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On a Generalization of the Kummer's Quadratic Transformation and a Resolution of an Isolated Case

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Abstract: The resolution of isolated cases of the well-known quadratic transformation due to Kummer was thoroughly examined in two recent publications by Atia as well as Atia and Al-Mohaimeed. The objective of this paper is twofold. We establish generalizations of the quadratic transformation due to Kummer in the most general case in the first section, and in the second section, an effort is made to discuss the resolution of an isolated case of a quadratic transformation contiguous to that of Kummer established by Choi and Rathie.

Keywords: hypergeometric function; Kummer's quadratic transformation; hypergeometric series with finitely many terms; hypergeometric series with infinitely many terms; quadratic transformation contiguous to that of Kummer; differential equation

MSC: 33C05; 33D15

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1. Introduction

Every one of us who has worked on classical orthogonal polynomials, special functions, or other related fields has, at some point, used the extensive list of quadratic transformation formulas for Gauss' hypergeometric function, ${}_2F_1(u,v;w;z)$, that many authors have so thoughtfully reproduced in their monographs, encyclopedias, and handbooks [1–3]. They are extremely useful for students and researchers alike. In 1812, Gauss [4] defined their famous infinite series as follows

$$1+\frac{uv}{w}\frac{z}{1!}+\frac{u(u+1)v(v+1)}{w(w+1)}\frac{z^2}{2!}+\dots \tag{1}$$

The infinite series (1) is usually denoted by the notation ${}_2F_1(u,v;w;z)$, or simply F, and is commonly known as Gauss's function or the hypergeometric series. Gauss's function or the hypergeometric series is a solution of a second-order differential equation. In this function, we have two numerator parameters, u and v, and one denominator parameter, w, which are quantities that may be real or complex, with one exception that w should not be zero or a negative integer and the quantity, z, is called the variable of the series. It is interesting to mention here that for $u=1$ and $v=w$ (or, equivalently, $v=1$ and $u=w$), the infinite series (1) reduces to the well-known "geometric series", and with this fact, the series (1) is called "hypergeometric series".

Moreover, in terms of Pochhammer's symbol, $(u)_m$, which is defined for a u complex number ($u\neq 0$) by

$$(u)_m=\begin{cases} u(u+1)\dots(u+m-1), & m\in\mathbb{N},\\ 1, & m=0,\end{cases}, \tag{2}$$

The infinite series (1) can be represented by

$$ {}_2F_1(u,v;w;z) = \sum_{m=0}^{\infty} \frac{(u)_m (v)_m}{(w)_m} \frac{z^m}{m!}. \tag{3} $$

Further, in terms of the well-known Gamma function, the Pochhammer symbol, $(u)_m$, is represented by

$$ (u)_m = \frac{\Gamma(u+m)}{\Gamma(u)}, \quad Re(u) > 0. \tag{4} $$

By using a ratio test, it is not difficult to verify that the infinite series (1):

- Is convergent for all values of z, provided $\mid z \mid < 1$, and is divergent when $\mid z \mid > 1$;
- Is convergent for $z = 1$, provided $\Re(w-u-v) > 0$, and is divergent when $\Re(w-u-v) \leq 0$;
- Is absolutely convergent for $z = -1$, provided $Re(w-u-v) > 0$; convergent, but not absolutely, for $-1 < Re(w-u-v) \leq 0$; and divergent when $Re(w-u-v) < -1$.

The limiting case of (3) is worthy mentioning here. For this, if we replace z by $\frac{z}{v}$ in (3) and take the limit as $v \to \infty$, then, since $\frac{(v)_m}{v^m} z^m \to z^m$, we arrive at the following infinite series, which is known in the literature either as a confluent hypergeometric function or as Kummer's function, ${}_1F_1$,

$$ {}_1F_1(u;w;z) = \sum_{m=0}^{\infty} \frac{(u)_m}{(w)_m} \frac{z^m}{m!}. \tag{5} $$

We remark in passing that almost all elementary functions of mathematics and mathematical physics are special cases or limiting cases of Gauss's hypergeometric function. For more details about Gauss's hypergeometric function, we refer to the standard text of Rainville [5].

Looking toward the definition of Gauss's hypergeometric function, it is self-evident that symmetry occurs in the numerator parameters of Gauss's hypergeometric function.

Moreover, it is evident that the transformation formulas (including quadratic and cubic) for the hypergeometric function play an important role in the theory of hypergeometric functions. A large number of very useful and interesting transformation formulas have been listed in the well-known paper by Goursat [6].

However, in our present investigation, we are interested in the following quadratic transformation due to Kummer [7]

$$ {}_2F_1(u,v;2v;z) = (1-\frac{z}{2})^{-u} \, {}_2F_1\left(\frac{u}{2}, \frac{u+1}{2}; v+\frac{1}{2}; (\frac{z}{2-z})^2\right) \tag{6} $$

provided that $\{2v+1,\ v+\frac{3}{2}\}$ are not natural numbers and $v-u$ is not an integer.

