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Application of Double Sumudu-Generalized Laplace Decomposition Method for Solving 2+1-Pseudoparabolic Equation

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Abstract: The main purpose of this research paper is to discuss the solution of the singular two-dimensional pseudoparabolic equation by employing the double Sumudu-generalized Laplace transform decomposition method (DSGLTDM). We establish two theorems related to the partial derivatives. Furthermore, to investigate the relevance of the proposed method to solving singular two-dimensional pseudo parabolic equations, three examples are provided.

Keywords: double Sumudu-generalized Laplace transform; double Sumudu transform; inverse double Sumudu-generalized Laplace transform; pseudoparabolic equation; decomposition methods

MSC: 35A44; 65M44; 35A22

1. Introduction

Partial differential equations have several implementations in mathematical physics. One of these implementations is the pseudoparabolic (also known as Sobolev-type differential equation) which is considered an important problem occurring in a kind of physical situation like the flow of fluid out of fissured rocks, thermodynamics, and wave propagation. The parabolic equation happens in many areas of applied mathematics physic, for example, the heat diffusion equation and fluid mechanics; for more details, see [1–4]. The Adomian decomposition and the series expansion methods are used in solutions of fractional diffusion equation problems in [5,6]. The existence, uniqueness, and continuous dependence of powerful solutions of one-dimensional pseudoparabolic equation were studied in [7]. Presently, several researchers have proposed a precise solution to a one-dimensional connected parabolic equation; see [8,9]. The author in [10] suggested the modification of the double Laplace decomposition method to find the analytical approximation solution of a coupled system of pseudoparabolic equations with initial conditions. Newly, in [11], the authors applied the three-dimensional Laplace Adomian decomposition method to solve singular pseudoparabolic equations. The convergence of the Adomian method was studied by several researchers (we refer the readers to see [12–15]). The author in [16] proposed Sumudu transform, later used by Belgacem et al. in [17] to generalize the existence of Sumudu differentiation, integration, and convolution theorems. It was also applied by the same authors for solving an integral production depreciation problem in [18]. In [19], the researcher employed the Sumudu transform to obtain the solution of Abel's integral equation, an integrodifferential equation, a dynamic system with delayed time signals, and a differential dynamic system. Moreover, the author in [20] expanded the single Sumudu transform to a double Sumudu transform with a focus on solutions to partial differential equations. Ahmeda et al. [21] discussed the convergence of the double Sumudu transformation and used it to obtain the solution of the Volterra integro-partial differential equation. The generalized Laplace transforms were used to study the solution of partial differential equations (PDEs) and also, the suggested ideas gave an easy solution to engineering problems by freely selecting integer α in the definition [22].

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Recently, the researchers in [23] studied the solution of fractional third-order dispersive partial differential equations and symmetric KdV by utilizing Sumudu-generalized Laplace transform decomposition.

The main goal of this work is to generate a new method by combining the decomposition method and double Sumudu-generalized Laplace transform, which is called the double Sumudu-generalized Laplace transform decomposition method, to solve 2+1-singular pseudoparabolic equations. Here, we list some definitions that are applied in this work.

2. Properties of Double Sumudu-Generalized Laplace Transform

The definitions and existence conditions of the double Sumudu-generalized Laplace transform are presented. Here, we work with the double Sumudu-generalized Laplace transform, which is defined by

$$S_xS_yG_t(f(x,y,t)) = F(\zeta_1,\zeta_2,s) = \frac{s^\alpha}{\zeta_1\zeta_2}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{x}{\zeta_1}+\frac{y}{\zeta_2}+\frac{t}{s}\right)} f(x,y,t)dtdydx \tag{1}$$

Existence Condition for the Double Sumudu-Generalized Laplace Transform

In the following, the conditions for the existence of the double Sumudu-generalized Laplace transform are offered. If $f(x,y,t)$ is an exponential order a_1, a_2 and b as $x \to \infty$, $y \to \infty, t \to \infty$, and if $\exists R > 0$ similarly applies for all $x > X$, $y > Y$ and $t > T$

$$|f(x,y,t)| \le Re^{a_1x+a_2y+bt}, \tag{2}$$

for some X, Y and T, then we write

$$f(x,y,t) = O\left(e^{a_1x+a_2y+bt}\right) \text{ as } y \to \infty,\ y \to \infty,\ t \to \infty,$$

and similarly,

$$\lim_{\substack{x\to\infty\\ y\to\infty\\ t\to\infty}} e^{-\frac{1}{\mu}x-\frac{1}{\eta}y-\frac{1}{\varepsilon}t}|f(x,y,t)| = R\lim_{\substack{x\to\infty\\ y\to\infty\\ t\to\infty}} e^{-\left(\frac{1}{\lambda_1}-a_1\right)x-\left(\frac{1}{\lambda_2}-a_2\right)y-\left(\frac{1}{\eta}-c\right)t} = 0, \tag{3}$$

whenever $\frac{1}{\lambda_1} > a$, $\frac{1}{\eta} > c$ and $\frac{1}{\lambda_2} > b$. The function $f(x,y,t)$ does not develop swifter than $K(x,y,t)$ as $x \to \infty, y \to \infty, t \to \infty$.

Theorem 1. *The function $f(x,y,t)$ is defined on $(0,X),(0,Y)$ and $(0,T)$ and is of exponential order (x,y,t), then the double Sumudu-generalized Laplace transform of $f(x,y,t)$ exists for all $\Re(\frac{1}{\zeta_1}) > \frac{1}{\lambda_1}$, $\Re(\frac{1}{\zeta_2}) > \frac{1}{\lambda_2}$ $\Re\frac{1}{s} > \frac{1}{\eta}$*

Proof. By putting Equation (1) into Equation (2), we obtain

$$\begin{aligned} |F(\zeta_1,\zeta_2,s)| &= \left|\frac{s^\alpha}{\zeta_1\zeta_2}\int_0^\infty\int_0^\infty e^{-\left(\frac{x}{\zeta_1}+\frac{y}{\zeta_2}+\frac{t}{s}\right)} f(x,y,t)dxdydt\right| \\ &\le R\left|\frac{s^\alpha}{\zeta_1\zeta_2}\int_0^\infty\int_0^\infty\int_0^\infty e^{\left(\frac{1}{u_1}-a\right)x-\left(\frac{1}{u_2}-b\right)y-\left(\frac{1}{v}-c\right)t} dxdy\,dt\right| \\ &= \frac{Rs^{\alpha+1}}{(1-a\zeta_1)(1-c\zeta_2)(1-bs)}. \end{aligned} \tag{4}$$

