle(\varphi)-\varepsilon})\right) = 0.$$

Since

$$\bigcup_{\varepsilon>0}\bigcap_{t\geq 0}\overline{\varphi}_t^{-1}(A_{\triangle(\varphi)-\varepsilon}) = \{(a,b)\in M^2 : \sup_{r\geq 0}\rho(\varphi_r(a),\varphi_r(b)) < \triangle(\varphi)\},$$

for almost everywhere measure $\mu^{\otimes 2}$ and $(a,b)\in M^2$, one has

$$\sup_{k\geq 0}\rho(\varphi_{r_k}(a),\varphi_{r_k}(b)) \geq \triangle(\varphi).$$

(3) Hypothesis that

$$\beta(\mu) < \triangle(\varphi).$$

Since φ is pairwise sensitive, then

$$\mu^{\otimes 2}\left(\bigcap_{t\geq 0}\overline{\varphi}_t^{-1}(A_r)\right) = 0$$

and $\mu(A_r) = 1$ for all $r\in(\beta(\mu),\triangle(\varphi)]$. This a contradiction. Therefore,

$$\beta(\mu) \geq \triangle(\varphi).$$

Consequently, by Lemma 4, (3) is true. □

The theorem presented below corresponds to the semi-flow version of Theorem 2.2 in [18].

Theorem 5. *Assume that M is a nontrivial metric space and φ is a weakly mixing semi-flow on $(M,\rho,\mathcal{B}(M),\mu)$. Then, φ is pairwise sensitive, and $\triangle(\varphi) =$ diam(supp μ).*

Proof. Let $\delta <$ diam(supp μ). Since φ is weakly mixing, then $\overline{\varphi}$ is weakly mixing, which implies that $\overline{\varphi}$ is ergodic [32]. By the definition and the ergodicity of $\overline{\varphi}$, one has

$$\mu^{\otimes 2}\left(\bigcap_{t\geq 0}\overline{\varphi}_t^{-1}(A_\delta)\right) \in \{0,1\}.$$

By Lemma 4,

$$\mu^{\otimes 2}(A_\delta) < 1.$$

So,

$$\mu^{\otimes 2}\left(\bigcap_{t\geq 0}\overline{\varphi}_t^{-1}(A_\delta)\right) = 0.$$

Consequently, φ is pairwise sensitive and $\triangle(\varphi)\geq$ diam(supp μ). By Theorem 4 (3), $\triangle(\varphi) =$ diam(supp μ). Thus, the theorem is true. □

Obviously, for almost everywhere measure $\mu^{\otimes 2}$ and $(a,b)\in M^2$, one has

$$\sup_{t\geq 0}\rho(\varphi_t(a),\varphi_t(b)) \leq \mathrm{diam}(\mathrm{supp}\,\mu).$$

Then, the following corollary is the semi-flow version of Corollary 2.1 in [18] and is a straightforward consequence of Theorem 4 (2) and Theorem 5.

Corollary 1. *Assume that M is a nontrivial metric space and φ is a weakly mixing semi-flow on $(M,\rho,\mathcal{B}(M),\mu)$. Then, for almost everywhere measure $\mu^{\otimes 2}$ and $(x,y)\in M^2$, one has*

$$\sup_{r\geq 0}\rho(\varphi_r(x),\varphi_r(y)) = diam(supp\mu).$$

Assume that ξ is a measurable countable partition of M, the metric entropy of transformation $f : M \to M$ with respect to the partition ξ is denoted by $h(f,\xi)$ (see [33]). It is widely accepted, although the definition of sensitivity may different, while positive metric entropy implies sensitivity (see [8,14,34]). The subsequent theorem provides a partial response in the context of pairwise sensitivity for semi-flows.

If $P_i \subset M(i = 1, 2, \dots, l, l \geq 2)$ are satisfying that $P_i \cap P_j = \emptyset, i \neq j, i, j \in \{1, 2, \dots, l\}$ and $\cup_{i=1}^{l} P_i = M$, then it is called that $\{P_1, P_2, \dots, P_l\}$ is a finite partition of M.

Theorem 6. *Assume that φ is a weakly mixing semi-flow on $(M, \rho, \mathcal{B}(M), \mu)$, where M is a nontrivial metric space. Taking $r \geq 0$ such that φ_r is ergodic. Within this context, a finite measurable partition*

$$\xi = \{P_1, P_2, \cdots, P_l\}$$

of M is considered.

Assuming that $P_1, P_2, \cdots, P_l$ are μ-continuity sets for ρ, and $h(\varphi_r, \xi) > 0$, then φ is pairwise sensitive.