The result (6) is also recorded, for example, in the standard text of I.S. Gradshteyn and I.M. Ryzhik [1] (9.134 and 9.134.1) and G. Andrews et al. [2] (3.1.7 page 127, with a slight modification), in the handbook by Abramowitz-Stegun [3] (15.3.20), and in DLMF: NIST Digital Library of Mathematical Functions, https://dlmf.nist.gov/ (accessed on 15 June 2023) 15.8.13 [8].

The transformation (1.6) is a quadratic transformation that relates two hypergeometric functions (with the linear argument in one and a quadratic in the other), which are true under some conditions. In fact, in [1], page 1008, 9.130, the authors wrote:

> *Generally speaking, the analytic function that is defined by the series ${}_2F_1(\alpha,\beta;\gamma;z)$ has singularities at z = 0, 1, and ∞. (There are branch points in the general case.) From $z = 1$ to $z = \infty$, we cut the z-plane along the real axis; in other words, we require that $|arg(-z)| < \pi$ for $|z| = 1$. The series ${}_2F_1(\alpha,\beta;\gamma;z)$ will then yield a single-valued analytic continuation on the cut plane, which can be obtained using the formulas below (provided $\gamma+1$ is not a natural number and $\alpha-\beta$ and $\gamma-\alpha-\beta$ are not integers). These equations allow for the calculation of ${}_2F_1(\alpha,\beta;\gamma;z)$ values in the specified region, even when $|z| > 1$. If the corresponding correlations between $\alpha, \beta,$ and γ*

are true, other closely related transformation formulas can also be employed to obtain the analytical continuation.

In 2011, Choi and Rathie [9] established the following two formulas closely related to Kummer's transformation (6)

$$\begin{aligned}&{}_2F_1(u,v;2u+1;2z)\\&=(1-z)^{-v}\,{}_2F_1(\frac{v}{2},\frac{v+1}{2};u+\frac{1}{2};(\frac{z}{1-z})^2)\\&-\frac{vz}{2u+1}(1-z)^{-v-1}\,{}_2F_1(\frac{v+1}{2},\frac{v}{2}+1;u+\frac{3}{2};(\frac{z}{1-z})^2),\end{aligned}\tag{7}$$

which is valid when $2u+1\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $|z|<\frac{1}{2}$, and $|\frac{z}{1-z}|<1$.

The aim of this paper is twofold: first, we will generalize the Choi and Rathie result to ${}_2F_1(u,v;2u\pm n;2z)$, $n\in\mathbb{N}$; then, we give the right expressions with the isolated cases $2u\pm n\in\mathbb{Z}_0^-$, $|z|<\frac{1}{2}$, and $|\frac{z}{1-z}|<1$.

Preliminaries and Main Notations

It is not out of place to mention here that Kummer [7] established transformation Formula (6) using the theory of differential equations. On the other hand, if, in (6), we replace z by $2z$, we obtain the following formula:

$${}_2F_1(u,v;2v;2z)=(1-z)^{-u}\,{}_2F_1\left(\frac{u}{2},\frac{u+1}{2};v+\frac{1}{2};(\frac{z}{1-z})^2\right).\tag{8}$$

It is shown in the standard text of Rainville [5] that transformation Formula (8) can be proven very quickly by employing the following classical summation theorem due to Gauss [4]:

$${}_2F_1(u,v;w;1)=\frac{\Gamma(w)\Gamma(w-u-v)}{\Gamma(w-u)\Gamma(w-v)},\tag{9}$$

provided $\Re(w-u-v)>0$.

In addition to this, by considering the case when $v-u$ is an integer and u is either an even or an odd integer, very recently, Atia [10] and Atia with Al-Mohaimeed [11] established two results, which are recorded here

(a) For $v\in\mathbb{Z}^-$, $u\in 2\mathbb{Z}$,

$$\begin{aligned}&{}_2F_1(\frac{u}{2},\frac{u+1}{2};v+\frac{1}{2};(\frac{z}{2-z})^2)=(1-\frac{z}{2})^u\,{}_2F_1(u,v;2v;z)\\&+\frac{2\sqrt{\pi}(\frac{u}{2})_{-v+1}}{\Gamma(-v+\frac{1}{2})}\frac{(4z-4)^{v-u}(2-z)^{u+1}}{z^{2v+1-u}}\,{}_2F_1\left(1-\frac{u}{2},\frac{1-u}{2}+v;\frac{3}{2};(\frac{2}{z}-1)^2\right).\end{aligned}\tag{10}$$

