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Moderate Deviation Principle for Linear Processes Generated by Dependent Sequences under Sub-Linear Expectation

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Abstract: We are interested in the linear processes generated by dependent sequences under sub-linear expectation. Using the Beveridge–Nelson decomposition of linear processes and the inequalities, the moderate deviation principle for linear processes produced by an m-dependent sequence is established. We also prove the upper bound of the moderate deviation principle for linear processes produced by negatively dependent sequences via different methods from m-dependent sequences. These conclusions promote and improve the corresponding results from the traditional probability space to the sub-linear expectation space.

Keywords: moderate deviation principle; linear process; m-dependent; negatively dependent; sub-linear expectation

MSC: 60F10; 60F05

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1. Introduction

An important method for analyzing the limiting behavior of a family of probability measures is the large (moderate) deviation principle (abbreviated as LDP (MDP)) which plays a significant role in probability theory. It can describe the tail probability with an exponential rate of convergence. It is widely used in various areas, such as statistical mechanics, partial differential equations, dynamical systems, statistics, stochastic processes and their related fields and so on. It has been widely studied by numerous scholars. For example, refs. [1,2] established the basic and groundbreaking work for the LDP (MDP) on a traditional probability space. We also refer to refs. [3–7] and the references therein for more information on the large deviation principle on a traditional probability space.

One of the research hotspots in risk measurement, assets pricing, cooperative game and decision theory is model uncertainty, see refs. [8–11], etc. Recently, by relaxing the linear property of the traditional linear expectation to subadditivity and positive homogeneity, ref. [12] introduced a systematic framework of the sub-linear expectation of random variables in a general function space. Obviously, the natural extension of the classical linear expectation is sub-linear expectation. Subsequently, many limit theorems were established in the sub-linear expectation space. At the same time, some results on the LDP (MDP) were established on a traditional probability space, we refer to ref. [13], which obtained the LDP for stochastic differential equations driven by G-Brownian motion; ref. [14], which established relative entropy and the LDP under sub-linear expectations; ref. [15], which proved the LDP for negatively dependent random sequences in the sense of upper probability; ref. [16], which studied the self-normalized MDP and laws of the iterated logarithm under G-expectation; ref. [17], which proposed the LDP for random variables under sub-linear expectations on $\mathbb{R}^d$; ref. [18], which discussed the MDP for independent and nonidentical distributed random variables under sub-linear expectation; ref. [19], which presented the

MDP for the sequence of m-dependent strictly stationary random variables in the sense of sub-linear expectation; and ref. [20], which found the LDP for linear processes produced by independent stationary sequences under upper probability. We also refer to refs. [21–28] for the other limit properties under the sub-linear expectation. Most work on the LDP (MDP) assumes that the random variables under discussion are independent despite the different definitions of independence. Furthermore, there is no work on the MDP for linear processes under the sub-linear expectation. Therefore, we will discuss the MDP for linear processes generated by a dependent sequence.

The paper is organized as follows. In Section 1, the significance of the research is introduced. Some basic settings and definitions under sub-linear expectations are given in Section 2. Some lemmas required to demonstrate the main results are presented in Section 3. In Section 4, the MDP for linear processes generated by an m-dependent sequence is stated and proven. In Section 5, the upper bound of the MDP for linear processes generated by a negatively dependent sequence is established and proven.

Throughout the paper, C denotes a positive constant, which may take different values wherever it appears in different expressions. I_A denotes the indicator function of the set A, $(\Omega, \mathcal{F}, P)$ stands for a traditional probability space, and $(\Omega, \mathcal{H}, \mathbb{E})$ stands for a sub-linear expectation space.

2. Basic Settings and Definitions

In this section, we will recall some notations and definitions on sub-linear expectation spaces. We use the framework and notations of ref. [29]. Let Ω be a given set, $\mathcal{B}(\Omega)$ be the $\sigma-$algebra of subsets of the set Ω, and $\mathcal{H}$ be a linear space of real measurable functions defined on $(\Omega, \mathcal{B}(\Omega))$. $\mathcal{H}$ contains all I_A, where $A \in \mathcal{B}(\Omega)$. If $X \in \mathcal{H}$, then $|X| \in \mathcal{H}$. The space $\mathcal{H}$ can be considered as a space of random variables.

Definition 1. *A function $\mathbb{E} : \mathcal{H} \to [-\infty, +\infty]$ is said to be a sub-linear expectation if it satisfies for $\forall\, X, Y \in \mathcal{H}$,*

1. Monotonicity: $X \geq Y$ implies $\mathbb{E}[X] \geq \mathbb{E}[Y]$.
2. Constant preserving: $\mathbb{E}[c] = c$, $\forall c \in \mathbb{R}$.
3. Subadditivity: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.
4. Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, $\forall \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sub-linear expectation space. Given a sub-linear expectation $\mathbb{E}$, let us denote the conjugate expectation $\mathcal{E}$ of $\mathbb{E}$ by $\mathcal{E}[X] := -\mathbb{E}[-X]$, $\forall X \in \mathcal{H}$.

Now, we give the definitions of independence, m-dependence and negative dependence on a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, which can be found in refs. [15,30].

Definition 2. *Let $X_1, X_2, \cdots, X_{n+1}$ be random variables in a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. X_{n+1} is described to be independent from $(X_1, X_2, \cdots, X_n)$ under $\mathbb{E}$ if*

$$\mathbb{E}\left[\prod_{i=1}^{n+1} \varphi_i(X_i)\right] = \mathbb{E}\left[\prod_{i=1}^{n} \varphi_i(X_i)\right] \mathbb{E}[\varphi_{n+1}(X_{n+1})],$$

for every nonnegative measurable function φ_i on $\mathbb{R}$ with $\mathbb{E}[\varphi_i(X_i)] < \infty$, $1 \leq i \leq n+1$. A sequence of random variables $\{X_n, n \geq 1\}$ is described to be independent if X_{n+1} is independent of $(X_1, X_2, \cdots, X_n)$ for all $n \in \mathbb{N}$.

A natural extension of independence is the following.

Definition 3. *If $(X_{n+m+1}, \cdots, X_{n+j})$ is independent from $(X_1, \cdots, X_n)$ for fixed integer m and every n and every $j \geqslant m+1$, then $\{X_n,\ n \geq 1\}$ is called a sequence of m-dependent random variables. Especially, $\{X_n,\ n \geq 1\}$ is called an independent sequence for $m = 0$.*

Definition 4. *X_{n+1} is described to be negatively dependent (abbreviated as ND) from $(X_1, X_2, \cdots, X_n)$ under $\mathbb{E}$ if*

$$\mathbb{E}\left[\prod_{i=1}^{n+1} \varphi_i(X_i)\right] \leq \mathbb{E}\left[\prod_{i=1}^{n} \varphi_i(X_i)\right]\mathbb{E}[\varphi_{n+1}(X_{n+1})],$$

for every nonnegative measurable function φ_i on $\mathbb{R}$ with $\mathbb{E}[\varphi_i(X_i)] < \infty$, $1 \leq i \leq n+1$. A sequence of random variables $\{X_n, n \geq 1\}$ is described to be ND if X_{n+1} is ND of $(X_1, X_2, \cdots, X_n)$ for all $n \in \mathbb{N}$.