Proof. Without loss of generality, it is possible to assume that

$$h(\varphi_r, \xi) < \infty.$$

Given any $A \in \mathcal{B}(M)$ and any $\varepsilon > 0$, let $A^{-\varepsilon}$ denote the internal ε-boundary of A, that is,

$$A^{-\varepsilon} = \{a \in A : d(a, A^c) < \varepsilon\}.$$

Let

$$K_\varepsilon = \exp\left(2l \sum_{i=1}^{l} \mu(P_i^{-\varepsilon})\right).$$

Since the P_i are μ-continuity sets, it follows that

$$\lim_{\varepsilon\to 0} K_\varepsilon = 0.$$

Therefore, one can select $\eta > 0$ such that

$$K_\eta < \sqrt{2^{h(\varphi_r,\xi)}}.$$

By Theorem 3.1 in [18] and its proof,

$$\mu^{\otimes 2}\left(\bigcap_{n\geq 0}(\overline{\varphi_r})^{-n}(A_\delta)\right) = 0.$$

Since

$$\bigcap_{n\geq 0}(\overline{\varphi_r})^{-n}(A_\delta) \supset \bigcap_{t\geq 0}(\overline{\varphi_t})^{-1}(A_\delta),$$

then

$$\mu^{\otimes 2}\left(\bigcap_{t\geq 0}(\overline{\varphi_t})^{-1}(A_\delta)\right) = 0.$$

Thus, by the definition, φ is pairwise sensitive. $\square$

4. Applications

Let a selfmap f on a compact metric space (M, ρ) be continuous. One can define an equivalence relation '$\sim$' in the product space $[0,1] \times M$ as follows. For any $(t_1, y_1), (t_2, y_2) \in [0,1] \times M$. $(t_1, y_1) \sim (t_2, y_2)$ if and only if one of the following conditions hold: (1) $(t_1, y_1) = (t_2, y_2)$; (2) $t_1 = 1, t_2 = 0$ and $y_2 = f(y_1)$.

Let $W = ([0,1] \times M)/\sim$. By [35], W is compact and metrizable. Then the suspended semi-flow $\varphi(f)$ induced by it on W is defined as $\varphi(f)(x_1, [(x_2, y)]) = x_1 + x_2 - k, f^k(y)$ for any $x_1 \geq 0$ and any $[(x_2, y)] \in W$ with $k \leq x_1 + x_2 < k + 1$ and k is a nonnegative integer.

Example 1. *For a continuous map $f : M \to M$, if $A \subset M$ is finitely chaotic with respect to a monotonically increasing sequence of positive integers, then for a suspended semi-flow $\varphi(f) : \mathbb{R}^+ \times M \to M$ of f, A is also finitely chaotic with respect to a positive integers sequence. On the contrary, for a semi-flow $\varphi(f)$, if A is finitely chaotic with respect to a monotonically increasing unbounded sequence $\{s_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^+$, then for f, A is finitely chaotic with respect to sequence $\{[s_j]\}_{j \in \mathbb{N}} \subset \mathbb{N}$, where $[s_j]$ is the integer part of s_j.*

Proof. Let $\{s_j\}_{j \in \mathbb{N}}$ is a monotonically increasing sequence of positive integers and A is finitely chaotic with respect to $\{s_j\}_{j \in \mathbb{N}}$. Then, for every finite subset $B \subset A$ and every continuous mapping $G : B \to M$, there exists a subsequence $\{t_j\}_{j \in \mathbb{N}} \in \mathbb{N}$ of $\{s_j\}_{j \in \mathbb{N}}$ such that $\lim\limits_{j \to \infty} f^{t_j}(u) = G(u)$ $(u \in B)$. By the definition of $\varphi(f)$, $\lim\limits_{j \to \infty} \varphi_{t_j}(u) = G(u)$ $(u \in B)$. Then for $\varphi(f)$, A is finitely chaotic with respect to $\{t_j\}_{j \in \mathbb{N}}$.