(b) For $v\in\mathbb{Z}^-$, $u\in 1+2\mathbb{Z}$,

$$\begin{aligned}&{}_2F_1(\frac{u}{2},\frac{u+1}{2};v+\frac{1}{2};(\frac{z}{2-z})^2)=(1-\frac{z}{2})^u\,{}_2F_1(u,v;2v;z)\\&-\frac{\sqrt{\pi}(\frac{u+1}{2})_{-v}}{\Gamma(-v+\frac{1}{2})}\frac{(4z-4)^{v-u}(2-z)^{u}}{z^{2v-u}}\,{}_2F_1\left(\frac{1-u}{2},v-\frac{u}{2};\frac{1}{2};(\frac{2}{z}-1)^2\right).\end{aligned}\tag{11}$$

For simplicity, let us denote by

$$F_1(u,v,z):=\,{}_2F_1\left(\frac{u}{2},\frac{u+1}{2};v+\frac{1}{2};(\frac{z}{2-z})^2\right)$$

$$F_2(u,v,z):=(1-\frac{z}{2})^u\,{}_2F_1(u,v;2v;z),$$

$$b_v^{(u)} = \frac{2\sqrt{\pi}(\frac{a}{2})_{-v+1}}{\Gamma(-v+\frac{1}{2})} \frac{(4z-4)^{v-u}(2-z)^{u+1}}{z^{2v+1-u}} \, {}_2F_1\left(1-\frac{u}{2}, \frac{1}{2}+v-\frac{u}{2}; \frac{3}{2}; (\frac{2}{z}-1)^2\right), \tag{12}$$

and

$$c_v^{(u)} = -\frac{\sqrt{\pi}(\frac{u+1}{2})_{-v}}{\Gamma(-v+\frac{1}{2})} \frac{(4z-4)^{v-u}(2-z)^{u}}{z^{2v-u}} \, {}_2F_1\left(\frac{1}{2}-\frac{u}{2}, v-\frac{u}{2}; \frac{1}{2}; (\frac{2}{z}-1)^2\right). \tag{13}$$

Atia and Al-Mohaimeed [11] have also proven that

$$F_1(u,v,z) = -c_v^{(u,even)}$$

and

$$F_1(u,v,z) = -b_v^{(u,odd)}.$$

Here, we return to Equation (6). If we replace z by $\frac{2z}{1+z}$, then it takes the following form

$$(1+z)^{-u} \, {}_2F_1\left(u, v; 2v; \frac{2z}{1+z}\right) = {}_2F_1\left(\frac{u}{2}, \frac{u+1}{2}; v+\frac{1}{2}; z^2\right) \tag{14}$$

provided $2v \notin \mathbb{Z}^-$, $\mid \frac{2z}{1+z} \mid < \frac{1}{2}$, and $\mid z \mid < 1$.

In 2011, Choi and Rathie [9] established the result closely related to (14) given below

$$\begin{aligned} &(1+z)^{-u} \, {}_2F_1\left(u, v; 2v+1; \frac{2z}{1+z}\right) \\ &= {}_2F_1\left(\frac{u}{2}, \frac{u+1}{2}; v+\frac{1}{2}; z^2\right) - \frac{uz}{2v+1} \, {}_2F_1\left(\frac{u+1}{2}, \frac{u+2}{2}; v+\frac{3}{2}; z^2\right) \end{aligned} \tag{15}$$

provided $2v+1 \notin \mathbb{Z}^-$, $\mid \frac{2z}{1+z} \mid < \frac{1}{2}$, and $\mid z \mid < 1$.

In the next section, we give interesting generalizations of the identities (14) and (15) in the most general form.

2. Generalization of Kummer's Transformation

In this section, we shall provide generalizations of (14) and (15) asserted in the following theorem.

Theorem 1. *For any integer $m \geq 0$, the following results hold true:*

$$\begin{aligned} {}_2F_1\left(u, v; 2v+m; \frac{2z}{1+z}\right) &= \sum_{k=0}^{m} (-\frac{2z}{1+z})^k \frac{(u)_k (v)_k \binom{m}{k}}{(2v+k-1)_k (2v+m)_k} \\ &\times {}_2F_1\left(u+k, v+k; 2(v+k); \frac{2z}{1+z}\right), \end{aligned} \tag{16}$$

and

$$\begin{aligned} {}_2F_1\left(u, v; 2v-m; \frac{2z}{1+z}\right) &= \sum_{k=0}^{m} (\frac{-2z}{1+z})^k \frac{(-1)^k (u)_k (v-m)_k \binom{m}{k}}{(2v-2m+k-1)_k (2v-m)_k} \\ &\times {}_2F_1\left(u+k, v+k-m; 2(v+k-m); \frac{2z}{1+z}\right). \end{aligned} \tag{17}$$

In order to prove this theorem, first, we shall derive the following Lemma.

