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A Solitonic Study of Riemannian Manifolds Equipped with a Semi-Symmetric Metric ξ-Connection

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Abstract: The aim of this paper is to characterize a Riemannian 3-manifold M^3 equipped with a semi-symmetric metric ξ-connection $\tilde{\nabla}$ with ρ-Einstein and gradient ρ-Einstein solitons. The existence of a gradient ρ-Einstein soliton in an M^3 admitting $\tilde{\nabla}$ is ensured by constructing a non-trivial example, and hence some of our results are verified. By using standard tensorial technique, we prove that the scalar curvature of $(M^3, \tilde{\nabla})$ satisfies the Poisson equation $\Delta R = \frac{4(2-\sigma-6\rho)}{\rho}$.

Keywords: Riemannian manifolds; ρ-Einstein solitons; Einstein manifolds; Poisson equation

MSC: 53E20; 53C25; 53C21

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1. Introduction

The Ricci and other geometric flows are active topics of current research in mathematics, physics and engineering. The Ricci flow [1] is defined on a Riemannian n-manifold (M^n, g) by an evolution equation for metric $g(t)$ of the form $\frac{\partial g}{\partial t} = -2S$, where S is the Ricci tensor of M^n and t indicates the time. The metric g on M^n satisfies the Ricci soliton (in short, RS) equation $\pounds_E g + 2\sigma g + 2S = 0$, where E is a vector field on M^n, $\sigma \in \mathcal{R}$ (the set of real numbers), and $\pounds_E$ represents the Lie derivative operator in the direction of E on M^n. A RS is called expanding (steady or shrinking) if $\sigma > 0$ ($\sigma = 0$ or $\sigma < 0$). If $E = 0$ or Killing, then the RS is called a trivial RS, and M^n becomes an Einstein manifold. Thus the RS is a basic generalization of an Einstein manifold [2]. If $\mathcal{F}$ is a smooth function such that $E = \mathcal{DF}$ for the gradient operator $\mathcal{D}$ of g, then the RS is described as a gradient Ricci soliton (GRS), E is referred to as the potential vector field, and $\mathcal{F}$ is called the potential function. Thus, the RS equation becomes $Hess\mathcal{F} + \sigma g + S = 0$, where $Hess\mathcal{F}$ is the Hessian of $\mathcal{F}$ and $(Hess\mathcal{F})(\zeta_1, \zeta_2) = g(\nabla_{\zeta_1} D\mathcal{F}, \zeta_2)$ for all vector fields ζ_1 and ζ_2 on M^n. Here, ∇ stands for the Levi–Civita connection.

The notion of Ricci–Bourguignon flow, a natural generalization of Ricci flow, has been proposed in [3] and is described on an M^n as:

$$\frac{\partial g}{\partial t} = -2(S - \rho R g), \quad g(0) = g_0, \tag{1}$$

where R is the scalar curvature and $\rho \in \mathcal{R}$. It is to be noticed that for the specific values of ρ, the following cases for the tensor $S - \rho R g$ appeared in (1) [4] are obtained:

(i) $\rho = \frac{1}{2}$, the Einstein tensor $S - \frac{R}{2}g$, (for Einstein soliton),
(ii) $\rho = \frac{1}{n}$, the trace-less Ricci tensor $S - \frac{R}{n}g$,
(iii) $\rho = \frac{1}{2(n-1)}$, the Schouten tensor $S - \frac{R}{2(n-1)}g$, (for Schouten soliton),
(iv) $\rho = 0$, the Ricci tensor S (for RS).

An (M^n, g), $n \geq 3$ is said to be a ρ-Einstein soliton (or ρ-ES) (g, E, σ, ρ) if

$$\pounds_E g + 2S + 2(\sigma - \rho R)g = 0. \tag{2}$$

Similar to the RS, a ρ-ES is called expanding (steady or shrinking) if $\sigma > 0$ ($\sigma = 0$ or $\sigma < 0$). If $E = \mathcal{DF}$, then (M^n, g) is called a gradient ρ-Einstein soliton (or gradient ρ-ES). Hence, (2) takes the form

$$Hess\mathcal{F} + S + (\sigma - \rho R)g = 0, \tag{3}$$

where $Hess\mathcal{F}$ denotes the Hessian of $\mathcal{F} \in C^\infty(M^n)$ and defined by $Hess\mathcal{F} = \nabla\nabla\mathcal{F}$. Recently, ρ-Einstein solitons have been studied by several authors, such as [5–12]. On the other hand, we recommend the papers [13–19] for the studies of Ricci, Yamabe, Ricci-Yamabe, η-Ricci-Yamabe and quasi-Yamabe solitons on different geometric structures.

In this paper, we have made an effort to the solitonic study of a 3-dimensional Riemannian manifold M^3 equipped with a semi-symmetric metric ξ-connection $\tilde{\nabla}$. To achieve the goal, we present our work as follows: In Section 2, we gather the basic information of a Riemannian 3-manifold equipped with a semi-symmetric metric ξ-connection $(M^3, \tilde{\nabla}, g)$, definitions and Lemmas. The properties of ρ-ES in $(M^3, \tilde{\nabla}, g)$ are studied in Section 3. We address the properties of gradient ρ-ES in $(M^3, \tilde{\nabla}, g)$ in Section 4. In the last section, we model a non-trivial example of $(M^3, \tilde{\nabla}, g)$ admitting a gradient ρ-ES, and prove our results.