From using the condition $\Re(\frac{1}{\zeta_1}) > \frac{1}{\lambda_1}$, $\Re(\frac{1}{\zeta_2}) > \frac{1}{\lambda_2}$ $\Re\frac{1}{s} > \frac{1}{\eta}$, we yield

$$\lim_{\substack{x\to\infty\\ y\to\infty\\ t\to\infty}} |F(u_1,u_2,v)| = 0 \text{ or } \lim_{\substack{x\to\infty\\ y\to\infty\\ t\to\infty}} F(u_1,u_2,v) = 0.$$

□

Theorem 2. *If the double Sumudu-generalized Laplace transform of the function $f(x,y,t)$ is presented by $S_xS_yG_t(f(x,y,t)) = F(\zeta_1,\zeta_2,s)$, then the double Sumudu-generalized Laplace transform of the function*

$$xyf(x,y,t),$$

is determined by

$$S_xS_yG_t[xyf(x,y,t)] = \zeta_1\zeta_2\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2F(\zeta_1,\zeta_2,s)). \tag{5}$$

Proof. By applying a partial derivative according to ζ_1 for Equation (1), we obtain

$$\begin{aligned}\frac{\partial F(\zeta_1,\zeta_2,s)}{\partial\zeta_1} &= \frac{\partial}{\partial\zeta_1}\int_0^\infty\int_0^\infty\int_0^\infty \frac{s^\alpha}{\zeta_1\zeta_2}e^{-\left(\frac{1}{\zeta_1}x+\frac{1}{\zeta_2}y+\frac{1}{s}t\right)}f(x,y,t)dxdydt,\\ &= \int_0^\infty\int_0^\infty \frac{s^\alpha}{\zeta_2}e^{-\left(\frac{1}{\zeta_2}y+\frac{1}{s}t\right)}\left(\int_0^\infty \frac{\partial}{\partial\zeta_1}\frac{1}{\zeta_1}e^{-\frac{1}{\zeta_1}x}f(x,y,t)dx\right)dydt,\end{aligned} \tag{6}$$

and by computing the partial derivative within the brackets, we obtain

$$\begin{aligned}\int_0^\infty \frac{\partial}{\partial\zeta_1}\frac{1}{\zeta_1}e^{-\frac{1}{\zeta_1}x}f(x,y,t)dx &= \int_0^\infty\left(\frac{1}{\zeta_1^3}x-\frac{1}{\zeta_1^2}\right)e^{-\frac{1}{\zeta_1}x}f(x,y,t)dx\\ &= \int_0^\infty \frac{1}{\zeta_1^3}xe^{-\frac{1}{\zeta_1}x}f(x,y,t)dx\\ &\quad -\int_0^\infty \frac{1}{\zeta_1^2}e^{-\frac{1}{\zeta_1}x}f(x,y,t)dx,\end{aligned} \tag{7}$$

then substituting Equation (7) into Equation (6), we obtain

$$\begin{aligned}\frac{\partial F(\zeta_1,\zeta_2,s)}{\partial\zeta_1} &= \int_0^\infty\int_0^\infty \frac{s^\alpha}{\zeta_2}e^{-\left(\frac{1}{\zeta_2}y+\frac{1}{s}t\right)}\left(\int_0^\infty \frac{1}{\zeta_1^3}xe^{-\frac{1}{\zeta_1}x}f(x,y,t)dx\right)dydt\\ &\quad -\int_0^\infty\int_0^\infty \frac{s^\alpha}{\zeta_2}e^{-\left(\frac{1}{\zeta_2}y+\frac{1}{s}t\right)}\left(\int_0^\infty \frac{1}{\zeta_1^2}e^{-\frac{1}{\zeta_1}x}f(x,y,t)dx\right)dydt,\end{aligned} \tag{8}$$

and by taking the derivative according to ζ_2 for Equation (8), we achieve

$$\begin{aligned}\frac{\partial^2 F(\zeta_1,\zeta_2,s)}{\partial\zeta_1\partial\zeta_2} &= \frac{s^\alpha}{\zeta_1^3}\int_0^\infty\int_0^\infty xe^{-\left(\frac{1}{\zeta_1}x+\frac{1}{s}t\right)}\left(\int_0^\infty e^{-\frac{1}{\zeta_2}y}\left(\frac{1}{\zeta_2^3}y-\frac{1}{\zeta_2^2}\right)f(x,y,t)\right)dxdydt\\ &\quad -\frac{s^\alpha}{\zeta_1^2}\int_0^\infty\int_0^\infty e^{-\left(\frac{1}{\zeta_1}x+\frac{1}{s}t\right)}\left(\int_0^\infty e^{-\frac{1}{\zeta_2}y}\left(\frac{1}{\zeta_2^3}y-\frac{1}{\zeta_2^2}\right)f(x,y,t)\right)dxdydt.\end{aligned} \tag{9}$$

Then, Equation (9), becomes

$$\begin{aligned}\frac{\partial^2 F(\zeta_1,\zeta_2,s)}{\partial\zeta_1\partial\zeta_2} &= \frac{1}{\zeta_1^2\zeta_2^2}S_xS_yG_t[xyf(x,y,t)] - \frac{1}{\zeta_1^2\zeta_2}S_xS_yG_t[xf(x,y,t)]\\ &\quad -\frac{1}{\zeta_1\zeta_2^2}S_xS_yG_t[yf(x,y,t)] + \frac{1}{\zeta_1\zeta_2}S_xS_yG_t[f(x,y,t)],\end{aligned} \tag{10}$$

by arranging the above equation, we achieve

$$\begin{aligned}S_xS_yG_t[xy(x,y,t)] &= \zeta_1^2\zeta_2^2\frac{\partial^2 F(\zeta_1,\zeta_2,s)}{\partial\zeta_1\partial\zeta_2} + \zeta_1^2\zeta_2\frac{\partial F(\zeta_1,\zeta_2,s)}{\partial\zeta_1}\\ &\quad +\zeta_1\zeta_2^2\frac{\partial F(\zeta_1,\zeta_2,s)}{\partial\zeta_2} + \zeta_1\zeta_2F(\zeta_1,\zeta_2,s),\end{aligned}$$

and thus, through simplifying, we obtain

$$S_xS_yG_t[xyf(x,y,t)] = \zeta_1\zeta_2\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2F(\zeta_1,\zeta_2,s)).$$

The proof is finished. □

The following theorem offers the double Sumudu-generalized Laplace transform of the partial derivatives $xy\psi_t$.