Ref. [20] gave the following definition of strictly stationary in a sub-linear expectation space.

Definition 5. *A sequence of random variables $\{X_n,\ n \geq 1\}$ is said to be strictly stationary on the $(\Omega, \mathcal{H}, \mathbb{E})$ if*

$$\mathbb{E}[\phi_n(\xi_1, \xi_2, \cdots, \xi_n)] = \mathbb{E}[\phi_n(\xi_{1+k}, \xi_{2+k}, \cdots, \xi_{n+k})], \quad \forall n \geq 1,\ k \geq 1,$$

for any function $\phi_n \in C_{l,Lip}(\mathbb{R}^n) : \mathbb{R}^n \to \mathbb{R}$, where $\varphi \in C_{l,Lip}(\mathbb{R}^n)$ denotes the linear space of local Lipschitz continuous functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq c(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|,\ \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

for some $c > 0$, $m \in \mathbb{N}$ depending on φ.

Next, we look back upon the definitions of the upper expectation and corresponding capacity. We also refer to Chapter 6 in [29] for more details. Let $\mathcal{B}(\Omega)$ be a σ-algebra on Ω. Let $\mathcal{M}$ and $\mathcal{B}(\Omega)$ be the set of all probability measures and a σ-algebra on Ω, respectively. Every non-empty subset $\mathcal{P} \subseteq \mathcal{M}$ defines an upper probability/expectation and a lower probability/expectation

$$\mathbb{V}(A) = \sup_{P \in \mathcal{P}} P(A) = \mathbb{E}[I_A],\ \ \mathcal{V}(A) = \inf_{P \in \mathcal{P}} P(A) = \mathcal{E}[I_A],\ A \in \mathcal{B}(\Omega),$$

$$\mathbb{E}[X] = \sup_{P \in \mathcal{P}} E_P[X],\ \ \mathcal{E}[X] = \inf_{P \in \mathcal{P}} E_P[X],\ X \in \mathcal{H}.$$

It is easy to check that $\mathbb{V}(\cdot)$ is a Choquet capacity (ref. [8]), which meets the following definition, $\mathcal{V}(\cdot)$ is the conjugate capacity of $\mathbb{V}(\cdot)$, $\mathbb{E}[\cdot]$ is a sub-linear expectation which is given in Definition 1 and $\mathcal{E}[\cdot]$ is the conjugate expectation $\mathbb{E}[\cdot]$.

Definition 6. $\mathbb{V}(\cdot)$ *is a Choquet capacity, i.e.,*

(1). $0 \leqslant \mathbb{V}(A) \leqslant 1,\ \ \forall A \in \mathcal{B}(\Omega)$.

(2). $\mathbb{V}(A) \leqslant \mathbb{V}(B), \forall A \subset B \in \mathcal{B}(\Omega)$.

(3). $\mathbb{V}\left(\bigcup_{n=1}^{\infty} A_n\right) \leqslant \sum_{n=1}^{\infty} \mathbb{V}(A_n)$, *for any* $A_n \in \mathcal{B}(\Omega),\ n \geq 1$.

(4). $\lim_{n\to\infty} \mathbb{V}(A_n) = \mathbb{V}(A)$ *if* $A_n \uparrow A = \bigcup_{n=1}^{\infty} A_n,\ A_n \in \mathcal{B}(\Omega),\ n \geq 1$.

Next, we provide the definition of regularity of $\mathbb{E}$ (ref. [29]), which will be used in the following.

Definition 7. *The sub-linear expectation $\mathbb{E}$ is regular for any sequence $\{X_n,\ n \geq 1\} \subset \mathcal{H}$ such that*

$$X_n \downarrow 0 \Longrightarrow \lim_{n\to\infty} \mathbb{E}[X_n] = 0.$$

Next, we will give the definitions of a good rate function (ref. [2]) and the LDP (ref. [14]) in a sub-linear expectation space.

Definition 8. *A function I: $\mathbb{R} \to [0, \infty)$ is called a good rate function if for all $l \geqslant 0$, the set $\{x : I(x) \leqslant l\}$ is a compact subset of $\mathbb{R}$.*

Definition 9. *Let Ω be a topological space and $\{a(n), n \geqslant 1\}$ be a sequence of positive functions satisfying $a(n) \to \infty$ as $n \to \infty$. A family of measurable maps $\{\mathbb{V}(V_n \in \cdot), n \geqslant 1\}$ satisfies the LDP with speed $a(n)$ and rate function $I(x)$ if for any open set $G \subset \mathcal{B}(\mathbb{R})$,*

$$\liminf_{n\to\infty} \frac{1}{a(n)} \log \mathbb{V}(V_n \in G) \geqslant - \inf_{x\in G} I(x), \tag{1}$$

and for any closed subset $F \subset \mathcal{B}(\mathbb{R})$,

$$\limsup_{n\to\infty} \frac{1}{a(n)} \log \mathbb{V}(V_n \in F) \leqslant - \inf_{x\in F} I(x). \tag{2}$$

Equations (1) and (2) are described as the lower bound of large deviations (LLD) and the upper bound of large deviations (ULD), respectively.

If Equations (1) and (2) are satisfied with $a(n) = \frac{b_n^2}{n}$ and $V_n = \frac{S_n}{b_n}$, we say that V_n satisfies the LDP with speed $\frac{n}{b_n^2}$ and rate function $I(\cdot)$, we also say that V_n satisfies the MDP, where $\{b_n,\ n \geq 1\}$ is a sequence of positive real numbers satisfying

$$\lim_{n\to\infty} \frac{b_n}{\sqrt{n}} = \infty, \lim_{n\to\infty} \frac{b_n}{n} = 0. \tag{3}$$

3. Preliminary Lemmas

Some useful lemmas which are needed to prove the main results are given in the following.