2. Riemannian Manifolds with a Semi-Symmetric Metric ξ-Connection

In 1970, Yano [20] investigated the properties of a semi-symmetric metric connection $\tilde{\nabla}$ on Riemannian n-manifolds M^n and defined by $\tilde{\nabla}_{\zeta_1}\zeta_2 = \nabla_{\zeta_1}\zeta_2 + \eta(\zeta_2)\zeta_1 - g(\zeta_1, \zeta_2)\xi$ for all ζ_1 and ζ_2 on M^n, where η is a 1-form associated with the unit vector field ξ such that $g(\xi, \xi) = \eta(\xi) = 1$ and $g(\zeta_1, \xi) = \eta(\zeta_1)$. Later, the properties of the semi-symmetric metric connection $\tilde{\nabla}$ have been explored by several researchers. One of these properties is the curvature invariant respecting to the semi-symmetric metric connection $\tilde{\nabla}$ and the Levi–Civita connection ∇. For example, the conformal curvature tensors corresponding to the semi-symmetric connection (Yano's sense) and the Levi–Civita connection coincide. Similar results for different curvature tensors have been established by many geometers. A connection $\tilde{\nabla}$ is said to be semi-symmetric metric ξ-connection if and only if $\tilde{\nabla}\xi = 0$. Afterwards, the properties of semi-symmetric metric ξ-connection have been studied in [21–24].

In an $(M^n, \tilde{\nabla}, g)$, we have [21]

$$\nabla_{\zeta_1}\xi = -\zeta_1 + \eta(\zeta_1)\xi, \;\; g(\xi, \xi) = 1, \text{ and } \eta(\zeta_1) = g(\zeta_1, \xi) \tag{4}$$

for any ζ_1 on M^n. Next, we have [21]

$$(\nabla_{\zeta_1}\eta)\zeta_2 = -g(\zeta_1, \zeta_2) + \eta(\zeta_1)\eta(\zeta_2), \tag{5}$$

$$K(\zeta_1, \zeta_2)\xi = \eta(\zeta_1)\zeta_2 - \eta(\zeta_2)\zeta_1, \tag{6}$$

$$K(\zeta_1, \xi)\zeta_2 = g(\zeta_1, \zeta_2)\xi - \eta(\zeta_2)\zeta_1, \tag{7}$$

$$S(\zeta_1, \xi) = -(n-1)\eta(\zeta_1) \iff Q\xi = -(n-1)\xi, \tag{8}$$

$$(\pounds_\xi g)(\zeta_1, \zeta_2) = 2\{-g(\zeta_1, \zeta_2) + \eta(\zeta_1)\eta(\zeta_2)\}, \tag{9}$$

for all ζ_1, ζ_2 on M^n. Here, K and Q represent the curvature tensor and the Ricci operator of M^n, respectively.

Definition 1. *An M^n is said to be quasi-Einstein if its $S(\neq 0)$ satisfies*

$$S(\zeta_1, \zeta_2) = \mathfrak{l}\, g(\zeta_1, \zeta_2) + \mathfrak{m}\, \eta(\zeta_1)\eta(\zeta_2),$$

where $\mathfrak{m}$ and $\mathfrak{l}$ are smooth functions on M^n. If $\mathfrak{m} = 0$, then the manifold is called an Einstein manifold.

Definition 2. *A partial differential equation $\Delta u = v$ on a complete M^n is called a Poisson equation for some smooth functions u and v.*

Remark 1 ([21,22])**.** *An $(M^3, \tilde{\nabla}, g)$ is a quasi-Einstein manifold of the form*

$$S(\zeta_1, \zeta_2) = \left(1 + \frac{R}{2}\right) g(\zeta_1, \zeta_2) - \left(3 + \frac{R}{2}\right) \eta(\zeta_1)\eta(\zeta_2). \tag{10}$$

Remark 2 ([21,22])**.** *In an $(M^3, \tilde{\nabla}, g)$, we have*

$$\xi(R) = 2(R + 6), \tag{11}$$

$$\eta(\nabla_\xi \mathcal{D}R) = 4(R + 6), \tag{12}$$

where $\mathcal{D}$ is the gradient operator of g. From (11), it is noticed that R of M^3 is constant if and only if $R = -6$.

3. ρ-ES on $(M^3, \tilde{\nabla}, g)$

First, we prove the following theorem.

Theorem 1. *If $(M^3, \tilde{\nabla}, g)$ admits a ρ-ES (g, E, σ, ρ), then its scalar curvature R satisfies the Poisson equation $\Delta R = \frac{4(2-\sigma-6\rho)}{\rho}$, provided $\rho \neq 0$.*

Proof. Let the metric of an $(M^3, \tilde{\nabla}, g)$ be a ρ-ES (g, E, σ, ρ), then in view of (10), (2) leads to

$$\begin{aligned}(\pounds_E g)(\zeta_1, \zeta_2) \quad &= \quad -2\{1 + \sigma + (\frac{1}{2} - \rho)R\} g(\zeta_1, \zeta_2) \\ &\quad\quad + (R + 6)\eta(\zeta_1)\eta(\zeta_2),\end{aligned} \tag{13}$$

for any vector fields ζ_1, ζ_2 on M^3.