Theorem 3. *The double Sumudu-generalized Laplace transform of the partial derivatives $xy\psi_t$ is presented by*

$$\begin{aligned} S_xS_yG_t[xy\psi_t] &= \frac{\zeta_1\zeta_2}{s}\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2\Psi(\zeta_1,\zeta_2,s)) \\ &\quad -\zeta_1\zeta_2 s^\alpha\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2\Psi(\zeta_1,\zeta_2,0)), \end{aligned} \tag{11}$$

Proof. By taking the partial derivative according to ζ_1 for Equation (22), we have

$$\begin{aligned} \frac{\partial}{\partial\zeta_1}(S_xS_yG_t[\psi_t]) &= \frac{\partial}{\partial\zeta_1}\int_0^\infty\int_0^\infty\int_0^\infty \frac{s^\alpha}{\zeta_1\zeta_2}e^{-\left(\frac{1}{\zeta_1}x+\frac{1}{\zeta_2}y+\frac{1}{s}t\right)}\psi_t\,dxdydt, \\ &= \int_0^\infty\int_0^\infty \frac{s^\alpha}{\zeta_2}e^{-\left(\frac{1}{\zeta_2}y+\frac{1}{s}t\right)}\left(\int_0^\infty \frac{\partial}{\partial\zeta_1}\frac{1}{\zeta_1}e^{-\frac{1}{\zeta_1}x}\psi_t\,dx\right)dydt, \end{aligned} \tag{12}$$

we calculate the partial derivative inside brackets as shown below:

$$\begin{aligned} \int_0^\infty \frac{\partial}{\partial\zeta_1}\frac{1}{\zeta_1}e^{-\frac{1}{\zeta_1}x}\frac{\partial\psi}{\partial t}dx &= \int_0^\infty\left(\frac{1}{\zeta_1^3}x-\frac{1}{\zeta_1^2}\right)e^{-\frac{1}{\zeta_1}x}\psi_t\,dx \\ &= \int_0^\infty \frac{1}{\zeta_1^3}xe^{-\frac{1}{\zeta_1}x}\psi_t\,dx \\ &\quad -\int_0^\infty \frac{1}{\zeta_1^2}e^{-\frac{1}{\zeta_1}x}\psi_t\,dx, \end{aligned} \tag{13}$$

substituting Equation (13) into Equation (12), we obtain

$$\begin{aligned} \frac{\partial}{\partial\zeta_1}(S_xS_yG_t[\psi_t]) &= \int_0^\infty\int_0^\infty \frac{s^\alpha}{\zeta_2}e^{-\left(\frac{1}{\zeta_2}y+\frac{1}{s}t\right)}\left(\int_0^\infty \frac{1}{\zeta_1^3}xe^{-\frac{1}{\zeta_1}x}\psi_t\,dx\right)dydt \\ &\quad -\int_0^\infty\int_0^\infty \frac{s^\alpha}{\zeta_2}e^{-\left(\frac{1}{\zeta_2}y+\frac{1}{s}t\right)}\left(\int_0^\infty \frac{1}{\zeta_1^2}e^{-\frac{1}{\zeta_1}x}\psi_t\,dx\right)dydt, \end{aligned} \tag{14}$$

by taking the partial derivative according to ζ_2 for Equation (14),

$$\begin{aligned} \frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(S_xS_yG_t[\psi_t]) &= \frac{\partial}{\partial\zeta_2}\left(\int_0^\infty\int_0^\infty \frac{s^\alpha}{\zeta_2}e^{-\left(\frac{1}{\zeta_2}y+\frac{1}{s}t\right)}\left(\int_0^\infty \frac{1}{\zeta_1^3}xe^{-\frac{1}{\zeta_1}x}\psi_t\,dx\right)dydt\right) \\ &\quad -\frac{\partial}{\partial\zeta_2}\left(\int_0^\infty\int_0^\infty \frac{s^\alpha}{\zeta_2}e^{-\left(\frac{1}{\zeta_2}y+\frac{1}{s}t\right)}\left(\int_0^\infty \frac{1}{\zeta_1^2}e^{-\frac{1}{\zeta_1}x}\psi_t\,dx\right)dydt\right), \end{aligned} \tag{15}$$

and therefore, Equation (15) becomes

$$
\begin{aligned}
\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}\left(S_xS_yG_t[\psi_t]\right) &= \frac{1}{\zeta_1^2\zeta_2^2}\left(\frac{s^\alpha}{\zeta_1\zeta_2}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{1}{\zeta_1}x+\frac{1}{\zeta_2}y+\frac{1}{s}t\right)}xy\psi_t\,dxdydt\right)\\
&+\frac{1}{\zeta_1\zeta_2}\left(\frac{s^\alpha}{\zeta_1\zeta_2}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{1}{\zeta_1}x+\frac{1}{\zeta_2}y+\frac{1}{s}t\right)}\psi_t\,dxdydt\right)\\
&-\frac{1}{\zeta_1\zeta_2^2}\left(\frac{s^\alpha}{\zeta_1\zeta_2}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{1}{\zeta_1}x+\frac{1}{\zeta_2}y+\frac{1}{s}t\right)}y\psi_t\,dxdydt\right)\\
&-\frac{1}{\zeta_1^2\zeta_2}\left(\frac{s^\alpha}{\zeta_1\zeta_2}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{1}{\zeta_1}x+\frac{1}{\zeta_2}y+\frac{1}{s}t\right)}x\psi_t\,dxdydt\right), \qquad (16)
\end{aligned}
$$

and hence,

$$
\begin{aligned}
\frac{\partial^2}{\partial u_1\partial u_2}\left(S_xS_yG_t[\psi_t]\right) &= \frac{1}{\zeta_1^2\zeta_2^2}S_xS_yG_t[xy\psi_t]+\frac{1}{\zeta_1\zeta_2}S_xS_yG_t[\psi_t]\\
&-\frac{1}{\zeta_1\zeta_2^2}S_xS_yG_t[y\psi_t]-\frac{1}{\zeta_1^2\zeta_2}S_xS_yG_t[x\psi_t], \qquad (17)
\end{aligned}
$$

by reordering Equation (17), we prove Equation (11):

$$
\begin{aligned}
S_xS_yG_t[xy\psi_t] &= \frac{\zeta_1\zeta_2}{s}\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}\left(\zeta_1\zeta_2\Psi(\zeta_1,\zeta_2,s)\right)\\
&-\zeta_1\zeta_2 s^\alpha\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}\left(\zeta_1\zeta_2\Psi(\zeta_1,\zeta_2,0)\right),
\end{aligned}
$$