Lemma 1 ([31])**.** *Let $f(x) \geq 0$ be a nondecreasing function defined on $\mathbb{R}$; then, for any x,*

$$\mathbb{V}(X \geq x) \leq \frac{\mathbb{E}[f(X)]}{f(x)}.$$

Lemma 2 ([20])**.** *Let $\{\eta_i, 1 \leq i \leq n\}$ be a sequence of random variables on $(\Omega, \mathcal{H}, \mathbb{E})$ and $\gamma_i \in [0, 1]$, $1 \leq i \leq n$, such that $\sum_{i=1}^n \gamma_i = 1$. Then,*

$$\log \mathbb{E} \exp\left\{ \sum_{i=1}^n \gamma_i \eta_i \right\} \leq \sum_{i=1}^n \gamma_i \log \mathbb{E} \exp\{\eta_i\}.$$

Lemma 3 ([20])**.** *Assume that a sequence of random variables $\{\eta_i, i \in \mathbb{Z}\}$ is strictly stationary on $(\Omega, \mathcal{H}, \mathbb{E})$ and $\{\beta_i, i \in \mathbb{Z}\}$ is a sequence of real numbers satisfying $\sum_{i\in\mathbb{Z}} |\beta_i| < \infty$. Then,*

$$\log \mathbb{E} \exp\left\{ \sum_{k\in\mathbb{Z}} |\beta_k||\eta_k| \right\} \leq \log \mathbb{E} \exp\left\{ |\eta_1| \sum_{k\in\mathbb{Z}} |\beta_k| \right\}.$$

Lemma 4 ([19])**.** *Suppose that $\{X_n,\ n \geq 1\}$ is a sequence of strictly stationary m-dependent random variable in the sense of sub-linear expectation satisfying*

$$\mathbb{E}[X_1] = \mathbb{E}[-X_1] = 0, \quad \lim_{n\to\infty} \frac{1}{n} \mathbb{E}[\sum_{i=1}^n X_i]^2 = \sigma^2 < \infty, \tag{4}$$

and

$$\mathbb{E}[e^{\delta|X_1|}] < \infty, \tag{5}$$

for some $\delta > 0$. *Then,* $\{\mathbb{V}(\frac{S_n}{b_n} \in \cdot), n \geq 1\}$ *satisfies the MDP with rate function* $I_1(x)$*:*

(1) For every closed set $F \subset \mathcal{B}(\mathbb{R})$ *such that*

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{S_n}{b_n} \in F\right) \leqslant - \inf_{x\in F} I_1(x);$$

(2) For every open set $G \subset \mathcal{B}(\mathbb{R})$ *such that*

$$\liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{S_n}{b_n} \in G\right) \geqslant - \inf_{x\in G} I_1(x),$$

where $I_1(x) := \sup_{y\in\mathbb{R}}\left\{xy - \frac{y^2\sigma^2}{2}\right\} = \frac{x^2}{2\sigma^2}$, $S_n = \sum_{k=1}^n X_k$ *and* $\{b_n, n \geq 1\}$ *is defined as in (3).*

Lemma 5 ([2])**.** *Let N be a fixed integer. Then, for every* $a_\epsilon^i \geq 0$*,*

$$\limsup_{\epsilon\to 0} \epsilon \log\left(\sum_{i=1}^N a_\epsilon^i\right) = \max_{1\leq i\leq N} \limsup_{\epsilon\to 0} \epsilon \log a_\epsilon^i.$$

The following lemma ensures that $\mathbb{E}$ is not linear; Cramér's method can also be used for the ULD. Let $\overline{C}(\cdot)$ be a family of upper probabilities, obviously it is also a Choquet capacity (ref. [13], Lemma A.2 in Appendix).

Lemma 6 ([13])**.** *Let* (S, ρ) *be a Polish space. Let* $(V^\varepsilon, \varepsilon > 0) : \Omega \to (S, \rho)$ *be a family of measurable mappings and let* $\{\lambda(\varepsilon), \varepsilon > 0\}$ *be a positive function with* $\lambda(\varepsilon) \to 0$ *as* $\varepsilon \to 0$*. Assume* $S = \mathbb{R}^d$*. For any* $y \in \mathbb{R}^d$*, if there exists* $\delta > 0$ *such that*

$$\Lambda(\delta y) := \limsup_{\varepsilon\to 0} \lambda(\varepsilon) \log \mathbb{E} \exp\left\{\frac{\langle V^\varepsilon, \delta y\rangle}{\lambda(\varepsilon)}\right\} \in \mathbb{R},$$

then $(\overline{C}(V^\varepsilon \in \cdot), \varepsilon > 0)$ *satisfies the ULD with speed* $\lambda(\varepsilon)$ *and good rate function* Λ^* *defined by*

$$\Lambda^*(x) = \sup_{y\in\mathbb{R}^d}\{\langle x, y\rangle - \Lambda(y)\}, \;\; x \in \mathbb{R}^d.$$

Lemma 7 ([15])**.** *Let X and Y be two random variables defined on* $(\Omega, \mathcal{H}, \mathbb{E})$ *with capacity* $\mathbb{V}$*. If for every* $\varepsilon > 0$*,*

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}(X - Y > \varepsilon) = -\infty,$$

and, for every closed set $F \in \mathcal{B}(\mathbb{R})$*,*

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}(X \in F) \leq - \inf_{x\in F} J(x),$$

where $J(x)$ *is a good rate function, then*

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}(Y \in F) \leq - \inf_{x\in F} J(x).$$

4. The MDP for Linear Processes Generated by an m-Dependent Sequence

In this section, we will give the following MDP for linear processes generated by an m-dependent sequence.

Theorem 1. *Consider the following linear processes,*

$$Y_k = \sum_{j=-\infty}^{\infty} a_j X_{k-j}, \quad k \geq 1, \tag{6}$$

where $\{X_k,\ k \in \mathbb{Z}\}$ is a sequence of strictly stationary m-dependent random variables defined on $(\Omega, \mathcal{H}, \mathbb{E})$ satisfying Equations (4) and (5), $\{a_j,\ j \in \mathbb{Z}\}$ is a sequence of real numbers satisfying $\sum_{j=-\infty}^{+\infty} |a_j| < \infty$ and $a = \sum_{j=-\infty}^{+\infty} a_j \neq 0$. Then, $\left\{\mathbb{V}(\frac{1}{b_n} T_n \in \cdot), n \geq 1\right\}$ satisfies the MDP with rate function $I(x)$ and speed n/b_n^2, that is, for any closed set $F \in \mathcal{B}(\mathbb{R})$,

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{1}{b_n} T_n \in F\right) \leq -\inf_{x\in F} I(x),$$

for any open set $G \in \mathcal{B}(\mathbb{R})$,

$$\liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{1}{b_n} T_n \in G\right) \geq -\inf_{x\in G} I(x),$$

where

$$I(x) = \sup_{y\in\mathbb{R}}\left\{xy - \frac{\sigma^2 a^2 y^2}{2}\right\} = \frac{x^2}{2\sigma^2 a^2}, \quad \forall x \in \mathbb{R},$$

and $\{b_n, n \geq 1\}$ is defined as in Equation (3) and $T_n = \sum_{k=1}^{n} Y_k,\ n \geq 1$.