On the contrary, let $\{s_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^+$ is a monotonically increasing unbounded sequence, and for suspend semi-flow $\varphi(f)$, A is finitely chaotic with respect to $\{s_j\}_{j \in \mathbb{N}}$. Then, for every finite subset $B \subset A$ and every continuous mapping $G : B \to M$, there exists a subsequence $\{t_j\}_{j \in \mathbb{N}}$ of $\{s_j\}_{j \in \mathbb{N}}$ such that $\lim\limits_{j \to \infty} \varphi_{t_j}(u) = G(u)$ $(u \in B)$. By the definition of $\varphi(f)$, $\lim\limits_{j \to \infty} f^{[t_j]}(u) = \lim\limits_{j \to \infty} \varphi_{t_j}(u) = G(u)$ $(u \in B)$. Thus, for f, A is finitely chaotic with respect to $\{[s_j]\}_{j \in \mathbb{N}}$. $\square$

Example 2. *A continuous map $f : M \to M$ is pairwise sensitive if and only if its suspended semi-flow $\varphi(f) : \mathbb{R}^+ \times M \to M$ is pairwise sensitive, and the sensitivity constants can be the same.*

Proof. Let f is pairwise sensitive with the sensitivity constant η. Then, for almost everywhere measure $\mu^{\otimes 2}$ and $(a, b) \in M^2$, there exists a integer $r \geq 0$ such that $\rho(f^r(a), f^r(b)) \geq \eta$. This implies that $\rho(\varphi_r(f)(a), \varphi_r(f)(b)) \geq \eta$. Thus, $\varphi(f)$ is pairwise sensitive with the sensitivity constant η.

On the contrary, if $\varphi(f)$ is pairwise sensitive with the sensitivity constant η, then for almost everywhere measure $\mu^{\otimes 2}$ and $(a, b) \in M^2$, there exists a real number $r \geq 0$ such that $\rho(\varphi_r(f)(a), \varphi_r(f)(b)) \geq \eta$. So, $\rho(f^{[r]}(a), f^{[r]}(b)) \geq \eta$. That is, f is pairwise sensitive with the sensitivity constant η. $\square$

Example 3. *Assume that the semi-flow $\varphi(f) : \mathbb{R}^+ \times M \to M$ of f satisfies the conditions of Theorem 4, and $\{r_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^+$ is a monotonically increasing unbounded sequence, then the conclusion of Theorem 4 also holds for f and integer sequence $\{[r_k]\}_{k \in \mathbb{N}}$.*

Proof. By the proof of Example 2, $\varphi(f)$ is pairwise sensitive if and only if f is pairwise sensitive. And because $\rho(\varphi_{r_k}(f)(a), \varphi_{r_k}(f)(b)) = \rho(f^{[r_k]}(a), f^{[r_k]}(b))$ $(r_k \in \mathbb{R}^+)$, then $\rho(\varphi_{r_k}(f)(a), \varphi_{r_k}(f)(b)) \geq \eta$ if and only if $\rho(f^{[r_k]}(a), f^{[r_k]}(b)) \geq \eta$. Thus, the proof is completed. $\square$

5. Conclusions

For semi-flows, the relationship between pairwise sensitivity and weak mixing, the relationship between pairwise sensitivity and positiveness of metric entropy are studied in this research. Moreover, the largest sensitivity constant for some semi-flows is computed.

While, one may object that the present study assumes idealized conditions, such as perfect precision, complete determinism, and infinite computational resources. Real-world systems are subject to various sources of noise, uncertainties, and limitations in measurement accuracy and computational resources. The impact of these factors on the studied phenomena may need to be carefully evaluated. Finite chaotic systems are highly sensitive to parameter values and initial conditions. Different parameter regimes or initial conditions can lead to substantially different behaviors. Therefore, care must be taken to explore a range of parameter values and initial conditions to ensure the robustness and generality of the findings. This important aspect deserves to be discussed.

In addition, as is well known, fractal wavelet analysis has wide applications in fields such as communication, signal processing, and information security [36–42]. Chaos theory also has a very rich application in these areas. Therefore, the integration of chaos theory and fractal wavelet analysis to study and solve problems will become a hot frontier problem, and we will explore in this direction.

Author Contributions: Conceptualization, R.L., J.P. and T.L.; validation, R.L., T.L. and Y.L.; formal analysis, R.L., J.P. and T.L.; investigation, R.L., Y.L., J.W. and X.D.; writing original draft, R.L.; writing review and editing, Y.L., J.P. and T.L.; supervision, T.L.; funding acquisition, Y.L. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Cooperative Education Project of the Ministry of Education (No. 220605115023220), the Natural Science Foundation of Sichuan Province (Nos. 2023NSFSC0070, 2022NSFSC1821), the Scientific Research Project of SUSE (No. 2020RC24), the Ministry of Education Science and Technology Development Center (No. 2020QT13).

Data Availability Statement: Not applicable.

Acknowledgments: Many thanks to experts.

Conflicts of Interest: The authors declare no conflict of interest regarding the publication of this paper.

Sample Availability: Not applicable.

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