Taking covariant derivative of (13) respecting to ζ_3, we find

$$\begin{aligned}(\nabla_{\zeta_3} \pounds_E g)(\zeta_1, \zeta_2) \quad &= \quad (\zeta_3 R)\{(2\rho - 1) g(\zeta_1, \zeta_2) + \eta(\zeta_1)\eta(\zeta_2)\} \\ &\quad\quad -(R + 6)\{g(\zeta_1, \zeta_3)\eta(\zeta_2) + g(\zeta_2, \zeta_3)\eta(\zeta_1) - 2\eta(\zeta_1)\eta(\zeta_2)\eta(\zeta_3)\}.\end{aligned} \tag{14}$$

As g is parallel with respect to ∇, then the formula [25]

$$(\pounds_E \nabla_{\zeta_1} g - \nabla_{\zeta_1} \pounds_E g - \nabla_{[E, \zeta_1]} g)(\zeta_2, \zeta_3) = -g((\pounds_E \nabla)(\zeta_1, \zeta_2), \zeta_3) - g((\pounds_E \nabla)(\zeta_1, \zeta_3), \zeta_2)$$

turns to

$$(\nabla_{\zeta_1} \pounds_E g)(\zeta_2, \zeta_3) = g((\pounds_E \nabla)(\zeta_1, \zeta_2), \zeta_3) + g((\pounds_E \nabla)(\zeta_1, \zeta_3), \zeta_2).$$

Since $\pounds_E \nabla$ is symmetric, therefore we have

$$2g((\pounds_E \nabla)(\zeta_1, \zeta_2), \zeta_3) = (\nabla_{\zeta_1} \pounds_E g)(\zeta_2, \zeta_3) + (\nabla_{\zeta_2} \pounds_E g)(\zeta_1, \zeta_3) - (\nabla_{\zeta_3} \pounds_E g)(\zeta_1, \zeta_2),$$

which in view of (14) gives

$$\begin{aligned} 2g((\pounds_E\nabla)(\zeta_1,\zeta_2),\zeta_3) &= (\zeta_1R)\{(2\rho-1)g(\zeta_2,\zeta_3)+\eta(\zeta_2)\eta(\zeta_3)\} \\ &\quad +(\zeta_2R)\{(2\rho-1)g(\zeta_1,\zeta_3)+\eta(\zeta_1)\eta(\zeta_3)\} \\ &\quad -(\zeta_3R)\{(2\rho-1)g(\zeta_1,\zeta_2)+\eta(\zeta_1)\eta(\zeta_2)\} \\ &\quad -2(R+6)\{g(\zeta_1,\zeta_2)\eta(\zeta_3)-\eta(\zeta_1)\eta(\zeta_2)\eta(\zeta_3)\}, \end{aligned}$$

from which it follows that

$$\begin{aligned} 2(\pounds_E\nabla)(\zeta_1,\zeta_2) &= (\zeta_1R)\{(2\rho-1)\zeta_2+\eta(\zeta_2)\xi\} \\ &\quad +(\zeta_2R)\{(2\rho-1)\zeta_1+\eta(\zeta_1)\xi\} \\ &\quad -(\mathcal{D}R)\{(2\rho-1)g(\zeta_1,\zeta_2)+\eta(\zeta_1)\eta(\zeta_2)\} \\ &\quad -2(R+6)\{g(\zeta_1,\zeta_2)\xi-\eta(\zeta_1)\eta(\zeta_2)\xi\}. \end{aligned} \tag{15}$$

Replacing ζ_2 by ξ and ζ_1 by ζ_2 in (15), we have

$$\begin{aligned} (\pounds_E\nabla)(\zeta_2,\xi) &= \rho g(\mathcal{D}R,\zeta_2)\xi-\rho(\mathcal{D}R)\eta(\zeta_2) \\ &\quad +(R+6)\{(2\rho-1)\zeta_2+\eta(\zeta_2)\xi\}. \end{aligned} \tag{16}$$

The covariant differentiation of (16) respecting to ζ_1 yields

$$\begin{aligned} (\nabla_{\zeta_1}\pounds_E\nabla)(\zeta_2,\xi) &= 2(\zeta_1R)\{(2\rho-1)\zeta_2+\eta(\zeta_2)\xi\} \\ &\quad +(\zeta_2R)\{(\rho-1)\zeta_1+\eta(\zeta_1)\xi\} \\ &\quad -(\mathcal{D}R)\{(\rho-1)g(\zeta_1,\zeta_2)+\eta(\zeta_1)\eta(\zeta_2)\} \\ &\quad -3(R+6)\{g(\zeta_1,\zeta_2)\xi-\eta(\zeta_1)\eta(\zeta_2)\xi\} \\ &\quad -(R+6)\{(2\rho-1)\eta(\zeta_1)\zeta_2+\eta(\zeta_2)\zeta_1\} \\ &\quad +\rho g(\nabla_{\zeta_1}\mathcal{D}R,\zeta_2)\xi-\rho(\nabla_{\zeta_1}\mathcal{D}R)\eta(\zeta_2), \end{aligned} \tag{17}$$

where (4), (5) and (16) being used.

Again from [25], we have

$$(\pounds_EK)(\zeta_1,\zeta_2)\zeta_3=(\nabla_{\zeta_1}\pounds_E\nabla)(\zeta_2,\zeta_3)-(\nabla_{\zeta_2}\pounds_E\nabla)(\zeta_1,\zeta_3), \tag{18}$$

which by putting $\zeta_3=\xi$ and using (17) becomes

$$\begin{aligned} (\pounds_EK)(\zeta_1,\zeta_2)\xi &= g(\mathcal{D}R,\zeta_1)\{(3\rho-1)\zeta_2+\eta(\zeta_2)\xi\} \\ &\quad -g(\mathcal{D}R,\zeta_2)\{(3\rho-1)\zeta_1+\eta(\zeta_1)\xi\} \\ &\quad +2(R+6)(\rho-1)\{\eta(\zeta_2)\zeta_1-\eta(\zeta_1)\zeta_2\} \\ &\quad +\rho g(\nabla_{\zeta_1}\mathcal{D}R,\zeta_2)\xi-\rho g(\nabla_{\zeta_2}\mathcal{D}R,\zeta_1)\xi \\ &\quad -\rho(\nabla_{\zeta_1}\mathcal{D}R)\eta(\zeta_2)+\rho(\nabla_{\zeta_2}\mathcal{D}R)\eta(\zeta_1). \end{aligned} \tag{19}$$