□

Double Sumudu transform of function $\psi(x,t)$ is given by $S_xS_t[\psi(x,t)] = \psi(\zeta_1,s)$. Moreover, the Sumudu-generalized Laplace transform of ψ_x, ψ_{xx}, ψ_{tt} and ψ_t is determined by

$$
\begin{aligned}
S_xG_t[\psi_x] &= \frac{\psi(\zeta_1,s)-\psi(0,s)}{\zeta_1},\\
S_xG_t(\psi_{xx}) &= \frac{\psi(\zeta_1,s)}{\zeta_1^2}-\frac{\psi(0,s)}{\zeta_1^2}-\frac{\psi_x(0,s)}{\zeta_1}
\end{aligned}
$$

and

$$
\begin{aligned}
S_xG_t[\psi_t] &= \frac{\psi(\zeta_1,s)}{s}-s^\alpha\psi(\zeta_1,0),\\
S_xG_t(\psi_{tt}) &= \frac{\psi(\zeta_1,s)}{s^2}-s^{\alpha-1}\psi(\zeta_1,0)-s^\alpha\psi_t(\zeta_1,0).
\end{aligned}
$$

Double Sumudu-generalized Laplace transform of the function $\psi(x,y,t)$ is presented by $S_xS_yG_t[\psi(x,y,t)] = \psi(\zeta_1,\zeta_2,s)$; therefore, the double Sumudu -generalized Laplace transform of ψ_x, ψ_{xx}, ψ_t and ψ_{tt} is provided by

$$
\begin{aligned}
S_xS_yG_t[\psi_x] &= \frac{\psi(\zeta_1,\zeta_2,s)-\psi(0,\zeta_2,s)}{\zeta_1}, \qquad (18)\\
S_xS_yG_t(\psi_{xx}) &= \frac{\psi(\zeta_1,\zeta_2,s)}{\zeta_1^2}-\frac{\psi(0,\zeta_2,s)}{\zeta_1^2}-\frac{\psi_x(0,\zeta_2,s)}{\zeta_1} \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
S_xS_yG_t\left[\psi_y\right] &= \frac{\psi(\zeta_1,\zeta_2,s)-\psi(\zeta_1,0,s)}{\zeta_2},\\
S_xS_yG_t\left(\psi_{yy}\right) &= \frac{\psi(\zeta_1,\zeta_2,s)}{\zeta_2^2}-\frac{\psi(\zeta_1,0,s)}{\zeta_2^2}-\frac{\psi_y(\zeta_1,0,s)}{\zeta_2} \qquad (20)
\end{aligned}
$$

and

$$\begin{aligned} S_xS_yG_t[\psi_t] &= \frac{\psi(\zeta_1,\zeta_2,s)}{s} - s^{\alpha}\psi(\zeta_1,\zeta_2,0), \\ S_xS_yG_t(\psi_{tt}) &= \frac{\psi(\zeta_1,\zeta_2,s)}{s^2} - s^{\alpha-1}\psi(\zeta_1,\zeta_2,0) - s^{\alpha}\psi_t(\zeta_1,\zeta_2,0). \end{aligned}$$

3. Double Sumudu-Generalized Laplace Decomposition Method and 2+1-Dimensional Linear Pseudoparabolic Equation

The following is the procedure showing two problems that are concerning to the linear and nonlinear singular 2+1-D pseudoparabolic equation.

The general singular 2+1-D pseudo parabolic equation is considered as follows:

$$\psi_t = \frac{1}{x}(x\psi_x)_x + \frac{1}{y}\left(y\psi_y\right)_y + \frac{1}{x}(x\psi_x)_{xt} + f(x,y,t), \tag{21}$$

dependent on the initial condition

$$\psi(x,y,0) = f_1(x,y), \tag{22}$$

where the functions $f_1(x,y)$ and $f(x,y,t)$. Firstly, we obtain the product of both sides of Equation (21) by xy, and implementing double Sumudu-generalized Laplace, we yield Equation (21), and implementing double Sumudu transform for Equation (22), we obtain

$$\begin{aligned} S_xS_yG_t[xy\psi_t] &= S_xS_yG_t\left[y(x\psi_x)_x + x\left(y\psi_y\right)_y\right] \\ &\quad + S_xS_yG_t[y(x\psi_x)_{xt} + xyf(x,y,t)], \end{aligned} \tag{23}$$

and by arranging Equation (23), it becomes

$$\begin{aligned} &\frac{\zeta_1\zeta_2}{s}\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2\Psi(\zeta_1,\zeta_2,s)) \\ = \; &\zeta_1\zeta_2 s^{\alpha}\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2F(\zeta_1,\zeta_2)) \\ &+ S_xS_yG_t\left[y(x\psi_x)_x + x\left(y\psi_y\right)_y + yy(x\psi_x)_{xt}\right] \\ &+ \zeta_1\zeta_2\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2F(\zeta_1,\zeta_2,s)), \end{aligned} \tag{24}$$

and therefore, Equation (24) becomes

$$\begin{aligned} &\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2\Psi(\zeta_1,\zeta_2,s)) \\ = \; &s^{\alpha+1}\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2F(\zeta_1,\zeta_2)) \\ &+ \frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[y(x\psi_x)_x + x\left(y\psi_y\right)_y + yy(x\psi_x)_{xt}\right] \\ &+ s\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2F(\zeta_1,\zeta_2,s)). \end{aligned} \tag{25}$$

By taking the integral for Equation (25), from 0 to ζ_1 and 0 to ζ_2 according to ζ_1 and ζ_2, we have

$$\begin{aligned} &\Psi(\zeta_1,\zeta_2,s) \\ = \; &\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2} s^{\alpha+1}\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2F(\zeta_1,\zeta_2))d\zeta_1 d\zeta_2 \\ &+ \frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[y(x\psi_x)_x + x\left(y\psi_y\right)_y\right]d\zeta_1 d\zeta_2 \\ &+ \frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[y(x\psi_x)_{xt}]d\zeta_1 d\zeta_2 \\ &+ \frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2} s\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2F(\zeta_1,\zeta_2,s))d\zeta_1 d\zeta_2. \end{aligned} \tag{26}$$