Lemma 1.

$$(i) \qquad {}_2F_1(u,v;2v+m,x) - {}_2F_1(u,v;2v+m-1;x) = -\frac{uvx}{(2v+m)(2v+m-1)}\,{}_2F_1(u+1,v+1;2v+m+1;x), \tag{18}$$

and

$$(ii) \qquad {}_2F_1(u,v;2v-m,x) - {}_2F_1(u,v;2v-m+1;x) = \frac{uvx}{(2v-m)(2v-m+1)}\,{}_2F_1(u+1,v+1;2v-m+2;x). \tag{19}$$

Proof. In order to prove (i), we start with the left hand side of (21) .

$$\begin{aligned}
L.H.S &={}_2 F_1(u,v;2v+m,x) - {}_2F_1(u,v;2v+m-1;x)\\
&= \sum_{k\geq 0} \frac{(u)_k(v)_k x^k}{(2v+m)_k k!} - \sum_{k\geq 0} \frac{(u)_k(v)_k x^k}{(2v+m-1)_k k!}\\
&= \sum_{k\geq 0} \frac{(u)_k(v)_k x^k}{k!}\left(\frac{1}{(2v+m)_k} - \frac{1}{(2v+m-1)_k}\right)\\
&= \sum_{k\geq 0} \frac{(u)_k(v)_k x^k}{k!}\frac{((2v+m-1)_k - (2v+m-1)_k)}{(2v+m)_k(2v+m-1)_k}\\
&= \sum_{k\geq 0} \frac{(u)_k(v)_k x^k}{k!}\frac{(2v+m)_{k-1}(2v+m-1-(2v+m+k-1))}{(2v+m)_k(2v+m-1)_k}\\
&= -\sum_{k\geq 0} \frac{k(u)_k(v)_k x^k}{k!}\frac{(2v+m)_{k-1}}{(2v+m)_k(2v+m-1)_k}\\
&= -\sum_{k\geq 1} \frac{u(u+1)_{k-1}v(v+1)_{k-1}x^k}{(k-1)!}\frac{(2v+m)_{k-1}}{(2v+m)(2v+m+1)_{k-1}(2v+m-1)(2v+m)_{k-1}}\\
&= -\frac{uvx}{(2v+m)(2v+m-1)}\sum_{k\geq 1}\frac{(u+1)_{k-1}(v+1)_{k-1}x^{k-1}}{(2v+m+1)_{k-1}(k-1)!}\\
&= -\frac{uvx}{(2v+m)(2v+m-1)}\,{}_2F_1(u+1,v+1;2v+m+1;x).
\end{aligned}$$

which is the right-hand side of (21) . □

The result (ii) can be proven on similar lines, so we prefer to avoid the details. Here, we are ready to establish the result (16) asserted in Theorem 1.

Proof. The proof is made by induction on m.

For $m=0$, it is obvious that we obtain

$${}_2F_1\left(u,v;2v;\frac{2z}{1+z}\right) = {}_2F_1\left(u,v;2v;\frac{2z}{1+z}\right).$$

For $m=1$, we obtain

$$\begin{aligned}
&{}_2F_1\left(u,v;2v+1;\frac{2z}{1+z}\right)\\
&= (-\frac{2z}{1+z})^0\frac{(u)_0(v)_0\binom{1}{0}}{(2v+1-0)_0(2v+1)_0}\,{}_2F_1(u+0,v+0;2(v+0);\frac{2z}{1+z})\\
&+(-\frac{2z}{1+z})^1\frac{(u)_1(v)_1\binom{1}{1}}{(2v+1-1)_1(2v+1)_1}\,{}_2F_1(u+1,v+1;2(v+1);\frac{2z}{1+z}),
\end{aligned}$$

which becomes

$$ {}_2F_1\left(u,v;2v+1;\frac{2z}{1+z}\right) = {}_2F_1(u,v;2v,\frac{2z}{1+z}) \\ -(\frac{2uvz}{(2v)(2v+1)(1+z)}\ {}_2F_1(u+1,v+1;2(v+1);\frac{2z}{1+z})), \tag{20} $$

After simplifying and using (6) by substituting z with $\frac{z}{1+z}$, we obtain Choi and Rathie identity.

Furthermore, using the previous lemma, one obtains

$$ {}_2F_1\left(u,v;2v+m+1;\frac{2z}{1+z}\right) = {}_2F_1\left(u,v;2v+m;\frac{2z}{1+z}\right) \\ -\frac{2uvz}{(2v+m+1)(2v+m)(1+z)}\ {}_2F_1\left(u+1,v+1;2v+m+2;\frac{2z}{1+z}\right), $$

which we write as

$$ {}_2F_1\left(u,v;2v+m+1;\frac{2z}{1+z}\right) = {}_2F_1\left(u,v;2v+m;\frac{2z}{1+z}\right) \\ -\frac{2uvz}{(2v+m+1)(2v+m)(1+z)}\ {}_2F_1\left(u+1,v+1;2(v+1)+m;\frac{2z}{1+z}\right). $$