Contracting (19) respecting to ζ_1 then using (4) and (11) we lead to

$$\begin{aligned} (\pounds_ES)(\zeta_2,\xi) &= (1-6\rho)\zeta_2(R)+2(R+6)(2\rho-1)\eta(\zeta_2) \\ &\quad +\rho g(\nabla_\xi\mathcal{D}R,\zeta_2)\xi-\rho(\Delta R)\eta(\zeta_2). \end{aligned} \tag{20}$$

By putting $\zeta_2=\xi$ in (20) then using (4), (11) and (12), we find

$$(\pounds_ES)(\xi,\xi) = -4\rho(R+6)-\rho(\Delta R). \tag{21}$$

The Lie derivative of (8) respecting to E leads to

$$(\pounds_ES)(\xi,\xi)=4\eta(\pounds_E\xi). \tag{22}$$

Putting $\zeta_1=\zeta_2=\xi$ in (13) infers

$$(\pounds_E g)(\xi,\xi) = -2\sigma + 2\rho R + 4. \tag{23}$$

The Lie derivative of $g(\xi,\xi) = 1$ gives

$$(\pounds_E g)(\xi,\xi) = -2\eta(\pounds_E \xi). \tag{24}$$

Now combining (21)–(25) we deduce

$$\Delta R = \frac{4(2-\sigma-6\rho)}{\rho}, \text{ provided } \rho \neq 0. \tag{25}$$

This completes the proof. □

It is well-known that the ρ-ES Equation (2) on M^n with the soliton constant $\rho = \frac{1}{2}, \frac{1}{n}, \frac{1}{2(n-1)}$ reduces to the Einstein soliton, traceless Ricci soliton, Schouten soliton, respectively. It is also known that a smooth function $\mathfrak{f}$ on an M^n is called harmonic, subharmonic or superharmonic if $\Delta\mathfrak{f} = 0$, ≥ 0 or ≤ 0, respectively. These facts together with Theorem 1 state the following:

Corollary 1. *Let $(M^3, \tilde{\nabla}, g)$ admit a ρ-ES, then we have*

Value of ρ	*Solitons*	*Poisson equation*	*Condition for R to be subharmonic and superharmonic*
$\rho = \frac{1}{2}$	*Einstein soliton*	$\Delta R = -8(\sigma+1)$	(i) *R is subharmonic if $\sigma \leq -1$,* (ii) *R is superharmonic if $\sigma \geq -1$,*
$\rho = \frac{1}{3}$	*traceless Ricci soliton*	$\Delta R = -12\sigma$	(i) *R is subharmonic if $\sigma \leq 0$,* (ii) *R is superharmonic if $\sigma \geq 0$,*
$\rho = \frac{1}{4}$	*Schouten soliton*	$\Delta R = 16(\frac{1}{2} - \sigma)$	(i) *R is subharmonic if $\sigma \leq \frac{1}{2}$,* (ii) *R is superharmonic if $\sigma \geq \frac{1}{2}$.*

Remark 3. *The ρ-ES on an M^n with $\rho = 0$ reduces to the RS. The properties of RS on $(M^3, \tilde{\nabla}, g)$ have been explored by Chaubey and De [22]. Thus, we can say that the Theorem 1 generalizes the study of Einstein soliton, traceless RS and the Schouten soliton on $(M^3, \tilde{\nabla}, g)$.*

It is well-known that the Poisson equation $\Delta u = v$ with $v = 0$ becomes a Laplace equation. Suppose that an $(M^3, \tilde{\nabla}, g)$ does not admit RS. Then, Theorem 1 and above discussion state:

Corollary 2. *If $(M^3, \tilde{\nabla}, g)$ admits a ρ-ES, which is not a RS ($\rho \neq 0$), then R of M^3 satisfies Laplace equation if and only if $\sigma = 2(1-3\rho)$.*

Let $(M^3, \tilde{\nabla}, g)$ admit a ρ-ES. If R of M^3 satisfies the Laplace equation, then $\sigma = 2(1-3\rho)$. The ρ-ES under consideration to be steady, shrinking or expanding if ρ is equal to, less than or greater than $\frac{1}{3}$. Thus, we write our corollary as

Corollary 3. *Let the metric of an $(M^3, \tilde{\nabla}, g)$ be ρ-ES, which is not a RS ($\rho \neq 0$). If R of M^3 satisfies the Laplace equation, then the ρ-ES is steady, shrinking or expanding if $\rho = \frac{1}{3}$, $\rho < \frac{1}{3}$ or $\rho > \frac{1}{3}$, respectively.*

4. Gradient ρ-ES on $(M^3, \tilde{\nabla}, g)$

Theorem 2. *Let $(M^3, \tilde{\nabla}, g)$ admit a gradient ρ-ES. Then, either M^3 is Einstein or the gradient ρ-ES is steady type gradient traceless RS.*

Proof. Let the metric of an $(M^3, \tilde{\nabla}, g)$ be a gradient ρ-ES. Then, (3) can be written as

$$\nabla_{\zeta_1} \mathcal{D}\mathcal{F} + Q\zeta_1 + (\sigma - \rho R)\zeta_1 = 0, \tag{26}$$

for all ζ_1 on M^3.