Proof. For $m, k \geq 1$, denote

$$\begin{aligned}
&d_m = \sum_{j=-m}^{m} a_j, \quad Y_{k,m} = \sum_{j=-m}^{m} a_j X_{k-j}, \\
&\widetilde{a}_m = 0, \ \ \widetilde{a}_j = \sum_{i=j+1}^{m} a_i, \quad j = 0, 1, 2, \cdots, m-1, \\
&\widetilde{\widetilde{a}}_{-m} = 0, \ \ \widetilde{\widetilde{a}}_j = \sum_{i=-m}^{j-1} a_i, \quad j = -m+1, -m+2, \cdots, 0, \\
&\widetilde{X_k} = \sum_{j=0}^{m} \widetilde{a}_j X_{k-j}, \quad \widetilde{\widetilde{X}}_k = \sum_{j=-m}^{0} \widetilde{\widetilde{a}}_j X_{k-j}.
\end{aligned}$$

By the above notation, it is obvious that

$$\begin{aligned}
Y_{k,m} &= \left(\sum_{j=-m}^{m} a_j\right) X_k - \left(\sum_{j=1}^{m} a_j\right) X_k + \sum_{j=1}^{m} a_j X_{k-j} - \left(\sum_{j=-m}^{-1} a_j\right) X_k + \sum_{j=-m}^{-1} a_j X_{k-j} \\
&= \left(\sum_{j=-m}^{m} a_j\right) X_k - \widetilde{a}_0 X_k + \sum_{j=1}^{m} (\widetilde{a}_{j-1} - \widetilde{a}_j) X_{k-j} - \widetilde{\widetilde{a}}_0 X_k + \sum_{j=-m}^{-1} (\widetilde{\widetilde{a}}_{j+1} - \widetilde{\widetilde{a}}_j) X_{k-j} \\
&= d_m X_k + \widetilde{X}_{k-1} - \widetilde{X}_k + \widetilde{\widetilde{X}}_{k+1} - \widetilde{\widetilde{X}}_k.
\end{aligned} \tag{7}$$

Then, by Equation (7), we have the following Beveridge–Nelson decomposition of linear processes

$$
\begin{aligned}
T_n = \sum_{k=1}^{n} Y_k = \sum_{k=1}^{n} Y_{k,m} + \left(\sum_{k=1}^{n} \sum_{|j|>m} a_j X_{k-j} \right) \\
= d_m \left(\sum_{k=1}^{n} X_k \right) + (\widetilde{X}_0 - \widetilde{X}_n) + (\widetilde{\widetilde{X}}_{n+1} - \widetilde{\widetilde{X}}_1) + \sum_{k=1}^{n} \sum_{|j|>m} a_j X_{k-j}.
\end{aligned} \tag{8}
$$

Theorem 1 will be proved by the following two steps.
Step 1 The upper bound of the MDP.
In this step, we want to show

$$
\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{1}{b_n} T_n \in F \right) \le - \inf_{x\in F} I(x), \tag{9}
$$

for any closed set $F \in \mathcal{B}(\mathbb{R})$.
By Equation (8) and Lemma 2, for every $y \in \mathbb{R}$, $m \ge 1$, $\epsilon > 0$, we obtain

$$
\begin{aligned}
\widetilde{\Lambda}(y) &:= \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E} \exp\left(y \frac{b_n}{n} T_n \right) \\
&\le \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{ y d_m \frac{b_n}{n} \left(\sum_{k=1}^{n} X_k \right) + |y| \frac{b_n}{n} |\widetilde{X}_0| + |y| \frac{b_n}{n} |\widetilde{X}_n| \right. \\
&\quad \left. + |y| \frac{b_n}{n} |\widetilde{\widetilde{X}}_{n+1}| + |y| \frac{b_n}{n} |\widetilde{\widetilde{X}}_1| + |y| \frac{b_n}{n} \left| \sum_{k=1}^{n} \sum_{|j|>m} a_j X_{k-j} \right| \right\} \\
&\le \limsup_{n\to\infty} \frac{1}{1+\epsilon} \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{ (1+\epsilon) y d_m \frac{b_n}{n} \left(\sum_{k=1}^{n} X_k \right) \right\} \\
&\quad + \limsup_{n\to\infty} \frac{\epsilon}{5(1+\epsilon)} \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{ \frac{5(1+\epsilon)}{\epsilon} |y| \frac{b_n}{n} |\widetilde{X}_0| \right\} \\
&\quad + \limsup_{n\to\infty} \frac{\epsilon}{5(1+\epsilon)} \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{ \frac{5(1+\epsilon)}{\epsilon} |y| \frac{b_n}{n} |\widetilde{X}_n| \right\} \\
&\quad + \limsup_{n\to\infty} \frac{\epsilon}{5(1+\epsilon)} \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{ \frac{5(1+\epsilon)}{\epsilon} |y| \frac{b_n}{n} |\widetilde{\widetilde{X}}_{n+1}| \right\} \\
&\quad + \limsup_{n\to\infty} \frac{\epsilon}{5(1+\epsilon)} \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{ \frac{5(1+\epsilon)}{\epsilon} |y| \frac{b_n}{n} |\widetilde{\widetilde{X}}_1| \right\} \\
&\quad + \limsup_{n\to\infty} \frac{\epsilon}{5(1+\epsilon)} \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{ \frac{5(1+\epsilon)}{\epsilon} |y| \frac{b_n}{n} \left| \sum_{k=1}^{n} \sum_{|j|>m} a_j X_{k-j} \right| \right\} \\
&:= E_1 + E_2 + E_3 + E_4 + E_5 + E_6.
\end{aligned} \tag{10}
$$

By Lemma 3.2 from [19], we can obtain

$$
\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E} \exp\left(y \frac{b_n}{n} \sum_{k=1}^{n} X_k \right) = \frac{\sigma^2 y^2}{2}. \tag{11}
$$

Then, following Equation (11), one can get

$$
E_1 = \frac{1}{1+\epsilon} \frac{\sigma^2}{2} [(1+\epsilon) y d_m]^2 = \frac{(1+\epsilon)\sigma^2 y^2 d_m^2}{2}. \tag{12}
$$