The solution is obtained by using the inverse double Sumudu-generalized Laplace for Equation (26):

$$\begin{aligned}
\psi(x,y,t) &= f_1(x,y) \\
&+ S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[y(x\psi_x)_x]d\zeta_1 d\zeta_2\right] \\
&+ S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[x(y\psi_y)_y\right]d\zeta_1 d\zeta_2\right] \\
&+ S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[y(x\psi_x)_{xt}]d\zeta_1 d\zeta_2\right] \\
&+ S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}s\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2F(\zeta_1,\zeta_2,s))d\zeta_1 d\zeta_2\right]. \qquad (27)
\end{aligned}$$

where $S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}$ indicates the inverse double Sumudu-generalized Laplace. The double Sumudu-generalized Laplace decomposition method (DSGLTDM) defines the solutions $\psi(x,y,t)$ with the help of infinite series as

$$\psi(x,y,t) = \sum_{n=0}^{\infty}\psi_n(x,y,t). \qquad (28)$$

By substituting Equation (28) into Equation (27), we receive

$$\begin{aligned}
\sum_{n=0}^{\infty}\psi_n(x,y,t) &= f_1(x,y) + S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}s\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2F(\zeta_1,\zeta_2,s))d\zeta_1 d\zeta_2\right] \\
&+ S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[y\left(x\sum_{n=0}^{\infty}\psi_{nx}\right)_x\right]d\zeta_1 d\zeta_2\right] \\
&+ S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[x\left(y\sum_{n=0}^{\infty}\psi_{ny}\right)_y\right]d\zeta_1 d\zeta_2\right] \\
&+ S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[y\left(x\sum_{n=0}^{\infty}\psi_{nx}\right)_{xt}\right]d\zeta_1 d\zeta_2\right]. \qquad (29)
\end{aligned}$$

By matching both sides of Equation (29), we obtain

$$\begin{aligned}
\psi_0(x,y,t) &= f_1(x,y) \\
&+ S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}s\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2F(\zeta_1,\zeta_2,s))d\zeta_1 d\zeta_2\right]. \qquad (30)
\end{aligned}$$

In general, the remaining terms are given by

$$\begin{aligned}
&\psi_{n+1}(x,y,t) = \\
&S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[y(x\psi_{nx})_x]d\zeta_1 d\zeta_2\right] \\
&+ S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[x(y\psi_{ny})_y\right]d\zeta_1 d\zeta_2\right] \\
&+ S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[y(x\psi_{nx})_{xt}]d\zeta_1 d\zeta_2\right]. \qquad (31)
\end{aligned}$$

where the inverse double Sumudu-generalized Laplace transform is given by $S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}$. Here, we assume that the inverse exists for Equations (30) and (31). In order to demonstrate the advantages and the preciseness of the (DSGLTDM) for solving singular 2+1-D parabolic equations, we employed the method proposed in the example below.

Example 1. *Consider a singular 2+1-D parabolic equation*

$$\psi_t = \frac{1}{x}(x\psi_x)_x + \frac{1}{y}\left(y\psi_y\right)_y + \left(x^2 - y^2\right)\cos t \tag{32}$$

subject to the initial condition

$$\psi(x,0) = 0, \tag{33}$$

and by utilizing Equation (27), we have

$$\begin{aligned} \psi(x,y,t) \;&=\; S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[y(x\psi_x)_x]d\zeta_1 d\zeta_2\right] \\ &+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[x\left(y\psi_y\right)_y\right]d\zeta_1 d\zeta_2\right] \\ &+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left(2\zeta_1^2-\zeta_2^2\right)\left[s^{\alpha+2}-s^{\alpha+4}+s^{\alpha+6}-s^{\alpha+8}+\dots\right]. \end{aligned} \tag{34}$$

Therefore,

$$\begin{aligned} \psi(x,y,t) \;&=\; S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[y(x\psi_x)_x]d\zeta_1 d\zeta_2\right] \\ &+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[x\left(y\psi_y\right)_y\right]d\zeta_1 d\zeta_2\right] \\ &+\left(x^2-y^2\right)\left[t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\dots\right]. \end{aligned}$$

our desired recursive relation is given by

$$\psi_0 = \left(x^2-y^2\right)\left[t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\dots\right],$$

and

$$\begin{aligned} \psi_{n+1}(x,t) \;&=\; S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[y(x\psi_{nx})_x]d\zeta_1 d\zeta_2\right] \\ &+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[x\left(y\psi_{ny}\right)_y\right]d\zeta_1 d\zeta_2\right], \end{aligned}$$

for $n = 0, 1, 2, \dots$; hence, at $n = 0$,

$$\begin{aligned} \psi_1(x,y,t) \;&=\; S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[y(x\psi_{0x})_x]d\zeta_1 d\zeta_2\right] \\ &+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[x\left(y\psi_{0y}\right)_y\right]d\zeta_1 d\zeta_2\right] \\ &=\; S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[4xy\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\dots\right)\right]d\zeta_1 d\zeta_2\right] \\ &-S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[4xy\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\dots\right)\right]d\zeta_1 d\zeta_2\right] \\ \psi_1(x,y,t) \;&=\; 0 \end{aligned}$$

at $n = 1$

$$\psi_2(x,y,t) = 0$$

and at $n = 2$,

$$\psi_2(x,y,t) = 0.$$

By applying Equation (28), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}\psi_n(x,y,t) &= \psi_0+\psi_1+\psi_2+\dots. \\
\psi(x,y,t) &= \left(x^2-y^2\right)\left[t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\dots\right].
\end{aligned}$$