The covariant differentiation of (26) with respect to ζ_2 leads to

$$\nabla_{\zeta_2}\nabla_{\zeta_1}\mathcal{DF} = -(\nabla_{\zeta_2}Q)\zeta_1 - Q(\nabla_{\zeta_2}\zeta_1) - (\sigma - \rho R)\nabla_{\zeta_2}\zeta_1 + \rho\zeta_2(R)\zeta_1. \tag{27}$$

Interchanging ζ_1 and ζ_2 in (27) leads to

$$\nabla_{\zeta_1}\nabla_{\zeta_2}\mathcal{DF} = -(\nabla_{\zeta_1}Q)\zeta_2 - Q(\nabla_{\zeta_1}\zeta_2) - (\sigma - \rho R)\nabla_{\zeta_1}\zeta_2 + \rho\zeta_1(R)\zeta_2. \tag{28}$$

By plugging of (26)–(28), we find

$$K(\zeta_1,\zeta_2)\mathcal{DF} = -(\nabla_{\zeta_1}Q)\zeta_2 + (\nabla_{\zeta_2}Q)\zeta_1 + \rho\{\zeta_1(R)\zeta_2 - \zeta_2(R)\zeta_1\}.$$

Contracting the forgoing equation along ζ_1, we obtain

$$S(\zeta_2,\mathcal{DF}) = \frac{(1-4\rho)}{2}\zeta_2(R). \tag{29}$$

In account of (10), we have

$$S(\zeta_2,\mathcal{DF}) = (1+\frac{R}{2})\zeta_2(\mathcal{F}) - (3+\frac{R}{2})\eta(\zeta_2)\xi(\mathcal{F}). \tag{30}$$

Thus, from (29) and (30), it follows that

$$(1-4\rho)\zeta_2(R) = (R+2)\zeta_2(\mathcal{F}) - (R+6)\eta(\zeta_2)\xi(\mathcal{F}). \tag{31}$$

By putting $\zeta_2 = \xi$ in (31), then using (4) and (11), we find

$$\xi(\mathcal{F}) = -\frac{1}{2}(1-4\rho)(R+6). \tag{32}$$

By using (32) and (31) turns to

$$(1-4\rho)\zeta_2(R) = (R+2)\zeta_2(\mathcal{F}) + \frac{1}{2}(R+6)^2(1-4\rho)\eta(\zeta_2). \tag{33}$$

The covariant differentiation of (33) along ζ_1 leads to

$$\begin{aligned}(1-4\rho)g(\nabla_{\zeta_1}\mathcal{D}R,\zeta_2) &= \zeta_1(R)\zeta_2(\mathcal{F}) + (R+2)g(\nabla_{\zeta_1}\mathcal{DF},\zeta_2)\\ &+(R+6)(1-4\rho)\zeta_1(R)\eta(\zeta_2)\\ &+\frac{1}{2}(R+6)^2(1-4\rho)\{\eta(\zeta_1)\eta(\zeta_2) - g(\zeta_1,\zeta_2)\}.\end{aligned} \tag{34}$$

Interchanging ζ_1 and ζ_2 in (34), we have

$$\begin{aligned}(1-4\rho)g(\nabla_{\zeta_2}\mathcal{D}R,\zeta_1) &= \zeta_2(R)\zeta_1(\mathcal{F}) + (R+2)g(\nabla_{\zeta_2}\mathcal{DF},\zeta_1)\\ &+(R+6)(1-4\rho)\zeta_2(R)\eta(\zeta_1)\\ &+\frac{1}{2}(R+6)^2(1-4\rho)\{\eta(\zeta_1)\eta(\zeta_2) - g(\zeta_1,\zeta_2)\}.\end{aligned} \tag{35}$$

Equating the left hand sides of last two equations gives

$$\begin{gathered}\zeta_1(R)\zeta_2(\mathcal{F}) + (R+6)(1-4\rho)\zeta_1(R)\eta(\zeta_2)\\ -\zeta_2(R)\zeta_1(\mathcal{F}) - (R+6)(1-4\rho)\zeta_2(R)\eta(\zeta_1) = 0,\end{gathered}$$

which by replacing $\zeta_2 = \xi$ then using (4), (11) and (32) takes the form

$$(R+6)\{(1-4\rho)\zeta_1(R) - 4\zeta_1(\mathcal{F}) - 4(R+6)(1-4\rho)\eta(\zeta_1)\} = 0.$$

Thus, we have either $R = -6$, or $(1-4\rho)\zeta_1(R) = 4\zeta_1(\mathcal{F}) + 4(R+6)(1-4\rho)\eta(\zeta_1)$. If we firstly suppose that $R \neq -6$ and $(1-4\rho)\zeta_1(R) = 4\zeta_1(\mathcal{F}) + 4(R+6)(1-4\rho)\eta(\zeta_1)$, which by virtue of (33) turns to

$$(R-2)\{2\zeta_1(\mathcal{F}) + (R+6)(1-4\rho)\eta(\zeta_1)\} = 0, \tag{36}$$

which refers that either $R = 2$ or $\zeta_1(\mathcal{F}) = -\frac{1}{2}(R+6)(1-4\rho)\eta(\zeta_1)$. From (11), it is obvious that if R is constant, then its value must be -6, which shows that $R = 2$ is inadmissible. Thus, we have $\zeta_1(\mathcal{F}) = -\frac{1}{2}(R+6)(1-4\rho)\eta(\zeta_1)$, which is equivalent to

$$\mathcal{D}\mathcal{F} = -\frac{1}{2}(R+6)(1-4\rho)\xi = \xi(\mathcal{F})\xi. \tag{37}$$

Thus, the gradient of $\mathcal{F}$ is pointwise collinear with ξ. Now, taking the covariant derivative of (37) with respect to ζ_1 and using (4), we have

$$\nabla_{\zeta_1}\mathcal{D}\mathcal{F} = \zeta_1(\xi(\mathcal{F}))\xi - \xi(\mathcal{F})(\zeta_1 - \eta(\zeta_1)\xi). \tag{38}$$

Therefore, from (26) and (38), we obtain

$$Q\zeta_1 + (\sigma - \rho R)\zeta_1 = -\zeta_1(\xi(\mathcal{F}))\xi + \xi(\mathcal{F})(\zeta_1 - \eta(\zeta_1)\xi). \tag{39}$$