Next, we will consider the second term E_2. It follows from the definition of $\widetilde{X}_0$, Lemma 3 and Equation (3) with Equation (5) that

$$\begin{aligned}
E_2 &\leq \limsup_{n\to\infty} \frac{\epsilon}{5(1+\epsilon)} \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{ \frac{5(1+\epsilon)}{\epsilon} |y| \frac{b_n}{n} \sum_{j=0}^{m-1} \sum_{i=j+1}^{m} |a_i||X_{-j}| \right\} \\
&\leq \limsup_{n\to\infty} \frac{\epsilon}{5(1+\epsilon)} \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{ \frac{5(1+\epsilon)}{\epsilon} |y| \frac{b_n}{n} |X_1| \sum_{j=0}^{m-1} \sum_{i=j+1}^{m} |a_i| \right\} \\
&\leq \limsup_{n\to\infty} \frac{\epsilon}{5(1+\epsilon)} \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{ \frac{5(1+\epsilon)}{\epsilon} |y| m \sum_{i=-\infty}^{+\infty} |a_i| \frac{b_n}{n} |X_1| \right\} \\
&= 0.
\end{aligned}$$

Therefore, by the same arguments as in E_2, we have

$$E_i \leq 0, \quad 2 \leq i \leq 5. \tag{13}$$

Next, we will estimate the last term E_6. Combining Lemmas 3 and 5, the stationarity of $\{X_k\}$, $e^{|x|} \leq e^x + e^{-x}$, $x \in \mathbb{R}$ and Equation (11), one can obtain

$$\begin{aligned}
E_6 &\leq \limsup_{n\to\infty} \frac{\epsilon}{5(1+\epsilon)} \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{ \frac{5(1+\epsilon)}{\epsilon} |y| \frac{b_n}{n} c_m \left| \sum_{k=1}^{n} X_{k-j} \right| \right\} \\
&\leq \limsup_{n\to\infty} \frac{\epsilon}{5(1+\epsilon)} \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{ \frac{5(1+\epsilon)}{\epsilon} |y| \frac{b_n}{n} \left| \sum_{k=1}^{n} X_k \right| c_m \right\} \\
&\leq \limsup_{n\to\infty} \frac{\epsilon}{5(1+\epsilon)} \frac{n}{b_n^2} \log \left[\mathbb{E} \exp\left\{ y \frac{5(1+\epsilon)}{\epsilon} \frac{b_n}{n} \sum_{k=1}^{n} X_k c_m \right\} \right. \\
&\quad \left. + \mathbb{E} \exp\left\{ -y \frac{5(1+\epsilon)}{\epsilon} \frac{b_n}{n} \sum_{k=1}^{n} X_k c_m \right\} \right] \\
&= \limsup_{n\to\infty} \frac{\epsilon}{5(1+\epsilon)} \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{ y \frac{5(1+\epsilon)}{\epsilon} \frac{b_n}{n} \sum_{k=1}^{n} X_k c_m \right\} \\
&\quad \vee \limsup_{n\to\infty} \frac{\epsilon}{5(1+\epsilon)} \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{ -y \frac{5(1+\epsilon)}{\epsilon} \frac{b_n}{n} \sum_{k=1}^{n} X_k c_m \right\} \\
&= \frac{\epsilon}{5(1+\epsilon)} \frac{\sigma^2}{2} \left\{ y \frac{5(1+\epsilon)}{\epsilon} c_m \right\}^2 \vee \frac{\epsilon}{5(1+\epsilon)} \frac{\sigma^2}{2} \left\{ -y \frac{5(1+\epsilon)}{\epsilon} c_m \right\}^2 \\
&= \frac{5(1+\epsilon)}{\epsilon} \frac{\sigma^2}{2} c_m^2.
\end{aligned} \tag{14}$$

$\sum_{j=-\infty}^{+\infty} |a_j| < \infty$ implies

$$\lim_{m\to\infty} d_m = a, \quad \lim_{m\to\infty} c_m = \sum_{|j|>m} |a_j| = 0. \tag{15}$$

Moreover, it follows from Equations (10), (12) and (13) with Equation (15) that

$$\begin{aligned}
\widetilde{\Lambda}(y) &\leq \lim_{\epsilon\to 0} \lim_{m\to\infty} [E_1 + E_2 + E_3 + E_4 + E_5 + E_6] \\
&\leq \lim_{\epsilon\to 0} \lim_{m\to\infty} \left[\frac{(1+\epsilon)\sigma^2 y^2 d_m^2}{2} + \frac{5(1+\epsilon)}{\epsilon} \frac{\sigma^2}{2} c_m^2 \right] \\
&= \lim_{\epsilon\to 0} \frac{(1+\epsilon)\sigma^2 y^2 a^2}{2} = \frac{\sigma^2 y^2 a^2}{2} < \infty.
\end{aligned} \tag{16}$$

Then, by Equation (16), one can calculate that

$$\widetilde{\Lambda}^*(x) = \sup_{y\in\mathbb{R}}\left\{xy - \widetilde{\Lambda}(y)\right\} \geq \sup_{y\in\mathbb{R}}\left\{xy - \frac{\sigma^2 y^2 a^2}{2}\right\} = \frac{x^2}{2\sigma^2 a^2} = I(x). \tag{17}$$

Therefore, the proof of Equation (9) is complete by combining Equation (17) with Lemma 6.

Step 2 The lower bound of the MDP.

In order to prove the lower bound of the MDP, it suffices to show that

$$\liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{1}{B_n} T_n \in B(x,\ \delta)\right) \geq -\frac{x^2}{2\sigma^2 a^2} = -I(x), \tag{18}$$

where $B(x,\ \delta) = \{y;\ |x-y| < \delta\} \in G$, $\delta > 0$ and $G \in \mathcal{B}(\mathbb{R})$ is an open set.

In fact, Equation (18) implies

$$\liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{1}{b_n} T_n \in G\right) \geq \liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{1}{b_n} T_n \in B(x,\ \delta)\right) \geq -\frac{x^2}{2\sigma^2 a^2} = -I(x).$$

Thus, by the arbitrariness of x, one can get

$$\liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{1}{b_n} T_n \in G\right) \geq -\inf_{x\in G} \frac{x^2}{2\sigma^2 a^2} = -\inf_{x\in G} I(x).$$

By Lemma 4, we know that $\{\frac{1}{b_n}\sum_{k=1}^n X_k, n \geq 1\}$ satisfies the MDP with rate function $I_1(x) = \frac{x^2}{2\sigma^2}$; hence,

$$\begin{aligned}
\liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(d_m \frac{1}{b_n} \sum_{k=1}^n X_k \in B\left(x,\ \frac{\delta}{6}\right)\right) &\geq -\inf_{d_m y \in B\left(x,\ \frac{\delta}{6}\right)} \left\{\frac{y^2}{2\sigma^2}\right\} \\
&= -\inf_{\frac{d_m y}{a} \in B\left(x,\ \frac{\delta}{6}\right)} \left\{\frac{y^2}{2\sigma^2 a^2}\right\} \\
&= -\inf_{y\in B\left(\frac{ax}{d_m},\ \frac{|a|\delta}{6|d_m|}\right)} \left\{\frac{y^2}{2\sigma^2 a^2}\right\}.
\end{aligned}$$