Taking into account (16), we obtain

$$ {}_2F_1\left(u,v;2v+m+1;\frac{2z}{1+z}\right) $$

$$ = \sum_{k=0}^{m}(-\frac{2z}{1+z})^k\frac{(u)_k(v)_k\binom{m}{k}}{(2v+k-1)_k(2v+m)_k}\ {}_2F_1\left(u+k,v+k;2(v+k);\frac{2z}{1+z}\right) $$

$$ -\frac{2uvz}{(2v+m+1)(2v+m)(1+z)}\sum_{k=0}^{m}(-\frac{2z}{1+z})^k\frac{(u+1)_k(v+1)_k\binom{m}{k}}{(2v+k+1)_k(2v+2+m)_k} \\ \times\ {}_2F_1\left(u+1+k,v+1+k;2(v+1+k);\frac{2z}{1+z}\right) $$

$$ = \sum_{k=0}^{m}(-\frac{2z}{1+z})^k\frac{(u)_k(v)_k\binom{m}{k}}{(2v+k-1)_k(2v+m)_k}\ {}_2F_1\left(u+k,v+k;2(v+k);\frac{2z}{1+z}\right) $$

$$ +\frac{1}{(2v+m)(2v+1+m)}\sum_{k=0}^{m}(-\frac{2z}{1+z})^{k+1}\frac{(u)_{k+1}(v)_{k+1}\binom{m}{k}}{(2v+k+1)_k(2v+m)_{k+2}} \\ \times\ {}_2F_1\left(u+1+k,v+1+k;2(v+1+k);\frac{2z}{1+z}\right) $$

$$ = \sum_{k=0}^{m}(-\frac{2z}{1+z})^k\frac{(u)_k(v)_k\binom{m}{k}}{(2v+k-1)_k(2v+m)_k}\ {}_2F_1\left(u+k,v+k;2(v+k);\frac{2z}{1+z}\right) $$

$$ +\frac{1}{(2v+m)(2v+1+m)}\sum_{k=1}^{m+1}(-\frac{2z}{1+z})^{k}\frac{(u)_k(v)_k\binom{m}{k-1}}{(2v+k+1)_{k-1}(2v+m)_{k+1}} \\ \times\ {}_2F_1\left(u+k,v+k;2(v+k);\frac{2z}{1+z}\right) $$

$$
\begin{aligned}
&= {}_2F_1\left(u, v; 2v; \frac{2z}{1+z}\right) + \sum_{k=1}^{m} (-\frac{2z}{1+z})^k \frac{(u)_k (v)_k \binom{m}{k}}{(2v+k-1)_k (2v+m)_k} \\
&\times {}_2F_1\left(u+k, v+k; 2(v+k); \frac{2z}{1+z}\right) + (-\frac{2z}{1+z})^{m+1} \frac{(u)_{m+1}(v)_{m+1}\binom{m}{m}}{(2v+m)_{m+1}(2v+m+1)_{m+1}} \\
&\times {}_2F_1\left(u+m+1, v+m+1; 2(v+m+1); \frac{2z}{1+z}\right) + \frac{1}{(2v+m)(2v+1+m)} \sum_{k=1}^{m} (-\frac{2z}{1+z})^k \\
&\times \frac{(u)_k (v)_k (2v+k-1)(2v+m+1)\binom{m}{k-1}}{(2v+k-1)_k (2v+m+1)_k} {}_2F_1\left(u+k, v+k; 2(v+k); \frac{2z}{1+z}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= {}_2F_1\left(u, v; 2v; \frac{2z}{1+z}\right) + (-\frac{2z}{1+z})^{m+1} \frac{(u)_{m+1}(v)_{m+1}\binom{m+1}{m+1}}{(2v+m)_{m+1}(2v+m+1)_{m+1}} \\
&\times {}_2F_1\left(u+m+1, v+m+1; 2(v+m+1); \frac{2z}{1+z}\right) + \sum_{k=1}^{m} (-\frac{2z}{1+z})^k (u)_k (v)_k \\
&\times {}_2F_1\left(u+k, v+k; 2(v+k); \frac{2z}{1+z}\right) \\
&\times \left(\frac{\binom{m}{k}}{(2v+k-1)_k (2v+m)_k} + \frac{(2v+k-1)(2v+m+1)\binom{m}{k-1}}{(2v+m)(2v+1+m)(2v+k-1)_k (2v+m+1)_k} \right)
\end{aligned}
$$

taking into account

$$
\begin{aligned}
&\frac{\binom{m}{k}}{(2v+k-1)_k (2v+m)_k} + \frac{(2v+k-1)(2v+m+1)\binom{m}{k-1}}{(2v+m)(2v+1+m)(2v+k-1)_k (2v+m+1)_k} \\
&= \frac{\binom{m+1}{k}}{(2v+k-1)_k (2v+m+1)_k}
\end{aligned}
$$

This completes the proof.