Now, by replacing ζ_1 by ξ in (39) then using (8), (11) and (32) we lead to

$$\sigma = (1-3\rho)(R+8). \tag{40}$$

Let us suppose that $\rho = \frac{1}{3}$, that is, the gradient ρ-ES on an M^3 is gradient traceless RS. This fact together with Equation (40) leads to $\sigma = 0$. Thus, the gradient traceless RS is steady. This completes the proof. □

Theorem 3. *Let an $(M^3, \tilde{\nabla}, g)$ be a non-gradient traceless RS. Then, the gradient ρ-ES is trivial soliton with constant $\sigma = 2(1-3\rho)$. Also, the ρ-ES is shrinking and expanding according to $\rho > \frac{1}{3}$ and $\rho < \frac{1}{3}$.*

Proof. Now, we suppose that $\rho \neq \frac{1}{3}$. Thus, (40) leads to

$$R = \frac{\sigma}{1-3\rho} - 8, \tag{41}$$

which informs that R is constant and hence (11) infers that $R = -6$. This contradicts our hypothesis $R \neq -6$.

Secondly, we consider that $R = -6$ and $(1-4\rho)\zeta_1(R) \neq 4\zeta_1(\mathcal{F}) + 4(R+6)(1-4\rho)\eta(\zeta_1)$. For $R = -6$, (33) informs that $\mathcal{F} \in \mathcal{R}$ and hence the GRBS on the manifold is trivial. Moreover, the Riemannian 3-manifold under assumption is an Einstein manifold with $\sigma = 2(1-3\rho)$. This completes the proof. □

Let us suppose that an $(M^3, \tilde{\nabla}, g)$ admits a proper gradient ρ-ES. Then, the ρ-ES reduces to the gradient traceless RS and $\rho = \frac{1}{3}, \sigma = 0$. Using these facts in (26) and then contracting the foregoing equation over ζ_1 gives $\Delta\mathcal{F} = 0$.

A smooth function $\mathfrak{h}$ on an M^n is called harmonic if $\Delta\mathfrak{h} = 0$.

The above discussions state the following:

Corollary 4. *Let a complete $(M^3, \tilde{\nabla}, g)$ admit a proper gradient ρ-ES. Then the gradient function of the gradient ρ-ES is harmonic.*

Contracting (38) over ζ_1, we find

$$\Delta\mathcal{F} = \xi(\xi(\mathcal{F})) - 2\xi(\mathcal{F}).$$

Again, considering $\sigma = 0,\ \rho = \frac{1}{3}$ and then contracting (26) over ζ_1, we conclude that

$$\Delta\mathcal{F} = 0.$$

The last two equations show that $\xi(\xi(\mathcal{F})) - 2\xi(\mathcal{F}) = 0$. Let $\xi = \frac{\partial}{\partial t}$. Thus, we notice that the potential function $\mathcal{F}$ satisfies the PDE

$$\frac{\partial^2\mathcal{F}}{\partial t^2} - 2\frac{\partial\mathcal{F}}{\partial t} = 0.$$

It is obvious that $\mathcal{F} = Ae^{2t} + B$ for smooth functions A and B, which are independent of t, is the solution of the above PDE. Now, we list our results in the following:

Corollary 5. *Let the metric of a complete* $(M^3, \tilde{\nabla}, g)$ *admit a proper gradient* ρ*-ES. Then, the potential function* $\mathcal{F}$ *of such soliton satisfies the PDE* $\frac{\partial^2\mathcal{F}}{\partial t^2} - 2\frac{\partial\mathcal{F}}{\partial t} = 0$*, and it can be evaluated by* $\mathcal{F} = Ae^{2t} + B$*.*

5. Example

We consider the manifold $M^3 = \{(w_1, w_2, w_3) \in \mathcal{R}^3\}$, where (w_1, w_2, w_3) are the usual coordinates in $\mathbb{R}^3$. Let u_1, u_2 and u_3 be the vector fields on M^3 given by

$$u_1 = e^{\mathfrak{b}w_3 + w_1}\frac{\partial}{\partial w_1}, \quad u_2 = e^{\mathfrak{b}w_3 + w_2}\frac{\partial}{\partial w_2}, \quad u_3 = \frac{1}{\mathfrak{b}}\frac{\partial}{\partial w_3} = \xi,$$

where $\mathfrak{b}(\neq 0) \in \mathcal{R}$. Then, $\{u_1, u_2, u_3\}$ forms a basis in the module of the vector fields of M^3.

Let the Riemannian metric g be defined by

$$g(u_p, u_q) = \begin{cases} 1, & 1 \leq p = q \leq 3, \\ 0, & otherwise. \end{cases}$$

Hence, M^3 is a Riemannian manifold of dimension 3. Let the 1-form η on M^3 be defined by $\eta(\zeta_1) = g(\zeta_1, u_3) = g(\zeta_1, \xi)$ for all ζ_1 on M^3. Now, by direct computations, we obtain

$$[u_1, u_2] = 0,\ \ [u_1, u_3] = -u_1,\ \ [u_2, u_3] = -u_2.$$

By using Koszul's formula, we obtain

$$\nabla_{u_p}u_q = \begin{cases} -u_p, & p = 1, 2, q = 3, \\ u_3, & 1 \leq p = q \leq 2, \\ 0, & otherwise. \end{cases}$$

Now we suppose that $\zeta_1 = \zeta_1^1 u_1 + \zeta_1^2 u_2 + \zeta_1^3 u_3$, then for $\xi = u_3$ it follows that $\nabla_{\zeta_1}\xi = -\zeta_1 + \eta(\zeta_1)\xi$. It can be easily seen that $\tilde{\nabla}$ defined on M^3 satisfies the conditions

$$\tilde{T}(\zeta_1, \zeta_2) = -\eta(\zeta_1)\zeta_2 + \eta(\zeta_2)\zeta_1,\ \ \tilde{\nabla}g = 0,\ \text{and}\ \ \tilde{\nabla}\xi = 0,$$

for arbitrary vector fields ζ_1 and ζ_2 on M^3, where $\tilde{T}$ indicates the torsion tensor of $\tilde{\nabla}$. Thus, we can say that $\tilde{\nabla}$ is a semi-symmetric metric ξ-connection on M^3.