Following Equation (15), we can obtain $x \in B(ax/d_m,\ |a|\delta/(6|d_m|))$ for sufficiently large m. Therefore,

$$\liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(d_m \frac{1}{b_n} \sum_{k=1}^n X_k \in B\left(x,\ \frac{\delta}{6}\right)\right) \geq -\frac{x^2}{2\sigma^2 a^2} = -I(x). \tag{19}$$

We can show, by triangle inequality and Equation (8), that

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(d_m \frac{1}{b_n} \sum_{k=1}^n X_k \in B\left(x,\ \frac{\delta}{6}\right)\right) \\
&= \liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\left(\frac{T_n}{b_n} - \frac{(\widetilde{X}_0 - \widetilde{X}_n)}{b_n} - \frac{(\widetilde{\widetilde{X}}_{n+1} - \widetilde{\widetilde{X}}_1)}{b_n} - \frac{\sum_{k=1}^n \sum_{|j|>m} \alpha_j X_{k-j}}{b_n}\right) \in B\left(x,\ \frac{\delta}{6}\right)\right) \\
&\leq \liminf_{n\to\infty} \frac{n}{b_n^2} \log \left\{\mathbb{V}\left(\frac{T_n}{b_n} \in B(x,\ \delta)\right) + \mathbb{V}\left(\frac{|\widetilde{X}_0|}{b_n} \geq \frac{\delta}{6}\right) + \mathbb{V}\left(\frac{|\widetilde{X}_n|}{b_n} \geq \frac{\delta}{6}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{V}\left(\frac{|\widetilde{\widetilde{X}}_{n+1}|}{b_n}\geq\frac{\delta}{6}\right)+\mathbb{V}\left(\frac{|\widetilde{\widetilde{X}}_1|}{b_n}\geq\frac{\delta}{6}\right)+\mathbb{V}\left(\frac{|\sum_{k=1}^{n}\sum_{|j|>m}\alpha_jX_{k-j}|}{b_n}\geq\frac{\delta}{6}\right)\Bigg\}\\
\leq{}& \liminf_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{V}\left(\frac{T_n}{b_n}\in B(x,\,\delta)\right)\\
&\vee\limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{V}\left(\frac{|\widetilde{X}_0|}{b_n}\geq\frac{\delta}{6}\right)\vee\limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{V}\left(\frac{|\widetilde{X}_n|}{b_n}\geq\frac{\delta}{6}\right)\\
&\vee\limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{V}\left(\frac{|\widetilde{\widetilde{X}}_{n+1}|}{b_n}\geq\frac{\delta}{6}\right)\vee\limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{V}\left(\frac{|\widetilde{\widetilde{X}}_1|}{b_n}\geq\frac{\delta}{6}\right)\\
&\vee\limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{V}\left(\frac{1}{b_n}|\sum_{k=1}^{n}\sum_{|j|>m}a_jX_{k-j}|\geq\frac{\delta}{6}\right)\\
=:{}&F_1\vee F_2\vee F_3\vee F_4\vee F_5\vee F_6.
\end{aligned}\tag{20}
$$

Thus, by Equation (20), in order to show Equation (18), it suffices to calculate

$$
F_i\leq-\frac{x^2}{2\sigma^2a^2}-1,\ 2\leq i\leq 6.\tag{21}
$$

Taking $y_0\geq 6(\frac{x^2}{2\sigma^2a^2}+1)/\delta$, by Lemma 1 and the same discussion as in E_2, it is easily seen that

$$
\begin{aligned}
F_2&=\limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{V}\left(\frac{1}{b_n}|\widetilde{X}_0|\geq\frac{\delta}{6}\right)\\
&=\limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{V}\left(\frac{y_0b_n}{n}|\widetilde{X}_0|\geq\frac{y_0\delta b_n^2}{6n}\right)\\
&\leq\limsup_{n\to\infty}\frac{n}{b_n^2}\log\left(\exp\{-\frac{y_0\delta b_n^2}{6n}\}\mathbb{E}\left[\exp\{\frac{y_0b_n}{n}|\widetilde{X}_0|\}\right]\right)\\
&\leq-\frac{y_0\delta}{6}+\limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\left[e^{\frac{y_0b_n}{n}|\widetilde{X}_0|}\right]\\
&=-\frac{y_0\delta}{6}\leq-\frac{x^2}{2\sigma^2a^2}-1.
\end{aligned}
$$

Therefore, by the same arguments as in F_2, we have

$$
F_i\leq-\frac{x^2}{2\sigma^2a^2}-1,\quad 2\leq i\leq 5.\tag{22}
$$

Finally, we want to show that $F_6\leq-\frac{x^2}{2\sigma^2a^2}-1$. Taking $y_0\geq 6(\frac{x^2}{2\sigma^2a^2}+2)/\delta$ and following Lemma 1 and the same argument as in Equation (14), then

$$
\begin{aligned}
J_6&=\limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{V}\left(\frac{1}{b_n}|\sum_{k=1}^{n}\sum_{|j|>m}a_jX_{k-j}|\geq\frac{\delta}{6}\right)\\
&=\limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{V}\left(\frac{y_0b_n}{n}|\sum_{k=1}^{n}\sum_{|j|>m}a_jX_{k-j}|\geq\frac{y_0\delta b_n^2}{6n}\right)
\end{aligned}
$$

$$\leq \limsup_{n\to\infty} \frac{n}{b_n^2} \log\left(\exp\{-\frac{y_0\delta b_n^2}{6n}\}\mathbb{E}\left[\exp\{\frac{y_0 b_n}{n}|\sum_{k=1}^{n}\sum_{|j|>m} a_j X_{k-j}|\}\right]\right)$$
$$\leq -\frac{y_0\delta}{6} + \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\left[\exp\{\frac{y_0 b_n}{n}|\sum_{k=1}^{n}\sum_{|j|>m} a_j X_{k-j}|\}\right]$$
$$\leq -\frac{x^2}{2\sigma^2 a^2} - 2 + \frac{\sigma^2}{2}\left\{\sum_{|j|>m}|a_j|\right\}^2. \tag{23}$$

Noting that $\sum_{|j|>m}|a_j| \to 0,\ m \to \infty$, then, by Equation (23), for sufficiently large m, one can get

$$F_6 \leq -\frac{x^2}{2\sigma^2 a^2} - 1. \tag{24}$$

Thus, the above arguments yield Equation (21). Therefore, the proof is complete. □

Remark 1. *If taking $a_0 = 1,\ a_j = 0,\ j \neq 0$ in Theorem 1, then $\{Y_k, k \geq 1\}$ is a special linear process. Thus, we need to emphasize that*

$$\mathbb{V}\left(\frac{1}{b_n}\sum_{k=1}^{n} X_k \in \cdot\right), \quad n \geq 1,$$

satisfies the MDP for an m-dependent sequence, that is Theorem 3.1 from [19]. Thus, our theorem extends the known result.