In exactly the same manner, the result (17) given in Theorem 1 can be proven. Thus, we left this as an exercise for the interested reader. □

We shall mention here that,for the first time, it is observed here that the transformation (15) still makes sense, even if $2v+1$ is a negative integer; thus, our aim in this paper is to discover a new formula for any negative integer, v, such that $2v \in \mathbb{Z}^-$, and for $u \in \mathbb{Z}$. Therefore, in order to find the results in the most general form, two cases have to be considered separately:

1. For any negative integer v and $u \in 2\mathbb{Z}$.
2. For any negative integer v and $u \in 1+2\mathbb{Z}$.

The details are given in the next section.

3. Resolution of an Isolated Case

In this section, we shall establish two new and interesting results asserted in the following theorem.

Theorem 2.

(i) *For $u \in 2\mathbb{Z}$ and $v \in \{\ldots, -4, -3, -2\}$, the following formula holds true:*

$$(1+z)^{-u}\,{}_2F_1\left(u,v;2v+1;\frac{2z}{1+z}\right)-\,{}_2F_1\left(\frac{1}{2}u,\frac{1}{2}u+\frac{1}{2};v+\frac{1}{2};z^2\right) \quad (21)$$
$$+\frac{uz}{2v+1}\,{}_2F_1\left(\frac{1}{2}u+\frac{1}{2},\frac{1}{2}u+1;v+\frac{3}{2};z^2\right)$$
$$=a_u^v(z)=b_{\frac{u}{2}}^{-v+1}(z)-\frac{uz}{2v+1}c_{\frac{u+1}{2}}^{-v}(z).$$

(ii) *For $u\in 2\mathbb{Z}+1$ and $v\in\{\ldots,-4,-3,-2\}$, the following formula holds true:*

$$(1+z)^{-u}\,{}_2F_1\left(u,v;2v+1;\frac{2z}{1+z}\right)-\,{}_2F_1\left(\frac{1}{2}u,\frac{1}{2}u+\frac{1}{2};v+\frac{1}{2};z^2\right) \quad (22)$$
$$+\frac{uz}{2v+1}\,{}_2F_1\left(\frac{1}{2}u+\frac{1}{2},\frac{1}{2}u+1;v+\frac{3}{2};z^2\right)$$
$$=a_u^v(z)=c_{\frac{u}{2}}^{-v+1}(z)-\frac{uz}{2v+1}b_{\frac{u+1}{2}}^{-v}(z).$$

Proof. In order to prove results (21) and (22), let us first denote by

$$G_1(u,v,z)=(1+z)^{-u}\,{}_2F_1\left(u,v;2v+1;\frac{2z}{1+z}\right),$$

$$G_2(u,v,z)=\,{}_2F_1\left(\frac{1}{2}u,\frac{1}{2}u+\frac{1}{2};v+\frac{1}{2};z^2\right),$$

and

$$G_3(u,v,z)=\frac{uz}{2v+1}\,{}_2F_1\left(\frac{1}{2}u+\frac{1}{2},\frac{1}{2}u+1;v+\frac{3}{2};z^2\right).$$

Then, let us express $G_1(u,v,z)$, $G_2(u,v,z)$, and $G_3(u,v,z)$ in terms of the functions of the function of $F_1(u,m,z)$, $F_2(u,m,z)$, $b_m^{(u)}$, and $c_m^{(u)}$.

For this, the first step is an easy task to see that the expression

$${}_2F_1(u,v;2v;z)-\,{}_2F_1(u,v;2v+1;z)=\frac{uz}{2(2v+1)}\,{}_2F_1(u+1,v+1;2v+2;z),$$

can be written in the following form:

$$F_2(u,v,z)-\frac{uz}{2(2v+1)(1-\frac{z}{2})}F_2(u+1,v+1,z)=(1-\frac{z}{2})^u\,{}_2F_1(u,v;2v+1;z).$$

where

$$G_1(u,v,z)=(1+z)^{-u}\,{}_2F_1\left(u,v;2v+1;\frac{2z}{1+z}\right)$$
$$=F_2\left(u,v,\frac{2z}{1+z}\right)-\frac{uz}{(2v+1)}F_2\left(u+1,v+1,\frac{2z}{1+z}\right).$$

$$G_2(u,v,z)=F_1\left(u,v,\frac{2z}{z+1}\right)$$

and

$$G_3(u,v,z)=\frac{uz}{2v+1}F_1\left(u+1,v+1,\frac{2z}{z+1}\right).$$

On the other hand, we have the following results

$$F_1(u,v,z)=F_2(u,v,z)+b_v^{(u)}(u,v,z),\textit{where } u \textit{ is an even integer},$$

as well as

$$F_1(u,v,z)=F_2(u,v,z)+c_v^{(u)}(u,v,z), \textit{where } u \textit{ is an odd integer}.$$