Thus, the solution of Equation (32) is given by

$$\psi(x,y,t)=\left(x^2-y^2\right)\sin t.$$

Example 2. *Consider the singular 2+1-D pseudoparabolic equation given by*

$$\begin{aligned}
\psi_t &= \frac{1}{x}(x\psi_x)_x+\frac{1}{y}\left(y\psi_y\right)_y \\
&\quad +\frac{1}{x}(x\psi_x)_{xt}+\left(x^2-y^2\right)e^t-4e^t,
\end{aligned} \tag{35}$$

subject to the initial condition

$$\psi(x,y,0)=x^2-y^2. \tag{36}$$

For the purpose of keep on with our method for Equation (35), we obtain

$$\begin{aligned}
\psi_0 &= \left(x^2-y^2\right)\left[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\dots\right] \\
&\quad -4\left[t+\frac{t^2}{2!}+\frac{t^3}{3!}+\dots\right],
\end{aligned}$$

the exponential function formula is defined by

$$e^t=1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\dots$$

and

$$\psi_{n+1}(x,y,t)=S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[\Delta_n]d\zeta_1d\zeta_2\right],$$

where

$$\Delta_n=y(x\psi_{nx})_x+x\left(y\psi_{ny}\right)_y+y(x\psi_{nx})_{xt},$$

at $n=0$, we have

$$\psi_1(x,y,t)=S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[\Delta_0]d\zeta_1d\zeta_2\right],$$

and therefore,

$$\begin{aligned}
\Delta_0 &= y(x\psi_{0x})_x+x\left(y\psi_{0y}\right)_y+y(x\psi_{0x})_{xt} \\
&= 4xy\left[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\dots\right],
\end{aligned}$$

hence

$$\begin{aligned}
\psi_1(x,y,t) &= S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[4\left(s^{\alpha+2}+s^{\alpha+3}+s^{\alpha+4}+s^{\alpha+5}+\dots\right)\right], \\
&= 4\left[t+\frac{t^2}{2!}+\frac{t^3}{3!}+\dots\right].
\end{aligned}$$

In a similar way, at $n=1$, we have

$$\psi_2(x,y,t)=0.$$

In the same way, at $n = 1$, we obtain

$$\psi_3(x, y, t) = 0.$$

Therefore, the approximate solution of Equation (35) is presented as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} \psi_n(x, y, t) &= \psi_0 + \psi_1 + \psi_2 + \dots. \\
\psi(x, y, t) &= \left(x^2 - y^2\right)\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right] \\
&\quad -4\left[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right] \\
&\quad +4\left[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right] \\
\psi(x, y, t) &= \left(x^2 - y^2\right)\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right].
\end{aligned}$$

Hence,

$$\psi(x, y, t) = \left(x^2 - y^2\right)e^t.$$

4. Double Sumudu-Generalized Laplace Decomposition Method and 2+1-Dimensional Nonlinear Pseudoparabolic Equation

We must illustrate the double Sumudu-generalized Laplace decomposition method to solve the singular 2+1-dimensional nonlinear pseudoparabolic equation:

Consider the following general form of the singular 2+1-dimensional nonlinear pseudoparabolic equation of the model:

$$\begin{aligned}
\psi_t &= \frac{1}{x}(x\psi_x)_x + \frac{1}{y}\left(y\psi_y\right)_y + \frac{1}{x}(x\psi_x)_{xt} \\
&\quad -2\mu(x)\psi_x\psi_{xx} + \nu(x)(\psi_x)^2 + f(x, y, t)
\end{aligned} \tag{37}$$

with initial condition

$$\psi(x, y, 0) = h_1(x, y), \tag{38}$$

where the functions $\mu(x)$ and $\nu(x)$ are arbitrary. With a view to procure the solution of Equation (37), first, obtain the product of both sides of Equation (37) by xy, and implementing double the Sumudu-generalized Laplace transform, we find

$$\begin{aligned}
&S_x S_y G_t[xy\psi_t] = S_x S_y G_t\left[y(x\psi_x)_x + x\left(y\psi_y\right)_y + y(x\psi_x)_{xt}\right] \\
&+S_x S_y G_t\left[-2xy\mu(x)\psi_x\psi_{xx} + xy\nu(x)(\psi_x)^2 + xyf(x, y, t)\right].
\end{aligned} \tag{39}$$

Second, applying theorem 1 and 2 into Equation (39), we obtain

$$\begin{aligned}
&\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2\Psi(\zeta_1, \zeta_2, s)) \\
= {}& s^{\alpha+1}\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2 H_1(\zeta_1, \zeta_2)) \\
&+\frac{s}{\zeta_1\zeta_2} S_x S_y G_t\left[y(x\psi_x)_x + x\left(y\psi_y\right)_y + y(x\psi_x)_{xt}\right] \\
&-\frac{s}{\zeta_1\zeta_2} S_x S_y G_t\left[2xy\mu(x)\psi_x\psi_{xx} - xy\nu(x)(\psi_x)^2\right] \\
&+s\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2 F(\zeta_1, \zeta_2, s)).
\end{aligned} \tag{40}$$

By taking the integral for Equation (40), from 0 to ζ_1 and 0 to ζ_2 according to ζ_1 and ζ_2, we have

$$\begin{aligned}
&\Psi(\zeta_1,\zeta_2,s) \\
=\ & \frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2} s^{\alpha+1}\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2 H_1(\zeta_1,\zeta_2))d\zeta_1 d\zeta_2 \\
&+\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[y(x\psi_x)_x + x\left(y\psi_y\right)_y\right]d\zeta_1 d\zeta_2 \\
&+\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[y(x\psi_x)_{xt}]d\zeta_1 d\zeta_2 \\
&-\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[2xy\mu(x)\psi_x\psi_{xx} - xy\nu(x)(\psi_x)^2\right]d\zeta_1 d\zeta_2 \\
&+\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2} s\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2 F(\zeta_1,\zeta_2,s))d\zeta_1 d\zeta_2. \qquad (41)
\end{aligned}$$

The third step, now using inverse double Sumudu-generalized Laplace for both sides of Equation (41), the solution of Equation (37) can be written as

$$\begin{aligned}
\psi(x,y,t) =\ & h_1(x,y) \\
&+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[y(x\psi_x)_x]d\zeta_1 d\zeta_2\right] \\
&+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[x\left(y\psi_y\right)_y\right]d\zeta_1 d\zeta_2\right] \\
&+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[y(x\psi_x)_{xt}]d\zeta_1 d\zeta_2\right] \\
&-S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[2xy\mu(x)\psi_x\psi_{xx}]d\zeta_1 d\zeta_2\right] \\
&+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[xy\nu(x)(\psi_x)^2\right]d\zeta_1 d\zeta_2\right] \\
&+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2} s\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2 F(\zeta_1,\zeta_2,s))d\zeta_1 d\zeta_2\right]. \qquad (42)
\end{aligned}$$