5. The Upper Bound of the MDP for Linear Processes Generated by an ND Sequence

In this section, we will discuss the upper bound of the MDP for linear processes generated by an ND sequence. The method of proof is different from that used in the above section depending on the existence of a logarithmic moment generating function; however, the limit of the logarithmic moment generating function of an ND sequence may not exist. Therefore, we adopt another method to prove the upper bound of the MDP.

Theorem 2. *Assume that $\{X_k,\ k \in \mathbb{Z}\}$ is a strictly stationary ND random variable sequence on $(\Omega, \mathcal{H}, \mathbb{E})$ satisfying*

$$\mathbb{E}[X_1] = \mathbb{E}[-X_1] = 0, \quad \mathbb{E}[|X_1|^{2+\eta}] < \infty, \tag{25}$$

and

$$\mathbb{E}[e^{t|X_1|}] < \infty, \tag{26}$$

for some $\eta > 0$ and all $t \in \mathbb{R}$. Suppose that

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log\left(\sum_{i=1}^{n} \mathbb{V}(|X_i| > b_n)\right) = -\infty, \tag{27}$$

and moreover, for all $t \in \mathbb{R}$ and $n \geq 1$,

$$\mathbb{E}\left[\exp\left(t\sum_{i=1}^{n} X_i I_{\{|X_i|<\frac{n}{b_n}\}}\right)\right] = \left(\mathbb{E}\left[\exp\left(t\sum_{i=1}^{n} X_1 I_{\{|X_1|<\frac{n}{b_n}\}}\right)\right]\right)^n. \tag{28}$$

Furthermore, let $\{a_j,\ j \in \mathbb{Z}\}$ be sequence of real numbers satisfying $\sum_{j=-\infty}^{+\infty}|a_j| < \infty$ and $a = \sum_{j=-\infty}^{+\infty} a_j \neq 0$ and $\{Y_k, k \geq 1\}$ be a linear process defined by Equation (6) and partial sum

$T_n = \sum_{k=1}^{n} Y_k$, $n \geq 1$. Then, $\left\{\mathbb{V}(\frac{1}{b_n}T_n \in \cdot), n \geq 1\right\}$ satisfies the upper bound of the MDP with speed n/b_n^2 and rate function $J(x/a)$, that is, for any closed set $F \in \mathcal{B}(\mathbb{R})$,

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{1}{b_n}T_n \in F\right) \leq -\inf_{x\in F} J(x/a), \tag{29}$$

where $\{b_n, n \geq 1\}$ is defined as in Equation (3) and

$$J(x) = \sup_{y\in\mathbb{R}}\left\{xy - \frac{y^2}{2}\mathbb{E}[X_1^2]\right\} = \frac{x^2}{2\mathbb{E}[X_1^2]}.$$

Proof. We will adopt the same notation defined in the proof of Theorem 1. By Lemma 7, in order to prove Equation (29), we only need to show that for all $\varepsilon > 0$

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{T_n}{b_n} - \frac{d_m}{b_n}\left(\sum_{k=1}^{n} X_k I_{\{|X_i| < \frac{n}{b_n}\}}\right) > \varepsilon\right) = -\infty \tag{30}$$

and

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{d_m}{b_n}\left(\sum_{k=1}^{n} X_k I_{\{|X_i| < \frac{n}{b_n}\}}\right) \in F\right) \leq -\inf_{x\in F} J(x/a). \tag{31}$$

Next, we will prove Equation (30). By Equation (8), it suffices to show that

$$G_1 := \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{d_m}{b_n}\left(\sum_{k=1}^{n} X_k I_{\{|X_i| \geq \frac{n}{b_n}\}}\right) > \varepsilon/6\right) = -\infty, \tag{32}$$

$$G_2 := \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{1}{b_n}\widetilde{X}_0 > \varepsilon/6\right) = -\infty, \tag{33}$$

$$G_3 := \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{1}{b_n}\widetilde{X}_n > \varepsilon/6\right) = -\infty, \tag{34}$$

$$G_4 := \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{1}{b_n}\widetilde{\widetilde{X}}_1 > \varepsilon/6\right) = -\infty, \tag{35}$$

$$G_5 := \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{1}{b_n}\widetilde{\widetilde{X}}_{n+1} > \varepsilon/6\right) = -\infty, \tag{36}$$

$$G_6 := \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{1}{b_n}\sum_{k=1}^{n}\sum_{|j|>m} a_j X_{k-j} > \varepsilon/6\right) = -\infty. \tag{37}$$

By Equation (26) and by the same argument as in page 409–410 of [15], one can get

$$G_1 = \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{d_m}{b_n}\left(\sum_{k=1}^{n} X_k I_{\{|X_i| \geq \frac{n}{b_n}\}}\right) > \varepsilon/6\right) = -\infty. \tag{38}$$

By the definition of $\widetilde{X}_0$, Lemma 3 and Equation (5), for every $\beta > 1$, we have

$$\begin{aligned}
G_2 &= \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{1}{b_n} \sum_{j=0}^{m-1} \sum_{i=j+1}^{m} a_i X_{-j} > \varepsilon/6 \right) \\
&\le \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{\beta}{b_n} \sum_{j=0}^{m-1} \sum_{i=j+1}^{m} |a_i||X_{-j}| > \varepsilon\beta/6 \right) \\
&\le \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{\beta}{b_n}|X_1| \sum_{j=0}^{m-1} \sum_{i=j+1}^{m} |a_i| > \varepsilon\beta/6 \right) \\
&\le \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{\beta}{b_n} m \sum_{i=-\infty}^{+\infty} |a_i||X_1| > \varepsilon\beta/6 \right) \\
&= \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{\beta b_n}{n} m \sum_{i=-\infty}^{+\infty} |a_i||X_1| > \frac{\varepsilon\beta b_n^2}{6n} \right) \\
&\le \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log\left\{ \exp\{-\frac{\varepsilon\beta b_n^2}{6n}\} \mathbb{E}[\exp\{\frac{\beta b_n}{n} m \sum_{i=-\infty}^{+\infty} |a_i||X_1|\}] \right\} \\
&= -\frac{\varepsilon\beta}{6} + \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}[\exp\{\frac{\beta b_n}{n} m \sum_{i=-\infty}^{+\infty} |a_i||X_1|\}] \\
&= -\frac{\varepsilon\beta}{6}.
\end{aligned}$$