The non-zero constituents of K are obtained as follows:

$$K(u_1, u_3)u_1 = u_3,\ \ K(u_1, u_2)u_1 = u_2,\ \ K(u_2, u_3)u_2 = u_3,$$

$$K(u_1, u_2)u_2 = K(u_1, u_3)u_3 = -u_1,\ \ K(u_2, u_3)u_3 = -u_2.$$

By using above components of the curvature tensor K we obtain

$$S(u_p, u_q) = -2, \;\; 1 \leq p = q \leq 3,$$

from which we obtain $R = -6$.

Now, by taking $\mathcal{DF} = (u_1\mathcal{F})u_1 + (u_2\mathcal{F})u_2 + (u_3\mathcal{F})u_3$, we have

$$\nabla_{u_1}\mathcal{DF} = (u_1(u_1\mathcal{F}) - u_3\mathcal{F})u_1 + (u_1(u_2\mathcal{F}))u_2 + (u_1(u_3\mathcal{F}) + u_1\mathcal{F})u_3,$$

$$\nabla_{\mathcal{E}_2}\mathcal{DF} = (u_2(u_1\mathcal{F}))u_1 + (u_2(u_2\mathcal{F}) - u_3\mathcal{F})u_2 + (u_2(u_3\mathcal{F}) + \mathcal{F}_2\mathcal{F})\mathcal{F}_3,$$

$$\nabla_{\mathcal{E}_3}\mathcal{DF} = (u_3(u_1\mathcal{F}))u_1 + (u_3(u_2\mathcal{F}))u_2 + (u_3(u_3\mathcal{F}))u_3.$$

Thus, by virtue of (26), we obtain

$$\begin{cases} u_1(u_1\mathcal{F}) - u_3\mathcal{F} = 2 - 6\rho - \sigma, \\ u_2(u_2\mathcal{F}) - u_3\mathcal{F} = 2 - 6\rho - \sigma, \\ u_3(u_3\mathcal{F}) = 2 - 6\rho - \sigma, \\ u_1(u_2\mathcal{F}) = 0, \\ u_2(u_1\mathcal{F}) = 0, \\ u_2(u_3\mathcal{F}) + u_2\mathcal{F} = 0. \end{cases} \tag{42}$$

Thus, the relations in (42) are, respectively, amounting to

$$e^{2(\mathfrak{b}w_3 + w_1)}\left[\frac{\partial^2\mathcal{F}}{\partial w_1^2} + \frac{\partial\mathcal{F}}{\partial w_1}\right] - \frac{1}{\mathfrak{b}}\frac{\partial\mathcal{F}}{\partial w_3} = 2 - 6\rho - \sigma,$$

$$e^{2(\mathfrak{b}w_3 + w_1)}\left[\frac{\partial^2\mathcal{F}}{\partial w_2^2} + \frac{\partial\mathcal{F}}{\partial w_2}\right] - \frac{1}{\mathfrak{b}}\frac{\partial\mathcal{F}}{\partial w_3} = 2 - 6\rho - \sigma,$$

$$\frac{1}{\mathfrak{b}^2}\frac{\partial^2\mathcal{F}}{\partial w_3^2} = 2 - 6\rho - \sigma,$$

$$\frac{\partial^2\mathcal{F}}{\partial w_1 \partial w_2} = 0,$$

$$\frac{\partial^2\mathcal{F}}{\partial w_2 \partial w_1} = 0,$$

$$\frac{1}{\mathfrak{b}}\left[\frac{\partial^2\mathcal{F}}{\partial w_2 \partial w_3} + \frac{\partial\mathcal{F}}{\partial w_2}\right] = 0.$$

From the above relations, it is noticed that $\mathcal{F} \in \mathcal{R}$ for $\sigma = 2 - 6\rho$. Hence, the Equation (26) is satisfied. Thus, g is a gradient ρ-ES with the soliton vector field $E = \mathcal{DF}$, where $\mathcal{F} \in \mathcal{R}$ and $\sigma = 2 - 6\rho$. For $\rho = \frac{1}{3}$, we obtain $\sigma = 0$, i.e., the gradient ρ-ES is trivial with constant $\sigma = 2 - 6\rho$. Thus, Theorem 2 is verified.

6. Results and Discussion

It is well known that the ρ-Einstein soliton Equation (2) with $\rho = 0$ becomes the Ricci soliton equation, which has been studied in [22]. Thus, we can say that the ρ-Einstein soliton is a natural generalization of Ricci soliton. In this manuscript, we have explored the properties of ρ-Einstein solitons in Riemannian geometry, which generalizes the results of [22].