By substituting Equation (28) into Equation (42), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}\psi_n(x,y,t) =\ & h_1(x,y) \\
&+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[y\left(x\sum_{n=0}^{\infty}\psi_{nx}\right)_x\right]d\zeta_1 d\zeta_2\right] \\
&+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[x\left(y\sum_{n=0}^{\infty}\psi_{ny}\right)_y\right]d\zeta_1 d\zeta_2\right] \\
&+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[y\left(x\sum_{n=0}^{\infty}\psi_{nx}\right)_{xt}\right]d\zeta_1 d\zeta_2\right] \\
&-S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[2xy\mu(x)A_n]d\zeta_1 d\zeta_2\right] \\
&+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[xy\nu(x)B_n]d\zeta_1 d\zeta_2\right] \\
&+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2} s\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2 F(\zeta_1,\zeta_2,s))d\zeta_1 d\zeta_2\right]. \qquad (43)
\end{aligned}$$

where $n = 0, 1, 2, \ldots$. Hence, from Equation (43) above, we have

$$\begin{aligned}\psi_0(x,y,t) &= h_1(x,y) + tg_2(x,y) \\ &+ S_{\zeta_1}^{-1} S_{\zeta_2}^{-1} G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2} s\frac{\partial^2}{\partial\zeta_1\partial\zeta_2}(\zeta_1\zeta_2 F(\zeta_1,\zeta_2,s))d\zeta_1 d\zeta_2\right]\end{aligned}$$

and

$$\begin{aligned}\psi_{n+1}(x,y,t) &= S_{\zeta_1}^{-1} S_{\zeta_2}^{-1} G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2} \frac{s}{\zeta_1\zeta_2} S_x S_y G_t[y(x\psi_{nx})_x] d\zeta_1 d\zeta_2\right] \\ &+ S_{\zeta_1}^{-1} S_{\zeta_2}^{-1} G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2} \frac{s}{\zeta_1\zeta_2} S_x S_y G_t\left[x\left(y\psi_{ny}\right)_y\right] d\zeta_1 d\zeta_2\right] \\ &+ S_{\zeta_1}^{-1} S_{\zeta_2}^{-1} G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2} \frac{s}{\zeta_1\zeta_2} S_x S_y G_t[y(x\psi_{nx})_{xt}] d\zeta_1 d\zeta_2\right] \\ &- S_{\zeta_1}^{-1} S_{\zeta_2}^{-1} G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2} \frac{s}{\zeta_1\zeta_2} S_x S_y G_t[2xy\mu(x)A_n] d\zeta_1 d\zeta_2\right] \\ &+ S_{\zeta_1}^{-1} S_{\zeta_2}^{-1} G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2} \frac{s}{\zeta_1\zeta_2} S_x S_y G_t[xy\nu(x)B_n] d\zeta_1 d\zeta_2\right],\end{aligned}$$

where nonlinear terms A_n and B_n are addressed as

$$A_n = \sum_{n=0}^{\infty} \psi_{nx}\psi_{nxx}, \quad B_n = \sum_{n=0}^{\infty} (\psi_{nx})^2, \tag{44}$$

where the nonlinear terms $\psi_x\psi_{xx}$ and $(\psi_x)^2$ are in the following forms:

$$\begin{aligned}&A_0 = \psi_{0x}\psi_{0xx} \\ &A_1 = \psi_{0x}\psi_{1xx} + \psi_{x1}\psi_{0xx}, \\ &A_2 = \psi_{0x}\psi_{2xx} + \psi_{1x}\psi_{1xx} + \psi_{2x}\psi_{0xx}, \\ &A_3 = \psi_{0x}\psi_{3xx} + \psi_{1x}\psi_{2xx} + \psi_{2x}\psi_{1xx} + \psi_{3x}\psi_{0xx}.\end{aligned} \tag{45}$$

and

$$\begin{aligned}B_0 &= (\psi_{0x})^2 \\ B_1 &= 2\psi_{0x}\psi_{1x} \\ B_2 &= 2\psi_{0x}\psi_{2x} + (\psi_{1x})^2 \\ B_3 &= 2\psi_{0x}\psi_{3x} + 2\psi_{1x}\psi_{2x}.\end{aligned} \tag{46}$$

To ascribe this method to a linear singular 2+1-dimensional nonlinear pseudoparabolic equation, we give the next example.

Example 3. *The singular 2+1-dimensional nonlinear pseudoparabolic equation is offered by*

$$\begin{aligned}\psi_t &= \frac{1}{x}(x\psi_x)_x + \frac{1}{y}\left(y\psi_y\right)_y + \frac{1}{x}(x\psi_x)_{xt} \\ &-2\mu(x)\psi_x\psi_{xx} + \nu(x)(\psi_x)^2 \\ &+2\left(x^2 - y^2\right)e^{2t} - 8e^{2t}\end{aligned} \tag{47}$$

with the initial condition

$$\psi(x,y,0) = x^2 - y^2. \tag{48}$$

In order to flow with our method for Equation (47), we obtain

$$\begin{aligned}\psi_0(x,y,t) &= \left(x^2-y^2\right)\left(1+2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\frac{(2t)^4}{4!}+\ldots\right)\\ &\quad -8\left(t+\frac{2t^2}{2!}+\frac{4t^3}{3!}+\frac{8t^4}{4!}+\ldots\right),\end{aligned}$$

and

$$\begin{aligned}\psi_{n+1}(x,y,t) &= S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[y(x\psi_{nx})_x]d\zeta_1 d\zeta_2\right]\\ &+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[x\left(y\psi_{ny}\right)_y\right]d\zeta_1 d\zeta_2\right]\\ &+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[y(x\psi_{nx})_{xt}]d\zeta_1 d\zeta_2\right]\\ &-S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[2xy\mu(x)A_n]d\zeta_1 d\zeta_2\right]\\ &+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[xy\nu(x)B_n]d\zeta_1 d\zeta_2\right],\end{aligned}$$

the first repeat at $n=0$ is denoted by

$$\begin{aligned}\psi_1(x,y,t) &= S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[y(x\psi_{0x})_x]d\zeta_1 d\zeta_2\right]\\ &+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[x\left(y\psi_{0y}\right)_y\right]d\zeta_1 d\zeta_2\right]\\ &+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[y(x\psi_{0x})_{xt}]d\zeta_1 d\zeta_2\right]\\ &-S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[2xy\mu(x)A_0]d\zeta_1 d\zeta_2\right]\\ &+S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t[xy\nu(x)B_0]d\zeta_1 d\zeta_2\right],\end{aligned}$$