By the arbitrariness of β, let $\beta \to \infty$, then we can obtain $G_2 = -\infty$. Since the arguments of $G_2 - G_5$ are essentially similar, we obtain the following:

$$G_i = -\infty, \quad 2 \le i \le 5. \tag{39}$$

By Lemmas 1, 3 and 5, noting that $e^{|x|} \le e^x + e^{-x},\ x \in \mathbb{R}$ and $e^x = 1 + x + x^2 + o(x^2), x \to 0$ and $\log(1+x) \le x, x \ge 0$ and $c_m = \sum_{|j|>m} |a_j| \to 0,\ m \to \infty$, the definitions of ND and stationary, then one can obtain

$$\begin{aligned}
G_6 &\le \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\frac{\beta b_n c_m}{n} |\sum_{k=1}^{n} X_{k-j}| > \frac{\varepsilon\beta b_n^2}{6n} \right) \\
&\le \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log\left\{ \exp\{-\frac{\varepsilon\beta b_n^2}{6n}\} \mathbb{E}[\exp\{\frac{\beta b_n c_m}{n} |\sum_{k=1}^{n} X_{k-j}|\}] \right\} \\
&\le -\frac{\varepsilon\beta}{6} + \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log\left\{ \mathbb{E}[\exp\{\frac{\beta b_n c_m}{n} \sum_{k=1}^{n} X_k|\}] \right\} \\
&\le -\frac{\varepsilon\beta}{6} + \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log\left\{ \mathbb{E}[\exp\{\frac{\beta b_n c_m}{n} \sum_{k=1}^{n} X_k\} + \exp\{-\frac{\beta b_n c_m}{n} \sum_{k=1}^{n} X_k\}] \right\} \\
&\le -\frac{\varepsilon\beta}{6} + \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log\left\{ \mathbb{E}[\exp\{\frac{\beta b_n c_m}{n} \sum_{k=1}^{n} X_k\}] \right\} \\
&\quad \vee \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n}{b_n^2} \log\left\{ \mathbb{E}[\exp\{-\frac{\beta b_n c_m}{n} \sum_{k=1}^{n} X_k\}] \right\} \\
&\le -\frac{\varepsilon\beta}{6} + \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n^2}{b_n^2} \log\left\{ \mathbb{E}[\exp\{\frac{\beta b_n c_m}{n} X_1\}] \right\} \\
&\quad \vee \lim_{m\to\infty} \limsup_{n\to\infty} \frac{n^2}{b_n^2} \log\left\{ \mathbb{E}[\exp\{-\frac{\beta b_n c_m}{n} X_1\}] \right\}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\varepsilon\beta}{6} + \lim_{m\to\infty}\limsup_{n\to\infty}\frac{n^2}{b_n^2}\log\left\{\mathbb{E}[1+\frac{\beta b_n c_m}{n}X_1+\frac{\beta^2 b_n^2 c_m^2}{2n^2}X_1^2+o(\frac{b_n^2 c_m^2}{n^2}X_1^2)]\right\} \\
&\quad \vee \lim_{m\to\infty}\limsup_{n\to\infty}\frac{n^2}{b_n^2}\log\left\{\mathbb{E}[1-\frac{\beta b_n c_m}{n}X_1+\frac{\beta^2 b_n^2 c_m^2}{2n^2}X_1^2+o(\frac{b_n^2 c_m^2}{n^2}X_1^2)]\right\} \\
&\le -\frac{\varepsilon\beta}{6} + \lim_{m\to\infty}\limsup_{n\to\infty}\frac{n^2}{b_n^2}\log\left\{1+\frac{\beta^2 b_n^2 c_m^2}{2n^2}\mathbb{E}[X_1^2]+o(\frac{b_n^2 c_m^2}{n^2})\right\} \\
&\le -\frac{\varepsilon\beta}{6} + \lim_{m\to\infty}\left\{\frac{\beta^2}{2}c_m^2\mathbb{E}[X_1^2]+o(c_m^2)\right\} \\
&= -\frac{\varepsilon\beta}{6}.
\end{aligned}
$$

Moreover, by the arbitrariness of β, let $\beta \to \infty$. Then, we can obtain

$$G_6 = -\infty. \tag{40}$$

Then, Equation (30) follows by putting Equations (32)–(41) together.

Finally, we want to prove Equation (31). Using the same arguments as in page 410–411 of [15], one can get

$$\lim_{n\to\infty}\frac{n}{b_n^2}\log\left\{\mathbb{E}[\exp\{\frac{td_m b_n}{n}\sum_{k=1}^{n}X_k I_{\{|X_i|<\frac{n}{b_n}\}}\}]\right\} = \frac{t^2 d_m^2}{2}\mathbb{E}[X_1^2] < \infty. \tag{41}$$

Then, by Equation (41) and Lemma 6, one can get

$$
\begin{aligned}
\limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{V}\left(\frac{d_m}{b_n}\left(\sum_{k=1}^{n}X_k I_{\{|X_i|<\frac{n}{b_n}\}}\right)\in F\right) &\le -\inf_{x\in F}\sup_{t\in\mathbb{R}}\left\{xt-\frac{t^2 d_m^2}{2}\mathbb{E}[X_1^2]\right\} \\
&= -\inf_{x\in F}\frac{x^2}{2d_m^2\mathbb{E}[X_1^2]}.
\end{aligned} \tag{42}
$$

Noting $d_m \to a$ as $m \to \infty$, Equation (31) follows by letting $m \to \infty$ in Equation (42). The proof is complete. □

Remark 2. *If taking $a_0 = 1$, $a_j = 0$, $j \neq 0$ in Theorem 2, then $\{Y_k, k \ge 1\}$ is a simple linear process. Then,*

$$\mathbb{V}\left(\frac{1}{b_n}\sum_{k=1}^{n}X_k \in \cdot\right), \quad n \ge 1,$$

satisfies the upper bound of the MDP, that is Theorem 4.1 in [15]. Thus, our theorem extends the known result.

Remark 3. *In this section, we obtain the upper bound of the MDP for linear processes generated by an ND sequence; the lower bound of MDP leaves the problem open. One of the methods to prove the lower bound is the central limit theorem. To our knowledge, the central limit theorem for ND random variables under sub-linear expectation is not yet detailed; we will consider this problem in future work.*

6. Conclusions

In this paper, using the Beveridge–Nelson decomposition of linear processes and the inequalities under sub-linear expectation, the authors establish the MDP for linear processes produced by an m-dependent sequence and the upper bound of the MDP for linear processes produced by an ND sequence. The results extend the MDP from the traditional probability space to the sub-linear expectation space. Furthermore, they also extend the MDP to linear processes. In the future, we will try to establish the lower bound of the MDP for ND sequences and other dependent sequences under sub-linear expectations.

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