7. Conclusions

To prove the curvatures invariant, Chauey et al. [23] defined the notion of semi-symmetric metric P-connection in Riemannian setting, which is a particular case of Riemannian concircular structure manifold [26]. This topic has great applications in differential equations. We proved that the scalar curvature of Riemannian 3-manifolds endowed with a semi-symmetric metric ξ-connection and Ricci–Bourguignon soliton satisfies the Poisson and Laplace equations. It is well known that the Poisson and Laplace equations play a crucial role in the development of engineering, physics, mathematics, etc. We have also established the conditions for which the scalar curvature is harmonic, sub-harmonic and super-harmonic. We also established the existence condition of a gradient ρ-Einstein soliton in the Riemannian 3-manifolds, and consequently we proved some results. To verify our results, we constructed a non-trivial example of a three-dimensional Riemannian manifold equipped with a semi-symmetric metric ξ-connection. These topics are modern and have a lot of scope for researchers.

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References

1. Hamilton, R.S. The Ricci Flow on Surfaces, Mathematics and General Relativity. *Contemp. Math.* **1988**, *71*, 237–262.
2. Besse, A.L. *Einstein Manifolds, Classics in Mathematics*; Springer: Berlin/Heidelberg, Germany, 2008.
3. Bourguignon, J.P. Ricci curvature and Einstein metrics. *Glob. Differ. Geom. Glob. Anal. Lect. Notes Math.* **1981**, *838*, 42–63.
4. Bourguignon, J.P.; Lawson, H.B. Stability and isolation phenomena for Yang-mills fields. *Commun. Math. Phys.* **1981**, *79*, 189–230. [CrossRef]
5. Dwivedi, S. Some results on Ricci-Bourguignon solitons and almost solitons. *Can. Math. Bull.* **2020**, *64*, 1–15. [CrossRef]
6. Huang, G. Integral pinched gradient shrinking ρ-Einstein solitons. *J. Math. Anal. Appl.* **2017**, *451*, 1045–1055. [CrossRef]
7. Mondal, C.K.; Shaikh, A.A. Some results on η-Ricci Soliton and gradient ρ-Einstein soliton in a complete Riemannian manifold. *Commun. Korean Math. Soc.* **2019**, *34*, 1279–1287.
8. Patra, D.S. Some characterizations of ρ-Einstein solitons on Sasakian manifolds. *Can. Math. Bull.* **2022**, *65*, 1036–1049. [CrossRef]
9. Shaikh, A.A.; Cunha, A.W.; Mandal, P. Some characterizations of ρ-Einstein solitons. *J. Geom. Phys.* **2021**, *166*, 104270. [CrossRef]
10. Shaikh, A.A.; Mandal, P.; Mondal, C.K. Diameter estimation of gradient ρ-Einstein solitons. *J. Geom. Phys.* **2022**, *177*, 104518. [CrossRef]
11. Suh, Y.J. Ricci-Bourguignon solitons on real hypersurfaces in the complex hyperbolic quadric. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.* **2022**, *116*, 110. [CrossRef]
12. Venkatesha, V.; Kumara, H.A. Gradient ρ-Einstein soliton on almost Kenmotsu manifolds. *Ann. Dell' Univ. Ferrara* **2019**, *65*, 375–388. [CrossRef]
13. Blaga, A.M. Some geometrical aspects of Einstein, Ricci and Yamabe solitons. *J. Geom. Symmetry Phys.* **2019**, *52*, 17–26. [CrossRef]
14. Chen, Z.; Li, Y.; Sarkar, S.; Dey, S.; Bhattacharyya, A. Ricci soliton and certain related metrics on a three-dimensional trans-Sasakian manifold. *Universe* **2022**, *8*, 595. [CrossRef]

15. Li, Y.; Haseeb, A.; Ali, M. LP-Kenmotsu manifolds admitting η-Ricci-Yamabe Solitons and spacetime. *J. Math.* **2022**, *2022*, 6605127. [CrossRef]
16. Suh, Y.J. Yamabe and gradient Yamabe solitons in the complex hyperbolic two-plane Grassmannians. *Rev. Math. Phys.* **2022**, *34*, 2250024. [CrossRef]
17. Suh, Y.J. Yamabe and quasi-Yamabe solitons on hypersurfaces in the complex hyperbolic space. *Mediterr. J. Math.* **2023**, *20*, 69. [CrossRef]
18. Turki, N.B.; Blaga, A.M.; Deshmukh, S. Soliton-type equations on a Riemannian manifold. *Mathematics* **2022**, *10*, 633. [CrossRef]
19. Yoldas, H.I. On Kenmotsu manifolds admitting η-Ricci-Yamabe solitons. *Int. J. Geom. Methods Mod. Phys.* **2021**, *18*, 2150189. [CrossRef]
20. Yano, K. On semi-symmetric metric connections. *Rev. Roumaine Math. Pures Appl.* **1970**, *15*, 1579–1586.
21. Chaubey, S.K.; De, U.C. Characterization of three-dimensional Riemannian manifolds with a type of semi-symmetric metric connection admitting Yamabe soliton. *J. Geom. Phys.* **2020**, *157*, 103846. [CrossRef]
22. Chaubey, S.K.; De, U.C. Three dimensional Riemannian manifolds and Ricci solitons. *Quaest. Math.* **2022**, *45*, 765–778. [CrossRef]
23. Chaubey, S.K.; Lee, J.W.; Yadav, S. Riemannian manifolds with a semisymmetric metric P-connection. *J. Korean Math. Soc.* **2019**, *56*, 1113–1129.
24. Haseeb, A.; Chaubey, S.K.; Khan, M.A. Riemannian 3-manifolds and Ricci-Yamabe solitons. *Int. J. Geom. Methods Mod. Phys.* **2023**, *20*, 2350015. [CrossRef]
25. Yano, K. *Integral Formulas in Riemannian Geometry, Pure and Applied Mathematics*; Marcel Dekker: New York, NY, USA, 1970; Volume I.
26. Chaubey, S.K.; Suh, Y.J. Riemannian concircular structure manifolds. *Filomat* **2022**, *36*, 6699–6711. [CrossRef]