$$\begin{aligned}\psi_1(x,y,t) &= S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[\frac{1}{\zeta_1\zeta_2}\int_0^{\zeta_1}\int_0^{\zeta_2}\frac{s}{\zeta_1\zeta_2}S_xS_yG_t\left[8xy\left[1+\frac{4t^2}{2!}+\frac{12t^3}{3!}+\frac{32t^4}{4!}+\ldots\right]\right]d\zeta_1 d\zeta_2\right]\\ &= S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}\left[8\left[s^{\alpha+2}+2s^{\alpha+3}+4s^{\alpha+4}+12s^{\alpha+5}+\ldots\right]\right]\\ &= 8\left(t+t+\frac{2t^2}{2!}+\frac{4t^3}{3!}+\frac{8t^4}{4!}+\ldots\right),\end{aligned}$$

at $n=1$, we have

$$\psi_2(x,y,t)=0$$

and letting $n=2$, we obtain

$$\psi_3(x,y,t)=0,$$

and therefore, by using Equation (28), the series solutions are denoted by

$$\begin{aligned}\sum_{n=0}^{\infty}\psi_n(x,y,t) &= \psi_0+\psi_1+\psi_2+\ldots.\\ &= \left(x^2-y^2\right)\left(1+2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\frac{(2t)^4}{4!}+\ldots\right)\end{aligned}$$

and therefore, the accurate solutions become

$$\psi(x,y,t) = \left(x^2 - y^2\right)e^{2t}.$$

5. Conclusions

In this study, we established a new hybrid method, which is named the double Sumudu-generalized Laplace transform decomposition method (DSGLTDM). This approach is successfully applied in singular 2+1-D pseudoparabolic equations. The (DSGLTDM) is an analytical process and works by utilizing the initial conditions only. Three examples are offered to examine the accuracy of the method.

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References

1. Gala, S.; Ragusa, M.A. A regularity criterion for 3D micropolar fluid flows in terms of one partial derivative of the velocity. *Ann. Pol. Math.* **2016**, *116*, 217–228. [CrossRef]
2. Qi, Q.; Chen, Y.J.; Wang, Q.S. Blow-up phenomena for a pseudo-parabolic system with variable exponents. *Electron. J. Qual. Theory Differ. Equ.* **2017**, *36*, 1–9. [CrossRef]
3. Ragusa, M.A. Commutators of fractional integral operators in Vanishing-Morrey Spaces. *J. Glob. Optim.* **2008**, *40*, 361–368. [CrossRef]
4. Zheng, Y.D.; Fang, Z.B. Qualitative properties for a pseudo-parabolic equation with nonlocal reaction term. *Bound. Value Probl.* **2019**, *2019*, 134. [CrossRef]
5. Wu, F.; Yang, X.J. Approximate solution of the non–linear diffusion equation of multiple orders. *Therm. Sci.* **2016**, *20*, 683–687. [CrossRef]
6. Yan, S.P.; Zhong, W.P.; Yang, X.J. A novel series method for fractional diffusion equation within Caputo fractional derivative. *Therm. Sci.* **2016**, *20*, 695–699. [CrossRef]
7. Mesloub, S. A nonlinear nonlocal mixed problem for a second order pseudo-parabolic equation. *J. Math. Anal. Appl.* **2006**, *316*, 189–209. [CrossRef]
8. Dehghan, M.; Hamidi, A.; Shakourifar, M. The solution of coupled Burgers' equations using Adomian Pade technique. *Appl. Math. Comput.* **2007**, *189*, 1034–1047. [CrossRef]
9. Kaya, D. An explicit solution of coupled viscous Burgers' equation by the decomposition method. *Int. J. Math. Math. Sci.* **2001**, *27*, 675–680. [CrossRef]
10. Gadain, H.E. Solving Coupled Pseudo-Parabolic Equation using a Modified double Laplace Decomposition method. *Acta Math. Sci.* **2018**, *38B*, 333–346. [CrossRef]
11. Eltayeb, H.; Elgezouli, D.E.; Kılıcman, A.; Bachar, I. Three-dimensional Laplace adomian decomposition method and singular pseudo-parabolic equations. *J. Funct. Spaces* **2021**, *2021*, 5563013.
12. Abbaoui, K.; Cherruault, Y. Convergence of Adomian's method applied to differential equations. *Comput. Math. Appl.* **1994**, *28*, 103–109. [CrossRef]
13. Abbaoui, K.; Cherruault, Y. Convergence of Adomian's method applied to nonlinear equations. *Math. Comput. Model.* **1994**, *20*, 69–73. [CrossRef]
14. Atangana, A.; Oukouomi Noutchie, S.C. On multi-Laplace transform for solving nonlinear partial differential equations with mixed derivatives. *Math. Probl. Eng.* **2014**, *2014*, 267843. [CrossRef]
15. Cherruault, Y.; Saccomandi, G.; Some, B. New results for convergence of Adomian's method applied to integral equations. *Math Comput. Model.* **1992**, *16*, 85–93. [CrossRef]
16. Watugala, G.K. Sumudu transform: A new integral transform to solve differential equations and control engineering problems. *Int. J. Math. Educ. Sci. Technol.* **1993**, *24*, 35–43. [CrossRef]
17. Belgacem, F.B.M.; Karaballi, A.A. Sumudu transform fundamental properties investigations and applications. *J. Appl. Math. Stoch. Anal.* **2006**, *2006*, 91083. [CrossRef]
18. Belgacem, F.B.M.; Karaballi, A.A.; Kalla, S.L. Analytical investigations of the Sumudu transform and applications to integral production equations. *Math. Probl. Eng.* **2003**, *3*, 103–118. [CrossRef]
19. Asiru, M.A. Further properties of the Sumudu transform and its applications. *Int. J. Math. Educ. Sci. Technol.* **2002**, *33*, 441–449. [CrossRef]
20. Watugala, G.K. The Sumudu transform for functions of two variables. *Math. Eng. Ind.* **2002**, *18*, 293–302.

21. Ahmeda, Z.; Idreesb, M.I.; Belgacemc, F.B.M.; Perveen, Z. On the convergence of double Sumudu transform. *J. Nonlinear Sci. Appl.* **2019**, *13*, 154–162. [CrossRef]
22. Kim, H.; Sattaso, S.; Kaewnimit, K.; Nonlaopon, K. An application of generalized Laplace transform in PDEs. *Adv. Dyn. Syst. Appl.* **2019**, *14*, 257–265. [CrossRef]
23. Eltayeb, H.; Alhefthi, R.K. Solution of Fractional Third-Order Dispersive Partial Differential Equations and Symmetric KdV via Sumudu–Generalized Laplace Transform Decomposition. *Symmetry* **2023**, *15*, 1540. [CrossRef]