Here, we are ready to establish the results, (3.1) and (3.2), asserted in Theorem 2. Therefore, for $u \in 2\mathbb{Z}$, we have

$$\begin{aligned}
&(1+z)^{-u}\,{}_2F_1\left(u,v;2v+1;\frac{2z}{1+z}\right)-{}_2F_1\left(\frac{1}{2}u,\frac{1}{2}u+\frac{1}{2};v+\frac{1}{2};z^2\right)\\
&+\frac{uz}{2v+1}\,{}_2F_1\left(\frac{1}{2}u+\frac{1}{2},\frac{1}{2}u+1;v+\frac{3}{2};z^2\right)\\
&=G_1(u,v,z)-G_2(u,v,z)+G_3(u,v,z)\\
&=F_2\left(u,v,\frac{2z}{1+z}\right)-\frac{uz}{(2v+1)}F_2\left(u+1,v+1,\frac{2z}{1+z}\right)-F_1\left(u,v,\frac{2z}{z+1}\right)\\
&+\frac{uz}{2v+1}F_1\left(u+1,v+1,\frac{2z}{z+1}\right)\\
&=F_2\left(u,v,\frac{2z}{1+z}\right)-F_1\left(u,v,\frac{2z}{z+1}\right)\\
&+\frac{uz}{2v+1}\left(F_1\left(u+1,v+1,\frac{2z}{z+1}\right)-F_2\left(u+1,v+1,\frac{2z}{1+z}\right)\right)\\
&=-b_v^u+\frac{uz}{2v+1}c_v^{u+1}.
\end{aligned}$$

Similarly, for $u \in 1+2\mathbb{Z}$, we have

$$\begin{aligned}
&(1+z)^{-u}\,{}_2F_1\left(u,v;2v+1;\frac{2z}{1+z}\right)-{}_2F_1\left(\frac{1}{2}u,\frac{1}{2}u+\frac{1}{2};v+\frac{1}{2};z^2\right)\\
&+\frac{uz}{2v+1}\,{}_2F_1\left(\frac{1}{2}u+\frac{1}{2},\frac{1}{2}u+1;v+\frac{3}{2};z^2\right)\\
&=G_1(u,v,z)-G_2(u,v,z)+G_3(u,v,z)\\
&=F_2\left(u,v,\frac{2z}{1+z}\right)-\frac{uz}{(2v+1)}F_2\left(u+1,v+1,\frac{2z}{1+z}\right)-F_1\left(u,v,\frac{2z}{z+1}\right)\\
&+\frac{uz}{2v+1}F_1\left(u+1,v+1,\frac{2z}{z+1}\right)\\
&=F_2\left(u,v,\frac{2z}{1+z}\right)-F_1\left(u,v,\frac{2z}{z+1}\right)\\
&+\frac{uz}{2v+1}\left(F_1(u+1,v+1,\frac{2z}{z+1})-F_2(u+1,v+1,\frac{2z}{1+z})\right)\\
&=-c_v^u+\frac{az}{2v+1}b_v^{u+1}.
\end{aligned}$$

This completes the proof. □

Remark 1. *For $v=-1$, the results (21) and (22), take the following form:*

(i) *For $u \in 2\mathbb{Z}$, we have*

$$\begin{aligned}
&(1+z)^{-u}\,{}_2F_1(u,-1;-1;\frac{2z}{1+z})-{}_2F_1(\frac{1}{2}u,\frac{1}{2}u+\frac{1}{2};-\frac{1}{2};z^2)\\
&+\frac{uz}{-1}\,{}_2F_1(\frac{1}{2}u+\frac{1}{2},\frac{1}{2}u+1;-1+\frac{3}{2};z^2)=-b^2_{\frac{u}{2}}(z)-uzc^1_{\frac{u+1}{2}}(z),
\end{aligned}$$

(ii) *For $u \in 2\mathbb{Z}+1$, we have*

$$\begin{aligned}
&(1+z)^{-u}\,{}_2F_1(u,-1;-1;\frac{2z}{1+z})-{}_2F_1(\frac{1}{2}u,\frac{1}{2}u+\frac{1}{2};-\frac{1}{2};z^2)\\
&+\frac{uz}{-1}\,{}_2F_1(\frac{1}{2}u+\frac{1}{2},\frac{1}{2}u+1;-1+\frac{3}{2};z^2)=-c^2_{\frac{u}{2}}(z)-uzb^1_{\frac{u+1}{2}}(z).
\end{aligned}$$

One can use the following Maple commands

```
restart;
F1 := proc (u, v, z) options
operator, arrow; hypergeom([(1/2)*u+1/2, (1/2)*u], [v+1/2],
z^2/(2-z)^2) end proc;
F2 := proc (u, v, z) options operator, arrow;
(1-(1/2)*z)^u*hypergeom([u, v], [2*v], z) end proc;
bvu := proc (u,
v, z) options operator, arrow;
2*(z/(2-z))^(u-1-2*v)*sqrt(Pi)*pochhammer((1/2)*u, 1-v)
*hypergeom([1-(1/2)*u, 1/2+v-(1/2)*u], [3/2], (2-z)^2/z^2)
*(z^2/(2-z)^2-1)^(v-u)/GAMMA(1/2-v) end proc;
cvu := proc (u, v, z) options operator, arrow;
-(z/(2-z))^(u-2*v)*sqrt(Pi)*pochhammer((1/2)*u+1/2, -v)
*hypergeom([v-(1/2)*u, 1/2-(1/2)*u], [1/2], (2-z)^2/z^2)
*(z^2/(2-z)^2-1)^(v-u)/GAMMA(1/2-v) end proc;
G1 := proc (u, v, z) options operator, arrow;
(1+z)^(-u)*hypergeom([u, v], [2*v+1], 2*z/(1+z)) end proc;
G2 :=proc (u, v, z) options operator, arrow; hypergeom([(1/2)*u,
(1/2)*u+1/2], [v+1/2], z^2) end proc;
G3 := proc (u, v, z) options
operator, arrow; u*z*hypergeom([(1/2)*u+1/2, (1/2)*u+1], [v+3/2],
z^2)/(2*v+1) end proc; simplify(F1(6, -2, z)-F2(6, -2, z)-bvu(6,
-2, z));
                                0
factor(simplify(F1(5, -2, z)-F2(5, -2, z)-cvu(5, -2, z)));
                                0
simplify(hypergeom([u, v], [2*v], z)-hypergeom([u, v], [2*v+1], z)
-(1/2)*z*u*hypergeom([v+1, u+1], [2*v+2], z)/(2*v+1));
                                0
simplify(F2(u, v, z)-u*z*F2(u+1, v+1, z)/((2*(2*v+1))*(1-(1/2)*z)));
(1-(1/2)*z)^u*hypergeom([u, v], [2*v+1], z):

simplify(G1(u, v, z)-F2(u, v, 2*z/(z+1))+u*z*F2(v+1, v+1 ,
2*z/(z+1))/(2*v+1)):; simplify(G2(u, v, z)-F1(u, v, 2*z/(z+1)));
                                0
simplify(G3(u, v, z)-u*z*F1(u+1, v+1, 2*z/(z+1))/(2*v+1));
                                0
u := 8; vs. := -3; simplify(G1(u, v, z)-G2(u, v, z)+G3(u, v, z)
+bvu(u, v, 2*z/(z+1))-u*z*cvu(u+1, v+1, 2*z/(z+1))/(2*v+1));
                                0
u := 8; vs. := -1; simplify(G1(u, v, z)-G2(u, v, z)+G3(u, v, z)
-bvu(u, v, 2*z/(z+1))+u*z*cvu(u+1, v+1, 2*z/(z+1))/(2*v+1));
                                0
u := 7; vs. := -3; simplify(G1(u, v, z)-G2(u, v, z)+G3(u, v, z)
+cvu(u, v, 2*z/(z+1))-u*z*bvu(u+1, v+1, 2*z/(z+1))/(2*v+1));
                                0
u := 1; vs. := -1; simplify(G1(u, v, z)-G2(u, v, z)+G3(u, v, z)
-cvu(u, v, 2*z/(z+1))+u*z*bvu(u+1, v+1, 2*z/(z+1))/(2*v+1));
                                0
```

Remark 2. *It is clear that the transformation formula*

$$(1+z)^{-u}\ {}_2F_1(u,v;2v+1;\frac{2z}{1+z})$$
$$= {}_2F_1(\frac{1}{2}u,\frac{1}{2}u+\frac{1}{2};v+\frac{1}{2};z^2)-\frac{uz}{2v+1}\ {}_2F_1(\frac{1}{2}u+\frac{1}{2},\frac{1}{2}u+1;v+\frac{3}{2};z^2)$$

obtained earlier by Choi and Rathie is valid, provided $2b\notin\mathbb{Z}^-$, $\mid\frac{2z}{1+z}\mid<\frac{1}{2}$, *and* $\mid z\mid<1$.

It is observed for the first time that the above-mentioned transformation formula still makes sense, even if v *is any negative integer. Therefore, in this paper, an attempt was made to establish two results for any negative integer,* v, *with* $u\in 2\mathbb{Z}$ *and* $u\in 1+2\mathbb{Z}$.

We conclude this paper by remarking the general result of the form $(1+z)^{-u}\ {}_2F_1(u,v;2v\pm m;\frac{2z}{1+z})$, *in the most general form for any* $m\in\mathbb{N}\cup\{0\}$, *by considering the following cases:*

1. *For* $2v\pm m$, *it is neither zero nor a negative integer and* $u\in 2\mathbb{Z}$.
2. *For* $2v\pm m$, *it is neither zero nor a negative integer and* $u\in 1+2\mathbb{Z}$,

these cases are under investigation and will form a subsequent paper in this direction.

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